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Convergence of full-order observers for the slow states of a singularly perturbed system (Part I: Theory)

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***Abstract*— Estimation of physical variables of nonlinear systems with two-time scales is a hard task to address. Whilst nonlinear systems exhibiting a singularly perturbed structure are common in engineering applications, current observer design results apply only to a specific class of plants and observers. We consider a broader class of plants and observers to generalise existing results on observer design for slow states of nonlinear singularly perturbed systems. Under reasonable assumptions, it is shown that the estimation error can be made semi-globally practically asymptotically stable in the singular perturbation parameter. This subsequently leads to appropriate conditions for the observer design for slow variables that guarantee satisfactory estimation error performance in the full system.**

I. INTRODUCTION

Systems that exhibit a time-scale separation are common in engineering practice, and they are typically analysed using singular perturbation techniques, see [1], [2]. A typical example is the class of electro-mechanical systems in which the electrical variables are much faster than the mechanical variables. There are situations in which it is only of interest to know the slow variables while the fast states are not needed. For instance, systems with fast sensors or actuators.

Estimation of process state variables, or the so-called observer design problem, has been of central importance in control theory. There is a well-developed framework for the linear case based on the Luenberger observer and Kalman filter. There is also a wide variety of nonlinear observers that address a number of diverse estimation problems, see [3], [4] - [9]. However, observer design for nonlinear singularly perturbed systems is an open research area.

Estimation of slow variables of nonlinear singularly perturbed systems was considered only for a specific class of plants and a special nonlinear observer for the reduced (slow) system in [3]. An observer design for the reduced system of a spring-mass-damper system was presented in [8] and [9]. In this paper, we generalise those results by considering a broader class of singularly perturbed plants and nonlinear observers.

Here, we provide a framework for nonlinear observer design for singularly perturbed systems where the observer is designed for the reduced (slow) model, ignoring the fast states, and then applied to the full system. We analyse the robustness of the observer with respect to perturbations due to the neglected fast dynamics. Our results apply in cases when the reduced model of the singularly perturbed system has a structure to which observer designs in [3], [4]- [9] can be applied. Estimation of fast variables may be necessary for some applications, but it is outside the scope of this paper.

We present our work via two manuscripts; first, in this paper (Part I: Theory), we state our assumptions and the main result that guarantees convergence of the estimation error. We show that the estimation error dynamics is semi-globally practically asymptotically (SPA) stable. Then, in our companion paper [10], we show that our assumptions hold for two important classes of plants and observers. We first demonstrate that our framework covers current results in [3], and then we also show that our results cover another class of systems and observers that cannot be covered by [3]. Although we present only two classes of systems and observers in [10], our results are more general and cover a broader class of plants and observers such as [3], [4] - [9].

In this manuscript, we show under a set of appropriate assumptions that the estimation error dynamics is SPA stable in the singular perturbation parameter. In other words, given an arbitrarily large set of initial conditions and an arbitrarily small offset, we can reduce the singular perturbation parameter sufficiently so that all solutions starting from the large set of initial conditions produce errors that asymptotically become smaller than the given offset. This result generalises existing results in the literature [3].

Notation: The (Euclidean) norm of a vector $x \in \mathbb{R}^n$ is denoted as $|x|$. We say that $s \in \mathcal{L}_\infty$ if $||s||_\infty < \infty$, where $||s||_\infty := \operatorname{ess\,sup}_t |s(t)|$. We use the following notation $||s[t_1, t_2]|| := \sup_{t \in [t_1, t_2]} |s(t)|$. A continuous function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, if it is strictly increasing and $\alpha(0) = 0$; additionally, if $\alpha(s) \to \infty$ as $s \to \infty$, $\alpha(\cdot)$ is a class-$\mathcal{K}_\infty$ function. A continuous function $\beta(\cdot, \cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, if $\beta(\cdot, s)$ is a class-$\mathcal{K}$ function for each $s \geq 0$ and $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to \infty$ for each $r \geq 0$.

II. PLANT DYNAMICS

We consider a class of plants in the singular perturbation standard form given by

$$\dot{x} = f_s(t, x, z, u, \epsilon), \tag{1a}$$

$$\epsilon \dot{z} = f_f(t, x, z, u, \epsilon), \tag{1b}$$

$$y = h(t, x, z, u, \epsilon), \tag{1c}$$

where $x \in \mathbb{R}^n$ corresponds to the slow state variables, $z \in \mathbb{R}^m$ represents the fast state variables of the system,

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$y \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^r$ is the control input and $\epsilon > 0$ is the perturbation parameter of the system.

Assumption 1: *The input of the system and its derivative belong to $\mathcal{L}_\infty$; i.e. $u, \dot{u} \in \mathcal{L}_\infty$.*

The Assumption 1 is common and useful in singular perturbation theory when we intend to establish a result over the full system from assumptions for the reduced and boundary layer systems. The nonlinear observer design for the slow part of the state of the singularly perturbed system (1) is studied in this work. We pick an existing observer and estimate the slow variable x by assuming that y and u are available. We use the standard technique on observer design for singularly perturbed systems; 1) we approximate the original system (1) by the reduced and the boundary layer systems, 2) we then choose and design an observer for the reduced system and implement it on the full system. Then, we analyse the robustness of the observer when it is applied to the original system. We prove that, under a set of appropriate assumptions, the estimation error has a SPA stability property.

Following the standard approach for the analysis of singularly perturbed systems, we set $\epsilon = 0$ to obtain the following algebraic equation

$$0 = f_f(t, x, z, u, 0). \tag{2}$$

Assumption 2: *The algebraic equation (2) has an isolated solution $z = H(t, x, u)$ that can be obtained analytically.*

Note that Assumption 2 is natural to the problem since it is needed for the analysis of the quasi-steady state behaviour of a singularly perturbed system. Now, substitute the isolated solution $z = H(t, x, u)$ in (1a) and (1c) at $\epsilon = 0$ to obtain the reduced dynamical system given by

$$\dot{x} = f_s(t, x, H(t, x, u), u, 0), \tag{3a}$$
$$y_s = h(t, x, H(t, x, u), u, 0). \tag{3b}$$

Assumption 3. *For the slow system (3), there exists a continuously differentiable function $V_1(t, x)$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_{V_1}(\cdot)$, $\overline{\alpha}_{V_1}(\cdot)$, $\alpha_{V_1}(\cdot)$, $\gamma_{V_1}(\cdot)$, and constants $\zeta_1 > 0$, $\delta_{V_1} > 0$, such that for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $t \geq 0$*

$$\underline{\alpha}_{V_1}(|x|) \leq V_1(t, x) \leq \overline{\alpha}_{V_1}(|x|), \tag{4}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} f_s(t, x, H(t, x, u), u, 0) \leq -\zeta_1 \alpha^2_{V_1}(|x|) + \gamma_{V_1}(|u|) + \delta_{V_1}. \tag{5}$$

The above assumption (Assumption 3) implies that, for any $u \in \mathcal{L}_\infty$, the system (3) is globally input-to-state practically stable with respect to u, see [11] and [12]. Assumption 3 is standard in nonlinear observer design theory. Moreover, it covers a wide range of plants; for instance, systems with globally stable limit cycles. Consider now the change of variables $\xi = z - H(t, x, u)$ and note that (1) becomes

$$\dot{x} = f_s(t, x, \xi + H(t, x, u), u, \epsilon), \tag{6a}$$
$$\epsilon\dot{\xi} = f_f(t, x, \xi + H(t, x, u), u, \epsilon) - \epsilon\frac{\partial H}{\partial t} - \epsilon\frac{\partial H}{\partial x} f_s(t, x, \xi + H(t, x, u), u, \epsilon) - \epsilon\frac{\partial H}{\partial u}\dot{u}, \tag{6b}$$
$$y = h(t, x, \xi + H(t, x, u), u, \epsilon), \tag{6c}$$

in which the quasi-steady-state of the fast dynamics is $\xi = 0$. We analyse the fast dynamics behaviour by considering (6). So, we write the system (6b) in the fast time scale $\tau = \frac{t - t_0}{\epsilon}$ as follows

$$\frac{d\xi}{d\tau} = f_f(t, x, \xi + H(t, x, u), u, \epsilon) - \epsilon\frac{\partial H}{\partial t} - \epsilon\frac{\partial H}{\partial x} f_s(t, x, \xi + H(t, x, u), u, \epsilon) - \epsilon\frac{\partial H}{\partial u}\dot{u}. \tag{7}$$

By following the standard technique for singularly perturbed systems, we set $\epsilon = 0$ and treat (t, x) as fixed parameters to obtain the boundary layer which is given by

$$\frac{d\xi}{d\tau} = f_f(t, x, \xi + H(t, x, u), u, 0). \tag{8}$$

Assumption 4. *For the Boundary-Layer System (8) there exists a Lyapunov function $W(t, x, \xi)$ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_W(\cdot)$, $\overline{\alpha}_W(\cdot)$ and $\alpha_W(\cdot)$, and a constant $\zeta_3 > 0$ such that for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $u \in \mathbb{R}^r$, $t \geq 0$*

$$\underline{\alpha}_W(|\xi|) \leq W(t, x, \xi) \leq \overline{\alpha}_W(|\xi|), \tag{9}$$

$$\frac{\partial W}{\partial \xi} f_f(t, x, \xi + H(t, x, u), u, 0) \leq -\zeta_3 \alpha^2_W(|\xi|). \tag{10}$$

Note that the model reduction on singularly perturbed systems is only possible if the boundary layer system (8) is uniformly asymptotically stable (Assumption 4). Although Assumption 4 seems to be a strong assumption due to the uniformity in u, it was previously used in singular perturbation analysis [12].

III. BOUNDEDNESS OF SOLUTIONS OF THE PLANT

In order to prove the convergence of the estimation error, we first state that the system (6) satisfies an input-to-state practical stability property in the perturbation parameter ϵ with respect to the input and its derivative. Note that such a property can be assumed. However, we decided to state this result since some of the needed assumptions for the main result are used to prove the boundedness result which is of interest in its own right. Since the derivatives of $V_1(t, x)$ and $W(t, x, \xi)$ must be computed along the trajectories of (6), some terms representing the interconnections between the slow and the fast dynamics arise during the analysis. So, we need appropriate conditions to bound such interconnection terms to further analyse their effect.

Assumption 5. *Consider $\alpha_{V_1}(\cdot)$ and $\alpha_W(\cdot)$ given in Assumption 3 and 4 respectively. Suppose that there exist nonnegative constants a_i ($i = 1, 2, 3$) and b_i ($i = 1, 2, 3$), and* *class-$\mathcal{K}_\infty$ functions $\gamma_i(\cdot)$ ($i = 1, \ldots, 4$), so that the following holds,* [1]

1)

$$\left|\frac{\partial V_1}{\partial x}[f_s(t,x,\xi+H,u,\epsilon) - f_s(t,x,H,u,0)]\right| \le \\ \epsilon a_1 \alpha_{V_1}^2(|x|) + \epsilon\gamma_1(|u|)\alpha_{V_1}(|x|) + b_1\alpha_{V_1}(|x|)\alpha_W(|\xi|), \tag{11}$$

2)

$$\left|\frac{\partial W}{\partial \xi}[f_f(t,x,\xi+H,u,\epsilon) - f_f(t,x,\xi+H,u,0)]\right| \le \\ \epsilon a_2 \alpha_W^2(|\xi|) + \epsilon\gamma_2(|u|)\alpha_W(|\xi|) + \epsilon b_2\alpha_{V_1}(|x|)\alpha_W(|\xi|), \tag{12}$$

3)

$$\left|\frac{\partial W}{\partial t} - \frac{\partial W}{\partial \xi}\frac{\partial H}{\partial t} - \frac{\partial W}{\partial \xi}\frac{\partial H}{\partial u}\dot{u} + \left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial \xi}\frac{\partial H}{\partial x}\right] \right. \\ \left. \times f_s(t,x,\xi+H,u,\epsilon)\right| \le \gamma_3(|u|)\alpha_W(|\xi|) \\ + \gamma_4(|\dot{u}|)\alpha_W(|\xi|) + a_3\alpha_W^2(|\xi|) + b_3\alpha_{V_1}(|x|)\alpha_W(|\xi|), \tag{13}$$

for all $(t,x,\xi) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m$ and for all $u, \dot{u} \in \mathbb{R}^r$.

The inequalities (11)-(13) are similar to the ones in [2]. Moreover, these inequalities can be verified in several examples by using quadratic-type Lyapunov functions in Assumption 3 and 4. We have checked that these interconnection conditions are satisfied in a number of cases; for instance, a suspension system [1], a biological reactor (predator-prey system) [13], a three-state SCR catalyst [14], and so on.

We now present Lemma 1 which states that for sufficiently small values of ϵ, the singularly perturbed system (6) is globally input-to-state practically stable with respect to the input and its derivative. Note that we do not present the proof of Lemma 1 due to the page limitation.

Lemma 1. *Consider the singularly perturbed system (6). If Assumptions 1-5 hold, there exists $\tilde{\epsilon}^* > 0$, $\Gamma_1(\cdot) \in \mathcal{K}_\infty$, class-$\mathcal{K}_\infty$ functions $\tilde{\Gamma}_\epsilon(\cdot)$ and $\hat{\Gamma}_\epsilon(\cdot)$ parametrized by ϵ, a constant $\overline{\mu} > 0$, and $\beta_L(\cdot,\cdot) \in \mathcal{KL}$, such that the trajectories of (6) satisfy*[2]

$$|(x(t),\xi(t))| \le \beta_L\left(|(x_0,\xi_0)|, t - t_0\right) + \Gamma_1(||u[t_0,t]||) \\ + \tilde{\Gamma}_\epsilon(||u[t_0,t]||) + \hat{\Gamma}_\epsilon(||\dot{u}[t_0,t]||) + \overline{\mu}, \tag{14}$$

for any $\epsilon \in (0, \tilde{\epsilon}^*)$ and for all $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $u, \dot{u} \in \mathcal{L}_\infty$, and $t \ge t_0 \ge 0$. Moreover, there exists a function $\beta_\xi(\cdot,\cdot) \in \mathcal{KL}$, such that for any $\tilde{\Delta} > 0$, $\tilde{\Delta}_{u_1} > 0$, $\tilde{\Delta}_{u_2} > 0$ and $\tilde{\mu} > 0$, there exists $\overline{\epsilon}^* > 0$ such that

$$|\xi(t)| \le \beta_\xi\left(|\xi_0|, \frac{t - t_0}{\epsilon}\right) + \tilde{\mu}, \tag{15}$$

for all $\epsilon \in (0, \overline{\epsilon}^*)$ and for all $|(x_0,\xi_0)| \le \tilde{\Delta}$, $||u||_\infty \le \tilde{\Delta}_{u_1}$, $||\dot{u}||_\infty \le \tilde{\Delta}_{u_2}$ and $t \ge t_0 \ge 0$.

Condition (14) implies that both slow and fast states are bounded since the input and its derivative belong to $\mathcal{L}_\infty$. On the other hand, (15) implies that the fast states rapidly converge to a small ball given by the ultimate bound $\tilde{\mu}$ which can be made arbitrarily small by reducing ϵ. We use Lemma 1 to prove the main result.

IV. MAIN RESULT

In this section, we analyse the robustness of a full-order nonlinear observer designed for the reduced system (3) and implemented on the full system (6). We give a general set of assumptions that cover a large class of observers to state that the estimation error is SPA stable.

We now assume that an existing nonlinear observer is designed for the reduced system (3). Consider the class of full-order observers described by

$$\dot{\hat{x}} = f_o(t, \hat{x}, y_s, u), \tag{16}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer's state and an estimate of x (slow variable), y_s and u are the output and input of the nonlinear reduced system (3). Define the estimation error as

$$e = x - \hat{x}. \tag{17}$$

Therefore, the error dynamics for the observer designed and implemented in the slow system (3) is given by

$$\dot{e} = f_e(t, x, e, H(t,x,u), y_s, u, 0). \tag{18}$$

where

$$f_e(t,x,e,H,y_s,u,0) = f_s(t,x,H,u,0) - f_o(t, x - e, y_s, u)$$

Note that the last argument in f_e and f_s corresponds to ϵ. So, we have set it to zero since the above error dynamics refers to the observer designed for the reduced system.

Assumption 6. *For the error dynamics in (18), there exists a continuously differentiable function $V_2(t,e)$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_{V_2}(\cdot)$, $\overline{\alpha}_{V_2}(\cdot)$, $\alpha_{V_2}(\cdot)$, and a constants $\zeta_2 > 0$, $\hat{\zeta}_2 > 0$, such that for all $x, e \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $t \ge 0$*

$$\underline{\alpha}_{V_2}(|e|) \le V_2(t,e) \le \overline{\alpha}_{V_2}(|e|), \tag{19}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial e} f_e(t,x,e,H,y_s,u,0) \le -\zeta_2 \alpha_{V_2}^2(|e|). \tag{20}$$

$$\left|\frac{\partial V_2}{\partial e}\right| \le \hat{\zeta}_2 \alpha_{V_2}(|e|). \tag{21}$$

Note that, in general, current observer design results prove an asymptotical stability property as the one given in Assumption 6, see [3], [4]- [9]. Moreover, condition (21) is common when one wants to analyse a robustness property.

The observer designed for the slow system (3) must be implemented on the original system (1). Due to the influence of the the perturbation parameter ϵ and the fast state ξ, the error dynamics is now given by

$$\dot{e} = f_e(t, x, e, \xi + H, y, u, \epsilon), \tag{22}$$

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[1] In the sequel, when it is necessary, we would suppress arguments of some functions to simplify the notation.

[2] In the sequel, $x_0 := x(0)$. The same apply for the other states.

where

$$f_e = f_s(t, x, \xi + H, u, \epsilon) - f_o(t, x - e, y, u).$$

Note that the extended state (x, e, ξ) represents the interconnection between the system (6), and the observer and error dynamics in (22). So, the full extended interconnected system is given by

$$\begin{aligned}
\dot{x} &= f_s(t, x, \xi + H(t, x, u), u, \epsilon), && \text{(23a)}\\
\dot{e} &= f_e(t, x, e, \xi + H(t, x, u), y, u, \epsilon), && \text{(23b)}\\
\epsilon\dot{\xi} &= f_f(t, x, \xi + H(t, x, u), u, \epsilon) - \epsilon\frac{\partial H}{\partial t} \\
&\quad - \epsilon\frac{\partial H}{\partial x} f_s(t, x, \xi + H(t, x, u), u, \epsilon) - \epsilon\frac{\partial H}{\partial u}\dot{u}, && \text{(23c)}\\
y &= h(t, x, \xi + H(t, x, u), u, \epsilon). && \text{(23d)}
\end{aligned}$$

It is observed that the error dynamics is in cascade with the state (x, ξ). Our goal is to ensure that the observer designed for the reduced model (3) will work well when applied to the full system (1). Since we compute the derivative of $V_2(t, e)$ along the solutions of (23) in our proof, we need a condition to bound the interconnection terms that arise in the derivative.

Assumption 7. *Consider $\alpha_{V_1}(\cdot)$, $\alpha_W(\cdot)$ and $\alpha_{V_2}(\cdot)$ given in Assumptions 3, 4 and 6 respectively. Suppose that there exit non-negative constants a_i $(i = 4, 5)$ and b_i $(i = 4, 5)$, and a class-$\mathcal{K}_\infty$ function $\gamma_5(\cdot)$, so that the following condition holds,*

$$\left|\frac{\partial V_2}{\partial e}[f_e(t, x, e, \xi + H, y, u, \epsilon) - f_e(t, x, e, H, y, u, 0)]\right| \leq \\ \epsilon a_4 \alpha_{V_1}(|x|)\alpha_{V_2}(|e|) + \epsilon a_5 \alpha_{V_2}(|e|)\alpha_W(|\xi|) \\ + \epsilon\gamma_5(|u|)\alpha_{V_2}(|e|) + b_4 \alpha_{V_2}(|e|)\alpha_W(|\xi|) \tag{24}$$

for all $(t, x, e, \xi) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ and $u, \dot{u} \in \mathbb{R}^r$.

The inequality in Assumption 7 is justified in a number of cases that we have considered. For instance, a car suspension system [1], a Selective Catalytic Reduction system [14], and so on. In our companion paper [10], we present a class of plants and nonlinear observers in which we show that all of the assumptions given here hold.

A. Stability Analysis

We now state our main result which implies that, under a set of appropriate assumptions, the estimation error is SPA stable in the perturbation parameter ϵ. Our result consider the cascade properties of the error dynamics to guarantee the convergence of the estimation error while the other states in (23) are bounded. The next theorem summarises our main result.

Theorem 1. *Consider the singularly perturbed system (23). If Assumptions 1-7 hold, there exists $\beta_{T_1}(\cdot, \cdot) \in \mathcal{KL}$, $\overline{\gamma}(\cdot) \in \mathcal{K}_\infty$, and functions $\tilde{\gamma}_\epsilon(\cdot), \hat{\gamma}_\epsilon(\cdot) \in \mathcal{K}_\infty$ parametrized by ϵ, such that for any $\Delta > 0$, $\Delta_{u_1} > 0$, and $\Delta_{u_2} > 0$, there exists $\hat{\epsilon}^* > 0$ such that*

$$|e(t)| \leq \beta_T(|e_0|, t - t_0) + \overline{\gamma}(||\xi[t_0, t]||) \\ + \tilde{\gamma}_\epsilon(||x[t_0, t]||) + \hat{\gamma}_\epsilon(||u[t_0, t]||), \tag{25}$$

for all $\epsilon \in (0, \hat{\epsilon}^*)$, $|(x_0, e_0, \xi_0)| \leq \Delta$, $||u||_\infty \leq \Delta_{u_1}$, $||\dot{u}||_\infty \leq \Delta_{u_2}$ and $t \geq t_0 \geq 0$. Moreover, for any $\mu > 0$, there exists $T^* > 0$ and $\epsilon^* > 0$ such that

$$|e(t)| \leq \beta_T(|e(T^* + t_0)|, t - T^* - t_0) + \mu, \tag{26}$$

for all $\epsilon \in (0, \epsilon^*)$, $|(x_0, e_0, \xi_0)| \leq \Delta$, $||u||_\infty \leq \Delta_{u_1}$, $||\dot{u}||_\infty \leq \Delta_{u_2}$ and $t \geq T^* + t_0$, where T^* is a constant of $O(\epsilon)$.

Remark 1. The global assumptions for the reduced and boundary layer systems, and for the error dynamics can be relaxed. In fact, if all assumptions hold on appropriate bounded sets, Theorem 1 holds in a given region defined by those sets.

Remark 2. Our approach is such that we can easily state local results if we relax our assumptions.

Remark 3. If the fast dynamics (1b) do not depend on the input u, there is no need of any conditions on $\dot{u}$

The proof for Theorem 1 is not given due to the page limitation. The statement of the Theorem 1 implies that for a given set of initial conditions and bounded inputs with bounded derivatives, the ultimate bound for the estimation error can be made arbitrarily small. This condition is only possible if ϵ is small enough to guarantee that the estimation error is ultimately bounded by μ.

Our result is semi-global because the stability property is restricted to a given set of initial conditions and bounded inputs with bounded derivatives. It is practical in the perturbation parameter since the ultimate bound μ in (26) can be made as small as wanted by reducing ϵ. And it is asymptotical because of the class-$\mathcal{KL}$ function in (25) and (26). These features give an important robustness property for the observer design problem.

The significance of Theorem 1 lies in the fact that one can choose any existing observer for the reduced system (3), that agrees with this framework, to estimate the slow states of a singularly perturbed system. Although the fast dynamics is neglected during the observer design, the observer performs well when implemented on the full system if it satisfies the given assumptions. To apply our results, we only have to verify that the plant satisfies Assumptions 1 to 5. Then, an existing observer must be designed, and one has to check that Assumptions 6 and 7 hold. We have verified that a number of existing full-order observers satisfy Assumptions 6 and 7, for example, observers in [3], [4] - [9]. Even though our results are not constructive, they cover a larger class of plants and nonlinear observers than existing results in literature [3].

V. CONCLUSIONS

We considered the observer design problem for the slow variables of nonlinear singularly perturbed systems. We presented a boundedness of solutions result over the states of the plant, and an ultimate boundedness property for the fast states of the system. Next, under reasonable and general assumptions, we stated that the estimation error is SPA stable. Since we assume that the observer exists, our results are not constructive. However, they justify the use of a large class of observers (see [3], [4]- [9]) within the singular perturbation framework. Our results are verified and applied to two classes of plants and two nonlinear observers in our companion paper [10].

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