

Sufficient Conditions for Stabilization of Sampled-data Linear Spatially Distributed Parameter Systems via Discrete time Approximations

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Abstract

In this paper we present preliminary results on sampled-data controller design for linear spatially distributed parameter systems (SDPS) via their discrete time approximations. Analogous to the results on sampled-data control of linear/nonlinear lumped parameter systems (LPS), we show that exponential stability of an approximate discrete time model implies exponential stability of the original sampled-data linear SDPS, provided that an appropriate consistency condition holds between the exact and approximate closed-loop systems. Consequently, the controller can be designed on the basis of an approximate discrete time model.

1. INTRODUCTION

Spatially distributed parameter systems arise in a range of different processes in optical telecommunications, fluid flows, thermal processes, biology, chemistry, environmental sciences, mechanical systems, and so on. SDPS are modelled by partial differential equations (PDE) or abstract differential equations, as opposed to lumped parameter systems (LPS) that are modelled by ordinary differential equations (ODE). PDE models often arise from Navier-Stokes and Maxwell-Boltzman equations that are fundamental in modelling the fluid flow processes and the propagation of electromagnetic waves respectively. For example, the PDE model of an optical Raman amplifier used in optical telecommunications can be derived under appropriate assumptions from the Maxwell-Boltzman equations [1].

Automatic control of SDPS is often used in engineering practice to achieve higher quality performance, productivity, reliability and efficiency, reduce costs, improve safety, and so on. Nowadays most control systems are implemented using digital technology since it is very cheap, fast, relatively easy to operate, flexible and reliable. This motivates an investigation of the so called *sampled-data systems* that consist of a continuous-time plant or process which is controlled by a discrete-time controller [2]. The plant and the controller are interconnected via the analog-to-digital (A-D) and digital-to-analog (D-A) converters. Consequently, the designed controller

needs to be time-discretized in order to be implemented using the prevalent digital technology. For LPS, general sufficient conditions which guarantee that controllers that stabilize an approximate discrete time model of a sampled-data LPS will also stabilize its exact model for sufficiently small sampling periods, were established in [3], [4], [5].

These results provide a prescriptive framework for controller design for LPS via their approximate discrete-time plant models were established in

Although sampled-data SDPS are as important as sampled-data LPS, we have a very modest understanding of how to address the controller design even for linear SDPS [6], [7], [8], [9], [10]. Emulation was investigated in [10] for linear SDPS in terms of the following abstract differential equation

$$\dot{x} = Ax + Bu \quad (1)$$

where x and u are elements of appropriate Hilbert spaces and A and B are appropriate linear operators on Hilbert spaces. Results similar to emulation of LPS are proved for above given linear SDPS with state feedback $u = Fx$, where F is a compact operator, if either the input operator B is bounded or the semigroup generated by A is analytic. In [9], it was shown that emulation does not work for general linear SDPS and a counterexample is provided. In particular, the authors prove for the given counterexample that there exists a sequence of sampling periods $T_k > 0$ with $\lim_{k \rightarrow \infty} T_k = 0$ such that the system with the emulated controller is unstable for each T_k . This indicates that the emulation method can not be used for certain classes of infinite dimensional systems and controllers. Note that this differs considerably from emulation results for LPS systems where it was shown for a large class of nonlinear systems that the emulation recovers the stability properties of the continuous-time closed loop in the limit as the sampling period goes to zero [5].

In this paper, we investigate exponential stabilization of linear SDPS via their discrete time approximations. This is different from emulation in [10] where the controller is designed in continuous time and then time discretized. For a given system (1), we design the controller, which stabilizes the discrete time approximate model of the system (1). We first

show (Theorem 1) that the controller that stabilizes an approximated discrete time model of the sampled-data plant will also stabilize its exact discrete-time model, provided the trajectories of the exact and approximate closed-loop systems can be made arbitrarily close on finite time intervals. The closeness of two trajectories of exact and approximate model can be interpreted as a kind of “convergence”, which implies that the approximate trajectory would converge to the exact trajectory on the finite time interval when sampling period T tends to zero. It is well-known that Lax-Richtmyer Equivalence Theorem [11] provides the sufficient and necessary condition to ensure the “convergence” (or closeness of the trajectories). With the help of Lax-Richtmyer Equivalence Theorem, we show (Theorem 2) that an appropriate consistency condition guarantees the closeness of the trajectories of exact and approximate model, thus the controller designed for the approximate model would stabilize the exact model under an appropriate consistency condition (Corollary 1). This result is quite similar to the result on LPS [3], in which the appropriate consistency conditions imply the stability properties. It is the purpose of this paper to demonstrate that recent results on stabilization of sampled-data LPS via their approximate discrete time models can be extended to linear sampled-data SDPS. For simplicity, we consider a situation where the discrete time model is infinite dimensional and we make the connection with the well-known Lax-Richtmyer theorem on convergence of finite element methods in numerical analysis of PDEs. While this problem is less interesting since the controller is infinite dimensional, the connection we made with Lax-Richtmyer Equivalence Theorem strongly indicates that this approach can be used to obtain finite dimensional discrete time controllers for SDPS via their finite dimensional discrete time approximate models. We are currently working on this extension.

This paper is organized as follows. In Section 2 we present the problem formulation and preliminaries. Main results are given in Section 3, followed by conclusions in Section 4.

2. PROBLEM FORMULATION AND PRELIMINARIES

We consider the following plant dynamics which is governed by an abstract differential equation:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (2)$$

on a Hilbert space X with $x_0 \in X$ and with u in some space of functions taking values in a Hilbert space U . $A : D(A) \rightarrow X$ is the generator of a strongly continuous semigroup $S(t)$ (C_0 -semigroup) on X , where $D(A)$ is the domain of A , which is a subset in X . $B : U \rightarrow X_{-1}$, where X_{-1} denotes the closure of X in the norm $\|x\|_X \triangleq \|(\lambda I - A)^{-1}x\|_X$. Here λ is any element of $\rho(A)$ and $\rho(A)$ denotes the resolvent set of A , i.e., the set of all $s \in \mathbb{C}$ such that $sI - A : D(A) \rightarrow X$ is bijective and $(sI - A)^{-1} \in \mathcal{L}(X)$, where $\mathcal{L}(X) := \mathcal{L}(X, X)$ is the space of all linear bounded operator from X to X . Since X_{-1} is a closure of X , it is clear that $\mathcal{L}(U, X) \subset \mathcal{L}(U, X_{-1})$. Following the notation in [10], the control operator B is called bounded if it maps boundedly into the state space X .

Otherwise B is called “unbounded” (with respect to the state space X).

In this paper, we consider the weak solution of (2) in the space X_{-1} , i.e., $S(t)$ extends to a strongly continuous semigroup on X_{-1} . The generator of $S(t)$ on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1} , i.e., $D(A) = X$). We shall use the same notation $S(t)$ (respectively, A) for the original semigroup (respectively, its generator) and the associated extensions.

First of all, we assume that (2) is well-posed as defined follows

Definition 1: The abstract differential equation (2) is well-posed if the operator A generates a C_0 -semigroup $S(\cdot)$ and B is a bounded control operator so that the solution of (2) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau \in X, \quad (3)$$

and (2) holds on X_{-1} .

Remark 1: According to [12, Theorem 2.2], since $S(\cdot)$ is a C_0 -semigroup, there exist constant $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \forall 0 \leq t < \infty. \quad (4)$$

The plant is assumed to be between a sampler (A/D converter) and zero-order-holder (D/A converter). The control input is assumed to be piecewise constant, i.e.,

$$u(s) = u(kT) =: u(k), \quad \forall s \in [kT, (k+1)T), \quad (5)$$

for all $k = 0, 1, \dots$. Here $T > 0$ is a sampling period. Moreover, we assume that the state measurement $x(k)$, where

$$x(k) := x(kT), \quad (6)$$

are available at sampling instants¹. The sampling period T is assumed to be a design parameter which can be arbitrarily assigned. Evaluating (3) at time $(k+1)T$, we obtain the exact discrete time model of (2):

$$x(k+1) = A_T^e x(k) + B_T^a u(k), \quad (7)$$

where $A_T^e \triangleq S(T)$ and $B_T^a u \triangleq \int_0^T S(t-kT-\tau)Bu d\tau, \forall u \in U$. According to [10, Lemma 2],

$$B_T^e \in \mathcal{L}(U, X), \quad \forall u \in U, \quad (8)$$

which indicates that there exists a constant K_B such that

$$\|B_T^e\|_{\mathcal{L}(X, U)} \leq K_B, \quad \text{for any } T \in (0, T^*).$$

Next, the following approximation of the well-posed abstract differential equation (2) is considered at the time instant $(k+1)T$,

$$x(k+1) = A_T^a x(k) + B_T^a u(k), \quad (9)$$

where A_T^a is a linear operator mapping X onto X and B_T^a is the linear operator mapping U onto X . In this paper, for simplicity, we just consider the state feedback controller. The

¹This is a strong assumption, but can be relaxed.

feedback controller is designed for the discretized system (9) as

$$u(k) = F_T x(k), \quad (10)$$

where $F_T : X \rightarrow U$. With the designed controller F_T , at time $(k+1)T$, the closed-loop of exact and the approximate model are respectively given by

$$x(k+1) = (A_T^e + B_T^e F_T)x(k), \quad x(0) = x_0, \quad (11)$$

$$x(k+1) = (A_T^a + B_T^a F_T)x(k), \quad x(0) = x_0. \quad (12)$$

We denote the solutions of the exact model (11) and (12) at k are $\phi_T^e(k, x_0)$ and $\phi_T^a(k, x_0)$ respectively. This paper focuses on investigating the stability properties of a family of the closed-loop exact plant models (11) from the stability properties of a family of closed-loop approximation plant models (12).

In this paper, the following definitions are employed:

Definition 2: The solution of approximate model (12) is bounded uniformly in small parameter T if the following holds. For any $k = 0, 1, 2, \dots$, there exists $L_a > 1$ and $T^* > 0$ such that for all $T \in (0, T^*]$ and $x_0 \in X$, we have

$$\|\phi_T^a(k, x_0)\|_X \leq L_a \|x_0\|_X, \quad \forall k = 0, 1, 2, \dots \quad (13)$$

Definition 3: The solution of $x(k+1) = A_T x(k)$ is exponentially stable uniformly in small parameter T if there exist $K, \lambda, T^* > 0$, such that, for all $T \in (0, T^*]$ and for all $x_0 \in X$, the following inequality holds:

$$\|\phi_T(k, x_0)\|_X \leq K e^{-\lambda k T} \|x_0\|_X \quad \forall k = 0, 1, \dots \quad (14)$$

Definition 4: We say that the solutions of the exact model (11) and the approximated model (12) can be made arbitrarily close on compact time intervals if the following holds. For any $t \geq 0$ and any $\delta > 0$, there exists $T^* > 0$, such that $\forall T \in (0, T^*)$ and $\forall x_0 \in X$, we have

$$\|\phi_T^e(k, x_0) - \phi_T^a(k, x_0)\|_X \leq \delta \|x_0\|_X, \quad \forall kT \in [0, t]. \quad (15)$$

Remark 2: If we choose $u(k) = 0$ in (11) and (12), the definition of the closeness of the two solutions of (11) and (12) are the same as the definition of “convergence” in [11], in which “convergence” implies that the exact solutions of the abstract differential equation converge to the solutions of the discrete (time or/and space) model when sampling in time or/and space domain tends to zero. Sufficient and necessary conditions were provided for “convergence” in [11]. In this paper, we extended the results of Lax-Richtmyer Equivalence Theorem that applies to open-loop systems to abstract differential equation with control feedback that applies to closed-loop systems. The “convergence” properties provide closeness between the solutions of the exact model and the approximate model, thus providing sufficient conditions to investigate the stability properties of the exact model from the approximate model.

3. MAIN RESULTS

In this section, we present sufficient conditions under which the stability of approximated model guarantees the stability of the exact model. These sufficient conditions provide a

framework for controller design for sampled-data systems vis their approximate discrete time models.

The first result is given in the following theorem:

Theorem 1: Suppose that the following conditions hold:

- i F_T in (10) is a bounded operator uniform in T , that is, for a given T^* , there exists K_F such that

$$\|F_T\|_{\mathcal{L}(U, X)} \leq K_F < \infty, \forall T \in (0, T^*) \quad (16)$$

- ii The solution of the approximate model (12) is exponentially stable uniformly in small parameter T ;
- iii The solutions of the exact model (11) and the approximate model (12) can be made arbitrarily close on compact time intervals.

Then, there exist $T^* > 0$, $K_1 > 0$ and $\lambda > 0$ such that for all $T \in (0, T^*)$ and all $x_0 \in X$, the solutions of (2) with controller shown in (5) satisfy the following inequality:

$$\|\phi(s, x_0)\|_X \leq K_1 \exp(-\lambda_1 s) \|x_0\|_X. \quad (17)$$

Proof: Let $c = (0, 1)$, $\delta \in (0, c)$, T_1^* , K and λ come from the item (ii) of the theorem. Let $T_2^* := \frac{1}{\lambda} \ln(\frac{K}{c-\delta})$. Given δ determine $T_3^* > 0$ via the item (iii) of the theorem, that is, for any $x_0 \in X$, finite time $t > 0$ and $T \in (0, T_3^*)$ and $k : kT \in [0, t]$

$$\|\phi_T^e(k, x_0)\|_X \leq \|\phi_T^a(k, x_0)\|_X + \delta \|x_0\|_X \quad (18)$$

Define $T^* = \min\{T_1^*, T_2^*, T_3^*, 1\}$ and $k_T^* = \max\{k \in \mathbb{N} : kT \leq t\}$, it is noted that k_T^* is a positive constant satisfying the following inequality,

$$k_T^* T > t - T > t - 1, \quad (19)$$

for all $T \in (0, T^*)$.

Consider the solution of the exact model (11) at kT , for all $k \in [k_T^*, 2k_T^*]$.

$$\begin{aligned} & \|\phi_T^e(k, x_0)\|_X \\ &= \|\phi_T^e(k - k_T^*, \phi_T^e(k_T^*, x_0))\|_X \\ &\leq \|\phi_T^a(k - k_T^*, \phi_T^e(k_T^*, x_0))\|_X + \delta \|\phi_T^e(k_T^*, x_0)\|_X \\ &\leq K e^{-\lambda(k - k_T^*)T} \|\phi_T^e(k_T^*, x_0)\|_X + \delta \|\phi_T^e(k_T^*, x_0)\|_X \\ &\leq (c - \delta) \|\phi_T^e(k_T^*, x_0)\|_X + \delta \|\phi_T^e(k_T^*, x_0)\|_X \\ &= c \|\phi_T^e(k_T^*, x_0)\|_X \end{aligned} \quad (20)$$

Introducing $k_i = ik_T^*, i = 0, 1, \dots$, we have

$$\begin{aligned} \|\phi_T^e(ik_T^*, x_0)\|_X &\leq c^i \|\phi_T^e(k_T^*, x_0)\|_X \\ &\leq c^i \|x_0\|_X \\ &\leq \exp(-\lambda_1 ik_T^*) \|x_0\|_X, \end{aligned} \quad (21)$$

where $\lambda_1 = \ln(1/c)$, for all $k = 0, 1, 2, \dots$

Finally, for all $s \in [kT, (k+1)T]$, according to (3) we have:

$$\begin{aligned} & \|\phi(s, x_0)\|_X \\ &= \|\{S(s - kT) + B_{s-kT}^e F_T\} \phi(s, x_k)\|_X \\ &\leq \|\{S(s - kT) + B_{s-kT}^e F_T\}\|_{\mathcal{L}(X)} \|\phi(s, x_k)\|_X \end{aligned} \quad (22)$$

where $B_{s-kT}^e u \triangleq \int_0^{s-kT} S(s-kT-\tau)Bu\tau$, for any $u \in U$. According to Remark 1, (8) as well as item (i), we have

$$\| \{S(s-kT) + B_{s-kT}^e F_T d\tau\} \|_{L(X)} \leq L, \quad (23)$$

where $L \triangleq Me^{\omega T} + K_L(T)K_F$.

Substituting (21) and (23) into (22) yields

$$\begin{aligned} \|\phi(s, x_0)\|_X &\leq L \exp(\lambda_1 kT) \|x_0\|_X \\ &= L \exp(\lambda_1(s-kT)) e^{-\lambda_1 s} \|x_0\|_X \\ &\leq L \exp(\lambda_1 T) \exp(-\lambda_1 s) \|x_0\|_X. \end{aligned} \quad (24)$$

Hence, the inequality (17) holds with $K_1 := L \exp(\lambda_1 T)$. ■

Theorem 1 shows that when the control operator F_T is a bounded operator uniform for T , if the solutions of the exact model (11) and the approximate model (12) are close on compact time intervals, the stability properties of the approximate model imply the similar stability properties of the exact model. The next question arises naturally: “how can we show that the solutions of the exact model (11) and the approximate model (12) are close on compact time intervals?” The closeness of solutions depends on the characterizations of the time-discretization (A_T^a, B_T^a) .

To characterize (A_T^a, B_T^a) , the following definitions, which are widely encountered in numerical analysis of the SDPS [11], are employed in this paper.

Definition 5: Given (A_T^e, B_T^e) in (7), a time-discretization scheme is a family of pairs (A_T^a, B_T^a) where

- 1) $A_T^a \in \mathcal{L}(X)$;
- 2) $B_T^a \in \mathcal{L}(U, X)$;
- 3) For a given $T_a > 0$, there exists a positive constant pair (α_C, α_D) such that

$$\|A_T^a\|_{\mathcal{L}(X)} \leq \alpha_C, \quad \|B_T^a\|_{\mathcal{L}(U, X)} \leq \alpha_D, \quad (25)$$

hold for every $T \in (0, T_a]$.

Definition 6: Given (A_T^e, B_T^e) in (7), a consistent time-discretization scheme method is a family of pairs (A_T^a, B_T^a) , such that, there exists $T_b > 0$ such that for any $T \in (0, T_b)$, we have

- 1) for any $x \in X$, there exist $\rho_c \in \mathcal{K}_\infty^2$ such that

$$\|\{A_T^a - A_T^e\}x\|_X \leq T\rho_c(T) \|x\|_X; \quad (26)$$

- 2) for any $u \in U$, there exists $\rho_d \in \mathcal{K}_\infty$ such that

$$\|\{B_T^a - B_T^e\}u\|_X \leq T\rho_d(T) \|u\|_U. \quad (27)$$

With above definitions, we have our second result.

Theorem 2: Given (A_T^e, B_T^e) in (7), assume that (A_T^a, B_T^a) is a consistent time-discretization scheme, given the controller (10) where F_T is a bounded operator uniform in T , if approximate model (12) is bounded uniformly in small parameter T , then the solution of the exact model (11) and (12) can be made arbitrarily close on compact time intervals.

²The continuous function $\rho : R_{\geq 0} \rightarrow R_{\geq 0}$ is of class $\mathcal{K}_\infty$ if $\rho(0) = 0$, $\rho(x)$ is strictly increasing and tends to infinity as $x \rightarrow \infty$.

Proof: Let $\delta > 0$ be arbitrary, $c_1 \in (0, 1)$, there exists a positive constant T_α^* such that, for all $T \in (0, T_\alpha^*)$ we have

$$\|T\rho_c(T) + \rho_d(T)K_F\|_X \leq \frac{c_1}{L_a + \delta} \delta \quad (28)$$

where ρ_c and ρ_d come from (26) and (27) respectively. K_F is the constant from (16) and L_a is from (13). Choosing $T_\gamma^* = \min\{T_\alpha^*, T_a, T_b, T^*\}$, where T^* is from Definition 2. Obviously, for all $T \in (0, T_\gamma^*)$, it follows that

$$\begin{aligned} &\|\phi_T^e(1, x_0) - \phi_T^a(1, x_0)\|_X \\ &= \|\{A_T^e + B_T^e F_T\}x_0 - \{A_T^a + B_T^a F_T\}x_0\|_X \\ &\leq \|\{A_T^e - A_T^a\}x_0\|_X + \|\{(B_T^e - B_T^a)F_T\}x_0\|_X \\ &\leq (T\rho_c(T) + \rho_d(T)K_F) \|x_0\|_X \\ &< \frac{c_1}{L_a + \delta} \delta \|x_0\|_X \\ &< \delta \|x_0\|_X \end{aligned} \quad (29)$$

We prove our result by Induction. Assume that for arbitrary k , for any $T \in (0, T^*)$, $(k+1)T \in [0, t]$, we have

$$\|\phi_T^e(k, x_0) - \phi_T^a(k, x_0)\|_X \leq \delta \|x_0\|_X, \quad (30)$$

which implies that $\|x(k)\|_X = \|\phi_T^e(k, x_0)\|_X \leq \|\phi_T^a(k, x_0)\|_X + \delta \|x_0\|_X$. Consequently, we have

$$\begin{aligned} &\|\phi_T^e(k+1, x_0) - \phi_T^a(k+1, x_0)\|_X \\ &= \|\phi_T^e(1, x_k) - \phi_T^a(1, x_k)\|_X \\ &= \|\{A_T^e + B_T^e F_T\}x(k) - \{A_T^a + B_T^a F_T\}x(k)\|_X \\ &\leq \|\{A_T^e - A_T^a\}x(k)\|_X + \|\{(B_T^e - B_T^a)F_T\}x(k)\|_X \\ &\leq (T\rho_c(T) + \rho_d(T)K_F) \|x(k)\|_X \\ &\leq (T\rho_c(T) + \rho_d(T)K_F) (\|\phi_T^a(k, x_0)\|_X + \delta \|x_0\|_X) \\ &\leq (T\rho_c(T) + \rho_d(T)K_F) (L_a \|x_0\|_X + \delta \|x_0\|_X) \\ &\leq \delta \|x_0\|_X \end{aligned} \quad (31)$$

which completes the proof. ■

Remark 3: Theorem 2 is an extension of Lax-Richtmyer Equivalence Theorem [11] with the consideration of the feedback $u(k) = F_T x(k)$, though in this paper, only time discretization is considered. The consistent time-discretization scheme (A_T^a, B_T^a) as well as the uniform boundedness of the approximate model (Definition 2) ensure that the solutions of the approximate model (12) will converge to the solution of the exact model (11) when the sampling $T \rightarrow 0$ (on compact time intervals).

Remark 4: Since Lax-Richtmyer Equivalence Theorem [11] is applicable not only time discretization but also space discretization (finite element method) or both, it is possible to extend the result in Theorem 2 to design a finite dimensional sampled-data controller via its finite dimensional discrete time approximate plant model. We are currently working on this extension.

Combining Theorem 1 and Theorem 2, we have the following corollary:

Corollary 1: Assume that (A_T^a, B_T^a) is a consistent time-discretization scheme for the well-posed abstract differential

equation (2), given the controller (10) where F_T is a linear operator uniform in T . If the solution of approximated model (12) is exponentially stable uniformly in small parameter T , then the solution of the exact model (7) is exponentially stable.

Remark 5: The result of Corollary 1 is analogous to the result in [3, Theorem 1], where the multi-step consistency condition indicates that the stability properties of the approximate model ensure the stability properties of the exact model. In [3], nonlinear LPS considered while in this paper linear SDPS is considered. However, under appropriate assumptions on the existence of the solution of nonlinear SDPS, it is quite possible that the result could be extended to nonlinear SDPS.

4. CONCLUSIONS

In this paper, we present preliminary results on stabilization of sampled-data linear SDPS and present sufficient conditions which ensure that the controller stabilizing the discrete time approximate model would also stabilize the original SDPS systems when sampling is fast enough. We are currently working on the extensions of these results for finite dimensional sampled-data control design via finite dimensional discrete time approximate plant models of sampled data SDPS.

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