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STABILITY OF HIGH ORDER POLYNOMIAL DYNAMICS AND MINIMUM PHASE DISCRETE-TIME SYSTEMS*

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Abstract

The definition of a minimum phase nonlinear system, as usually found in the literature, is not general enough to be used for some classes of systems. The nonlinearity may yield a variety of different behaviors that are not addressed and analyzed in the literature. We provide a constructive method to test several different minimum phase properties for classes of explicit and implicit discrete-time polynomial systems. The method is based on a symbolic computation package called QEPCAD. Our results can also be interpreted as a constructive approach to stability and stabilizability of explicit and implicit discrete-time polynomial systems.

1 Introduction

The minimum phase property of nonlinear systems plays an important role for a number of control problems, such as I-O linearization and output dead-beat control [7, 9, 4]. It appears that the notion of minimum phase systems in the nonlinear context has inherited from linear theory not only its name but also the definition which tries to mimic and capture the behavior which is typical of linear systems. In order to exploit the possibilities that the nonlinearity has to offer, we introduce a number of different definitions for minimum phase systems. Some of them are based on [1] and some generalize the definition found in [9].

In this paper we consider input-output discrete-time polynomial system [5] of the form

$$y(k+1) = F(y(k), \dots, y(k-s), u(k-t), \dots, u(k)) \quad (1)$$

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where $y(k) \in \mathbb{R}$ and $u(k) \in \mathbb{R}$ are respectively output and input of the system at time instant k. The integers s and t are positive and the nonlinearity F is a polynomial in all its arguments. They represent a natural generalization of linear ARMAX models and can be identified using the so called identification techniques for block oriented models [5]. Models for a large number of systems, such as a heat exchanger [5], can be identified in this form.

It turns out that for these systems one needs to consider *a constrained stabilizability problem* and not a stability problem when checking minimum phaseness. Indeed, if we assume that for the above given system the output is kept at a constant value y^* we obtain:

$$y^* = F(y^*, y^*, \dots, y^*, u(k-t), \dots, u(k)) \quad (2)$$

which represents an *implicit difference equation.* Notice that in general at each time k, the above given equation may have several possible $u(k)$ that satisfy it. This fact in general gives us a possibility to choose the most appropriate value $u(k)$ at each time step. Hence, there is some design freedom that one can exploit when dealing with the "stability" of zero dynamics. We refer to [1] as a first reference in which this issue has been raised.

Here, we concentrate on the problem of checking minimum phase properties for classes of explicit and implicit discrete-time polynomial systems. Our method is based on the use of a software package for quantifier elimination in the first-order theory of real closed fields, which is called QEPCAD [2, 3]. We have recently shown how QEPCAD can be used to decide state and output dead-beat controllability of polynomial systems [11, 10]. The package has been used for number of other linear and nonlinear control theoretic problems and it can in a very natural way tackle constraints on controls and states. The price we pay for such a general methodology is the increase in the computational cost, which is in general prohibitive. At the moment only modest problems can be solved using the algorithm. However, by constraining the structure of general polynomial systems one can solve problems for which no other analytic results or a method exist.

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2 Preliminaries

QEPCAD is a symbolic computation package for quantifier elimination (QE) in the first order theory of real closed fields. It is based on the cylindrical algebraic decomposition (CAD) algorithm [2, 3]. The input to the algorithm is an expression which consists of polynomial equations and inequalities, Boolean operators and ($\wedge$), or ($\vee$), implies ($\rightarrow$) and not ($\neg$), as well as quantifiers $\exists$ and $\forall$. The solution (output formula) is a quantifier free expression. We note that all variables are assumed to be real. For more on QEPCAD see [2, 3, 8]. The simple examples we present illustrate the operation of the algorithm.

Example 1 If the input expression is $(\exists t)[a_2t^2 + a_1t + a_0 = 0]$, the output expression $(a_1^2 - 4a_2a_0 \geq 0)$ is obtained.

Example 2 [8] If the input expression is $(\forall x)(\forall y)[(x^2 + y^2 < 1) \rightarrow (y > x^4 - 2)]$, the output expression $TRUE$ is obtained.

A set $S \subset \mathbb{R}^n$ is called semi-algebraic if it can be constructed by finitely many applications of union, intersection and complementation operations on sets of the form $\{x \in \mathbb{R}^n : f_i(x) \geq 0\}$, where f_i are polynomials in x with real coefficients. Given a semi-algebraic set S we denote its defining expression as $S(x)$.

Euclidean norm of a vector x is denoted as $||x||$. The distance between points $x, y \in \mathbb{R}^t$ and the distance between a set $A \subset \mathbb{R}^t$ and a point $x \in \mathbb{R}^t$ are respectively denoted as $d(x, y) = ||x-y||$ and $d(x, A) = \inf_{y \in A} ||x-y||$. A hyper-ball centred at a point z^* with a radius $p > 0$ is denoted as

$$\mathcal{B}_p(z^*) = \{z \in \mathbb{R}^t : d(z, z^*) < p\}$$

and the hyper-cube centered at a point z^* with sides $2r > 0$ is denoted as

$$\mathcal{C}_r(z^*) = \{z : (|z_1 - \zeta| < r) \wedge \ldots \wedge (|z_t - \zeta| < r)\}$$

where $|a|$ is the absolute value of the scalar a.

We denote $F(y^*, \ldots, y^*, u(k-t), \ldots, u(k)) - y^*$ as $G(u(k-t), \ldots, u(k))$ and consider implicit recursive polynomial dynamics:

$$G(u(k-t), \ldots, u(k)) = 0 \tag{3}$$

A subclass of the implicit dynamics (3) are explicit dynamics:

$$u(k) = G_e(u(k-t), \ldots, u(k-1)) \tag{4}$$

Definition 1 A criterion of choice is a single valued function $c : \mathbb{R}^t \rightarrow \mathbb{R}$ (denoted also as $u(k) = c(u(k-t), \ldots, u(k-1)))$ such that

$$G(u(k-t), \ldots, u(k-1), c(u(k-t), \ldots, u(k-1))) = 0, \\ \forall u(k-1), \ldots, u(k-t) \in \mathbb{R}. \tag{5}$$

We introduce state variables $z_i = u(k-t-1+i), i = 1, \ldots, t$ and obtain the system:

$$\begin{aligned} z_1(k+1) &= z_2(k) \\ \ldots &\quad \ldots \\ z_t(k+1) &= u(k) \end{aligned} \tag{6}$$

with the constraint $G(z_1(k), \ldots, z_t, u(k)) = 0$. This system can be regarded as a linear system defined on a variety.

Definition 2 Consider a criterion of choice applied to the system (3):

$$\begin{aligned} z_1(k+1) &= z_2(k) \\ \ldots &\quad \ldots \\ z_t(k+1) &= c(z_1(k), \ldots, z_t(k)) \end{aligned} \tag{7}$$

We call the system (7) the "c"-resulting system.

Definition 3 The equilibrium z^* of the "c"-resulting system (7) is:

1. stable if $\forall \varepsilon > 0, \exists \delta > 0, \delta = \delta(\varepsilon)$ such that if $z(0) \in \mathcal{B}_\delta(z^*)$ then $z(k, z(0)) \in \mathcal{B}_\varepsilon(z^*), \forall k = 1, 2, \ldots$.
2. attractive if $\exists \Delta > 0$ such that if $z(0) \in \mathcal{B}_\Delta(z^*)$, then $\lim_{k\rightarrow\infty} ||z(k, z(0)) - z^*|| \rightarrow 0$
3. locally asymptotically stable if 1 and 2 hold
4. globally asymptotically stable if 1 holds and 2 is satisfied $\forall z(0) \in \mathbb{R}^t$.

Definition 4 Consider a criterion of choice c and the "c"-resulting system. A bounded set A ($\sup_{x,y \in A} d(x, y) < \infty$) is invariant if $\forall z(0) \in A$ we have that $z(k) \in A, \forall k$.

We note that an invariant set A is not unique in general and we usually do not work with the smallest such set but rather with the one which is simple to analyze.

Definition 5 An invariant set $A \neq \emptyset$ of a "c"-resulting system is

1. stable if $\forall \varepsilon > 0, \exists \delta > 0, \delta = \delta(\varepsilon)$ such that if $d(z(0), A) < \delta$ it follows that $d(z(k, z(0)), A) < \varepsilon, \forall k$
2. attractive if $\exists \Delta > 0$ such that if $d(z(0), A) < \Delta$ it follows that $\lim_{k\rightarrow\infty} d(z(k, z(0)), A) \rightarrow 0$
3. asymptotically stable if 1 and 2 hold
4. globally asymptotically stable if 1 holds and 2 holds with $\Delta = \infty$.

Below we propose definitions of minimum phase systems, which incorporate the criterion of choice.

Definition 6 The system (1) is:

1. point-minimum phase (set-minimum phase) if *there exists* a criterion of choice c such that the equilibrium z^* (bounded invariant set A) of the "c"-resulting system is asymptotically stable
2. uniformly point-minimum phase (uniformly set-minimum phase) if *for any* criterion of choice c the equilibrium z^* (bounded invariant set A) of the "c"-resulting system is asymptotically stable
3. non minimum phase if it is neither point nor set-minimum phase

The above given definitions of minimum phase systems may be generalized in many ways to suit a particular problem or a property we are dealing with. We point out that when we talk about point-minimum phase property, we assume that the equilibrium of interest is known a priory. For set-minimum phase properties we normally show how to find an invariant set A. This reflects that point-minimum phase property is used in local analysis and set-minimum phase for global analysis.

Example 3 Consider the following system:

$$\begin{aligned} x_1(k+1) &= (u(k)+x_1(k)+0.5x_2(k)) \\ &\quad (u(k)+x_1(k)-3x_2(k)) \\ x_2(k+1) &= u(k) \\ y(k) &= x_1(k) \end{aligned} \tag{8}$$

The system has relative degree $r = 1$ in the sense of [9]. Observe that the partial derivative in vanishes at the particular point $(x^*, u^*)=(0,0)$. We emphasize that this situation has not been analyzed in the literature.

We show below that the I-O linearizing feedback law exists and it is not unique. The system is already in the form which does not require a change of coordinates and only the state feedback is needed to I-O linearize the system. Consider the following continuous feedback law:

$$\begin{aligned} u(k) &= -x_1(k)+1.25x_2(k) \pm [0.25(x_1(k)-2.5x_2(k))^2 \\ &\quad -(x_1(k)+0.5x_2(k))(x_1(k)-3x_2(k))-v(k)]^{0.5} \end{aligned} \tag{9}$$

where $v(k)$ is a new control variable. Notice that we have a freedom of choosing "+" or "−" sign in the above control law. Any choice I-O linearizes the system. Suppose we choose the control law with the choice "+". Straightforward calculations show that the corresponding zero dynamics are $x_2(k+1) = 3x_2(k)$, and they are not stable. If we use the definition of minimum phase systems in [9], we conclude that the system is not minimum phase. However, by applying the choice "−" in the I-O control law the zero dynamics become $x_2(k+1) = -0.5x_2(k)$. Obviously they are stable. We conclude that if one wanted to blindly apply the known definition of minimum phase to this system, it would not be clear whether the system is minimum phase or not.

From the output dead beat control point of view, the important feature that needs to be captured by the notion of minimum phase should be that we can keep the inputs bounded while output is kept constant. Hence, it is clear that the choice of the feedback law $u(k) = g(x(k), v(k))$ must be taken into account when considering the minimum phase property. Notice, however, that we are no longer dealing with a stability problem when analyzing minimum phase systems but with a constrained stabilizability problem.

The given example shows that the known definition of minimum phase systems found in the literature relies heavily on the use of implicit function theorem. Moreover, in many situations the definition may be inadequate to capture what is going on.

3 A point-minimum phase test

We present below several tests for point-minimum phase properties of I-O polynomial systems. Without loss of generality we assume in this subsection that the equilibrium around which we are working is the origin $z^* = 0$.

Fix a number $U < 1$ such that $|1-U| \ll 1$, for instance $U = 0.9999$. In the sequel we exploit the following sets

$$\begin{aligned} S_1^{z_j} &= \{z \in \mathbb{R}^t : \exists u(0) \in \mathbb{R}, |u(0)| < U|z_j|, \\ &\quad G(z_1, \ldots, z_t, u(0)) = 0\} \\ S_2^{z_j} &= \{z \in \mathbb{R}^t : \exists u(0), u(1) \in \mathbb{R}, |u(1)| < U|z_j|, \\ &\quad G(z_1, \ldots, z_t, u(0)) = 0, \\ &\quad G(z_2, \ldots, z_t, u(0), u(1)) = 0\} \\ \ldots &\quad \ldots \end{aligned} \tag{10}$$

where $j = 1, 2, \ldots, t$. Hence, sets $S_k^{z_j}$ represent states in $\mathbb{R}^t$ for which there is a sequence of controls (criterion of choice) yielding $|z_t(k)| < U|z_j(0)|$. The above given sets can be used to check minimum phase properties of the system (1). Notice that the sets can be computed using QEPCAD since the inequality $|u(0)| < U|z_j|$ can be rewritten as four inequalities without absolute values. We, however, use absolute values to shorten notation. These sets are semi-algebraic. We use the notation $S_1^{z_j}(z)$ to denote the expression which defines the set $S_1^{z_j}$.

The expression $S_1^{z_j}(z)$ can be computed by considering the decision problem

$$(\exists u(0))[|u(0)| < U|z_j| \wedge G(z, u(0)) = 0].$$

The above defined sets can be given in certain cases a nice interpretation based on Lyapunov functions. Indeed, assume that the set $S_1^{z_1}$ is a neighborhood of the origin in the case of the explicit zero dynamics (4) with the function G continuous, and define the Lyapunov function:

$$V(z(k)) = \sum_{i=1}^{t} |z_i(k)|$$

which is positive definite. By considering the difference:

$$V(z(k+1)) - V(z(k)) = \sum_{i=2}^{t} |z_i(k)| + |G_e(z(k))|$$

$$- \sum_{i=1}^{t} |z_i(k)|$$

we obtain $V(z(k+1)) - V(z(k)) = |G_e(z(k))| - |z_1(k)|$. By definition of the set $S_1^{z_1}$ we have that $|G(z(k))| < |z_1(k)|$ on the set. Hence, we obtain

$$V(z(k+1)) - V(z(k)) < 0, \forall z \in S_1^{z_1}$$

and since $S_1^{z_1}$ is a neighborhood of the origin, the origin of the "c"-resulting system is asymptotically stable. Notice that in this case we could use the quadratic Lyapunov function $V(z(k)) = \sum_{i=1}^{t} (z_i(k))^2$ to arrive at the same conclusion.

This result can be generalized to implicitly defined zero dynamics (3) and when the criterion of choice is a discontinuous mapping. We show below that the sets can be used to prove stability properties without having to resort to Lyapunov functions, namely *by definition*.

Theorem 1 *There exists a criterion of choice c such that the origin of the "c"-resulting system is stable if the set $N = \cup_j S_1^{z_j}$ is a neighborhood of the origin.*

Proof of Theorem 1: Notice first that if the set N contains a neighborhood of the origin this guarantees that Assumption 1 is satisfied on the neighborhood.

Given a positive number $s > 0$, we consider the hyper-cube

$$\mathcal{C}_s(0) = \{z \in \mathbb{R}^t : |z_1| < s \wedge \ldots \wedge |z_t| < s\}$$

Notice that if the conditions of Theorem 1 are satisfied, there exists $s^* > 0$ such that $\mathcal{C}_{s^*}(0) \subset N$. Then, there exists a criterion of choice c such that any hyper-cube $\mathcal{C}_s(0), s \in]0, s^*[$ satisfies that if $z(0) \in \mathcal{C}_s$ then $z(k) \in \mathcal{C}_s(0), \forall k = 1, 2, \ldots$. Indeed, if

$$|z_1(0)| < s \wedge \ldots \wedge |z_t(0)| < s$$

then we have from the structure of the system that

$$|z_1(1)| = |z_2(0)| < s \wedge \ldots |z_{t-2}(1)| = |z_t(0)| < s$$

Moreover, by definition of sets (10) we have that there exists a criterion of choice c such that

$$|z_t(1)| < U|z_j(0)| < s, j \in \{1, 2, \ldots, t\}$$

and hence we conclude that $z(1) \in \mathcal{C}_s(0)$. Notice that this holds for arbitrary $z(0) \in \mathcal{C}_s(0)$ and hence we have that $z(k) \in \mathcal{C}_s(0), \forall k$.

Consider now any hyper-ball $\mathcal{B}_\varepsilon(0), \varepsilon > 0$ and define $\delta = \delta(\varepsilon) = \min(\varepsilon/2, s^*/2)$. Then if $z(0) \in \mathcal{B}_\delta$ we have that $z(k) \in \mathcal{C}_\delta(0), \forall k$ since $\delta \in]0, s^*[$. Moreover, we have that $\mathcal{C}_\delta(0) \subset \mathcal{B}_\varepsilon(0), \forall \varepsilon > 0$ and hence $z(k) \in \mathcal{B}_\varepsilon(0), \forall k$. Therefore there exists a criterion of choice c such that the "c"-resulting system is stable by definition. Q.E.D.

Comment 1 QEPCAD can be used to check whether a semi-algebraic set S with the defining expression $S(x)$ is a neighborhood of a point x^*. Indeed, this can be done by considering the decision problem:

$$(\exists q)[S(x) \wedge ||x|| < q]$$

Theorem 2 *Suppose that $\exists j \in \{1, \ldots, t\}$ such that the set $S_1^{z_j}$ is a neighborhood of the origin. Then the system is point-minimum phase.*

Proof of Theorem 2: Stability follows from Theorem 1. We show now that the system is also locally attractive. We know that there exists a number s^* such that any hyper-cube $\mathcal{C}_s(0), s \in]0, s^*[$ is invariant with respect to the solutions $z(k), \forall k$. Hence, we have that if $z(0) \in \mathcal{C}_s(0)$

$$|z_t(k + t - j)| < |z_j(k)|, \forall k$$

If $k = 0$ we have that $|z_t(t - j)| = p_0|z_j(0)|, p_0 < U < 1$. For $k = 1$ we have that $|z_t(1 + t - j)| = p_1|z_j(1)| = p_1 p_0 |z_j(0)|, p_1 < U < 1$. In general we obtain that

$$|z_t(N + t - j)| = \prod_{i=0}^{N} p_i |z_j(0)|, p_i < U < 1$$

and by taking the limit we obtain $\lim_{N\to\infty} |z_{2,t}(N + t - j)| \to 0$. Since $z_l(k + 1) = z_{l+1}(k), l = 1, \ldots, t-2$ we conclude that

$$\lim_{N\to\infty} |z_l(N)| \to 0, \forall l = 1, \ldots, t-2.$$

In other words, we have that $\lim_{N\to\infty} ||z(N)|| \to 0$. We can therefore take $\Delta = s^*/2$ and the attractivity of the zero dynamics follows by definition. Q.E.D.

An obvious consequence of the above results is:

Corollary 1 *Suppose there exists $j \in \{1, 2, \ldots, t\}$ such that $S_1^{z_j} = \mathbb{R}^t$. Then there exists a criterion of choice such that the origin of the "c"-resulting system is globally asymptotically stable.*

Now we need the following assumptions:

Assumption 1 *$\forall z \in \mathbb{R}^t$, all the solutions u_i to the equation $G(z, u) = 0$ satisfy $|u_i| < \infty$.*

Assumption 2 $\forall z_1, \ldots, z_{t-2} \in \mathbb{R}$ there exists a real solution u^* to the equation $G(z, u) = 0$ satisfying

$$\lim_{|z_t|\to 0} |u^*| \to 0$$

We use the notation $H = \{z \in \mathbb{R}^t : z_t = 0\}$. We state now a result for

Theorem 3 *Suppose that Assumptions 1 and 2 are satisfied for the implicit polynomial dynamics (3). There exists a criterion of choice such that the origin of the "c"-resulting system is globally attractive if there is an integer N such that $\cup_{i=1}^{N} S_i^{z_j} = \mathbb{R}^t - H$ for some $j \in \{1, \ldots, t\}$.*

Proof of Theorem 3: Suppose that conditions of Theorem 3 are satisfied. Consider any initial state $z(0) \in \mathbb{R}^t$. If $z(0) \in H$ then by simply applying $u(k) = 0, \forall k$ we have that $z(k) = 0, \forall k \geq t$. If $z(0) \in \mathbb{R}^t - H$, then we have that $z(0) \in S_{k_1}^{z_j}, k_1 \in \{1, \ldots, N\}$. By definition of the set $S_{k_1}^{z_j}$ we have that

$$|z_t(k_1)| = p_{k_1}|z_j(0)|, 0 \leq p_{k_1} < U < 1$$

If $z(k_1) \in H$ we trivially have attraction to the origin. Suppose that $z(k_1) \notin H$. Then, we have that $z(k_1) \in S_{k_2}^{z_j}, k_2 \in \{1, \ldots, N\}$ and by definition

$$|z_t(k_2)| = p_{k_2}|z_t(k_1)|, 0 \leq p_{k_2} < U < 1$$

Therefore, if we suppose that $z(k_i) \notin H, \forall i = 1, 2, \ldots$ we have that

$$|z_t(k_N)| = \prod_{i=1}^{N} p_{k_i}|z_j(0)|, 0 \leq p_{k_i} < U < 1, \forall i$$

and by taking the limit of the above expression we obtain that

$$\lim_{N\to\infty} |z_t(k_N)| \to 0$$

Because of the Assumption 2 and since $z_t(k_N) \to 0$ we have that $z_{t-j}(k_N + j) \to 0, j = 1, 2, \ldots, t-2$ and therefore $||z(k)|| \to 0$. The boundedness of $z(k), \forall k$ follows trivially from the boundedness of solutions (Assumption 1). Q.E.D.

Corollary 2 *If the conditions of Theorems 1 and 3 are satisfied the system (3) is point-minimum phase. Moreover, there exists a criterion of choice c such that the origin of the "c"-resulting system is globally asymptotically stable.*

Notice that Assumption 1 is not essential for the global attractivity result and is only used to guarantee that there are no finite escape times.

Comment 2 The computational complexity of the decision rules used to define the sets $S_k^{z_j}$ may be prohibitive and hence it is of utmost importance to investigate ways in which the complexity can be reduced. The required computations may be drastically reduced by first decomposing the polynomial G which defines the implicit zero dynamics (3)

$$G(z_1, \ldots, z_t, u) = \prod_{i=1}^{M} f_i(z_1, \ldots, z_t, u) \qquad (11)$$

where f_i are all irreducible polynomials. For example, we can use "factor" command in Maple. Notice that $G = 0$ if $f_i = 0$ for some i and if any of the newly defined implicit zero dynamics

$$f_i(z_1, \ldots, z_t, u) = 0$$

satisfies conditions of some of the Theorems 1, 2 or 3, the zero dynamics (3) have at least the same properties as the newly defined zero dynamics.

It is important to emphasize that the *uniform point-minimum phase property* can be tested using a very similar method, by redefining the decision problems used to compute the sets.

4 A set-minimum phase test

The following sufficient condition for set minimum phaseness is easily proved.

Theorem 4 *Suppose that $S_1^{z_1} = \mathbb{R}^t - L$, where L is a bounded set ($\sup_{x,y\in L} d(x, y) < \infty$). Then the system is set-minimum phase.*

Proof of Theorem 4: From the definition of the set $S_1^{z_1}$ we have that there exists a criterion of choice c such that the "c"-resulting system has the property $||z(1)|| < ||z(0)||, \forall z(0) \in S_1^{z_1}$. Moreover, because of Assumption 1 we have that $||z(1)|| < \infty, \forall z(0)$. Hence, we can find a number $s < \infty$ such that the hyper-ball $\mathcal{B}_s(0), L \subset \mathcal{B}_s(0)$ is an invariant set. Global asymptotic stability of $\mathcal{B}_s(0)$ follows directly from the fact that $\mathbb{R}^t - \mathcal{B}_s(0) \subset S_1^{z_1}$. Q.E.D.

Comment 3 In a very similar way we can prove other stability properties and also we can use other sets for this purpose. For example, we could work with sets

$$S_1 = \{z : \exists u \in \mathbb{R}, ||z(1)||^2 < U||z(0)||^2,$$
$$G(z(0), z_t(1)) = 0\}, etc.$$

With these sets all proofs become much easier but notice that the degrees of input polynomials are higher than for the sets that we used and hence they are more difficult to compute.

QEPCAD can be used to compute (or construct) Lyapunov functions if we know from converse Lyapunov theorems [6] that they belong to a class of polynomial functions. For example, all polynomial systems which have stable linearization allow for quadratic Lyapunov functions [6]. So suppose we want to check stability of the system:

$$x(k+1) = F(x(k)) \qquad (12)$$

with $x \in \mathbb{R}^n$ and F is a vector polynomial function in all its arguments. We can choose the quadratic function:

$$V(x(k)) = x^T(k)Hx(k),\ H = H^T,\ H > 0,\ H \in \mathbb{R}^{n\times n}$$

By considering the following set of formulas:

$$(\exists H)[(H - H^T = 0) \wedge (H > 0) \wedge (F^T(x)HxF(x) - x^T Hx < 0)$$

we can compute the existence of a quadratic Lyapunov function. Moreover, *all possible matrices H* which define Lyapunov functions can be found using QEPCAD. With any such Lyapunov function we can then estimate the domain of attraction which is very important in applications.

Observe that all systems (12) whose equilibrium is attractive must necessarily have the property that $\exists N$ such that $\cup_j S_N^{z_j}$ is a neighborhood of the origin. Suppose that this is satisfied. Then to check stability, we can just consider the formulas:

$$(\forall\varepsilon)(\exists\delta)[(\varepsilon > 0)\wedge(\delta > 0)\wedge(||x|| < \delta)\wedge(||F(x)|| < \delta)\wedge\ldots$$
$$\wedge(||F^{N-1}(x)|| < \delta)]$$

which qualify as input formulas to the QEPCAD algorithm and hence can be decided in finite time.

5 Conclusion

We presented some results on the problem of testing the minimum phase property of polynomial systems. We revisited the definitions of minimum phase systems with the aim of showing that they need to be generalized in order to be well defined for general nonlinear systems. The fact that we have choice over many controllers when I-O linearizing the system had not been studied in the literature. This introduces design procedures into the problem of stability of zero dynamics and in a sense we are talking about "stabilizability of zero dynamics".

We proposed QEPCAD to test different minimum phase properties. We point out that our intention was just to illustrate how it is possible to use this tool (QEPCAD) and we have not presented the most comprehensive or the most general solutions. We showed that stability properties of some classes of polynomial systems can be checked by definition. The main hindrance to the method is computational complexity but by constraining the structure of the system and using some analytical results we can reduce computations considerably.

An interesting question arises:

Is it less complex to check stability by constructing Lyapunov functions or by definition?

At this moment, it seems that the answer to the above question would depend on the class of systems that we are considering. We emphasize that one can easily construct examples for which testing stability by definition is easier to compute! This is a rather unexpected result which sheds a new light on the stability problem.

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