

The probability that all eigenvalues are real for products of truncated real orthogonal random matrices

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Abstract The probability that all eigenvalues of a product of m independent $N \times N$ sub-blocks of a Haar distributed random real orthogonal matrix of size $(L_i + N) \times (L_i + N)$, ($i = 1, \dots, m$) are real is calculated as a multi-dimensional integral, and as a determinant. Both involve Meijer G-functions. Evaluation formulae of the latter, based on a recursive scheme, allow it to be proved that for any m and with each L_i even the probability is a rational number. The formulae furthermore provide for explicit computation in small order cases.

Keywords Random matrix products · Truncated orthogonal matrices · Probability of real eigenvalues · Meijer G-functions · Arithmetic structures

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1 Introduction

In general an $N \times N$ real matrix may have both real and complex eigenvalues. A natural question is thus to ask for the probability $p_{N,k}^X$ that a random real matrix X chosen from a particular distribution has a specific number k of real eigenvalues. Due to the complex eigenvalues coming in complex conjugate pairs, k must have the same parity as N for $p_{N,k}^X$ to be non-zero.

This question is of interest from a number of different viewpoints. In probability theory, for large N , the distribution of $p_{N,k}^X$ in the case that $X = G_1^{-1}G_2$, where G_1, G_2 are standard real Gaussian matrices, can be proved [12] to satisfy a local central limit theorem with

$$\text{Var}_N \sim (2 - \sqrt{2})\mu_N, \quad \mu_N \asymp \sqrt{N}. \quad (1.1)$$

The recent work [25] proves a central limit theorem for polynomial functionals of the real eigenvalues in this setting, and the relation (1.1) is found. This latter relation was first observed in [13] in the case of matrices X drawn from the real Ginibre ensemble, and thus with independent standard real Gaussian

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entries. Very recently (1.1) has been observed in numerical simulations to hold for a wide class of random matrix ensembles, [and with a corresponding distribution consistent with a local limit theorem](#) [16].

The large N form of the probability $p_{N,0}^X$ (N even) that no eigenvalues are real has been observed in [18] to be intimately related to the large s form of the probability that the interval $[-s/2, s/2]$ of the real line is free of eigenvalues with the limit $N \rightarrow \infty$ already taken. For X a member of the real Ginibre ensemble, the latter asymptotic form was computed in [11]. [Paper \[17\] also calculates the large \$N\$ asymptotics of \$P_{N,k}^X\$ for \$k \ll \sqrt{N}/\log N\$.](#) The other extreme, the probability $p_{N,N}^X$ that all eigenvalues are real, is known to be related to large deviation principles [15], allowing for its asymptotic determination in the case of X equal to the product of m real Ginibre matrices [10].

From the viewpoint of applications, the probability $p_{N,N}^X$ in the case X of the form $G_1^{-1}G_2$ gives the probability that random elements from a certain tensor structure have minimal rank [24]. With G_1, G_2 real Ginibre matrices this probability can be computed exactly [12,3]. The probability $p_{N,N}^X$ with $N = 2$ and $X = G_1G_2$ was shown in [21] to have an interpretation in quantum entanglement, quantifying when two-qubits $|\phi_1\rangle$ and $|\phi_2\rangle$ chosen from a uniform distribution on the 3-sphere are an optimal pair.

Another interest in $p_{N,k}^X$, for X a real Ginibre matrix [5,2,23], the inverse of the real Ginibre matrix times a real Ginibre matrix [12], and the product of two real Ginibre matrices [10,20,8], are its special arithmetic properties. The ability to probe these properties relies on integrable structures associated with the computation of probabilities in these cases. In addition, the proof in the case of the product of two real Ginibre matrices relies on new evaluation formulae for certain Meijer G-functions [20]. Integrable structures are also present in the computation of statistical distributions relating to products of truncated real orthogonal matrices [19,9,17]. It is our aim in this paper to make use of the integrable structures to give a formula for the probability $p_{N,N}^X$ in the case that X is formed from the product of m truncated real orthogonal matrices, and to isolate arithmetic properties in the case $m = 1$ (general N , L_1), the case $m = 2$ ($L_1 = 1$, $L_2 = 2$, small N), and for $m \geq 2$ (general N , all L_i even). The latter requires the derivation of some further evaluation formulae for certain Meijer G-functions. The evaluations of $p_{N,N}^X$, both in the form of a multi-dimensional integral, and a determinant, are given in Section 2. The multi-dimensional integral can be evaluated as a product of gamma functions in the case $m = 1$, making the arithmetic properties immediate. Section 3 contains the evaluation formulae for certain Meijer G-functions in terms of recurrences, and from this it follows that $p_{N,N}^X$ is rational when the L_i are even. We conclude in Section 4 by implementing the recurrences in some low order cases (equivalent to $m = 1$ and $m = 2$) to give some explicit evaluations of $p_{N,N}^X$.

2 The probability $p_{N,N}^X$ for products of truncated real orthogonal matrices

2.1 Multidimensional integral formula

Let R be a Haar distributed random real orthogonal matrix of size $(L + N) \times (L + N)$, and let D be an $N \times N$ sub-block of R . A peculiarity of this setting is that only for $L \geq N$ is the corresponding probability density function of D free of delta function constraints, and given by the smooth function (see e.g. [9, Eq. (3.113)])

$$P(D) = \frac{1}{C_{N,L}} \det(\mathbb{I} - D^T D)^{(L-N-1)/2}, \quad (2.1)$$

where

$$C_{N,L} = \pi^{N^2/2} \prod_{j=0}^{N-1} \frac{\Gamma((L-N+1+j)/2)}{\Gamma((L+1+j)/2)}. \quad (2.2)$$

Our interest is in the real eigenvalues of the product of random matrices

$$P_m = D_1 D_2 \cdots D_m, \quad (2.3)$$

where each D_i is of size $N \times N$, but constructed as a sub-block of a $(L_i + N) \times (L_i + N)$ random real orthogonal matrix. To be able to make use of (2.1) we will require each $L_i \geq N$, however the final formulae to be obtained are well defined for all $L_i \geq 0$ and remain valid in this range.

Proposition 1. *Let P_m be specified as in (2.3). Define*

$$w_m(x) = G_{m,m}^{m,0} \left(\begin{matrix} L_1/2, L_2/2, \dots, L_m/2 \\ 0, 0, \dots, 0 \end{matrix} \middle| x^2 \right), \quad (2.4)$$

where $G_{m,m}^{m,0}$ denotes a particular Meijer G-function as specified in e.g. [22]. Then for $L_i \geq 0$ ($i = 1, \dots, m$),

$$p_{N,N}^{P_m} = \prod_{i=1}^m \prod_{s=0}^{N-1} \frac{\Gamma((L_i + 1 + s)/2)}{\Gamma((s + 1)/2)} \int_{\lambda_1 > \lambda_2 > \dots > \lambda_N} \prod_{l=1}^N w_m(\lambda_l) \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) d\lambda_1 \dots d\lambda_N. \quad (2.5)$$

Proof. Following an idea in [1], we decompose each D_i so that $D_i = Q_i R_i Q_{i+1}^T$, where each Q_i is real orthogonal and R_i is upper triangular,

$$R_i = \begin{bmatrix} \lambda_1^{(i)} & & & \\ & \lambda_2^{(i)} & & \\ & & \ddots & \\ & & & \lambda_N^{(i)} \end{bmatrix} + T_i,$$

with the matrix T_i being strictly upper triangular.

To keep the working succinct, we suppose for the time being that $L_i \geq N$ ($i = 1, \dots, m$), in which each D_i has probability density function (2.1). Ignoring the label (i) for the time being, we know from workings in [23] that

$$\begin{aligned} & \frac{1}{C_{N,L}} \int \det(\mathbb{I} - D^T D)^{(L-N-1)/2} (dT)(dQ) \\ &= \frac{1}{C_{N,L}} \frac{\pi^{N(N+1)/4}}{\prod_{j=1}^N \Gamma(j/2)} \prod_{s=1}^N \pi^{(N-s)/2} \frac{\Gamma((L-N+s)/2)}{\Gamma(L/2)} \prod_{s=1}^N (1 - \lambda_s^2)^{L/2-1} \\ &= \prod_{s=0}^{N-1} \frac{\Gamma((L+1+s)/2)}{\Gamma((s+1)/2)\Gamma(L/2)} \prod_{s=1}^N (1 - \lambda_s^2)^{L/2-1}. \end{aligned}$$

We now re-instate the (i) by writing $L \mapsto L_i$, $\lambda_s \mapsto \lambda_s^{(i)}$, and we require that $\lambda_s = \prod_{i=1}^m \lambda_s^{(i)}$, where $\{\lambda_s\}$ are the eigenvalues of the product (P_m). In changing variables to $\{\lambda_s\}$, and integrating out over (dQ) and (dT), we obtain for the joint probability density of these eigenvalues the functional form

$$\begin{aligned} & \prod_{i=1}^m \prod_{s=0}^{N-1} \frac{\Gamma((L_i + 1 + s)/2)}{\Gamma((s + 1)/2)\Gamma(L_i/2)} \\ & \times \prod_{l=1}^N \int_{-1}^1 d\lambda_l^{(1)} \dots \int_{-1}^1 d\lambda_l^{(m)} \delta\left(\lambda_l - \prod_{i=1}^m \lambda_l^{(i)}\right) \prod_{i=1}^m (1 - (\lambda_l^{(i)})^2)^{L_i/2-1} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k). \quad (2.6) \end{aligned}$$

Now write

$$F(\lambda) = \int_{-1}^1 d\lambda^{(1)} \dots \int_{-1}^1 d\lambda^{(m)} \delta\left(\lambda - \prod_{i=1}^m \lambda^{(i)}\right) \prod_{i=1}^m (1 - (\lambda^{(i)})^2)^{L_i/2-1}. \quad (2.7)$$

Taking the Mellin transform, we then have that

$$\int_{-\infty}^{\infty} F(\lambda) |\lambda|^{p-1} d\lambda = \int_{-1}^1 \cdots \int_{-1}^1 \prod_{i=1}^m |\lambda^{(i)}|^{p-1} (1 - (\lambda^{(i)})^2)^{L_i/2-1} d\lambda^{(i)}.$$

After a change of variables, these are all Euler beta integrals, and so

$$\int_0^{\infty} F(\lambda) \lambda^{p-1} d\lambda = \phi(p), \quad \phi(p) = \frac{1}{2} \prod_{i=1}^m \frac{\Gamma(p/2) \Gamma(L_i/2)}{\Gamma((p+L_i)/2)}.$$

Taking the inverse Mellin transform, we thus have that

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{\infty} x^{-2s} \prod_{l=1}^m \frac{\Gamma(s) \Gamma(L_l/2)}{\Gamma(s+L_l/2)} ds = \prod_{l=1}^m \Gamma(L_l/2) w_m(x),$$

where $w_m(x)$ is given by (2.4).

Recalling the definition of $F(x)$ as given by (2.7), and substituting in (2.6), we obtain (2.5), although with the restriction $L_i \geq N$. This restriction was imposed because of the essential use of the density (2.1). However, at the expense of more complex and lengthy working, it is possible to do without (2.1), making use instead of the fact that with D the top $N \times N$ sub-block of the $(L+N) \times (L+N)$ Haar distributed real orthogonal matrix, and B the bottom $L \times L$ sub-block, is proportional to $\delta(D^T D + B^T B - \mathbb{I}_{N+L})$. In the case $m=1$, the necessary working is sketched in [19], with the full details, including a generalisation to so called induced ensembles [6], given in the thesis [7]. This has the consequence of allowing (2.5) to be established for all $L_i \geq 0$.

In the special case $m=1$ we have

$$G_{1,1}^{1,0} \left(\begin{matrix} L_1/2 \\ 0 \end{matrix} \middle| x^2 \right) = \frac{1}{\Gamma(L_1/2)} (1-x^2)^{L_1/2-1} \chi_{|x|<1}. \quad (2.8)$$

Substituting in (2.5), the resulting multidimensional integral can be evaluated in terms of a product of gamma functions, allowing the arithmetic properties of $p_{N,N}^{P_1}$ to be specified.

Corollary 2. *We have*

$$p_{N,N}^{P_1} = \prod_{j=0}^{N-1} \frac{\Gamma(L_1+j) \Gamma((L_1+j)/2)}{\Gamma(L_1+(N+j-1)/2) \Gamma(L_1/2)}. \quad (2.9)$$

As a consequence, for L_1 even, $p_{N,N}^{P_1}$ is rational, while for L_1 odd, it is equal to a rational number times $1/\pi^{\lfloor N/2 \rfloor}$.

Proof. Substituting (2.8) in (2.5) gives

$$\begin{aligned} p_{N,N}^{P_1} &= \prod_{s=0}^{N-1} \frac{\Gamma((L_1+1+s)/2)}{\Gamma((s+1)/2) \Gamma(L_1/2)} \int_{\lambda_1 > \lambda_2 > \cdots > \lambda_N} \prod_{l=1}^N (1-\lambda_l^2)^{L_l/2-1} \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) d\lambda_1 \cdots d\lambda_N \\ &= \frac{1}{N!} \prod_{s=0}^{N-1} \frac{\Gamma((L_1+1+s)/2)}{\Gamma((s+1)/2) \Gamma(L_i/2)} \int_{-1}^1 \cdots \int_{-1}^1 \prod_{l=1}^N (1-\lambda_l^2)^{L_l/2-1} \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k| d\lambda_1 \cdots d\lambda_N. \end{aligned}$$

Changing variables in the integral shows that it is equal to

$$2^{NL_1} 2^{N(N-3)/2} \int_0^1 \cdots \int_0^1 \prod_{j=1}^N x_j^{L_1/2-1} (1-x_j)^{L_1/2-1} \prod_{1 \leq j < k \leq N} |x_k - x_j| dx_1 \cdots dx_N.$$

This is a particular Selberg integral (see e.g. [9, Ch. 4]) and so can be evaluated to give

$$2^{NL_1} 2^{N(N-3)/2} \prod_{j=0}^{N-1} \frac{(\Gamma(L_1/2 + j/2))^2 \Gamma(1 + (j+1)/2)}{\Gamma(L_1 + (N+j-1)/2) \Gamma(3/2)}.$$

Substituting and simplifying, (2.9) results. \square

Remark 3. Scaling the matrix D in (2.1) by $1/\sqrt{L}$ and taking the limit $L \rightarrow \infty$ shows that the distribution $P(D)$ tends to a Gaussian, now being proportional to $e^{-\frac{1}{2}\text{Tr} X^2}$. It has been known for some time that the probability of all eigenvalues being real for a real standard Gaussian matrix is equal to $2^{-N(N-1)/4}$ [5]. Indeed taking the limit $L_1 \rightarrow \infty$ in (2.9) reclaims this value.

2.2 Determinant formula

It is a standard exercise in random matrix theory to write the [multidimensional](#) integral in terms of a Pfaffian, [using a method based on integration over alternate variables due to de Bruijn](#) [4]. The details depend on the parity of N . As noted in earlier studies relating to the computation of probabilities relating to real eigenvalues for certain random matrix ensembles [13, 14, 12, 10, 8], the fact that the resulting matrix entries vanish in a checkerboard fashion allows the Pfaffian, when expressed as the square root of an anti-symmetric matrix, to be then expressed as a determinant of half the size.

Proposition 4. *With $w_m(x)$ given by (2.4), define*

$$\alpha_{j,k} = \int_{-1}^1 dx \int_{-1}^1 dy w_m(x) w_m(y) x^{j-1} y^{k-1} \text{sgn}(y-x), \quad \nu_j = \int_{-1}^1 w_m(x) x^{j-1} dx, \quad (2.10)$$

$$p_{N,N}^{P_m} = \prod_{i=1}^m \prod_{s=0}^{N-1} \frac{\Gamma((L_i + 1 + s)/2)}{\Gamma((s+1)/2)} \det A, \quad (2.11)$$

where for N even

$$A = [\alpha_{2j-1, 2k}]_{j,k=1, \dots, N/2}, \quad (2.12)$$

while for N odd

$$A = \left[[\alpha_{2j-1, 2k}]_{\substack{j=1, \dots, (N+1)/2 \\ k=1, \dots, (N-1)/2}} [\nu_{2j-1}]_{j=1, \dots, (N+1)/2} \right]. \quad (2.13)$$

Moreover, the matrix elements (2.10) permit the evaluations

$$\alpha_{2j-1, 2k} = G_{2m+1, 2m+1}^{m+1, m} \left(\begin{matrix} 3/2 - j, \dots, 3/2 - j; 1, L_1/2 + k, L_2/2 + k, \dots, L_m/2 + k \\ 0, k, \dots, k; 3/2 - j - L_1/2, \dots, 3/2 - j - L_m/2 \end{matrix} \middle| 1 \right), \quad (2.14)$$

as well as

$$\nu_{2j-1} = \prod_{l=1}^m \frac{\Gamma(j-1/2)}{\Gamma(L_l/2 + j-1/2)}. \quad (2.15)$$

Proof. In addition to the original paper [4], the de Bruijn formulae for the expression of the multiple integral (2.5) as a Pfaffian are given in e.g. [9, Prop. 6.3.4 (N even), Exercises 6.3 q.1 (N odd)]. Explicitly, for N even this gives (2.11) with $\det A$ replaced by $\text{Pf} [\alpha_{j,k}]_{j,k=1, \dots, N}$. The fact that $w_m(x)$ is even reveals from the definition (2.10) that

$$\alpha_{2j, 2k} = \alpha_{2j-1, 2k-1} = 0 \quad (j, k = 1, \dots, N/2),$$

so every alternate element in the matrix $[\alpha_{j,k}]$ is zero. Interchanging rows and columns so that the zero elements are all in the top left and bottom right block and noting $\alpha_{2k, 2j-1} = -\alpha_{2j-1, 2k}$ shows that $\text{Pf} [\alpha_{j,k}]_{j,k=1, \dots, N} = \det A$ as required. The N odd case is similar. The evaluations (2.14) and (2.15) follow from standard Meijer G-function formulae (see e.g. [22]). \square

Remark 5. Taking the limits $L_1, \dots, L_m \rightarrow \infty$, it follows from the definition of the Meijer G-function as a contour integral [22] that

$$p_{N,N}^{P_m} \rightarrow \prod_{s=0}^{N-1} \left(\frac{1}{\Gamma((s+1)/2)} \right)^m \begin{cases} \det[\tilde{\alpha}_{2j-1,2k}]_{j,k=1,\dots,N/2}, & N \text{ even} \\ \det \left[[\tilde{\alpha}_{2j-1,2k}]_{\substack{j=1,\dots,(N+1)/2 \\ k=1,\dots,(N-1)/2}} [\tilde{\nu}_{2j-1}]_{j=1,\dots,(N+1)/2} \right], & N \text{ odd} \end{cases}$$

with

$$\tilde{\alpha}_{j,k} = G_{m+1,m+1}^{m+1,m} \left(\begin{matrix} 3/2 - j, \dots, 3/2 - j; 1 \\ 0, k, \dots, k \end{matrix} \middle| 1 \right), \quad \tilde{\nu}_{2j-1} = (\Gamma(j-1/2))^m.$$

In keeping with Remark 3, this is the functional form derived in [10] for the probability that all eigenvalues are real for a product of m real standard Gaussian matrices of size $N \times N$.

Remark 6. Using the working used to derive [10, Prop. 6], we can show that, assuming $L_i \neq 0$ for all i ,

$$\lim_{m \rightarrow \infty} \frac{\prod_{i=1}^m \Gamma(L_i + 2 + j - 1/2) \Gamma(L_i/2 + k)}{(\Gamma(j-1/2) \Gamma(k))^m} \alpha_{2j-1,2k} = \begin{cases} 1, & j \leq k \\ 0, & j > k \end{cases}$$

and thus $\lim_{m \rightarrow \infty} p_{N,N}^{P_m} = 1$, in accordance with an effect first noted in [21].

3 Evaluation of some Meijer G-functions

Proposition 7. Consider, for positive integers μ, ν, j, k , the expression

$$K_{j,k}^{\mu,\nu} := \frac{\Gamma(j-1/2)}{\Gamma(\mu) \Gamma(\mu + \nu + j + k - 3/2)} \sum_{r=1}^{\nu} \frac{\Gamma(j+k+r-3/2) \Gamma(\mu + \nu - r)}{\Gamma(j+r-1/2) \Gamma(\nu - r + 1)}. \quad (3.1)$$

Then we have the following finite-sum results:

$$\begin{aligned} & G_{2m+1,2m+1}^{m+1,m} \left(\begin{matrix} 3/2 - j, \dots, 3/2 - j; 1, \mu + k, k, \dots, k \\ 0, k, \dots, k; 3/2 - j - \nu, 3/2 - j, \dots, 3/2 - j \end{matrix} \middle| 1 \right) \\ &= G_{3,3}^{2,1} \left(\begin{matrix} 3/2 - j; 1, \mu + k \\ 0, k; 3/2 - j - \nu \end{matrix} \middle| 1 \right) = K_{j,k}^{\mu,\nu}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} & G_{2m+1,2m+1}^{m+1,m} \left(\begin{matrix} 3/2 - j, \dots, 3/2 - j; 1, \mu + k, \nu + k, k, \dots, k \\ 0, k, \dots, k; 3/2 - j - \mu, 3/2 - j - \nu, 3/2 - j, \dots, 3/2 - j \end{matrix} \middle| 1 \right) \\ &= G_{5,5}^{3,2} \left(\begin{matrix} 3/2 - j, 3/2 - j; 1, \mu + k, \nu + k \\ 0, k, k; 3/2 - j - \mu, 3/2 - j - \nu \end{matrix} \middle| 1 \right) \\ &= \sum_{\xi=1}^{\mu} \sum_{\eta=1}^{\nu} \frac{\Gamma(2\mu - \xi) \Gamma(\xi + j + k - 3/2) \Gamma(2\nu - \eta) \Gamma(\eta + j + k - 3/2)}{\Gamma(\mu) \Gamma(\mu - \xi + 1) \Gamma(2\mu + j + k - 3/2) \Gamma(\nu) \Gamma(\nu - \eta + 1) \Gamma(2\nu + j + k - 3/2)} \\ &\quad \times \left(K_{j,k}^{\xi,\eta} + K_{j,k}^{\eta,\xi} + \frac{\Gamma^2(j-1/2)}{\Gamma(\xi + j - 1/2) \Gamma(\eta + j - 1/2)} \right). \end{aligned} \quad (3.3)$$

We note that these Meijer G-functions are rational numbers. These results will be used in Section 4 to obtain explicit evaluation of the probability $p_{N,N}^{P_m}$ for $m = 2$ and L_1, L_2 even.

Proof. We begin with the following three-term recurrence relation which is satisfied by Meijer G-functions:

$$\begin{aligned} & G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_n; a_{n+1}, \dots, a_{p-1}, a_p - 1 \\ b_1, \dots, b_m; b_{m+1}, \dots, b_q \end{matrix} \middle| z \right) + G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_n; a_{n+1}, \dots, a_{p-1}, a_p \\ b_1, \dots, b_m; b_{m+1}, \dots, b_{q-1}, b_q + 1 \end{matrix} \middle| z \right) \\ &= (a_p - b_q - 1) G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_n; a_{n+1}, \dots, a_p \\ b_1, \dots, b_m; b_{m+1}, \dots, b_q \end{matrix} \middle| z \right); \quad n < p, m < q. \end{aligned} \quad (3.4)$$

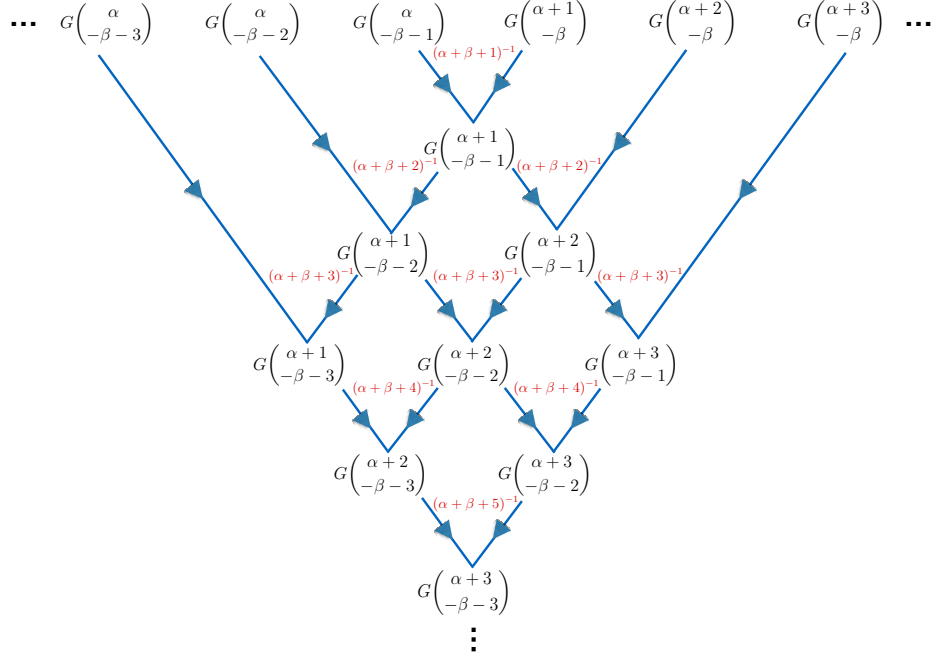


Fig. 1 Diagram facilitating the derivation of recurrence relation (3.5).

With the aid of this relation we can construct the diagram as shown in Fig. 1. In this diagram we use the shorthand notation G_{β}^{α} to represent the Meijer G-function $G_{2m+1,2m+1}^{m+1,m} \left(\begin{matrix} a_1, \dots, a_m; a_{m+1}, \dots, a_{2m}, \alpha \\ b_1, \dots, b_{m+1}; b_{m+2}, \dots, b_{2m}, \beta \end{matrix} \middle| 1 \right)$. The arrows show how successively new Meijer G-functions can be constructed from the previous ones. The ‘inverse terms’ (in red – colour-on-line) are the weights that have to be considered while constructing the Meijer G-functions. For example, $G_{-\beta-1}^{\alpha+1} = (\alpha + \beta + 1)^{-1} [G_{-\beta-1}^{\alpha} + G_{-\beta}^{\alpha+1}]$, $G_{-\beta-2}^{\alpha+1} = (\alpha + \beta + 2)^{-1} [G_{-\beta-2}^{\alpha} + G_{-\beta-1}^{\alpha+1}]$, etc. All the Meijer G’s below the topmost line can be constructed using those at the topmost line by following the arrows. A careful observation leads to the following recurrence relation:

$$G_{-\beta-\nu}^{\alpha+\mu} = \sum_{r=1}^{\nu} C_{\mu,\nu+1-r} \left(\prod_{s=r}^{\mu+\nu-1} (\alpha + \beta + s)^{-1} \right) G_{-\beta-r}^{\alpha} + \sum_{r=1}^{\mu} C_{\nu,\mu+1-r} \left(\prod_{s=r}^{\mu+\nu-1} (\alpha + \beta + s)^{-1} \right) G_{-\beta}^{\alpha+r}. \quad (3.5)$$

Here the coefficients $C_{i,j}$ are given by

$$C_{i,j} = \binom{i+j-2}{j-1} = \frac{\Gamma(i+j-1)}{\Gamma(i)\Gamma(j)} = \frac{1}{(i+j-1)\mathbf{B}(i,j)}. \quad (3.6)$$

Interestingly, these coefficients satisfy the recurrence relation

$$C_{1,j} = 1, \quad C_{i,j} = \sum_{r=1}^j C_{i-1,r}, \quad (3.7)$$

and form a tilted Pascal’s triangle, as depicted in Fig. 2.

$j \backslash i$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	4	5
3	1	3	6	10	15
4	1	4	10	20	35
5	1	5	15	35	70

Fig. 2 Some of the coefficients as defined in (3.6). These coefficients constitute a tilted Pascal's triangle.

Now, for non-negative integers l_1, \dots, l_m , when not all of them are 0, we have the following identities that follow from the standard contour integral formula for Meijer G-function [22]:

$$G_{2m+1, 2m+1}^{m+1, m} \left(\begin{matrix} 3/2 - j, \dots, 3/2 - j; 1, l_1 + k, \dots, l_m + k \\ 0, k, \dots, k; 3/2 - j, \dots, 3/2 - j \end{matrix} \middle| 1 \right) = 0, \quad (3.8)$$

$$\begin{aligned} G_{2m+1, 2m+1}^{m+1, m} \left(\begin{matrix} 3/2 - j, \dots, 3/2 - j; 1, k, \dots, k \\ 0, k, \dots, k; 3/2 - j - l_1, \dots, 3/2 - j - l_m \end{matrix} \middle| 1 \right) &= \prod_{s=1}^m \prod_{r_s=1}^{l_s} \frac{1}{(j + r_s - 3/2)} \\ &= \prod_{s=1}^m \frac{\Gamma(j - 1/2)}{\Gamma(j + l_s - 1/2)}. \end{aligned} \quad (3.9)$$

We note that (3.9) is related to (2.15). When $l_1 = \dots = l_m = 0$, the Meijer G-function $G_{2m+1, 2m+1}^{m+1, m} \left(\begin{matrix} 3/2 - j, \dots, 3/2 - j; 1, k, \dots, k \\ 0, k, \dots, k; 3/2 - j, \dots, 3/2 - j \end{matrix} \middle| z \right)$ reduces to $G_{1,1}^{1,0} \left(\begin{matrix} 1 \\ 0 \end{matrix} \middle| z \right)$, which is a theta function involving $|z|$,

$$G_{1,1}^{1,0} \left(\begin{matrix} 1 \\ 0 \end{matrix} \middle| z \right) = \Theta(1 - |z|) = \begin{cases} 1, & |z| < 1, \\ 0, & |z| > 1, \end{cases} \quad (3.10)$$

and hence discontinuous at $z = 1$. It turns out that taking its value to be $1/2$ at $z = 1$ gives correct result for certain probability, as observed in Section 4 ahead.

With the above results at our hands, (3.2) follows as

$$\begin{aligned} &G_{2m+1, 2m+1}^{m+1, m} \left(\begin{matrix} 3/2 - j, 3/2 - j; 1, \mu + k, k, \dots, k \\ 0, k, \dots, k; 3/2 - j - \nu, 3/2 - j, \dots, 3/2 - j \end{matrix} \middle| 1 \right) = G_{3,3}^{2,1} \left(\begin{matrix} 3/2 - j; 1, \mu + k \\ 0, k; 3/2 - j - \nu \end{matrix} \middle| 1 \right) \\ &= \sum_{r=1}^{\nu} C_{\mu, \nu+1-r} \left(\prod_{s=r}^{\mu+\nu-1} (j + k + s - 3/2)^{-1} \right) G_{3,3}^{2,1} \left(\begin{matrix} 3/2 - j; 1, k \\ 0, k; 3/2 - j - r \end{matrix} \middle| 1 \right) \\ &+ \sum_{r=1}^{\mu} C_{\nu, \mu+1-r} \left(\prod_{s=r}^{\mu+\nu-1} (j + k + s - 3/2)^{-1} \right) G_{3,3}^{2,1} \left(\begin{matrix} 3/2 - j; 1, k + r \\ 0, k; 3/2 - j \end{matrix} \middle| 1 \right) \\ &= \sum_{r=1}^{\nu} C_{\mu, \nu+1-r} \left(\prod_{s=r}^{\mu+\nu-1} (j + k + s - 3/2)^{-1} \right) \left(\prod_{t=1}^r (j + t - 3/2)^{-1} \right) + 0 \\ &= \frac{\Gamma(j - 1/2)}{\Gamma(\mu)\Gamma(j + k + \mu + \nu - 3/2)} \sum_{r=1}^{\nu} \frac{\Gamma(j + k + r - 3/2) \Gamma(\mu + \nu - r)}{\Gamma(j + r - 1/2) \Gamma(\nu - r + 1)}. \end{aligned}$$

In the second-last step we used Eqs. (3.8) and (3.9).

Now, let us consider

$$\begin{aligned}
U_{j,k}^{\mu,\nu} &:= G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \mu+k, k \\ 0, k, k; 3/2-j-\mu, 3/2-j-\nu \end{matrix} \middle| 1 \right) = G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, k, \mu+k \\ 0, k, k; 3/2-j-\nu, 3/2-j-\mu \end{matrix} \middle| 1 \right) \\
&= \sum_{\xi=1}^{\mu} C_{\mu, \mu+1-\xi} \left(\prod_{s=\xi}^{2\mu-1} (j+k+s-3/2)^{-1} \right) \left(G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, k, k \\ 0, k, k; 3/2-j-\nu, 3/2-j-\xi \end{matrix} \middle| 1 \right) \right. \\
&\quad \left. + G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, k, k+\xi \\ 0, k, k; 3/2-j, 3/2-j-\nu \end{matrix} \middle| 1 \right) \right) \\
&= \sum_{\xi=1}^{\mu} \frac{\Gamma(2\mu-\xi)\Gamma(\xi+j+k-3/2)}{\Gamma(\mu)\Gamma(\mu-\xi+1)\Gamma(2\mu+j+k-3/2)} \left(\frac{\Gamma^2(j-1/2)}{\Gamma(\nu+j-1/2)\Gamma(\xi+j-1/2)} + K_{j,k}^{\xi,\nu} \right), \quad (3.11)
\end{aligned}$$

$$\begin{aligned}
V_{j,k}^{\mu,\nu} &:= G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \mu+k, \nu+k \\ 0, k, k; 3/2-j-\mu, 3/2-j \end{matrix} \middle| 1 \right) = G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \nu+k, \mu+k \\ 0, k, k; 3/2-j, 3/2-j-\mu \end{matrix} \middle| 1 \right) \\
&= \sum_{\xi=1}^{\mu} C_{\mu, \mu+1-\xi} \left(\prod_{s=\xi}^{2\mu-1} (j+k+s-3/2)^{-1} \right) \left(G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \nu+k, k \\ 0, k, k; 3/2-j, 3/2-j-\xi \end{matrix} \middle| 1 \right) + 0 \right) \\
&= \sum_{\xi=1}^{\mu} \frac{\Gamma(2\mu-\xi)\Gamma(\xi+j+k-3/2)}{\Gamma(\mu)\Gamma(\mu-\xi+1)\Gamma(2\mu+j+k-3/2)} K_{j,k}^{\nu,\xi}. \quad (3.12)
\end{aligned}$$

Finally, the result (3.3) follows as

$$\begin{aligned}
&G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \mu+k, \nu+k \\ 0, k, k; 3/2-j-\mu, 3/2-j-\nu \end{matrix} \middle| 1 \right) = G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \nu+k, \mu+k \\ 0, k, k; 3/2-j-\nu, 3/2-j-\mu \end{matrix} \middle| 1 \right) \\
&= \sum_{\xi=1}^{\mu} C_{\mu, \mu+1-\xi} \left(\prod_{s=\xi}^{2\mu-1} (j+k+s-3/2)^{-1} \right) \left(G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \nu+k, k \\ 0, k, k; 3/2-j-\nu, 3/2-j-\xi \end{matrix} \middle| 1 \right) \right. \\
&\quad \left. + G_{5,5}^{3,2} \left(\begin{matrix} 3/2-j, 3/2-j; 1, \nu+k, \xi+k \\ 0, k, k; 3/2-j-\nu, 3/2-j \end{matrix} \middle| 1 \right) \right) \\
&= \sum_{\xi=1}^{\mu} \frac{\Gamma(2\mu-\xi)\Gamma(\xi+j+k-3/2)}{\Gamma(\mu)\Gamma(\mu-\xi+1)\Gamma(2\mu+j+k-3/2)} (U_{j,k}^{\nu,\xi} + V_{j,k}^{\nu,\xi}) \\
&= \sum_{\xi=1}^{\mu} \sum_{\eta=1}^{\nu} \frac{\Gamma(2\mu-\xi)\Gamma(\xi+j+k-3/2)\Gamma(2\nu-\eta)\Gamma(\eta+j+k-3/2)}{\Gamma(\mu)\Gamma(\mu-\xi+1)\Gamma(2\mu+j+k-3/2)\Gamma(\nu)\Gamma(\nu-\eta+1)\Gamma(2\nu+j+k-3/2)} \\
&\quad \times \left(K_{j,k}^{\xi,\eta} + K_{j,k}^{\eta,\xi} + \frac{\Gamma^2(j-1/2)}{\Gamma(\xi+j-1/2)\Gamma(\eta+j-1/2)} \right).
\end{aligned}$$

□

Remark 8. We used the identities (3.8) and (3.9) along with the recurrence relation (3.5) to obtain relevant Meijer G-functions for calculating $p_{N,N}^m$ for $m = 1, 2$ when the L_i are even. It turns out that actually (3.8), (3.9) and (3.5) can be used repeatedly in a systematic manner to obtain Meijer G-functions for any m and even L_i . This is facilitated by the diagram shown in Fig. 3. In this diagram we have used the notation $G_{2m+1, 2m+1}^{m+1, m} \left(\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_m \end{matrix} \right)$ to represent $G_{2m+1, 2m+1}^{m+1, m} \left(\begin{matrix} 3/2-j, \dots, 3/2-j; 1, a_1, \dots, a_m \\ 0, k, \dots, k; 3/2-b_1, \dots, 3/2-b_m \end{matrix} \middle| 1 \right)$. The arrows emanating from two Meijer G-functions and terminating at a third suggest the former two can be implemented in the recurrence (3.5) to obtain the third one. The α 's and β 's are positive integers that may serve as dummy summation variables when used in (3.5). The Meijer G-functions with unequal set of indices, say $G_{5,5}^{3,2} \left(\begin{matrix} k, k \\ -j-\beta_1, -j-\beta_2 \end{matrix} \right)$ and $G_{3,3}^{2,1} \left(\begin{matrix} k+\alpha_1 \\ -j-\beta_1 \end{matrix} \right)$ can be used in the recurrence relation by noting that

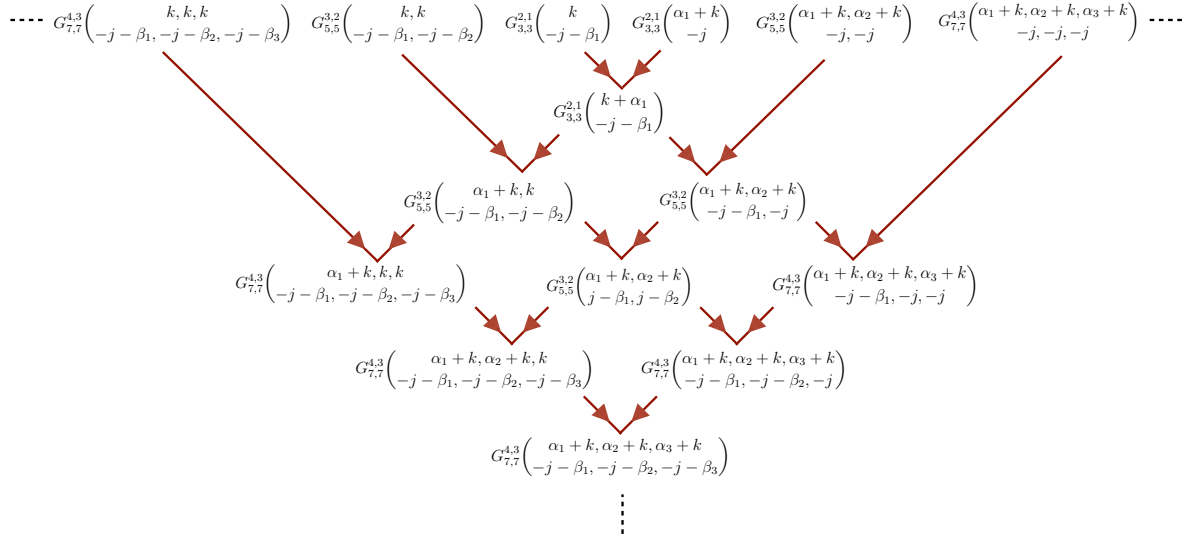


Fig. 3 Diagram showing how higher order Meijer G-functions can be obtained from the lower order ones using the recurrence (3.5).

the latter is same as $G_{5,5}^{3,2}\left(\begin{smallmatrix} k+\alpha_1, k \\ -j-\beta_1, -j \end{smallmatrix}\right)$. We also note that the Meijer G-functions at the top are already known from (3.8) and (3.9).

Now, since the recurrence relation involves multiplication of the Meijer G's with rational coefficients, and the initial functions at the top are all rationals, it follows that the repeated application of the recurrence relation also produces rational numbers. Therefore, eventually the probability $p_{N,N}^{F_m}$ for any m with each L_i even is always a rational number.

Remark 9. Computer algebra, using Mathematica [26], suggests the following results:

$$G_{3,3}^{2,1}\left(\begin{smallmatrix} 3/2-j; 1, \mu+1/2+k \\ 0, k; 3/2-j-1/2 \end{smallmatrix} \middle| 1\right) = \frac{\Gamma(k)}{\sqrt{\pi} \Gamma(j+k+\mu-1/2)} \sum_{\alpha=1}^k \frac{\Gamma(\alpha+\mu)\Gamma(j+k-\alpha-1/2)}{\Gamma(\alpha+\mu+1/2)\Gamma(k-\alpha+1)}, \quad (3.13)$$

$$G_{3,3}^{2,1}\left(\begin{smallmatrix} 3/2-j; 1, 1/2+k \\ 0, k; 3/2-j-\nu-1/2 \end{smallmatrix} \middle| 1\right) = \frac{\Gamma(k)}{\Gamma(\nu+1/2)\Gamma(j+k+\nu-1/2)} \sum_{\alpha=1}^k \frac{\Gamma(\alpha+\nu)\Gamma(j+k-\alpha-1/2)}{\Gamma(\alpha+1/2)\Gamma(k-\alpha+1)}. \quad (3.14)$$

It is clear that, when evaluated, these two yield $1/\pi$ times a rational number. Also, when used in a recurrence relation similar to (3.5), these two lead to $G_{3,3}^{2,1}\left(\begin{smallmatrix} 3/2-j; 1, \mu+1/2+k \\ 0, k; 3/2-j-\nu-1/2 \end{smallmatrix} \middle| 1\right)$, and hence we have for positive integer μ ,

$$G_{3,3}^{2,1}\left(\begin{smallmatrix} 3/2-j; 1, \mu+1/2+k \\ 0, k; 3/2-j-\mu-1/2 \end{smallmatrix} \middle| 1\right) = \sum_{r=1}^{\mu} \sum_{\alpha=1}^k \frac{\Gamma(k)\Gamma(r+\alpha)\Gamma(2\mu-r)\Gamma(j+k-\alpha-1/2)}{\Gamma(\mu)\Gamma(k-\alpha+1)\Gamma(\mu-r+1)\Gamma(j+k+2\mu-1/2)} \\ \times \left(\frac{1}{\sqrt{\pi}\Gamma(r+\alpha+1/2)} + \frac{1}{\Gamma(\alpha+1/2)\Gamma(r+1/2)} \right). \quad (3.15)$$

Clearly, this also equals $1/\pi$ times a rational number. This result can be used in the determinantal formula (2.11) for $m=1$ and odd L .

It appears that

$$G_{5,5}^{3,2} \left(\begin{matrix} 3/2 - j, 3/2 - j; 1, \mu + 1/2 + k, \nu + 1/2 + k \\ 0, k; 3/2 - j - \mu - 1/2, 3/2 - j - \nu - 1/2 \end{matrix} \middle| 1 \right) \text{ and } G_{5,5}^{3,2} \left(\begin{matrix} 3/2 - j, 3/2 - j; 1, \mu + k, \nu + 1/2 + k \\ 0, k; 3/2 - j - \mu, 3/2 - j - \nu - 1/2 \end{matrix} \middle| 1 \right),$$

with the first being the analog of (3.3), do not possess such simple arithmetic structures. For instance, we find that for $m = 2, L_1 = 1, L_2 = 2$, (2.4) reduces to $w_m(x) = (2/\sqrt{\pi}) \tanh^{-1}(\sqrt{1-x^2})\Theta(1-|x|^2)$, which when used in (2.14) leads to the Meijer G-values $\alpha_{1,2} = (20 + 8\mathcal{G})/(3\pi)$, $\alpha_{1,4} = (181 + 162\mathcal{G})/(90\pi)$, $\alpha_{3,2} = (17 - 6\mathcal{G})/(15\pi)$, and $\alpha_{3,4} = (1157 + 450\mathcal{G})/(3780\pi)$. Here $\mathcal{G} \approx 0.915966$ is Catalan's constant. Using these in (2.11) gives the probability values as $p_{2,2}^{P_2} = (2\mathcal{G} + 5)/(3\pi)$, $p_{3,3}^{P_2} = (38\mathcal{G} - 1)/(30\pi)$, and $p_{4,4}^{P_2} = (29412\mathcal{G}^2 + 10612\mathcal{G} - 6767)/(25200\pi^2)$.

4 Some special cases

• $L_1 = \dots = L_m = 0$

This case corresponds to the product of m orthogonal matrices each of dimension N , which is again an N -dimensional orthogonal matrix. For the trivial case of $N = 1$, (2.11) gives the probability as 1, as expected. For $N = 2$, with $G_{1,1}^{1,0} \left(\begin{matrix} 1 \\ 0 \end{matrix} \middle| 1 \right) := 1/2$, as discussed near (3.10), we get the probability value $1/2$. This is in conformity with the fact that all 2×2 orthogonal matrices are either rotations (almost surely complex eigenvalues) or reflections (real eigenvalues ± 1) with equal probability. For $N \geq 3$, the determinant in (2.11) vanishes and hence gives the probability as 0. To summarize, the probability of all eigenvalues real for the product of m number of N -dimensional orthogonal matrices is

$$p_{N,N}^{P_m} = \begin{cases} 1, & N = 1, \\ 1/2, & N = 2, \\ 0, & N \geq 3. \end{cases}$$

• $L_1 > 0, L_2 = \dots = L_m = 0$

This scenario gives the probability of all eigenvalues real for product of the N -dimensional block of a $L_1 + N$ dimensional orthogonal matrix and $m - 1$ orthogonal matrices of dimension N . This probability turns out to be same as the probability of all eigenvalues real for the N -dimensional block of a $L_1 + N$ dimensional orthogonal matrix, i.e., the product with other $m - 1$ orthogonal matrices does not change the probability. This can be seen from (2.4) that, when only one of the L_i is nonzero (here L_1), the weight function $w_m(x)$ reduces from $G_{m,m}^{m,0} \left(\begin{matrix} L_1/2, \dots, L_m/2 \\ 0, \dots, 0 \end{matrix} \middle| x^2 \right)$ to $G_{1,1}^{1,0} \left(\begin{matrix} L_1/2 \\ 0 \end{matrix} \middle| x^2 \right)$. Thus the probability, as given by (2.5), becomes same as that for the case $m = 1$, and is obtained using (2.9).

We also note that if we consider $L_1 = 2\mu > 0$ then using (3.2) we have

$$G_{2m+1,2m+1}^{m+1,m} \left(\begin{matrix} 3/2 - j, 3/2 - j; 1, \mu + k, k, \dots, k \\ 0, k, \dots, k; 3/2 - j - \mu, 3/2 - j, \dots, 3/2 - j \end{matrix} \middle| 1 \right) = G_{3,3}^{2,1} \left(\begin{matrix} 3/2 - j; 1, \mu + k \\ 0, k; 3/2 - j - \mu \end{matrix} \middle| 1 \right) = K_{j,k}^{\mu,\mu},$$

where $K_{j,k}^{\mu,\mu}$ is given by (3.1). This result can be used in the determinantal formula (2.11) to calculate the probability $p_{N,N}^{P_m}$ with $L_1 = 2\mu, L_2 = \dots = L_m = 0$. Similarly, equations (3.13) and (3.15) can be used to calculate the probability for $L_1 = 2\mu + 1, L_2 = \dots = L_m = 0$. However, these yield the same value for $p_{N,N}^{P_1}$ as given by (2.9), as they must.

- $L_1 > 0, L_2 > 0, L_3 = \dots = L_m = 0$

Here we consider the case when all but two of the L_i are nonzero, say L_1 and L_2 . Applying reasoning similar to that in the preceding case, we find that the probability $p_{N,N}^{P_m}$ in the present scenario is the same as the probability for $m = 2$, i.e., the probability $p_{N,N}^{P_2}$ of all eigenvalues real for product of N -dimensional blocks of orthogonal matrices of dimensions $L_1 + N$ and $L_2 + N$, respectively. In this case we do need to calculate the determinant and therefore need the values of Meijer G-functions. Equation (3.2) can be used to obtain the explicit answers when the L_i are even: $L_1 = 2\mu, L_2 = 2\nu$. As clear from the form of (3.2) these probabilities are rational numbers. When one of L_1, L_2 is odd, or both are odd, such a simple arithmetic structure does not seem to exist, as already indicated in Remark 9.

In Table 1 we present probability values for various combinations of N and even L_1, L_2 , giving the explicit rational numbers.

Table 1 Exact values and numerical values (6 significant digits) for some probabilities $p_{N,N}^{P_2}$.

N	L_1	L_2	$p_{N,N}^{P_2}$	
			Exact	Numerical value
2 (3)	2	2	$\frac{20}{27} \left(\frac{1312}{3375} \right)$	0.740741 (0.388741)
2 (3)	2	4	$\frac{1184}{1575} \left(\frac{4544}{11025} \right)$	0.751746 (0.412154)
2 (3)	2	6	$\frac{6112}{8085} \left(\frac{665216}{1576575} \right)$	0.755968 (0.421937)
2 (3)	4	4	$\frac{97984}{128625} \left(\frac{1504768}{3472875} \right)$	0.761780 (0.433292)
2 (3)	4	6	$\frac{649984}{848925} \left(\frac{161046016}{364188825} \right)$	0.765655 (0.442205)

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