Author/s:
Tan, Y;Nesic, D

Title:
A note on robustness of linear spatially distributed parameter systems and their numerical approximations

Date:
2007-01-01

Citation:
Tan, Y. & Nesic, D. (2007). A note on robustness of linear spatially distributed parameter systems and their numerical approximations. PROCEEDINGS OF THE 46TH IEEE CONFERENCE ON DECISION AND CONTROL, VOLS 1-14, pp.5595-5600. IEEE. https://doi.org/10.1109/CDC.2007.4434920.

Persistent Link:
https://hdl.handle.net/11343/299668

A note on robustness of Linear Spatially Distributed Parameter Systems and Their Numerical Approximations

Ying Tan and Dragan Nešić
The Department of Electrical and Electronics Engineering,
The University of Melbourne, Parkville, VIC 3010

***Abstract*— In this paper, we investigate a relationship between robust stability properties of linear spatially distributed parameter systems (LSDPS) with disturbances and robust stability properties of their numerical approximations. Since it is hard to analytically find solutions of a partial differential equation, numerical methods, such as finite-difference methods, are always used to approximately find the solutions. Moreover, it is crucial the the numerical method reproduces (approximately) the behaviour of the actual system model. For instance, if the actual system is stable in some sense, then the numerical method should possess (approximately) the same stability property and vice versa. Our results show that input-to-state exponential stability (ISES) properties of the numerical approximation with respect to disturbances imply *practical ISES* of the LSDPS provided that: (i) the finite-difference approximation is consistent with the model; (ii) an appropriate uniform boundedness condition holds for the numerical method. A similar result can be stated under the same sufficient conditions where ISES of the actual system implies practical ISES of the numerical scheme. Our results can be regarded as an extension of the celebrated Lax theorem to systems with disturbances, as well as its application to analysis of ISES. This question is typically not considered in the numerical analysis literature and yet it often arises in control applications.**

I. INTRODUCTION

Linear spatially distributed parameter systems (LSDPS) arise in a range of different control applications in optical telecommunications, fluid flows, thermal processes, biology, chemistry, environmental sciences, mechanical systems, and so on [2], [3]. LSDPS are modelled by linear partial differential equations (PDE) (or abstract linear differential equations), as opposed to linear lumped parameter systems (LLPS) that are modelled by linear ordinary differential equations (ODE). As disturbances are inevitable in most control applications, robustness of LSDPS with respect to disturbances is one of the central issues in stability analysis of these systems.

For systems with disturbances, there are many different ways to analyze stability robustness of the system. One of the most popular methods is the $\mathcal{H}_2/\mathcal{H}_\infty$ stability framework. $\mathcal{H}_\infty$ results have been extended from LLPS to LSDPS (see [15]). Input-to-state (ISS) stability notion provides an alternative framework for robust stability analysis, see for instance [4], [5], [6], [7].

Providing results and tools for analyzing different forms of robust stability is one of the central themes in control theory. An important (but often neglected) question in this context is to establish a relationship between the robustness properties of the system model and the robustness properties of its numerical approximations. Indeed, since numerical methods are ubiquitous in simulation studies, it is crucial to understand to what extend and under which conditions a numerical scheme can reproduce robust stability properties of the system itself. It is worthwhile to note that there are many numerical methods available in numerical analysis of LSDPS, for example, finite-difference method [13], [2], [14], finite element methods [14] and so on. Finite-difference methods are discussed in this paper for two reasons. On one hand, finite-difference methods are quite simple to implement in practice. On the other hand, the well-known Lax-Richtmyer Equivalence theorem [13] provides a powerful tool for characterizing the approximating properties of finite difference methods. In this paper, we revisit the Lax theorem for systems with disturbances and we use the result to establish a link between the robust stability properties of the LSDPS and their numerical approximations.

Our first main result shows that when a numerical approximation of LSDPS is ISES, then LSDPS is *practically*-ISES if the solutions of LSDPS and its numerical approximation can be made arbitrarily (weakly) close on the compact time intervals. Furthermore, when a stronger type of closeness of solutions is ensured, the solution of LSDPS will recover the same ISES properties when the numerical approximation is ISES. The second main result discusses how to ensure the weak or strong closeness of solution. It is shown that when the finite-difference approximation is consistent with the model of LSDPS and it is uniformly bounded in an appropriate sense, then the solutions of LSDPS and the finite-difference approximation can be made arbitrarily weakly/strongly close on compact time intervals. This last result can be viewed as a version of the celebrated Lax theorem whose form is useful in stability robustness analysis.

This paper is organized as follows. In Section 2, we present the preliminaries. Section 3 contains the main results followed by conclusions in Section 4. Proofs of Theorems 2 and 1 are provided in the Appendix.

II. PRELIMINARIES

In this paper, the set of real numbers is denoted as $\mathbb{R}$, the set of complex numbers is denoted as $\mathbb{C}$. X and W are both Banach spaces with their respective norms denoted as $\|\cdot\|_X$

and $\|\cdot\|_W$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ if it is zero at zero, continuous and nondecreasing. The continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if $\rho(0) = 0$, $\rho(z)$ is strictly increasing and tends to infinity as $z \to \infty$. Given a measurable function $w : \mathbb{R}_{\geq 0} \to W$, we define its infinity norm $\|w\|_\infty := \operatorname{ess\,sup}_{t\geq 0} \|w(t)\|_W$. If we have $\|w\|_\infty < \infty$, then we write $w \in \mathcal{L}_\infty$. We denote $\mathcal{L}(X, Y)$ as the space of all linear bounded operator from X to Y where both X and Y are Banach spaces and $\mathcal{L}(X) := \mathcal{L}(X, X)$.

We consider systems governed by partial differential equations with appropriate initial and boundary conditions that can be represented using the following abstract differential equation (for more details on how to carry out such transformations see [15], [14], [13]):

$$\begin{aligned} \frac{du}{dt} &= Au + Bw \qquad t \geq 0 \\ u(0) &= u_0, \end{aligned} \tag{1}$$

where $A : D(A) \to X$ is the infinitesimal generator of a continuous (C_0) semigroup $S(t)$ in X and $D(A)$ is the domain of A, which is a subset in X (by considering the domains $Y \subset D(A)$, we can incorporate homogeneous boundary conditions, see [14]). $w \in W$ is disturbance. $B : W \to X_{-1}$, where X_{-1} denotes the closure of X in the norm $\|x\|_{X_{-1}} := \left\| (\lambda I - A)^{-1} x \right\|_X$. Here λ is any element of $\rho(A)$ and $\rho(A)$ denotes the resolvent set of A, i.e., the set of all $s \in \mathbb{C}$ such that $sI - A : D(A) \to X$ is bijective and $(sI - A)^{-1} \in \mathcal{L}(X)$. Since X_{-1} is a closure of X, it is clear that $\mathcal{L}(W, X) \subset \mathcal{L}(W, X_{-1})$. The control operator B is called bounded if it maps boundedly into the state space X, that is, there exists a positive constant M_B such that

$$\|B\|_{\mathcal{L}(W,X)} \leq M_B. \tag{2}$$

Otherwise B is called "unbounded" (with respect to the state space X). We consider the weak solution of (1) in the space X_{-1}, i.e., $S(t)$ extends to a strongly continuous semigroup on X_{-1}. The generator of $S(t)$ on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}, i.e., $D(A) = X$). We shall use the same notation $S(t)$ (respectively, A) for the original semigroup (respectively, its generator) and the associated extensions.

First of all, we assume that the system (1) is well-posed. The well-poseness is defined as follows

Definition 1: Then abstract differential equation (1) is well-posed, if the operator A generates a C_0-semigroup $S(\cdot)$ and B is a bounded operator so that the solution of (1) is given by

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)Bw(\tau)d\tau \in X, \tag{3}$$

and (1) holds on X_{-1} for any $t \geq 0$.

Remark 1: Note that $S(t)$ is a C_0-semigroup, which has the following properties [15, Theorem 2.16]

$$\frac{1}{t}\int_0^t S(\tau)ud\tau \to u, \text{ as } t \to 0^+, \forall u \in X \tag{4}$$

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \qquad \forall 0 \leq t < \infty. \tag{5}$$

where $M > 1$ and $\omega > 0$ are positive constants.

Usually the solutions of system (1) is hard to find analytically. The numerical methods are widely used to approximate the solutions of the system (1). There are many numerical methods available in literature, for example, finite difference methods, finite element methods and so on. In this paper, we focus on a finite difference method, whose form is taken as follows:

$$u^{n+1} = E(\Delta t, \Delta\mathbf{x})u^n + F(\Delta t, \Delta\mathbf{x})w^n, \qquad u^0 = u_0 \tag{6}$$

where $u_0 \in X$, $w^n \in W$. Here $E(\Delta t, \Delta\mathbf{x})$ and $F(\Delta t, \Delta\mathbf{x})$ denote linear finite difference operators which depend on the size of the time increment Δt and on the sizes of the space increments $\Delta\mathbf{x}$. Followed the same concepts [13], we assume that E is a bounded linear transformations whose domain is the whole space X and F is also a bounded linear operator whose domain is the whole space W. Moreover, there exists a continuous map g_i, $i = 1, \ldots, n$ with $g_i(0) = 0$ such that

$$\Delta x_i = g_i(\Delta t)$$

which tells how the space increments approache zero as the time increment geos to zero along the sequence. Denote $C(\Delta t) := E(\Delta t, \mathbf{g}(\Delta t))$ and $D(\Delta t) := F(\Delta t, \mathbf{g}(\Delta t))$, where $\mathbf{g}(\cdot) = [g_1(\cot), g_2(\cdot), \ldots, g_n(\cdot)]^T$. Therefore (6) can be re-written as

$$u^{n+1} = C(\Delta t)u^n + D(\Delta t)w^n, \qquad u^0 = u_0 \tag{7}$$

where $C(\Delta t) \in \mathcal{L}(X)$ and $D(\Delta t) \in \mathcal{L}(W, X)$.

In what follows, we will denote the solutions of (1) at time $t = n\Delta t$ starting at an initial condition u_0 and with a disturbance $w(t), t \in [0, n\Delta t]$ as $u(n\Delta t)$. Similarly, the solution of (7) at time n starting at an initial condition u_0 and with a disturbance $w^k, k \in [0, n]$ as u^n. Moreover, we assume that $w^k = w(k\Delta t)$ for all k.

III. CLOSENESS OF SOLUTIONS

In [13], it was shown that uniform boundedness and consistency of a numerical method is necessary and sufficient for closeness of solutions of the actual system and the numerical method on arbitrary compact time intervals[1]. The results in [13] apply to systems without disturbances and the notion of closeness of solutions that is used in [13] is slightly weaker than the notion that we need. It is the purpose of this section to revisit results in [13] for systems with disturbances and prove an appropriate closeness of solutions result that can be used in the next section to analyze how ISES properties of the actual system are related to the ISES properties of the numerical method. Our results will be stated for a set of disturbances satisfying the following:

Assumption 1: The disturbances $w(\cdot)$ are continuously differentiable functions (their derivative is denoted as $\dot{w}(\cdot)$),

---

[1] We note that we use a different terminology from [13]. What we call "uniform boundedness" is referred to as "stability" of the numerical method in [13] and what we call "closeness of solutions" is referred to as "convergence" in [13]. The reason for this change of terminology comes from the fact that we prefer to reserve the term "stability" (that is ISES) for dynamical properties of solutions of the system on infinite time intervals.

such that $w, \dot{w} \in \mathcal{L}_\infty$ (in particular, we assume that $\|\dot{w}\|_\infty \leq r$ for some $r > 0$). We denote the set of all such disturbances as $\mathcal{F}$.

A direct consequence of Assumption 1 and the Lebourg's Mean Value Theorem [1, Theorem 2.4, pg. 75] is the following

$$\|w(\tau_1) - w(\tau_2)\|_W \leq \|\dot{w}\|_\infty |\tau_1 - \tau_2|, \tag{8}$$

where τ_1, τ_2 are arbitrary.

Condition (8) is can be relaxed but we use it to simplify our presentation. It means that the disturbances are globally Lipschitz and bounded functions (in the $\mathcal{L}_\infty$ sense). The following definitions, which are widely encountered in numerical analysis [13] are employed in this paper.

Definition 2: The numerical method (7) is said to be a consistent finite-difference scheme (with the system (1)) if there exist $\rho_1, \rho_2 \in \mathcal{G}$ and $T^* > 0$ such that for all $\Delta t \in (0, T^*)$, $u(t) \in X$ and $w(t) \in W$ we have

$$\|\{C(\Delta t) - S(\Delta t)\} u(t)\|_X \leq \Delta t \rho_1(\Delta t) \|u(t)\|_X \ . \tag{9}$$

$$\left\| \left\{ \frac{D(\Delta t)}{\Delta t} - B \right\} w(t) \right\|_X \leq \Delta t \rho_2(\Delta t) \|w(t)\|_W \ . \tag{10}$$

Remark 2: The consistency conditions (9) and (10) are stronger than those in [13] and [2]. We need these stronger notions of consistency in order to obtain a stronger closeness of solutions property that is needed in our results given in the next section.

Definition 3: The numerical method (7) is said to be *uniformly bounded* (in small Δt) if for any $t > 0$ there exist $T^* > 0$ and $K > 0$ such that for all $\Delta t \in (0, T^*)$ and all n such that $n\Delta t \in [0, t]$ we have that[2]

$$\|C(\Delta t)^n\| \leq K \ . \tag{11}$$

Definition 4: We say that the solutions of (1) and the solutions of the numerical method (7) can be made arbitrarily close on compact time intervals if the following holds. For any $t \geq 0$ and any positive constant δ, there exists a positive constant $T^* > 0$, such that for all $\Delta t \in (0, T^*)$, $u_0 \in X$, $w \in \mathcal{F}$ and $n : n\Delta t \in [0, t]$ we have[3]

$$\|u(n\Delta) - u^n\|_X \leq \delta(\|u_0\|_X + \|w\|_\infty + \|\dot{w}\|_\infty)$$

Theorem 1: Suppose that Assumption 1 and the following conditions hold:

(i) the numerical method (7) is consistent with (1);
(ii) the numerical method (7) is uniformly bounded.

Then, the solutions of (1) and the solutions of the numerical method (7) can be made arbitrarily close on compact time intervals.

Proof of Theorem 1: Let arbitrary $t \geq 0$ and $\delta > 0$ be given. Let T_1^* and ρ_1, ρ_2 come from Definition 2. Let T_2^* and K come from Definition 3. Next we define

$$T_3^* := \rho_1^{-1} \left(\frac{\delta}{KMe^{\omega t} t} \right) \ , \tag{12}$$

where M and ω come from (5);

$$T_4^* := \frac{\delta}{tMe^{\omega t} M_B} \ , \tag{13}$$

where M_B comes from (2). Let $T_5^* > 0$ be such that for all $\Delta t \in (0, T_5^*)$ we have that

$$\|S(n\Delta t - \tau) - S((n-j)\Delta)\| \leq \frac{\delta}{3tM_B} \tag{14}$$

for all $j : j\Delta t \in [0, t]$ and $\tau \in [j\Delta t, (j+1)\Delta t]$ (this T_5^* always exists because of (4)). Finally, we introduce

$$T_6^* := \rho_1^{-1} \left(\frac{\delta}{3KM_B M e^{\omega t} t} \right) \tag{15}$$

$$T_7^* := \rho_2^{-1} \left(\frac{\delta}{3Kt} \right) \tag{16}$$

and, finally, we define

$$T^* := \min_{i \in \{1, \ldots, 7\}} T_i^* \tag{17}$$

In the sequel, we assume that $\Delta t \in (0, T^*)$ and we consider arbitrary n such that $n\Delta t \in [0, t]$. Hence, we can write:

$$\begin{aligned} & \|u(n\Delta t) - u^n\|_X \\ = \ & \left\| S(n\Delta t) u_0 + \int_0^{n\Delta t} S(n\Delta t - \tau) B w(\tau) d\tau \right. \\ & \left. - C^n(\Delta t) u_0 - \sum_{j=0}^{n-1} C^{n-1-j}(\Delta t) D(\Delta t) w^j \right\|_X \end{aligned} \tag{18}$$

By adding and subtracting the following terms $\sum_{j=0}^{n-1} \int_{j\Delta t}^{(j+1)\Delta t} S(n\Delta t - \tau) B w^j d\tau$, $\Delta t \sum_{j=0}^{n-1} S((n - j)\Delta t) B w^j$ and $\Delta t \sum_{j=0}^{n-1} C^{n-1-j}(\Delta t) B w^j d\tau$ to the right hand side of (18), we can write:

---

[2] This condition is typically referred to as "stability" of the numerical method in the numerical analysis literature. As the term "stability" is reserved in our paper for characterizing the dynamical properties of the system and the numerical method, we adopted this new terminology.

[3] We assume the same initial condition u_0 for the solutions of (1) and (7).

$$\begin{aligned} & \|u(n\Delta t)-u^n\|_X \\ \le\ & \|S(n\Delta t)u_0 - C^n(\Delta t)u_0\|_X \\ & + \left\| \sum_{j=0}^{n-1}\int_{j\Delta t}^{(j+1)\Delta t} S(n\Delta t-\tau)B(w(\tau)-w^j)d\tau \right\|_X \\ & + \left\| \sum_{j=0}^{n-1}\int_{j\Delta t}^{(j+1)\Delta t} S(n\Delta t-\tau)Bw^j \right. \\ & \left. -\Delta t\sum_{j=0}^{n-1} S((n-j)\Delta t)Bw^j \right\|_X \\ & + \left\|\Delta t\sum_{j=0}^{n-1}S((n-j)\Delta t)Bw^j\right. \\ & \left. -\Delta t\sum_{j=0}^{n-1}C^{n-1-j}(\Delta t)Bw^j\right\|_X \\ & + \left\|\Delta t \sum_{j=0}^{n-1} C^{n-1-j}(\Delta t)\left(B-\frac{D(\Delta t)}{\Delta t}\right)w^j\right\|_X . \end{aligned} \tag{19}$$

We complete the proof by bounding separately each of the terms on the right hand side of (19). First note that (see [14, pg. 92])

$$\begin{aligned} & \{S(n\Delta t)-C^n(\Delta t)\}\,u_0 \\ =\ & \sum_{j=0}^{n-1} C^j(\Delta t)\,[C(\Delta t)-S(\Delta t)]\,S((n-1-j)\Delta t)u_0, \end{aligned}$$

Denote $v_j = [C(\Delta t)-S(\Delta t)]\,S((n-1-j)\Delta t)u_0$. Then, from item (i) of the theorem and (5) we have that

$$\|v_j\|_X \le \Delta t\rho_1(\Delta t)Me^{\omega t}\|u_0\|_X .$$

Moreover, because of our choice in (12) and using the item (ii) of the theorem, we can bound the first term in (19) as follows:

$$\begin{aligned} & \left\|\sum_{j=0}^{n-1} C^j(\Delta t)v_j\right\|_X \\ \le\ & \sum_{j=0}^{n-1} K\|v_j\|_X \\ \le\ & n\Delta t\rho_1(\Delta t)Me^{\omega t}\|u_0\|_X \\ \le\ & \delta\|u_0\|_X . \end{aligned} \tag{20}$$

Now we bound the second term in (19). Note that we can write the second term as follows:

$$\left\|\sum_{j=0}^{n-1}\int_{j\Delta t}^{(j+1)\Delta t} S(n\Delta t-\tau)B\left(w(\tau)-w^j\right)d\tau\right\|_X \tag{21}$$

Using Assumption 1, the inequalities (5) and (2), as well as our choice (13), we can bound (21) by

$$n\Delta t Me^{\omega t}M_B\Delta t\|\dot w\|_\infty \le \delta\|\dot w\|_\infty . \tag{22}$$

Next, we bound the third term in (19). Using (4), (2) and our choice (14), we can bound the third term in (19) by

$$\frac{\delta}{3tM_B}\cdot M_B t\|w\|_\infty = \frac{\delta}{3}\|w\|_\infty . \tag{23}$$

We bound now the fourth term in (19). In this case, we use the calculations similar to the ones we used to obtain (20), (2) and our choice of (15) to obtain that this term can be bounded by

$$KMe^{\omega t}\rho_1(\Delta t)n\Delta tM_B\|w\|_\infty = \frac{\delta}{3}\|w\|_\infty . \tag{24}$$

Finally, we use items (i) and (ii) of the theorem and our choice of (16) to bound the fifth term in (19) by

$$K\rho_2(\Delta t)\Delta tn\|w\|_\infty = \frac{\delta}{3}\|w\|_\infty . \tag{25}$$

The proof is completed by combining the bounds obtained in (20), (22), (23), (24), (25) to write:

$$\|u(n\Delta t)-u^n\|_X \quad\le\quad \delta(\|u_0\|_X + \|w\|_\infty + \|\dot w\|_\infty) .$$

IV. INPUT-TO-STATE EXPONENTIAL STABILITY

Now we define the stability properties of the system (1)

Definition 5: Let $L,\lambda,\gamma > 0$. The solutions of well-posed initial value problem (1) are said to be *input-to-state exponentially stable (ISES)* with the disturbance gain γ if for any $u(t_0)=u_0\in X$, and any $w(\cdot)$ the solutions of (1) exist and satisfy

$$\|u(t)\|_X \le Le^{-\lambda(t-t_0)}\|u_0\|_X + \gamma\left\|w_{[t_0,t]}\right\|_\infty , \tag{26}$$

$\forall t\ge t_0\ge 0$, where $w_{[t_0,t]}$ denotes the values of disturbance on the time interval $[t_0,t]$.

Definition 6: Let $\gamma > 0$ be given. The solutions of (7) are said to be practically *input-to-state exponentially stable (ISES)* with disturbance gain γ if for any $\delta > 0$ there exist $L,\lambda,T^* > 0$ such that for all $\Delta t\in(0,T^*)$, $u(n_0)=u_0\in X$ and w the solutions of (7) exist and satisfy

$$\|u^n\|_X \le Le^{-\lambda(n-n_0)\Delta t}\|u_0\|_X + (\gamma+\delta)\left\|w_{[n_0,n]}\right\|_\infty + \delta \tag{27}$$

$\forall n\ge n_0\ge 0$ ◦

Theorem 2: Suppose that Assumption 1 and the following conditions hold:

(i) the numerical method (7) is consistent with (1);
(ii) the numerical method (7) is uniformly bounded.

Then, (1) is ISES with disturbance gain γ if and only if (7) is practically ISES with disturbance gain γ. ◦

Remark 3: We note that while the gains in ISES properties in Theorem 2 are the same for the systems (1) and (7), we do not show in our proof that the exponentially decaying terms that involve initial conditions are the same.

V. CONCLUSIONS

VI. ACKNOWLEDGMENT

This research was supported by the Australia Research Council under the Discovery Project DP0664501:“Finite-dimensional Sampled-data Control of Nonlinear Spatially Distributed Parameter Systems”

REFERENCES

[1] F.H. Clarke, Yu. S. Ledyaev, R.J. Stern and P.R. Wolenski, *Nonsmooth analysis and control theory*, Springer-Verlag: New York, 1998.

[2] R.D. Richtmyer and K.W. Morton, *Difference Methoods For Initial-Value Problems, Second Edition*, Interscience Publishers, a division of John Wiley & Sons: New York, London, Sydney. 1967.

[3] M.N.Islam, *Raman Amplifiers for Telecommunications*, Springer-Verlag: New York. 2004.

[4] E.D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property", *Systems & Control Letters* **24**, 351–359, 1995.

[5] P.D. Christofides and A. R. Teel, "Singular perturbation and input-to-state stability", *IEEE Transactions on Automatic Control* **41**, 1645–1650, 1996.

[6] D. Nešić, A. R. Teel and E. D. Sontag, " On stability and input-to-state stability KL estimates of discrete-time and sampled-data nonlinear systems", *Syst. Contr. Lett.* **38**,49–60, 1999.

[7] E. Sontag, *Input to State Stability: Basic Concepts and Results*. Springer Lecture Notes in Mathematics (CIME Course, Cetraro, June 2004), 2005.

[8] T.-J.Tarn, J.R.Zavgren Jr. and Z. Zeng, "Stabilization of infinite dimensional systems with periodic feedback gains and sampled output", *Automatica*, vol. 24, pp. 95-99. 1988.

[9] Nešić, D. and A. R. Teel, "Input-to-state stability for nonlinear time-varying systems via averaging", *Math. Contr. Sign. Sys.* **14**, 524–533, 2001.

[10] L.Moreau, A. R. Teel and D. Nešić, " A trajectory based approach for robustness of input-to-state stability". In: *In Proc. American Control Conference, Arlington, Virginia*. Arlington, Virginia,USA. pp. 2570–3575, 2001.

[11] Teel, A. R., L. Moreau and D. Nešić (2003). "A unified framework for input-to-state stability in systems with two time scales", *IEEE Transactions on Automatic Control* **48**, 1526–1544.

[12] Logemann, H., R. Rebarber and S.Townley, "Stability of infinite-dimensional sampled-data systems", *ITrans. Amer. Math. Soc.*, vol. 35, pp. 319-328. 2003

[13] Lax, P.D. and R.D. Richtmyer, "Survey of the stability of linear finite difference equations", *ICommunications on Pure and Applied Mathematics*, vol. 4, pp. 267-293, 1956.

[14] C. A. Hall and T. Porsching, *Numerical Analysis of Partial Differential Equations*, Prince-Hall, Inc. A Devision of Simon & Schuster Englewood Cliff, New Jersy, 1990

[15] R.F. Curtain and H.J. Zwart, An Introduction To Infinite-Dimensional Linear Systems Theory, Springer-Verlag: New York, 1995.

[16] A. Pazy, Semigroups of Linear Operator and Applications to Partial Differential Equations, Springer-Verlag: New York. 1983.

VII. APPENDIX

Proof of Theorem 2: $\Longrightarrow$ Note that conditions of Theorem 2 imply that all conditions of Theorem 1 hold. Hence, the solutions of (1) and (7) can be made arbitrarily close on compact time intervals. Moreover, since (1) is ISES, we can write that for arbitrary $t > 0$ and $\delta > 0$ there exists $T^* > 0$ such that for all $\Delta t \in (0, T^*)$ and all $k \geq k_0$ such that $(k - k_0)\Delta t \in [0, t]$ then:

$$\begin{aligned} \left\|u^k\right\|_X &\leq Le^{-\lambda\Delta t(k-k_0)} \left\|u_0\right\|_X + \gamma \left\|w_{k_0,k}\right\| \\ &\quad +\delta[\left\|u_0\right\|_X + \left\|w_{[k_0,k]}\right\|_\infty + \left\|\dot{w}_{[k_0,k]}\right\|_\infty] \end{aligned} \tag{28}$$

Let arbitrary $\delta_1 > 0$ be given. Let $\delta = \min\{\delta_1, \delta_1/r\}$, where r comes from Assumption 1 (i.e. $\left\|\dot{w}\right\|_\infty \leq r$. Let t be such that for all $s \geq t$ we have that $Le^{-\bar{\lambda}s} \leq \frac{\delta}{2}$. Let $\left(2t, \frac{\delta}{2}\right)$ generate T^* and let $\Delta t \in (0, T^*)$. Introduce k_i such that $(k_{i+1} - k_i)\Delta t = t$ (we assume without loss of generality that $t/\Delta t$ is an integer - this assumption can be relaxed but it would complicate slightly the presentation). Then, we have that for all $k \in [k_0, k_2]$ that

$$\begin{aligned} \left\|u^k\right\|_X &\leq Le^{-\lambda\Delta t(k-k_0)} \left\|u_0\right\|_X + \gamma \left\|w_{k_0,k}\right\| \\ &\quad +\frac{\delta}{2}[\left\|u_0\right\|_X + \left\|w_{[k_0,k]}\right\|_\infty + \left\|\dot{w}_{[k_0,k]}\right\|_\infty] \end{aligned} \tag{29}$$

and, moreover, for $k \in [k_1, k_2]$ we have:

$$\left\|u^k\right\|_X \leq \delta \left\|u_0\right\|_X + \left(\gamma + \frac{\delta}{2}\right) \left\|w_{[k_0,k]}\right\| + \frac{\delta}{2} \left\|\dot{w}_{[k_0,k]}\right\|_\infty$$

In particular, we have that

$$\left\|u^{k_1}\right\|_X \leq \delta \left\|u_0\right\|_X + \left(\gamma + \frac{\delta}{2}\right) \left\|w_{[k_0,k_1]}\right\| + \frac{\delta}{2} \left\|\dot{w}_{[k_0,k_1]}\right\|_\infty$$

Note that we can reinitialize the two systems at u^{k_1} and using similar arguments and closeness of solutions we can show that the following holds for all $k \in [k_2, k_3]$:

$$\left\|u^k\right\|_X \leq \delta^2 \left\|u_0\right\|_X + \left(\gamma + \frac{\delta}{2}\right) \left\|w_{[k_0,k]}\right\| + \frac{\delta}{2} \left\|\dot{w}_{[k_0,k]}\right\|_\infty$$

Using the induction, we conclude that the following holds for all $i = 1, 2, \ldots$ and all $k \in [k_i, k_{i+1}]$

$$\begin{aligned} \left\|u^k\right\|_X &\leq e^{-\bar{\lambda}i} \left\|u_0\right\|_X + \left(\gamma + \frac{\delta}{2}\right) \left\|w_{[k_0,k]}\right\| + \frac{\delta}{2} \left\|\dot{w}_{[k_0,k]}\right\|_\infty \\ &\leq e^{\bar{\lambda}} e^{-\bar{\lambda}\frac{\Delta t}{t}(k-k_0)} \left\|u_0\right\|_X + (\gamma + \delta_1) \left\|w_{[k_0,k]}\right\| + \delta_1 \end{aligned}$$

where $\bar{\lambda} = \ln\left(\frac{1}{\delta}\right)$. Moreover, from (29) we have that for $k \in [k_0, k_1]$, we have:

$$\left\|u^k\right\|_X \leq L_1 e^{-\lambda\Delta t(k-k_0)} \left\|u_0\right\|_X + (\gamma + \delta_1) \left\|w_{[k_0,k]}\right\|_\infty + \delta_1 \ ,$$

where $L_1 := L + e^{\lambda t}\frac{\delta}{2}$. Combining the last two inequalities we obtain that

$$\left\|u^k\right\|_X \leq \widetilde{L} e^{-\widetilde{\lambda}\Delta t(k-k_0)} \left\|u_0\right\|_X + (\gamma + \delta_1) \left\|w_{[k_0,k]}\right\|_\infty + \delta_1 \ ,$$

where $\widetilde{L} := \max\{e^{\bar{\lambda}}, L_1\}$ and $\widetilde{\lambda} = \min\{\lambda, \frac{\bar{\lambda}}{t}\}$.

$\Longleftarrow$ The proof follows almost the same steps as the proof of sufficiency. However, note that the system (1) does not depend on Δt and we obtain that for arbitrary $\delta > 0$ we can adjust Δt in (7) so that we conclude that the solutions of (1) satisfy

$$\left\|u(t)\right\|_X \leq \widetilde{L} e^{-\widetilde{\lambda}(t-t_0)} \left\|u_0\right\|_X + (\gamma + \delta) \left\|w\right\|_\infty + \delta \ .$$

Since the above bound holds for each $\delta > 0$, we conclude that it holds for $\delta = 0$.