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# Input-to-state stabilization of linear systems with quantized feedback

Daniel Liberzon and Dragan Nešić

***Abstract*— We consider the problem of achieving input-to-state stability (ISS) with respect to external disturbances for control systems with linear dynamics and quantized state measurements. Quantizers considered in this paper take finitely many values and have an adjustable "zoom" parameter. Extending an approach developed previously for systems with no disturbances, we present a control methodology that counteracts an unknown disturbance by switching repeatedly between "zooming in" and "zooming out". Two specific control strategies that yield ISS are described. The first one is implemented in continuous time and analyzed with the help of a Lyapunov function, similarly to earlier work. The second strategy incorporates time sampling, and its analysis is novel in that it is completely trajectory-based and utilizes a cascade structure of the closed-loop hybrid system. We discover that in the presence of disturbances, time-sampling implementation requires an additional modification which has not been considered in previous work.**

## I. INTRODUCTION

The subject of this paper is feedback control of linear continuous-time systems with quantized state measurements. Control problems of this kind are motivated by numerous applications where communication between the plant and the controller is limited due to capacity or security constraints. This is a very active and expanding research area; see, e.g., [1]–[11].

The starting point for this paper is the approach developed in [3], [9] (see also [12, Chap. 5]) which we now briefly recall. The quantizer is assumed to take a finite set of values and incorporates an adjustable "zoom" parameter. The control strategy is composed of two stages. The first, "zooming-out" stage consists in increasing the range of the quantizer until the state of the system can be adequately measured; at this stage, the system is open-loop. The second, "zooming-in" stage involves applying feedback and at the same time decreasing the quantization error in such a way as to drive the state to the origin. This results in a hybrid control law, in which discrete transitions are triggered by the values of a suitable Lyapunov function.

The method of [3], [9] was shown to achieve global asymptotic stability (GAS). The focus of the present work is on achieving robustness with respect to disturbances. We characterize the desired robustness by an ISS-like property (see [13]) which involves bounded nonlinear gains from the initial state and the supremum norm of the disturbance to the supremum norm of the state and also from the supremum limit of the disturbance to the supremum limit of the state. The contributions cited earlier deal with stability only, with the notable exception of [11] which treats disturbances in the stochastic setting. However, the control strategy presented in [11] utilizes statistical information about the disturbance, while we assume the disturbance to be completely unknown to the controller.

Our first main result (Theorem 1 in Section II) is that the ISS property in the presence of disturbances can be achieved by extending the method of [3], [9]. An extension is necessary because an unknown disturbance may force the state outside the range of the quantizer after it has already been inside. Thus we develop a control strategy that switches multiple times between the zooming-in and zooming-out stages. This strategy is still Lyapunov-based, and its analysis is a natural but nontrivial extension of that from [9]. When no disturbances are present, the earlier stabilization result is recovered as a special case.

Next, we turn to the problem of achieving the same robustness property using sampled-data quantized feedback. Time-sampling implementation is important because it guarantees a finite data rate (cf. [7]) and exposes the issue of robustness with respect to time delays. We demonstrate that unless proper care is taken, the straightforward sampled-data adaptation of the continuous-time control strategy in general fails to provide ISS (Section III-B). We then describe a modification to the zooming-out procedure which allows us to obtain ISS in the time-sampling context; this is our second main result (Theorem 5 in Section III-C).

The proof of Theorem 5 sharply differs from that of Theorem 1 in that it does not use a Lyapunov function and instead is based entirely on trajectory analysis. Thus another principal contribution of this work is a novel alternative method for analyzing stability and robustness of quantized feedback control schemes (this method can be applied in continuous time as well). In particular, an important component of this time-based analysis consists in recognizing and utilizing a cascade structure[1] of the hybrid closed-loop system.

Some proofs are omitted due to space constraints and can be found in the full on-line version of this paper [15].

## II. LYAPUNOV-BASED CONTINUOUS-TIME APPROACH

We consider the linear system

$$\dot{x} = Ax + Bu + Dd \tag{1}$$

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[1] This can be viewed as a special instance of the general small-gain approach to stability analysis of hybrid systems proposed in [14].

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $d \in \mathbb{R}^s$ is a disturbance ($u$ and $d$ are taken to be Lebesgue measurable and locally bounded). We assume that $A$ is a nonzero, non-Hurwitz matrix. We assume this system is stabilizable, so there exist matrices $K$ and $P = P^T > 0$ such that $A + BK$ is Hurwitz and

$$(A + BK)^T P + P(A + BK) \leq -2I. \tag{2}$$

Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. In what follows, $|\cdot|$ denotes the Euclidean norm, $\|\cdot\|$ denotes the corresponding matrix induced norm, and $\|\cdot\|_J$ denotes the supremum norm of a signal on an interval $J$. For $x \in \mathbb{R}^n$, $z = \lceil x \rceil$ is the smallest integer $z \in \mathbb{N}$ such that $z \geq x$. We use the notation $(x, y) := (x^T \; y^T)^T$ for arbitrary vectors $x, y$. A continuous function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is zero at zero and strictly increasing.

A *quantizer* is a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$. We assume that there exist real numbers $M > \Delta > 0$ such that the following two conditions hold:

$$|z| \leq M \Rightarrow |q(z) - z| \leq \Delta, \tag{3}$$

$$|z| > M \Rightarrow |q(z)| > M - \Delta. \tag{4}$$

The first condition gives a bound on the quantization error when the quantizer does not saturate, while the second one provides a way to detect the possibility of saturation.

We also assume that $q(x) = 0$ on some neighborhood of the origin. This assumption can be stated as follows.

**Assumption 1** *There exists a number $\Delta_0 > 0$ such that for all $|z| \leq \Delta_0$ we have $q(z) = 0$.*

We will be using the one-parameter family of quantizers

$$q_\mu(x) := \mu q\Big(\frac{x}{\mu}\Big), \qquad \mu > 0.$$

Here $\mu$ is an adjustable parameter, which can be viewed as a "zoom" variable.

At each time $t$, the quantized measurement $q_{\mu(t)}(x(t))$ will represent the information about $x(t)$ which is available to the controller. This quantity takes on a finite number of values (equal to the cardinality of the set $\mathcal{Q}$). Geometrically, $\mathbb{R}^n$ is divided into a finite number of quantization regions (each corresponding to a fixed value of $q$) and the controller knows which of these regions contains the state $x$ at every given time. The variable $\mu$ is an adjustable parameter which we will vary in a discrete fashion in order to extract more information about the state (see [3], [9]).

The problem of interest is to design a quantized feedback control law and a scheme for updating $\mu$ to achieve the following goal: there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial condition $x_0 = x(t_0)$ and every bounded disturbance $d$ we have

$$|x(t)| \leq \gamma_1(|x_0|) + \gamma_2\left(\|d\|_{[t_0,\infty)}\right) \qquad \forall\, t \geq t_0, \tag{5}$$

$$\limsup_{t\to\infty} |x(t)| \leq \gamma_3\Big(\limsup_{t\to\infty} |d(t)|\Big). \tag{6}$$

We note that the gain functions $\gamma_1, \gamma_2, \gamma_3$ may depend on the choice of the initial value $\mu_0 = \mu(t_0)$ of the zoom variable $\mu$, but do not depend on $x_0$ or $d$. Since the closed-loop dynamics will not explicitly depend on time $t$, all bounds will also be uniform with respect to the initial time $t_0$.

We know that for continuous systems of the form $\dot{x} = f(x, d)$, the property expressed by these two inequalities is equivalent to *input-to-state stability* (ISS) with respect to $d$ [13]. In the present case, the closed-loop system is a hybrid system, because it contains an additional discrete state $\mu$. With some abuse of terminology, we will refer to the above property as ISS of the continuous closed-loop dynamics.

This ISS property also implies that in the disturbance-free case ($d \equiv 0$), the origin is a GAS equilibrium of the continuous closed-loop dynamics (for a fixed $\mu_0$). Thus we recover as a special case the property achieved by the algorithms developed in [3], [9] for the case of no disturbances. In fact, the algorithm presented next is a natural extension of the ones from [3], [9].

The overall closed-loop system will be hybrid. It will contain both continuous states (states taking values in a continuum) and discrete states (states taking values in a discrete set). Both will be functions of the continuous time $t \in [t_0, \infty)$. The continuous variables will be comprised of the system state $x$ and two auxiliary reset clock variables $\tau_{\text{in}}$ and $\tau_{\text{out}}$, both initialized at 0. They will take values in the intervals $[0, T_{\text{in}}]$ and $[0, T_{\text{out}}]$, respectively, where $T_{\text{in}}$ and $T_{\text{out}}$ are positive numbers. The discrete variables will be comprised of the zoom variable $\mu$ and an auxiliary logical variable `capt`. The variable $\mu$ will be initialized at some $\mu_0 > 0$ and will take values in a discrete subset of $(0, \infty)$ which depends on $\mu_0$. The variable `capt` will take values in the set {"yes", "no"} and will be initialized at "no"; it is needed to distinguish the "capture" (open-loop) stage from the control (closed-loop) stage.

The control law is defined by

$$u(t) = \begin{cases} 0 & \text{if } \texttt{capt} = \text{“no”} \\ Kq_{\mu(t)}(x(t)) & \text{if } \texttt{capt} = \text{“yes”} \end{cases} \tag{7}$$

The state dynamics describing the evolution of the system variables with respect to time are composed of *continuous evolution* and *discrete events*. During continuous evolution (i.e., while no discrete events occur), $\mu$ is held constant, $x$ satisfies (1) with $u$ defined by (7), and the clock variables satisfy

$$\dot{\tau}_{\text{in}} = \begin{cases} 1 & \text{if } \tau_{\text{in}} < T_{\text{in}} \\ 0 & \text{if } \tau_{\text{in}} = T_{\text{in}} \end{cases}, \qquad \dot{\tau}_{\text{out}} = \begin{cases} 1 & \text{if } \tau_{\text{out}} < T_{\text{out}} \\ 0 & \text{if } \tau_{\text{out}} = T_{\text{out}} \end{cases}.$$

We now describe the discrete events. Given an arbitrary time $t$, we will denote by $\mu^-(t)$, or simply by $\mu^-$ when the time arguments are omitted, the quantity $\lim_{s \nearrow t} \mu(s)$, and similarly for all other variables. All system variables will be continuous from the right by construction (and of course $x$ is continuous).

Let numbers $\Omega_{\text{out}} > 1$, $\Omega_{\text{in}} \in (0,1)$, $T_{\text{c}} \in (0, T_{\text{out}}/2)$, and $\ell_{\text{out}} > \ell_{\text{in}} > 0$ be given. The discrete events are of three types. They are governed by the following rules, which we write in the form "if <conditions> then <actions>". The conditions are mutually exclusive and are assumed to be checked continuously in time. Variables for which no actions are specified remain constant during the events.

**Zoom-out:** If

$$\begin{gathered}(\tau_{\text{out}}^- = T_{\text{out}} \text{ and } \mathtt{capt}^- = \text{“no”}) \quad \text{or} \\ (|q_{\mu^-}(x)| \geq \ell_{\text{out}}\mu^- \text{ and } \mathtt{capt}^- = \text{“yes”})\end{gathered} \tag{8}$$

then let $\mu = \Omega_{\text{out}}\mu^-$ and $\tau_{\text{out}} = 0$.

**Capture:** If

$$|q_{\mu^-}(x)| \leq \ell_{\text{out}}\mu^-,\ \tau_{\text{out}}^- \in [T_{\text{c}}, T_{\text{out}} - T_{\text{c}}] \text{ and } \mathtt{capt}^- = \text{“no”} \tag{9}$$

then let $\mu = \Omega_{\text{out}}\mu^-$ and $\mathtt{capt}$=“yes”.

**Zoom-in:** If

$$|q_{\mu^-}(x)| \leq \ell_{\text{in}}\mu^-,\ \tau_{\text{out}}^- = \tau_{\text{in}}^- = T_{\text{in}} \text{ and } \mathtt{capt}^- = \text{“yes”} \tag{10}$$

then let $\mu = \Omega_{\text{in}}\mu^-$ and $\tau_{\text{in}} = 0$.

The functioning of the clocks can be understood as follows. While $\mathtt{capt}$ = “no”, we wait at least $T_{\text{out}}$ units of time after a zoom-out before executing another zoom-out. Moreover, we wait at least $T_{\text{in}}$ units of time after the last zoom-in or zoom-out before executing another zoom-in. For convenience, the clock $\tau_{\text{out}}$ is also used to ensure that the capture event is separated in time from the zoom-outs.

For each fixed value of $\mu$, chattering on the boundaries between the quantization regions may occur, and solutions are to be interpreted in the sense of Filippov (this issue doesn't affect the Lyapunov-based analysis that follows). Solutions of the overall hybrid system are defined as usual, from one discrete event to the next. The only potential issue is the possibility of infinitely many zoom-in/out events in finite time (Zeno behavior), which in principle can happen since a minimal time between zoom-outs is not enforced while $\mathtt{capt}$ = “yes”. However, when the disturbance is bounded, such behavior is ruled out by the result proved next, which guarantees that $\mu$ remains bounded for all time. Indeed, first note that the variable $\mathtt{capt}$ cannot change its value more than once and hence can be ignored. Now suppose that on a finite interval $[t_1, t_2]$ we have Zeno behavior and $\mu$ is bounded. We have $\mu(t_2) = \Omega_{\text{in}}^{k_1}\Omega_{\text{out}}^{k_2}\mu(t_1)$, where $k_1$ and $k_2$ denote the (possibly infinite) number of zoom-ins and zoom-outs on the interval $[t_1, t_2]$, respectively. Our algorithm enforces that $k_1$ is finite, since $[t_1, t_2]$ is bounded and each zoom-in is preceded by a time interval of length at least $T_{\text{in}}$. Hence, only $k_2$ can be infinite. But this would contradict the boundedness of $\mu(t_2)$ since $\Omega_{\text{out}} > 1$ and $\mu(t_1) > 0$.

**Theorem 1** *Consider the system* (1)*. Pick matrices $K$ and $P = P^T > 0$ satisfying* (2)*. Let $q$ be a quantizer satisfying the conditions* (3) *and* (4)*, where $M$ and $\Delta$ satisfy*

$$M > \left(2 + 2\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|PBK\|\right)\Delta. \tag{11}$$

*Let $\Omega_{\text{in}}, \Omega_{\text{out}}, T_{\text{in}}, T_{\text{out}}, T_{\text{c}}$ be positive numbers satisfying the inequalities $\Omega_{\text{in}} < 1$, $T_{\text{c}} < T_{\text{out}}/2$,*

$$\Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - 2\Delta > \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|PBK\|\Delta, \tag{12}$$

$$\Omega_{\text{out}} > \frac{M}{\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta)}, \tag{13}$$

$$T_{\text{out}} < \frac{\log \Omega_{\text{out}}}{\|A\|} \tag{14}$$

*($T_{\text{in}} > 0$ is arbitrary). Define*

$$\ell_{\text{out}} := M - \Delta, \qquad \ell_{\text{in}} := \Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M - 2\Delta) - \Delta. \tag{15}$$

*Let the control be defined by* (7) *and let the evolution of $\mu$ be as described above, with an arbitrary fixed initial condition $\mu_0 = \mu(t_0) > 0$. Then there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial state $x_0 = x(t_0)$ and every bounded disturbance $d$ the closed-loop system has the properties that $\mu$ remains bounded and the continuous dynamics are ISS in the sense of satisfying* (5) *and* (6)*.*

The proof of the theorem relies on several lemmas, whose proofs can be found in [15].

**Lemma 1** *There exist a time $t_1 \geq t_0$ and functions $\rho_x, \rho_\mu : [0,\infty) \to [0,\infty)$ such that*

$$\|x\|_{[t_0,t_1]} \leq \rho_x(|x_0| + \|D\|\|d\|_{[t_0,\infty)}), \tag{16}$$

$$\mu(t_1) \leq \rho_\mu(|x_0| + \|D\|\|d\|_{[t_0,\infty)}) \tag{17}$$

*and for all $t \geq t_1$ we have $\mathtt{capt}(t)$ = “yes” and $|x(t)| \leq M\mu(t)$.*

**Lemma 2** *Define $V(x) := \frac{1}{2}x^T Px$. Then for $t \geq t_1$ we have*

$$|x| > \|PBK\|\Delta\mu + \|PD\|\|d| \quad \Rightarrow \quad \dot{V} < 0 \tag{18}$$

*along the continuous dynamics (i.e., on every subinterval of $[t_1, \infty)$ on which $\mu$ remains constant).*

**Lemma 3** *Consider some $t \geq t_1$ such that $x(t) \in \mathcal{R}_1(\mu(t))$, where*

$$\mathcal{R}_1(\mu) := \left\{x : V(x) < \lambda_{\min}(P)(M - 2\Delta)^2\mu^2\right\}.$$

*Suppose that $\mu(t)$ satisfies*

$$\begin{aligned}&\sqrt{\lambda_{\min}(P)}(M - 2\Delta)\mu(t) \\ &\quad > \sqrt{\lambda_{\max}(P)}(\|PBK\|\Delta\mu(t) + \|PD\|\|d\|_{[t,\infty)}).\end{aligned} \tag{19}$$

*Then the next discrete event can only be a zoom-in, and if*

$$\sqrt{\lambda_{\min}(P)}\left(\Omega_{\text{in}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}(M-2\Delta)-2\Delta\right)\mu(t) \tag{20}$$
$$> \sqrt{\lambda_{\max}(P)}\left(\|PBK\|\Delta\mu(t)+\|PD\|\|d\|_{[t,\infty)}\right)$$

*then this zoom-in will happen in finite time.*

**Lemma 4** *For every $\varepsilon > 0$ there exists a $\delta > 0$ with the property that if $|x_0| \leq \delta$ and $\|d\|_{[t_0,\infty)} \leq \delta$ then there exists a time $t_2 \geq t_1$ such that:*
*1. $\mathcal{R}_1(\mu(t_2)) \subset \{x : |x| \leq \varepsilon\}$.*
*2. $x(t) \in \mathcal{R}_1(\mu(t_2))$ for all $t \in [t_0, t_2]$.*
*3. The inequality* (19) *holds with $t = t_2$.*

PROOF OF THEOREM 1. Define

$$\hat{\mu} := \frac{\sqrt{\lambda_{\max}(P)}\|PD\|\|d\|_{[t_0,\infty)}}{\sqrt{\lambda_{\min}(P)}(M-2\Delta)-\sqrt{\lambda_{\max}(P)}\|PBK\|\Delta}.$$

It is straightforward to check that (19) holds whenever $\mu(t) > \hat{\mu}$.
*Claim 1*: for all $t \geq t_1$ we have $\mu(t) \leq \Omega_{\text{out}} \max\{\mu(t_1), \hat{\mu}\}$. If the claim is not true, then a zoom-out must have occurred after $t_1$ with $\mu^- > \max\{\mu(t_1), \hat{\mu}\}$. This in turn implies that the discrete event prior to that was either a zoom-out or a zoom-in which also occurred after $t_1$ and resulted in $\mu > \max\{\mu(t_1), \hat{\mu}\}$. By Lemma 1 we have $|x(t)| \leq M\mu(t)$ for $t \geq t_1$. It is easy to see from (13) and (15) that after a zoom-out or a zoom-in with $|x| \leq M\mu^-$ we necessarily have $|x| \in \mathcal{R}_1(\mu)$. Therefore, Lemma 3 tells us that the next discrete event could not be a zoom-out, and the resulting contradiction proves the claim.

Combining Claim 1 and the definition of $\hat{\mu}$ with the bounds (16) and (17) from Lemma 1, we see that the estimate (5) holds with some functions $\gamma_1$ and $\gamma_2$ which can be made continuous and nondecreasing, but not necessarily 0 at 0. Moreover, for every $\varepsilon > 0$ we can apply Lemma 4 to find a $\delta > 0$ with the three properties stated in that lemma. Lemma 3 then implies that the first discrete event after $t_2$ (if one occurs) is a zoom-in. It follows that $\mu(t) \leq \mu(t_2)$ for all $t \geq t_2$, because if $\mu$ returns to the value $\mu(t_2)$ then Lemma 3 again applies. This means that for $|x_0|$ and $\|d\|_{[t_0,\infty)}$ sufficiently small, Lemmas 1 and 4 yield an arbitrarily small bound for $|x(t)|$ for all time. Therefore, we can modify the functions $\gamma_1$ and $\gamma_2$ to make them 0 at 0, hence class $\mathcal{K}_\infty$, and the first ISS estimate (5) is established.

Next, pick an arbitrary $\tilde{\varepsilon} > 0$ and define

$$\tilde{\mu} := \lambda_{\max}(P)\|PD\|(\limsup_{t\to\infty}|d(t)| + \tilde{\varepsilon})/\Theta,$$

$$\Theta := \Omega_i\lambda_{\min}(P)(M-2\Delta) - \lambda_{\max}(P)\|PBK\|\Delta$$
$$- \sqrt{\lambda_{\min}(P)\lambda_{\max}(P)}2\Delta.$$

There exists a time $t_{\tilde{\varepsilon}} \geq t_1$ such that $|d(t)| \leq \limsup_{t\to\infty}|d(t)| + \tilde{\varepsilon}$ for all $t \geq t_{\tilde{\varepsilon}}$. It is straightforward to check that (20) holds whenever $t \geq t_{\tilde{\varepsilon}}$ and $\mu(t) > \tilde{\mu}$.

*Claim 2*: $\exists\, \tilde{t}_{\tilde{\varepsilon}} \geq t_{\tilde{\varepsilon}}$ such that $\mu(t) \leq \Omega_{\text{out}}\tilde{\mu}$ for all $t \geq \tilde{t}_{\tilde{\varepsilon}}$. If $\mu(t) \leq \tilde{\mu}$ for all $t \geq t_{\tilde{\varepsilon}}$, then the claim is trivially true. Otherwise, pick some $t \geq t_{\tilde{\varepsilon}}$ such that $\mu(t) > \tilde{\mu}$. If $x \notin \mathcal{R}_1(\mu(t))$, then Lemma 2 guarantees that either $x$ will enter $\mathcal{R}_1(\mu(t))$ before the next discrete event occurs or a zoom-out will occur and we will have $x \in \mathcal{R}_1(\mu)$ for the new value of $\mu$, i.e., $\Omega_{\text{out}}\mu(t)$. After that, Lemma 3 ensures that as long as $\mu > \tilde{\mu}$, zoom-ins will keep occurring. Therefore, we will eventually have $\mu \leq \tilde{\mu}$. This proves the claim, because if $\mu$ returns to a value in $(\tilde{\mu}, \Omega_{\text{out}}\tilde{\mu}]$, then the same argument applies and a further zoom-out is not possible.

In view of Claim 2, the definition of $\tilde{\mu}$, the bound $|x(t)| \leq M\mu(t)$ for $t \geq t_1$ provided by Lemma 1, and the fact that $\tilde{\varepsilon} > 0$ was arbitrary, the second ISS estimate (6) is also established, with a linear gain function $\gamma_3$. ☐

**Remark 1** As a corollary, we have that if $d \equiv 0$ then the continuous closed-loop dynamics are GAS. In fact, the rate of convergence of $x(t)$ to 0 is exponential. This can be deduced from the above proof, but also follows from the fact that the convergence is no slower than that obtained in [9] via dwell-time switching. ☐

The last claim of Lemma 1 was crucial in the above analysis, and was achieved by triggering a zoom-out immediately when the second condition in (8) becomes true. It is clear that this aspect of the above scheme makes it sensitive to time delays and not implementable in the sampled-data framework. Also, in general we cannot rule out Zeno behavior if the disturbance is not bounded. Thus the issue of designing a suitable zooming-out procedure will be central as we turn to the time-sampling scenario.

## III. TRAJECTORY-BASED SAMPLED-DATA APPROACH

In this section, we introduce a new sampled-data stabilization scheme that can be regarded as an alternative to the scheme from the previous section. We first discuss the simpler disturbance-free case to illustrate the new proof technique. Then, we present an example of a controller and zooming protocol that do not have robustness in an ISS sense. Finally, in the last part of the section we give a result on ISS of the closed-loop system with disturbances with a modified zooming protocol.

### *A. Disturbance-free case*

We consider the continuous-time linear system (1) and assume that $A$ is a non-zero non-Hurwitz matrix. In this subsection, we assume that $d(\cdot) \equiv 0$. We will control this system with quantized hybrid feedback that is defined next. Let $T > 0$ be a given sampling period and let $t_k := kT$ for $k \in \mathbb{N}$. We define $x(t_k) := x_k$ and a sequence $x_0, \ldots, x_k$ is denoted as $x_{[0,k]}$. Closed-loop dynamics will consist of:

$$\text{Plant: } \dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \in \mathbb{R}^n \tag{21}$$

$$\text{Controller: } u(t) = U(\Omega_k, \mu_k, x_k),\ t \in [t_k, t_{k+1}) \tag{22}$$

$$\text{Protocol: } \mu_{k+1} = G(\Omega_k, \mu_k, x_k),\ \mu_0 \in \mathbb{R}_{>0} \tag{23}$$

$$\text{Switching law: } \Omega_k = H(\Omega_{k-1}, \mu_k, x_k),\ \Omega_{-1} = \Omega_{\text{out}} \tag{24}$$

Let $\ell_{\text{out}} > \ell_{\text{in}} > 0$ be strictly positive numbers to be defined below. To simplify the notation, we introduce $q_k := q_{\mu_k}(x_k)$ for arbitrary $k \in \mathbb{N}$, where $q_\mu(\cdot)$ is the one-parameter family of quantizers defined in the previous section. The variable $\Omega$ determines the switching rules for the controller and the zooming protocol. It can only take two values $\Omega_{\text{out}}$ and $\Omega_{\text{in}}$, with the initial value $\Omega_{-1} = \Omega_{\text{out}}$. Then, we define the following hysteresis control law and zooming protocol:

$$U(\Omega_k, \mu_k, x_k) := \begin{cases} 0 & \text{if } \ \Omega_k = \Omega_{\text{out}} \\ Kq_k & \text{if } \ \Omega_k = \Omega_{\text{in}} \end{cases} \tag{25}$$

$$G(\Omega_k, \mu_k, x_k) := \begin{cases} \Omega_{\text{out}}\mu_k & \text{if } \ \Omega_k = \Omega_{\text{out}} \\ \Omega_{\text{in}}\mu_k & \text{if } \ \Omega_k = \Omega_{\text{in}} \end{cases} \tag{26}$$

$$H(\Omega_{k-1}, \mu_k, x_k) = \begin{cases} \Omega_{\text{out}} & \text{if } q_k > \ell_{\text{out}}\mu_k \\ \Omega_{\text{in}} & \text{if } q_k < \ell_{\text{in}}\mu_k \\ \Omega_{k-1} & \text{if } q_k \in [\ell_{\text{in}}\mu_k, \ell_{\text{out}}\mu_k] \end{cases} \tag{27}$$

where $\Omega_{\text{in}}$ and $\Omega_{\text{out}}$ are strictly positive constants to be defined below. We introduce some notation. Note that for all $k \geq 0$ we have that $\Omega_k = \Omega_{\text{out}}$ or $\Omega_k = \Omega_{\text{in}}$. In the former case, we say that the zoom-out condition is triggered at time $k$ and in the latter case we say that the zoom-in condition is triggered at time $k$. Given an initial condition (and a disturbance), there is a sequence of intervals on which we zoom in or out, i.e., we can introduce $k_j \in \mathbb{N}$ such that

$$\begin{aligned} \Omega_k &= \Omega_{\text{out}} \quad \text{if } \ k \in [k_{2i}, k_{2i+1} - 1] \\ \Omega_k &= \Omega_{\text{in}} \quad \text{if } \ k \in [k_{2i+1}, k_{2(i+1)} - 1] \end{aligned}$$

where $i = 0, 1, \ldots$, with $N \in \mathbb{N}$ and either $N$ is finite or $N = +\infty$ (we may have either infinitely many zoom-in/out switchings or finitely many). For notational purposes we will always let $k_0 = 0$ and if we actually have that the zoom-in condition is triggered at $k_0 = k = 0$, then we let $k_1 = k_0$ and we have that the first zoom-out interval is $[k_0, k_1 - 1] = [0, -1] = \emptyset$. In this way, all the proofs will start with a zoom-out interval knowing that this interval may actually be empty. This simplifies the presentation.

The above system induces the following discrete-time system that is more amenable to analysis:

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma U(\Omega_k, \mu_k, x_k) & x(0) &= x_0 & (28) \\ \mu_{k+1} &= G(\Omega_k, \mu_k, x_k) & \mu_0 &\in \mathbb{R}_{>0} & (29) \\ \Omega_{k+1} &= H(\Omega_{k-1}, \mu_k, x_k) & \Omega_{-1} &= \Omega_{\text{out}} & (30) \end{aligned}$$

where

$$\Phi := e^{AT}; \quad \Gamma := \int_0^T e^{As} B ds;$$

We will also need an auxiliary system of the following form:

$$\xi_{k+1} = \frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)\xi_k + \frac{1}{\Omega_{\text{in}}}\Gamma K \nu_k \ , \tag{31}$$

and we can state the following two standard results.

**Lemma 5** *Suppose that $\Phi + \Gamma K$ is Schur. Then, there exists $\Omega^*_{\text{in}} \in (0, 1)$, such that for all $\Omega_{\text{in}} \in [\Omega^*_{\text{in}}, 1)$*

$$\frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K) \tag{32}$$

*is Schur. Moreover, for any such $\Omega_{\text{in}}$, there exist strictly positive $L_1, \lambda_1, \gamma_1$ such that the solutions of the system (31) satisfy the following:*

$$|\xi_k| \leq L_1 \exp(-\lambda_1 k)|\xi_0| + \gamma_1 \|\nu\| \qquad \forall k \geq 0 \ .$$

Note that Lemma 5 imposes a lower bound on $\Omega_{\text{in}}$, which is similar to the condition (12) from the previous section.

**Corollary 2** *Let $\Omega_{\text{in}}$ come from Lemma 5. Then, there exist strictly positive $M, \Delta$ and $\Delta_M$, with $\Delta_M - \Delta > 0$ such that whenever $|\xi_0| \leq \Delta_M$ and $\|\nu\| \leq \Delta$, we have*

$$q_{\mu_k}(x_k) \leq (M - \Delta)\mu_k \text{ and } |\xi_k| \leq M \qquad \forall k \geq 0 \ .$$

Corollary 2 has an appropriate interpretation via Lyapunov functions that links results of this section with the previous section. Indeed, since we assume that $\frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)$ is Schur, there exists a quadratic Lyapunov function $V(\xi) := \xi^T P \xi$ such that for some $a > 0$ the solutions of (31) satisfy

$$|\xi_k| \geq a\,|\nu_k| \implies V(\xi_{k+1}) < V(\xi_k)$$

Suppose that $\Delta$ is given. Then, one possible choice of $M, \Delta_M, \Delta$ is $\Delta_M > \Delta$ and

$$M - 2\Delta > \max\{\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}, a\} \cdot \Delta_M \ .$$

A geometrical interpretation of this condition is that the ball of radius $M - 2\Delta$ contains a level set of $V$ that contains the ball of radius $\Delta_M$. This is similar to the condition (11) in the previous section.

**Theorem 3** *Consider the system (21) and suppose Assumption 1 holds. Suppose that for the given $T > 0$ the pair $(\Phi, \Gamma)$ is stabilizable. Let $K$ be such that $(\Phi + \Gamma K)$ is Schur. Let $\Omega_{\text{in}}$ be such that (32) is Schur and let $\Omega_{\text{out}} > |\Phi|$. Let the range $M$ of the quantizer be sufficiently larger than the error $\Delta$ of the quantizer so that Corollary 2 holds with $M$, $\Delta$ and some $\Delta_M$. Define $\ell_{\text{out}} := M - \Delta$ and $\ell_{\text{in}} := \Delta_M - \Delta$ in (25), (26) and (27). Then, $\mu_k$ is bounded for all $k \geq 0$ and the system (21), (22), (23), (24), (25), (26), (27) is globally asymptotically stable. More precisely, there exists $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that it is of class $\mathcal{K}_\infty$ in its first argument for any fixed value of its second argument and such that for all $x_0 \in \mathbb{R}^n$ and any $\mu_0$ we have*

$$|x_k| \leq \varphi(|x_0|, \mu_0) \qquad \forall k \geq 0 \tag{33}$$

*and $\lim_{k\to\infty} |x_k| = 0$, exponentially fast.*

**Remark 2** It is not hard to show that the stability bound valid only at sampling instants $t_k$ that is proved in Theorem 3 can be extended to all $t \geq 0$. The same is true for our ISS results in the next section. For similar results see [16]. ☐

The proof of Theorem 3 is omitted. A more general result for the disturbance case is presented in the next section.

The control law and protocol (25), (26), (27) are novel in that hysteresis switching is used to switch between the zoom-in and zoom-out stages. This switching strategy simplifies analysis of the time-driven sampling scheme. For instance, the underlying cascaded structure of the system during the zoom-in stage is obtained and used for the first time to prove stability.

While it can be shown that for any fixed $\mu > 0$ we can take $\varphi(\cdot,\mu) \in \mathcal{K}_\infty$ from the proof of Theorem 3, we have at the same time that for any fixed $s > 0$,

$$\lim_{\mu\to 0} \varphi(s,\mu) = \infty \ .$$

Hence, the overshoot of the $x$-subsystem is non-uniform in small $\mu_0$. While it is true that initializing the system at a particular $\mu_0$ gives a constant overshoot for the $x$ variable and one can prove stability of the $x$-subsystem, the lack of uniformity of the overshoot leads to an inherent lack of robustness as the following example illustrates.

### *B. Example: lack of robustness*

Consider the plant

$$x_{k+1} = \Phi x_k + \Gamma u_k + w_k$$

with (22), (23), (25), (26) and suppose that all conditions of Theorem 3 hold for this closed-loop system. Note that we assume that the system is controllable from disturbance in one step to simplify the analysis.

We show that the closed loop system does not have a finite ISS gain from an additive plant disturbance to $x$. We do this by showing that for any $C_1 > 0$, any $\varepsilon > 0$, any $x_0 \in R^n$ and any $\mu_0 > 0$ there exists an additive plant disturbance $w^\varepsilon$ such that $\|w^\varepsilon\| \le \varepsilon$ and the following holds:

$$\limsup_{k\to\infty} |x(k, x_0, \mu_0, w^\varepsilon_{[0,k]})| > C_1 \ .$$

Let $C_1 > 0$ and $\varepsilon > 0$ be arbitrary. Suppose without loss of generality that there exists a positive real eigenvalue $\lambda_m$ of $\Phi$ larger than one and let $\zeta_m$ be its corresponding eigenvector with $|\zeta_m| = 1$. Let $\hat{\varepsilon} > 0$ and $\overline{\varepsilon}_1 > 0$ be such that

$$\overline{\varepsilon}_1 \left(|\Phi + \Gamma K| + |\Gamma K|\Delta\right) + \hat{\varepsilon} < \varepsilon \ . \tag{34}$$

Let $C_1$ and $\hat{\varepsilon}$ generate

$$\overline{T} := \left\lceil \frac{\ln\left(\frac{C_1}{\hat{\varepsilon}}\right)}{\ln(\lambda_m)} \right\rceil \ . \tag{35}$$

Let $\overline{T}$ generate $C_2 > 0$ via

$$C_2 > \max\left\{ \ell_{\text{in}} \cdot \left| \left( \frac{\Omega_{\text{out}}^{\overline{T}}}{\Phi^{\overline{T}}} \right) \right|, \ell_{\text{out}} \right\} \ . \tag{36}$$

Let $C_2$ and $\hat{\varepsilon}$ generate $\bar{\varepsilon}_2$ as follows:

$$\bar{\varepsilon}_2 := \frac{\hat{\varepsilon}}{\Omega_{\text{in}} C_2} \ . \tag{37}$$

Finally, using $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ define

$$\bar{\varepsilon} := \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\} \ . \tag{38}$$

Note that since the system without disturbance is stable, as shown in the previous section, then for any $x_0 \in \mathbb{R}^n$, $\mu_0 > 0$ there exists $k_0^* > 0$ such that with $w_k \equiv 0$ we have

$$\max\{|x_{k_0^*}|, \mu_{k_0^*}\} \le \bar{\varepsilon} \text{ and } |\xi_{k_0^*}| \le M. \tag{39}$$

We now start the construction of the disturbance. Let the disturbance satisfy

$$w_k^\varepsilon = 0 \qquad k = [0, k_0^* - 1] \tag{40}$$

Hence, (39) holds. Let now

$$w_{k_0^*}^\varepsilon = -(\Phi + \Gamma K) x_{k_0^*} - \Gamma K \mu_{k_0^*} (q_{k_0^*} - \xi_{k_0^*}) + \hat{\varepsilon}\zeta_m \ .$$

This disturbance will yield $x_{k_0^*+1} = \hat{\varepsilon}\zeta_m$. The conditions (34) and (38) guarantee that $|w_{k_0^*}^\varepsilon| \le \varepsilon$. The conditions (37) and (38) guarantee that

$$\left|\xi_{k_0^*+1}\right| = \left| \frac{x_{k_0^*+1}}{\Omega_{\text{in}} \mu_{k_0^*}} \right| \ge \frac{\hat{\varepsilon}}{\Omega_{\text{in}} \bar{\varepsilon}_2} = C_2 \ , \tag{41}$$

and hence at time $k_0^*+1$ the zoom-out condition is triggered. Since the $\xi$ dynamics with $w_k \equiv 0$ evolve according to

$$\xi_{k+1} = \frac{\Phi}{\Omega_{\text{out}}} \xi_k \ ,$$

there exists an integer $k_1^*$ such that if the disturbance satisfies

$$w_k^\varepsilon = 0 \qquad \forall k \in [k_0^* + 1, k_1^* - 1] \ ,$$

then $\left|\xi_{k_1^*}\right| \le \ell_{\text{in}}$ and the zoom-in condition is triggered at $k = k_1^*$. Moreover, from (35) and (36) we have $k_1^* - k_0^* - 1 \ge \overline{T}$, which implies together with (41) that

$$|x_{k_1^*}| = \left| \lambda_m^{k_1^* - k_0^* - 1} \zeta_m \hat{\varepsilon} \right| \ge \lambda_m^{\overline{T}} \hat{\varepsilon} \ge C_1 \ .$$

Again via stability of the disturbance-free $(x, \mu)$ system, there exists $k_2^*$ such that if

$$w_k^\varepsilon = 0 \qquad \forall k \in [k_1^*, k_2^* - 1] \ ,$$

then we have

$$\max\{|x_{k_2^*}|, \mu_{k_2^*}\} \le \bar{\varepsilon} \text{ and } |\xi_{k_2^*}| \le M. \tag{42}$$

Continuing in a similar manner, we construct the disturbance which satisfies $\|w^\varepsilon\| \le \varepsilon$ and yields

$$|x_{k_{2j+1}^*}| > C_1 \qquad \forall j \in \mathbb{N}.$$

**Remark 3** The possible non-robustness of the control law in the above example holds for a much larger class of plants, control laws and zooming protocols. Indeed, the crucial ingredients of closed-loop systems that will exhibit this type of non-robustness are as follows:

1) The closed-loop system has to have a property that in the absence of disturbances, both $x$ and $\mu$ converge to zero. Moreover, given any initial conditions $x_0$ and $\mu_0 > 0$ the zoom-out stage is bounded;

2) The closed-loop system is such that the $x$ component is completely controllable locally around the origin with arbitrarily small disturbances $\|w\| \le \varepsilon$;

3) For all time $k \geq 0$, the zooming protocol takes the form $\mu_{k+1} = \gamma_k(\mu_k)$ where $\gamma_k$ are continuous, zero at zero, locally invertible and uniformly lower and upper bounded. 4) If the measurement overflows, the controller is switched off.

Hence, a suitable modification in the zooming-out procedure needs to be adopted in order to achieve ISS. We provide a modification of the zooming-out procedure (see (46) below) in the next section and prove that the closed loop system with the modified scheme is ISS. In particular, our modification violates the above item 3 and we show that this is sufficient to guarantee ISS. □

### *C. Input-to-state stability*

Consider the plant (1) with disturbance with the controller and zooming protocol introduced in the previous section. The corresponding discrete-time system is

$$x_{k+1} = \Phi x_k + \Gamma U(\Omega_k, \mu_k, x_k) + w_k,\ x(0) = x_0 \quad (43)$$

$$\mu_{k+1} = G(\Omega_k, \mu_k, x_k),\ \mu_0 > 0 \quad (44)$$

$$\Omega_k = H(\Omega_{k-1}, \mu_k, x_k),\ \Omega_{-1} = \Omega_{\text{out}} \quad (45)$$

where $U$ and $H$ are defined in (25), (27) and $w_k := \int_{kT}^{(k+1)T} e^{A((k+1)T-s)} Dd(s)ds$. We use here a new zooming protocol:

$$G(\Omega_k, \mu_k, x_k) := \begin{cases} \Omega_{\text{out}}[\mu_k + c] & \text{if } \ \Omega_k = \Omega_{\text{out}} \\ \Omega_{\text{in}}\mu_k & \text{if } \ \Omega_k = \Omega_{\text{in}} \end{cases} \quad (46)$$

where $c > 0$. For simplicity, we take $c = 1$ in the sequel.

Next we introduce a discrete-time version of the definition of ISS. This will suffice for our analysis in this section since the discrete-time ISS can be used to prove an appropriate version of continuous-time ISS that takes inter-sample behavior into account (see Remark 2). The system (43), (44) is said to be ISS if there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that the solutions of the system satisfy the following for all $x_0 \in \mathbb{R}^n$ and all $w$:

$$|x_k| \leq \gamma_1(|x_0|) + \gamma_2(\|w\|) \quad \forall k \geq 0, \quad (47)$$

$$\limsup_{k\to\infty}|x_k| \leq \gamma_3\left(\limsup_{k\to\infty}|w_k|\right) \ . \quad (48)$$

We note that the $\gamma_i$ depend on $\mu_0 > 0$ but not on $x_0$ or $w$.

**Lemma 6** *Suppose that* $\frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)$ *is Schur*[2]. *Then, there exist strictly positive* $L_1, \lambda_1, \gamma_1, \gamma_2$ *such that the solutions of the system*

$$\xi_{k+1} = \frac{1}{\Omega_{\text{in}}}(\Phi + \Gamma K)\xi_k + \frac{1}{\Omega_{\text{in}}}\Gamma K \nu_k + \frac{1}{\Omega_{\text{in}}}\zeta_k \ , \quad (49)$$

*satisfy the following:*

$$|\xi_k| \leq L_1 \exp(-\lambda_1 k)|\xi_0| + \gamma_1 \|\nu\| + \gamma_2 \|\zeta\| \qquad \forall k \geq 0 \ .$$

**Corollary 4** *Let* $\Omega_{\text{in}}$ *come from Lemma 6. Suppose that* $M$ *is sufficiently larger than* $\Delta$. *Then, there exist strictly positive* $\Delta_M$ *and* $\Delta_w$, *with* $\Delta_M - \Delta > 0$ *such that whenever* $|\xi_0| \leq \Delta_M$, $\|\nu\| \leq \Delta$ *and* $\|\zeta\| \leq \Delta_w$ *then*

$$q(\mu_k, x_k) \leq (M - \Delta)\mu_k \text{ and } |\xi_k| \leq M \qquad \forall k \geq 0 \ .$$

We can calculate $M, \Delta, \Delta_M, \Delta_w$ if we have a Lyapunov function for the system (see our earlier discussion).

**Theorem 5** *Consider the system (43), (44), (45) and suppose that Assumption 1 holds. Suppose that for the given* $T > 0$ *the pair* $(\Phi, \Gamma)$ *is stabilizable. Let* $K$ *be such that* $(\Phi + \Gamma K)$ *is Schur. Let* $\Omega_{\text{in}}$ *be such that (32) is Schur and let* $\Omega_{\text{out}} > |\Phi|$. *Let the range* $M$ *of the quantizer be sufficiently larger than the error* $\Delta$ *of the quantizer so that Corollary 4 holds with* $M$, $\Delta$ *and some* $\Delta_M, \Delta_w$. *Define* $\ell_{\text{out}} := M - \Delta$ *and* $\ell_{\text{in}} := \Delta_M - \Delta$ *in (25), (46) and (27). Then,* $\mu_k$ *is bounded for all* $k \geq 0$ *and the system (43), (44), (45) is ISS.*

The proof of Theorem 5 is carried out using several lemmas that are stated next (see [15] for proofs).

**Lemma 7** *Suppose that all conditions of Theorem 5 hold. Then, there exist* $\rho_1, \rho_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$ *such that for any* $i \in \mathbb{N}$, $x_{k_{2i}} \in \mathbb{R}^n$, $\mu_{k_{2i}} > 0$, $w$

$$k_{2i+1} - k_{2i} \leq 1 + \varphi_1(|x_{k_{2i}}|) + \varphi_2\left(\|w\|_{[k_{2i}, k_{2i+1}-1]}\right) \quad (50)$$

*and, moreover,*

$$|x_k| \leq \rho_1(|x_{k_{2i}}|) + \rho_2\left(\|w\|_{[k_{2i}, k_{2i+1}-1]}\right) \ \forall k \in [k_{2i}, k_{2i+1}]. \quad (51)$$

**Remark 4** Note that Lemma 7 implies that the zoom-out condition can be only triggered for finitely many time steps. Hence, if $N$ is finite, then $k_{2N+2} = \infty$. In other words, there exists $k_{2N+1} \in \mathbb{N}$ such that the zoom-in condition is triggered on the interval $[k_{2N+1}, \infty)$. □

**Lemma 8** *There exists a continuous bounded function* $\rho_\mu^{out}$ *such that for any* $\mu > 0$ *we have that* $\rho_\mu^{out}(\mu, 0, 0) > 0$ *and the following is true for all* $i \in \{0, 1, \ldots, N\}$ *and all* $\mu_{k_{2i}} > 0$, $x_{k_{2i}} \in \mathbb{R}^n$, $w \in l_\infty$:

$$\Omega_{\text{out}} \leq \mu_{k_{2i+1}} \leq \rho_\mu^{out}(\mu_{k_{2i}}, |x_{k_{2j}}|, \|w\|_{[k_{2i}, k_{2i+1}-1]}) \ . \quad (52)$$

**Lemma 9** *There exist positive* $K, \lambda, \gamma$ *such that for any* $s, t \in [k_{2i+1}, k_{2i+2}]$ *with* $s \geq t$, *any* $x_s$, $\mu_s$ *and* $w \in l_\infty$

$$|x_s| \leq K \exp(-\lambda(s - t))(|x_t| + \mu_t) + \gamma \|w\|_{[t, s-1]} \ . \quad (53)$$

*In particular, we have from (53) that for all* $k \in [k_{2i+1}, k_{2i+2}]$ *the following holds:*

$$\begin{aligned} |x_k| \leq\ & K \exp(-\lambda(k - k_{2i+1}))(|x_{2i+1}| + \mu_{2i+1}) \\ & + \gamma \|w\|_{[k_{2i+1}, k-1]} \ . \end{aligned} \quad (54)$$

**Lemma 10** *There exists a continuous function* $\rho_x^{in} : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, *with* $\rho_x^{in}(\mu, 0, 0) = 0$ *for all* $\mu > 0$, *and such that for any* $s \geq 0$, $\rho_x^{in}(\cdot, \cdot, s)$ *is nondecreasing in*

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[2] In Lemma 5 we showed that we can find appropriate $\Omega_{\text{in}} \in (0, 1)$ so that this holds whenever $(\Phi + \Gamma K)$ is Schur.

*its first two arguments and for any* $i \in \{0, 1, \ldots, N\}$ *the following holds for all* $\mu_{k_{2i+1}}$, $x_{k_{2i+1}}$, $w$, $k \in [k_{2i+1}, k_{2i+2}]$*:*

$$|x_k| \le \rho_x^{in}\left(\mu_{k_{2i+1}}, |x_{k_{2i+1}}|, \|w\|_{[k_{2i+1}, k_{2(i+1)}-1]}\right) .$$

**Lemma 11** *Consider arbitrary* $i \in \{0, 1, 2, \ldots, N\}$*. If* $k_{2i+2} < +\infty$*, then* $i < N-1$ *and there exists* $\tilde{\gamma} > 0$ *such that*

$$\max\{|x_{k_{2i+2}}|, \mu_{k_{2i+2}}\} \le \tilde{\gamma}\, \|w\|_{[k_{2i+1}, k_{2i+2}-1]} . \qquad (55)$$

PROOF OF THEOREM 5. First, we prove that there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that (47) holds. The proof is carried out using the previous lemmas and induction.

Step $i = 0$: Let $k_0 = 0$. Without loss of generality we suppose that $\Omega_0 = \Omega_{\text{out}}$ and we zoom out on the non-empty interval $[k_0, k_1 - 1]$. We have from Lemma 7 that

$$|x_k| \le \rho_1(|x_0|) + \rho_2(\|w\|_{[k_0, k_1-1]}) \qquad \forall k \in [0, k_1] . \quad (56)$$

Then, using the fact that $\rho_x^{in}$ is non-decreasing in its first two arguments, Lemmas 8 and 10, and (50), we can write for all $k \in [k_1, k_2]$:

$$\begin{aligned} |x_k| &\le \rho_x^{in}(\mu_{k_1}, |x_{k_1}|, \|w\|_{[k_1, k-1]}) \\ &\le \rho_x^{in}(\rho_\mu^{out}, \rho_1 + \rho_2, \|w\|_{[k_1, k-1]}) \\ &\le \overline{\gamma}_1(|x_0|) + \overline{\gamma}_2(\|w\|_{[0, k-1]}) \ , \qquad (57) \end{aligned}$$

for some $\overline{\gamma}_1, \overline{\gamma}_2 \in \mathcal{K}_\infty$.

We either have that $k_2 = +\infty$ or $k_2 < +\infty$. If the former is true, the proof is complete. If the latter is true, then we have from Lemma 11 that (55) holds, i.e.,

$$\max\{|x_{k_2}|, \mu_{k_2}\} \le \tilde{\gamma}\, \|w\|_{[k_1, k_2-1]} \ . \qquad (58)$$

Step $i = 1$: Using Lemma 7 and (58), it follows that for all $k \in [k_2, k_3]$

$$|x_k| \le \overline{\gamma}\left(\|w\|_{[k_1, k-1]}\right) \ ,$$

where $\overline{\gamma}(s) := \rho_1(\tilde{\gamma}s) + \rho_2(s)$. Moreover, using Lemma 10, (50) and (58) we can write for all $k \in [k_3, k_4]$:

$$\begin{aligned} |x_k| &\le \rho_x^{in}(\mu_{k_3}, |x_{k_3}|, \|w\|_{[k_3, k-1]}) \\ &\le \rho_x^{in}(\rho_\mu^{out}, \rho_1 + \rho_2, \|w\|_{[k_3, k-1]}) \\ &\le \overline{\gamma}_1(|x_{k_2}|) + \overline{\gamma}_2(\|w\|_{[k_2, k-1]}) \\ &\le \widehat{\gamma}(\|w\|_{[k_1, k-1]}) \ , \qquad (59) \end{aligned}$$

where $\widehat{\gamma}(s) := \overline{\gamma}_1(\tilde{\gamma}s) + \overline{\gamma}_2(s)$. Either $k_4 = +\infty$, in which case we have completed the proof, or $k_4 < +\infty$, in which case

$$\max\{|x_{k_4}|, \mu_{k_4}\} \le \tilde{\gamma}\, \|w\|_{[k_3, k_4-1]} \ ,$$

and hence we can repeat the argument.

Step $i \ge 1$: Repeating the above argument, it follows that for any $i \in \{1, 2, \ldots, N\}$ the following holds:

$$\begin{aligned} |x_k| &\le \overline{\gamma}(\|w\|_{[k_{2i-1}, k-1]}) \qquad k \in [k_{2i}, k_{2i+1}] \\ |x_k| &\le \widehat{\gamma}(\|w\|_{[k_{2i-1}, k-1]}) \qquad k \in [k_{2i+1}, k_{2i+2}] \ . \end{aligned}$$

The proof follows by induction and (47) holds with

$$\begin{aligned} \gamma_1(s) &:= \max\{\rho_1(s), \overline{\gamma}_1(s)\}, \\ \gamma_2(s) &:= \max\{\rho_2(s), \overline{\gamma}_2(s), \overline{\gamma}(s), \widehat{\gamma}(s)\} \ . \end{aligned}$$

The proof of (48) is completed in a similar fashion. In particular, if $N = \infty$ then we have already proved that for $k \ge k_2$ we have $|x_k| \le \gamma_2(\|w\|)$. When $N$ is finite, the last stage is zoom-in and Lemma 9 guarantees (48). ☐

**Remark 5** It is worth noting that the modified zooming protocol of the form (46) can be used in the event based scheme and it would not change the ISS properties of the system. Actually, this modification will have added benefits of reducing the number of zoom-outs and may provide robustness of the event based scheme to time delays. ☐

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