Periodic event-triggered supervisory control of nonlinear systems: dwell-time switching logic

W. Wang, D. Nešić and I. Shames

Abstract— **We consider supervisory control of nonlinear systems which are implemented on digital networks. In particular, two candidate controllers are orchestrated by a supervisor to stabilize the origin of the plant by following a dwell time logic, i.e. evaluating a control-mode switching rule at instants which are at least spaced by some dwell time interval. The plant, the controllers and the supervisor communicate via a network and the transmissions are triggered by a mechanism at the discrete sampling instants, which leads to periodic event-triggered control. Thus, there are possibly two kinds of events generated at the sampling instants: the control-mode switching event to activate another control law and the transmission event to update the control input. We propose a systematic design procedure for periodic event-triggered supervisory control for nonlinear systems. We start from a supervisory control scheme which robustly stabilizes the system in the absence of the network. We then implement it over the network and design event-triggering rules to preserve its stability properties. In particular, for each candidate controller, we provide a lower bound for the control-mode dwell time, design criterion to generate transmission events and present an explicit bound on the maximum sampling period with which the triggering rules are evaluated, to ensure stability of the whole system. We show that there exist relationships among the control-mode dwell time, a parameter used to define the transmission event-triggering condition and the bound of the sampling period. An example is given to illustrate the results.**

I. Introduction

It is a challenge when controlling systems of highly complexity or in the presence of large modeling uncertainty and supervisory control is one possible solution. The so-called *supervisory control* employs logic-based switching among a suitably defined family of candidate controllers to achieve better control performances, compared with the more traditional single controller, see more details in [1], [8], [9], [10], [12], [13] and reference therein. The switching schemes applied in the above references can be classified into three types: I) the dwell-time switching logic in [1], [12], [13], II) the hysteresis switching logic in [8], [9], [10] and III) the state-dependent dwell-time logic in [11], [14]. For the dwell-time switching logic, consecutive switching instants are separated by (at least) a pre-specified time interval, called the dwell time, which is sufficiently large so that the switching does not destabilize the system. In contrast, the hysteresis switching logic associates each possible value of some unknown parameters with a monitoring signal. The latter is designed in such a way that a small value of this signal indicates a high likelihood that the corresponding parameters are close to the actual unknown values. The switching algorithm then selects, from time to time, a controller that has been designed for the parameter values associated with the smallest monitoring signal [9]. These two kinds of switching schemes have some disadvantages. For the dwell-time logic, the performance of the currently active controller might deteriorate to an unacceptable level before the next switch is permitted, as the dwell time is pre-fixed. For the hysteresis switching logic, the stability analysis relies on the property that the hysteresis switching logic stops switching in finite time, and this might not hold in a realistic scenario. Consequently, the state-dependent dwell-time logic is proposed, for continuous-time systems in [14] and discrete-time systems in [11], where the control-mode switching rule is evaluated only after a state-dependent dwell time is elapsed from the previous switch.

We consider the scenario that the supervisory control system, which possibly employs the switching scheme I) or III), is implemented on a digital network, which is new in this context. This does bring advantages like installation costs reduction, greater flexibility and easy maintainability. On the other hand, it is a major challenge to design control strategies which do not "overuse" the network, to limit the transmission delays and the occurrence of packet losses, which may destroy the desired closed-loop system properties. *Event-triggered control* (ETC) adapts the transmissions to the current state of the plant to save network bandwidth, see [6] and the references therein. The idea of ETC is to use the network only when this is needed by generating a transmission event whenever a state or output-dependent condition is verified. To implement the supervisory control with switching logic I) or III), we have two kinds of events: a control-mode switching event to activate another control law, and a transmission event to update the control signal. The implementation is digital and the triggering criteria can only be evaluated at some sampling instants, which leads to the *periodic event-triggered control* (PETC), see [17], [4], [5], [7], [15].

We design the supervisory PETC control law for nonlinear systems by proceeding the emulation approach in the following sense. We start with a supervisory control scheme, consisting of two candidate controllers and a switching logic, which stabilizes the plant in the absence of network. Note that our results can be easily generalized to multiple controllers and we consider the case of two controllers for the brevity of the presentation. We assume that the controller

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W. Wang, D. Nešić and I. Shames are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia wei.wang@unimelb.edu.au, dnesic@unimelb.edu.au, iman.shames@unimelb.edu.au

keeps activated for at least a positive dwell time when it is chosen by the supervisor. We then provide lower bounds of the control-mode dwell time for two candidate controllers. For each controller and a given control-mode dwell time, we design transmission event-triggering conditions and provide explicit upper bounds on the sampling period to ensure the stability of the overall system. We also show that there exists a trade-off among the control-model dwell time, a parameter used to define the transmission event-triggering condition, and the bound of the maximum sampling period. We model for that purpose the overall system as a hybrid system using the formalism of [3]. Our results are consistent with the non-switched PETC control strategy in [17], and indeed, we recover it as a special case.

II. Preliminaries

Let $\mathbb{Z}_{>0} := \{1, 2, \cdots\}$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \cdots\}$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. Let $|x|$ denote the Euclidean norm of the vector $x \in \mathbb{R}^n$. For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, (x, y) stands for $[x^T,\, y^T]^T$. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define the distance of x to $\mathcal{A}$ as $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded if, for any $x \in \mathbb{R}^m$, there exists a neighborhood U_x of x such that $M(U_x)$ is a bounded set. A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semi-continuous when its graph $\{(y, z) \in \mathbb{R}^{m+n} : z \in M(y)\}$ is closed, see Lemma 5.10 in [3]. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$, if it is continuous, zero at zero and strictly increasing and it is of class-$\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$, if it is continuous, for each $r \in \mathbb{R}_{\geq 0}$, $\gamma(\cdot, r)$ is of class-$\mathcal{K}$, and, for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s, \cdot)$ is decreasing to zero. By $\vee$ and $\wedge$ we denote the logical 'or' and 'and' respectively. For $x, v \in \mathbb{R}^n$ and locally Lipschitz $U : \mathbb{R}^n \to \mathbb{R}$, let $U^\circ(x; v)$ be the Clarke derivative of the function U at x in the direction v, *i.e.* $U^\circ(x; v) := \limsup_{y \to x, \lambda \downarrow 0} \frac{U(y + \lambda v) - U(y)}{\lambda}$. This notion will be useful as we will be working with locally Lipschitz Lyapunov functions, which are not differentiable everywhere.

Consider the following hybrid system [3]

$$\begin{aligned} \dot{\xi} &= \mathcal{F}(\xi) \quad \xi \in C \\ \xi^+ &\in \mathcal{G}(\xi) \quad \xi \in D \end{aligned} \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state, $C, D \subset \mathbb{R}^n$ are respectively the flow and the jump sets. We assume that: the sets C and D are closed; $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function; $\mathcal{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded; and $\mathcal{G}(\xi)$ is nonempty for each $\xi \in D$.

We omit the definitions on the hybrid domain and solutions to (1), for which refer the readers to [3], and only define the *uniform global asymptotic stability* (UGAS) of a closed set for system (1).

Definition 1: The closed set $\mathcal{A} \subset \mathbb{R}^n$ is called UGAS for system (1) if there exists $\beta \in \mathcal{KL}$ such that all solutions ξ to system (1) satisfy

$$|\xi(t, j)|_{\mathcal{A}} \leq \beta(|\xi(0, 0)|_{\mathcal{A}}, t + j) \quad \forall (t, j) \in \mathrm{dom}\, \xi \tag{2}$$

and all maximal solutions to system (1) are complete. □

III. Model

We consider the plant model

$$\dot{x}_p = f_p(x_p, u_m), \tag{3}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state and $u_m \in \mathbb{R}^{n_u}$, $m \in \{1, 2\}$, is the control input. We assume that the full state vector x_p is measured, and $u_m \in \mathbb{R}^{n_u}$, $m \in \{1, 2\}$, are generated by the following controllers,

$$\begin{aligned} \dot{x}_c &= f_{c_m}(x_c, x_p) \\ u_m &= g_{c_m}(x_c, x_p), \end{aligned} \tag{4}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state of (4). When the controller is static, (4) becomes $u = g_{c_m}(x_p)$ and there is no need to introduce the state x_c. The functions f_p and f_{c_m}, $m \in \{1, 2\}$, are assumed to be continuous, and g_{c_m} is assumed to be continuously differentiable. We assume that a control-mode switching rule is given, with which the supervisor orchestrates the switching between u_1 and u_2. We consider two candidate controllers to keep the exposition concrete. However, the results presented later can be easily generalized to the bank of more than two candidate control laws, and to investigate the output feedback control by using the technique in [18] when only partial state measurements are available.

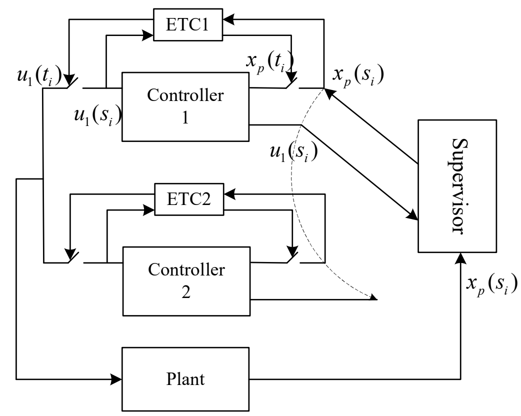

Fig. 1: Block diagram of the setup

We consider the scenario that the plant (3), the controllers (4) and supervisor communicate via a network, as showed in Figure 1. We assume that transmission delays and quantization effects are negligible. At the sampling instants s_i, where $\varepsilon \leq s_{i+1} - s_i \leq T_m$, $i \in \mathbb{Z}_{\geq 0}$, $m \in \{1, 2\}$, the control-mode switching rule and the transmission triggering conditions are evaluated to possibly trigger a control-mode switch or/and a transmission that corresponds to the current mode. The parameter $T_m > 0$ is the upper bound associated with u_m and $\varepsilon \in (0, \min\{T_1, T_2\}]$ is the minimum time between two successive evaluations, which reflects the hardware constraints. As two successive transmissions are spaced by at least ε units of time, Zeno is avoided.

We next model the control-mode switching and transmission triggering behaviours, and provide a model for the overall system.

A. Control mode switching

We introduce clock variables $\tau_1, \tau_2 \in \mathbb{R}_{\geq 0}$ to denote the time elapsed since the last sampling instant and since the last control-mode switch, respectively. We also introduce a variable $q \in \{1, -1\}$ to assist us modeling the control-mode switching. Let u_{m_q} denote the activated control law, where $m_q := \frac{3-q}{2}$. Let $x := (x_p, x_c) \in \mathbb{R}^{n_x}$, $n_x := n_p + n_c$, be the concatenation of the plant and the controller state, and $\Delta : \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ denote the control-mode switching function. Before modelling the control-mode switching behaviour, we list its implementation condition as follows, which says that the activated control law u_m is required to keep working for at least $T_{\text{dwell},m}$ time units.

Condition 1: The control law u_{m_q}, $q \in \{1, -1\}$, is switched to u_{3-m_q} when the control-mode switching condition $q\Delta(x, \tau_2) > 0$ is satisfied at the sampling instants s_i, $i \in \mathbb{Z}_{\geq 0}$, only after its being activated for at least $T_{\text{dwell},m} > 0$ time units. □

Hence, (τ_1, τ_2, q) agree with the following dynamics,

$$\begin{pmatrix} \dot{\tau}_1 \\ \dot{\tau}_2 \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \tau_1 \in \bigcup_{q \in \{1,-1\}} [0, T_{m_q}] \tag{5}$$

$$\begin{pmatrix} \tau_1^+ \\ (\tau_2^+ \;\; q^+) \end{pmatrix} \in \begin{pmatrix} 0 \\ Q(x, \tau_2, q) \end{pmatrix} \quad \tau_1 \in \bigcup_{q \in \{1,-1\}} [\varepsilon, T_{m_q}], \tag{6}$$

where

$$Q(x, \tau_2, q) := \tag{7}$$
$$\begin{cases} \left\{ \begin{pmatrix} 0 \\ -q \end{pmatrix} \right\} \\ \text{when } \left(q\Delta(x,\tau_2) > 0 \wedge \tau_2 \geq T_{\text{dwell},m_q}\right) \\ \left\{ \begin{pmatrix} \tau_2 \\ q \end{pmatrix} \right\} \\ \text{when } \left(q\Delta(x,\tau_2) < 0 \vee \tau_2 \leq T_{\text{dwell},m_q}\right) \\ \left\{ \begin{pmatrix} 0 \\ -q \end{pmatrix}, \begin{pmatrix} \tau_2 \\ q \end{pmatrix} \right\} \\ \text{when } \left(q\Delta(x,\tau_2) = 0 \wedge \tau_2 \geq T_{\text{dwell},m_q}\right). \end{cases}$$

We can see that, q is reset to $-q$ when $q\Delta(x, \tau_2) > 0$ and $\tau_2 \geq T_{\text{dwell},m_q}$ at the sampling instants, noting $\tau_1^+ = 0$ denotes the sampling behaviour. This indeed justify Condition 1. In view of (7), when u_1 is the activated control law ($q = 1$ and $\Delta(x, \tau_2) < 0$), $q = 1$ is reset to $q = -1$ and then u_2 is employed to replace u_1 when $\tau_2 \geq T_{\text{dwell},1}$ and $\Delta(x, \tau_2)$ changes sign since $q\Delta(x, \tau_2) > 0$ is satisfied. We can derive similar conclusions for the case when u_2 is the activated control law. When $\Delta(x, \tau_2) = 0$ and $\tau_2 \geq T_{\text{dwell},m_q}$, we allow both possibilities, i.e., $q^+ = q$ and $q^+ = -q$. This construction ensures that the jump map in (6) is outer semi-continuous, which is essential for the hybrid model presented below to be (nominally) well-posed, see Chapter 6 in [3] for more details.

We next provide an example which shows that the control-mode switching scheme from [14] can be modelled by (6).

Example 1: Consider an imprecisely modelled process $\mathbb{P}$,

$$\dot{x}_{\mathbb{P}} = \bar{f}\left(x_{\mathbb{P}}, u\right), \tag{8}$$

where $x_{\mathbb{P}} \in \mathbb{R}^{\bar{n}_p}$ is the state and $u \in \mathbb{R}^{\bar{n}_u}$ is the control law. The process $\mathbb{P}$ is a family of dynamical systems of the form $\widetilde{\mathcal{F}} = \bigcup_{k \in \mathcal{P}} \widetilde{\mathcal{F}}_p$, where $\widetilde{\mathcal{F}}_k$ is a given nominal process model $\mathbb{N}_k$ and $\mathcal{P} := \{1, \cdots, m\}$.

The estimator-based supervisor in [10], [14] consists of three subsystems, a multi-estimator $\mathbb{E}$, a bank of monitoring signal generators $\mathbb{M}_k$, $k \in \mathcal{P}$, and a switching logic S. The multi-estimator $\mathbb{E}$ is given by a single dynamical system

$$\dot{x}_{\mathbb{E}} = f_E(x_{\mathbb{E}}, x_{\mathbb{P}}, u), \tag{9}$$

where $x_{\mathbb{E}} := (x_{\mathbb{E},1}, \cdots, x_{\mathbb{E},m})$ and $x_{\mathbb{E},k}$ corresponds to the process model $\mathbb{N}_k$. It is ensured in [10], [14] that $x_{\mathbb{E},k}$, $k \in \mathcal{P}$, asymptotically converge to $x_{\mathbb{P}}$ provided $\mathbb{N}_k$ is the actual process model. The monitor signal generator $\mathbb{M}_k$ corresponding to $\mathbb{N}_k$ is such that

$$\dot{\mu}_k = -\lambda_k \mu_k + \bar{\gamma}_k(x_{\mathbb{P}} - x_{\mathbb{E},k}) \quad \mu_k(0) > 0, \tag{10}$$

where $\lambda_k > 0$ and $\bar{\gamma}_k : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$ is continuously differentiable, see their design in Section 9 of [10]. The controller u_k, $k \in \mathcal{P}$, is designed to robustly stabilize $\mathbb{N}_k$ and given by

$$\dot{x}_c = f_{c,k}(x_c, x_{\mathbb{P}}, x_{\mathbb{E},k}) \text{ and } u_k = g_{c,k}(x_c, x_{\mathbb{P}}, x_{\mathbb{E},k}). \tag{11}$$

The switching logic S works in a way such that, u_{k_1} is chosen if μ_{k_1} is smallest for all $k \in \mathcal{P}$ and keeps being activated until there exists $\mu_{k_2}(\tau_2) < \mu_{k_1}(\tau_2)$ for $\tau_2 \geq \tau_D$ with $\tau_D > 0$ being the state-dependent dwell time from [14].

We then rewrite (8) and (9) into (3) with letting $x_p := (x_{\mathbb{P}}, x_{\mathbb{E}})$ and $f_p := (\bar{f}, f_{\mathbb{E}})$. Note that $\mu_k(\tau_2) = \mu_k(0)\exp(-\lambda_k \tau_2) + \int_0^{\tau_2} \exp(\lambda_k s)\bar{\gamma}(x_{\mathbb{P}}(s) - x_{\mathbb{E},k}(s))ds$ for any $k \in \mathcal{P}$. We consider the case that $m = 2$ and we have that (6) model the switching logic S with letting $\Delta(x, \tau_2) := \mu_1(\tau_2) - \mu_2(\tau_2)$. □

B. Transmission triggered events

We consider the scenario that different transmission triggered conditions and maximal allowable sampling periods are assigned to the network when two control laws are respectively closed in the loop. Moreover, a transmission is enforced when the control-mode switches from one to the other, as formally given as below.

Condition 2:

1) For $m \in \{1, 2\}$, a transmission is granted when the event-triggering condition $\Upsilon_m > 0$ is satisfied, at the sampling instants s_i, $i \in \mathbb{Z}_{\geq 0}$, where $s_{i+1} - s_i \leq T_m$.
2) A transmission is enforced whenever the control law is switched from u_m to u_{3-m}, $m \in \{1, 2\}$. □

Because of the network, plant (3) no longer has access to u, but only to its networked version $\hat{u}$, and controller (4) has access to $\hat{x}_p$, the networked version of x_p. Between two successive sampling instants, $s_i, s_{i+1} \geq 0$, $i \in \mathbb{Z}_{\geq 0}$, and for any $q \in \{1, -1\}$, $\hat{x}_p$ and $\hat{u}_{m_q}$ are governed by

$$\left.\begin{aligned} \dot{\hat{x}}_p &= \hat{f}_p\left(t, x_p, u_{m_q}, \hat{x}_p, \hat{u}_{m_q}\right) \\ \dot{\hat{u}}_{m_q} &= \hat{f}_{c,m_q}\left(t, x_p, u_{m_q}, \hat{x}_p, \hat{u}_{m_q}\right) \end{aligned}\right\} \quad t \in (s_i, s_{i+1}), \tag{12}$$

where $m_q = \frac{3-q}{2}$ with q defined in (5) and (6), $\hat{f}_p$ and $\hat{f}_{c,m_q}$ are the holding functions. For instance, Zero-order-hold devices correspond to $\hat{f}_p = \hat{f}_{c,m_q} = 0$.

We now model the dynamics of $\hat{x}_p$ and $\hat{u}$ at sampling instants. Let $e := (e_p, e_u) \in \mathbb{R}^{n_e}$, where $e_p := \hat{x}_p - x_p$ is the network-induced error on the plant state measurement, $e_u := \hat{u} - u$ is the network-induced error on the control input, and $n_e := n_p + n_u$. At each sampling instant s_i, $i \in \mathbb{Z}_{\geq 0}$, the update of $\hat{x}_p$ and $\hat{u}$ satisfies

$$\begin{pmatrix} \hat{x}_p(s_i^+) \\ \hat{u}_{m_q}(s_i^+) \end{pmatrix} = \begin{cases} \begin{pmatrix} x_p(s_i) \\ u_{m_q}(s_i) \end{pmatrix} \text{when} \Upsilon_{m_q}(e(s_i), x(s_i)) \geq 0 \\ \begin{pmatrix} \hat{x}_p(s_i) \\ \hat{u}_{m_q}(s_i) \end{pmatrix} \text{when} \Upsilon_{m_q}(e(s_i), x(s_i)) \leq 0. \end{cases} \tag{13}$$

The function $\Upsilon_m : \mathbb{R}^{n_x+n_e} \to \mathbb{R}$, $m \in \{1, 2\}$, models the triggering condition when u_m is activated, which is evaluated at sampling instants by the event-triggering mechanism. We explain later how to construct Υ_m. In view of (13), when $\Upsilon_m(e, x) > 0$, a transmission occurs at s_i, and $\hat{x}_p$ and $\hat{u}_m$ are reset to the actual value of x_p and u, respectively. When $\Upsilon_m(e, x) < 0$, no transmission occurs and $\hat{x}_p$ and $\hat{u}$ remain unchanged. When $\Upsilon_m(e, x) = 0$, the model allows two possibilities: either a transmission occurs or not.

For any $q \in \{1, -1\}$, when no control-mode switching events is triggered at s_i, $i \in \mathbb{Z}_{\geq 0}$, e^+ only depends on the transmissions triggering conditions:

$$e(s_i^+) = h_{m_q}(e(s_i), x(s_i)), \tag{14}$$

where $h_{m_q}(e, x) := \big(1 - \Gamma_{m_q}(e, x)\big)e$. The functions $\Gamma_1, \Gamma_2 : \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \to \{0, 1\}$ are used to indicate if a transmission occurs or not. In particular, for $m \in \{1, 2\}$, $\Gamma_m(e, x) = \{1\}$ when $\Upsilon_m(e, x) > 0$, which corresponds to a transmission; $\Gamma_m(e, x) = \{0\}$ when $\Upsilon_m(e, x) < 0$ which corresponds to no transmission and $h_m(e, x) = e$ in this case. When $\Upsilon_m(e, x) = 0$, $\Gamma_m(e, x) = \{0, 1\}$ covers the above two possibilities.

Recall that a transmission is enforced when the control-mode is switching, as required by item 2) of Condition 2. In view of (3), (4), (14), we have that when $\tau_1 \in \bigcup_{q \in \{1,-1\}} [0, T_{m_q}]$,

$$\dot{\tau}_1 = 1 \quad \dot{e} = g_{m_q}(x, e) \tag{15}$$

where g_{m_q}, $q \in \{1, -1\}$, can be derived from (3), (4) and (12), and when $\tau_1 \in \bigcup_{q \in \{1,-1\}} [\varepsilon, T_{m_q}]$,

$$\tau_1^+ = 0 \tag{16}$$

$$e^+ = \begin{cases} 0 & \text{when } \big(q\Delta(x, \tau_2) \geq 0 \wedge \tau_2 \geq T_{\text{dwell},m_q}\big) \\ h_{m_q}(e, x) & \text{when } \big(q\Delta(x) \leq 0 \vee \tau_2 \leq T_{\text{dwell},m_q}\big). \end{cases}$$

C. Supervisory networked control system

We now rewrite the models defined by (3)-(6), (15) and (16) as hybrid system (1) with

$$\begin{aligned} \mathcal{F}(\xi) &:= F_{m_q}(\xi) \\ \mathcal{G}(\xi) &:= G_{m_q}(\xi) \end{aligned} \tag{17}$$

for all $\xi := (x, e, \tau_1, \tau_2, q) \in \mathbb{X}$, $\mathbb{X} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \{1, -1\}$, $C := C_1 \cup C_2$, $D := \bigcup_{m=1}^{2} (D_m^{cn} \cup D_m^{tr})$. For $q \in \{1, 2\}$, $C_{m_q} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times [0, T_{m_q}] \times \mathbb{R}_{\geq 0} \times \{q\}$, $D_{m_q}^{cn} := \{\xi \in \mathbb{X} | q\Delta(x) \geq 0 \wedge \tau_2 \geq T_{\text{dwell},m_q}, \tau_1 \in [\varepsilon, T_{m_q}] \cup \{0\}, q \in \{1, -1\}\}$, and $D_{m_q}^{tr} := \{\xi \in \mathbb{X} | q\Delta(x) \leq 0 \vee \tau_2 \leq T_{\text{dwell},m_q}, \tau_1 \in [\varepsilon, T_{m_q}], q \in \{1, -1\}\}$. $F_{m_q}(\xi) := \big(f_{m_q}(x, e), g_{m_q}(x, e), 1, 1, 0\big)$, where $m_q = \frac{3-q}{2}$, f_{m_q} and g_{m_q} are derived from (3), (4), (12), and (15), respectively. The jump map G_{m_q} is defined as

$$G_{m_q}(\xi) := \begin{cases} \mathcal{G}_{m_q}^{\text{tr}}(\xi) & \xi \in D_{m_q}^{tr} \\ \mathcal{G}_{m_q}^{\text{cn}}(\xi) & \xi \in D_{m_q}^{cn} \\ \emptyset & \xi \notin (D_{m_q}^{tr} \cup D_{m_q}^{cn}). \end{cases} \tag{18}$$

where $\mathcal{G}_{m_q}^{\text{tr}}(\xi) := (x, h_{m_q}, 0, \tau_2, q)$ corresponds to a transmission jump, and $\mathcal{G}_{m_q}^{\text{cn}}(\xi) := (x, 0, 0, 0, -q)$ corresponds to a control-mode jump. When $\xi \in D_{m_q}^{tr} \cap D_{m_q}^{cn}$, there are two situations might occur: I) $\mathcal{G}_{m_q}^{\text{cn}}$ enforces a jump as it maps the vector (e, τ_1, τ_2, q) to $(0, 0, 0, -q)$, II) $\mathcal{G}_{m_q}^{\text{tr}}$ with $\mathcal{G}_{m_q}^{\text{cn}}$ enforces two consecutive jumps. For case II, $\mathcal{G}_{m_q}^{\text{tr}}$ maps the vector (e, τ_1, τ_2, q) to $(0, 0, \tau_2, q)$, and another jump will immediately occur as (e, τ_1, τ_2, q) enters $D_{m_q}^{cn}$ and then it is reset to $(0, 0, 0, -q)$. These two cases all reflect the fact that a transmission jump is enforced when control-mode switches.

IV. Main Results

A. Assumptions

We first state the assumptions we make on the NCS defined by (1) and (17).

Assumption 1: For $m \in \{1, 2\}$, there exist locally Lipschitz functions $V_m : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $\underline{\alpha}_{V_m}, \overline{\alpha}_{V_m} \in \mathcal{K}_\infty$, and $a_m, \gamma_m > 0$ such that the following hold.

(i) For all $x \in \mathbb{R}^{n_x}$, $\underline{\alpha}_{V_m}(|x|_{\mathcal{A}}) \leq V_m(x) \leq \overline{\alpha}_{V_m}(|x|_{\mathcal{A}})$ with $\mathcal{A} := \{\xi \in C \cup D : x = 0, e = 0\}$.

(ii) For almost all $x \in \mathbb{R}^{n_x}$ and all $e \in \mathbb{R}^{n_e}$, $\langle \nabla V_m(x), f_m(x, e) \rangle \leq -a_m V_m(x) + \gamma_m^2 W^2(e)$. □

Assumption 1 implies that the system $\dot{x} = f_m(x, e)$, $m \in \{1, 2\}$, is $\mathcal{L}_2$ stable from $W(e)$ to $\sqrt{V_m(x)}$. The fact that the Lyapunov function V_m has an exponential decay rate in item (ii) of Assumption 1 is not restrictive as any input-to-state stable Lyapunov function can be modified accordingly in view of [16].

We also need the conditions on the e-system.

Assumption 2: For $m \in \{1, 2\}$, there exist a locally Lipschitz function $W : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, $\underline{\alpha}_W, \overline{\alpha}_W \in \mathcal{K}_\infty$, $L_{W_m} > 0$ and $L_{V_m} \geq 0$ such that the following hold.

(i) For any $e \in \mathbb{R}^{n_e}$, $\underline{\alpha}_W(|e|) \leq W(e) \leq \overline{\alpha}_W(|e|)$.

(ii) For $m \in \{1, 2\}$, for almost all $e \in \mathbb{R}^{n_e}$ and all $x \in \mathbb{R}^{n_x}$, $\langle \nabla W(e), g_m(x, e) \rangle \leq L_{W_m} W(e) + L_{V_m} \sqrt{V_m(x)}$, where V_m come from Assumption 1. □

Item (i) of Assumption 2 says that W is positive definite and radially unbounded, and item (ii) is an exponential growth condition of the e-system between two consecutive transmission instants.

We assume that V_1 and V_2 from Assumption 1 are bounded by each other with a linear gain, as given below. This condition can be justified by the properties of V_1 and V_2 given in item (i) of Assumption 1.

Assumption 3: There exist $\pi_{12}, \pi_{21} \geq 1$ such that

$$V_1(x) \leq \pi_{12} V_2(x),\ V_2(x) \leq \pi_{21} V_1(x) \quad \forall\, x \in \mathbb{R}^{n_x}, \quad (19)$$

where V_1, V_2 come from Assumption 1. □

B. Dwell time and transmission event-triggering condition

We now provide the minimum dwell time $T_{\mathrm{dwell},m}$, the event-trigged condition Υ_m and the *maximum allowable sampling period* (MASP) $T_{\mathrm{MASP},m}$, $m \in \{1,2\}$, to ensure stability of the system.

For $m \in \{1,2\}$, we select a $c_m \in [0, a_m)$ for $a_m > 0$ from Assumption 1 and let

$$T_{\mathrm{dwell},1} := \frac{1}{c_1} \ln(\pi_{21}) \qquad T_{\mathrm{dwell},2} := \frac{1}{c_2} \ln(\pi_{12}), \qquad (20)$$

where π_{21}, $\pi_{21} \geq 1$ come from Assumption 3. Then, for any $e \in \mathbb{R}^{n_e}$ and $x \in \mathbb{R}^{n_x}$, we define Υ_m in (13) by

$$\Upsilon_m(e,x) = W^2(e) - \lambda_m V_m(x), \qquad (21)$$

where V_m and W come from Assumptions 1 and 2, respectively, and $\lambda_m \geq 0$ is a design parameter.

We then select λ_m such that $\lambda_m < \lambda_m^*$, where

$$\lambda_m^* := \frac{a_m - c_m}{\gamma_m^2}, \qquad (22)$$

a_m and $\gamma_m > 0$ come from Assumption 1 and $c_m \in [0, a_m)$ from (20).

For each $\lambda_m \in [0, \lambda_m^*)$, we select T_m in (17) such that $T_m < T_{\mathrm{MASP},m}$, with $T_{\mathrm{MASP},m}$ being the MASP defined as

$$T_{\mathrm{MASP},m} := \qquad (23)$$
$$\frac{2}{L_m} \operatorname{arctanh}\left(\frac{L_m\left(\sqrt{a_m - c_m} - \gamma_m \sqrt{\lambda_m}\right)}{(L_m\sqrt{\lambda_m} + 2L_{V_m})\gamma_m + L_m\sqrt{a_m - c_m}} \right),$$

where $L_m := L_{W_m} + c_m$, $a_m, \gamma_m, L_{W_m} > 0$ and $L_{V_m} \geq 0$ come from Assumptions 1 and 2, and $c_m \geq 0$ from (20). The numerator $L_m(\sqrt{a_m - c_m} - \gamma_m\sqrt{\lambda_m})$ in (23) is non-negative in view of (22).

We can also see from (20), (22) and (23) that λ_m^*, $T_{\mathrm{MASP},m}$ and $T_{\mathrm{dwell},m}$ increase when $c_m \geq 0$ is reduced. For the special case when $c_m = 0$ for some $m \in \{1,2\}$, it follows that $T_{\mathrm{dwell},m} = \infty$ and the expressions of λ_m^* and $T_{\mathrm{MASP},m}$ are consistent to [17]. We then recover the non-switched PETC control in [17] as a special case. On the other hand, λ_m^* and $T_{\mathrm{MASP},m}$ decrease when c_m grows, which in general means more transmissions will be triggered. Note that the larger c_m the smaller $T_{\mathrm{dwell},m}$. We then have that there exists a trade off between the minimum control-mode dwell time and the transmission frequency. When the control-modes are triggered to change often, caused by exogenous disturbances for example, sufficient samplings and transmissions are essential to guarantee stability of the system, as implied by Theorem 1 in the following section. Moreover, for a determined c_m, $T_{\mathrm{MASP},m}$ is decreasing in λ_m. In other words, the larger the λ_m, the smaller $T_{\mathrm{MASP},m}$ and vice versa.

C. Stability analysis

We show that system defined in (1) and (17) is uniformly globally asymptotically stable (UGAS) when Assumptions proposed above are verified and parameters λ_m and T_m are chosen properly. The proofs are omitted due to the space limitation.

Theorem 1: Consider the system defined in (1) and (17) with $T_{\mathrm{dwell},m}$, $m \in \{1,2\}$, defined in (20). Suppose the following hold.

1) Assumptions 1-2 are satisfied.
2) For $m \in \{1,2\}$, $\lambda_m < \lambda_m^*$ with λ_m^* defined in (22).
3) For $m \in \{1,2\}$, $T_m < T_{\mathrm{MASP},m}$ with $T_{\mathrm{MASP},m}$ defined in (23).

Then, the closed set $\mathcal{A} := \{\xi \in C \cup D : x = 0, e = 0\}$ is UGAS. □

Theorem 1 provides a schematic design procedure for periodic event-triggered supervisory control of nonlinear systems. In particular, we first design a dwell-time supervisory control law in the absence of the network, by following the technique from [1], [12], [13], [14] for example. We then implement it on the network, as illustrated in Example 1 and Section III, we can model its periodic event-triggered implementation as the system defined in (1) and (17). Thus, Theorem 1 can be used to study the conditions on minimum control-mode dwell time and the network to ensure asymptotic stability of PETC supervisory control system.

Remark 1: Note that the switching schemes in [1], [12], [13], [14] enforce a dwell time on their own, say $\tau_D > 0$, we then need to let $T_m > \max\{\tau_D, T_{\mathrm{MASP},m}\}$ to implement the supervisory control. □

V. Numerical Example

We consider an example from [2], for which the plant is given by $\dot{x}_p = A_p x_p + B_p u_m$, $A_p := \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $B_p := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and for $m \in \{1,2\}$, $u_1 = K x_p$ with $K := [1 \;\; -4]$ and u_2 is given by $\dot{x}_c = A_{c,2} x_c + B_{c,2} x_p$ with $u_2 = C_2 x_c$, $A_{c,2} := \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}$, $B_{c,2} := \begin{bmatrix} 0 & 0 \\ -1 & 4 \end{bmatrix}$, $C_2 := [1 \;\; -4]$.

As shown in [2], when the plant and its controller u_2 communicate via a network and transmissions are triggered by continuous evaluating a rule, the inter-event times approach zero when the solution $x := (x_p, x_c)$ is close to the origin. Furthermore, when we enforce some minimum inter-event time to avoid Zeno, x only converges to a set $\{x \in \mathbb{R}^4 : |x| \leq 6.4\}$ not the origin. Here, we employ both u_1 and u_2 and let $\Delta(x) := 6 - |x|$ so that u_1 is activated when $\Delta(x) < 0$ and u_2 is employed when $\Delta(x) > 0$.

We consider the scenario that the plant state x_p and control signal u_m, $m \in \{1,2\}$, are transmitted via a digital network and received as $\hat{x}_p$ and $\hat{u}_m$, and the control system is implemented on a ZOH device. Then, we can follow Section III to formulate the overall networked control system as (1). We next verify Assumption 1 with $V_m(x) = x^T P_m x$,

$W(e) = |e|$, for $m \in \{1,2\}$, $P_1 := \begin{bmatrix} 1.5 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $P_2 := \begin{bmatrix} 37 & -0.5 & 34.5 & -3.5 \\ -0.5 & 27 & 3.5 & 21.5 \\ 34.5 & 3.5 & 35 & -0.5 \\ -3.5 & 21.5 & -0.5 & 18 \end{bmatrix}$. In particular, item (i) of Assumption 1 holds with $\underline{\alpha}_{V_1} = 0.691s^2$, $\overline{\alpha}_{V_1} = 1.809s^2$, $\underline{\alpha}_{V_2} = 0.317s^2$ and $\overline{\alpha}_{V_2} = 70.69s^2$ for all $s \geq 0$. This indeed ensure that Assumption 3 holds with $\pi_{12} = 5.71$ and $\pi_{21} = 102.3$. Item (ii) of Assumption 1 holds with $a_1 = 0.5196$, $\gamma_1 = 0.2429$, $a_2 = 0.016$ and $\gamma_2 = 10.34$. Assumption 2 holds with $\underline{\alpha}_W = \overline{\alpha}_W = s$ for all $s \geq 0$, $L_{W_1} = 5$, $L_{V_1} = 2.8944$, $L_{W_2} = 4.123$ and $L_{V_2} = 47.9$.

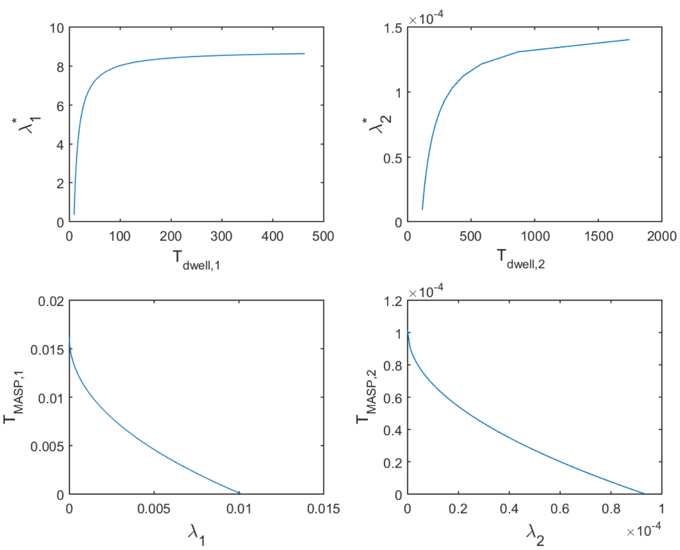

Fig. 2: $T_{dewell,m}$ v.s. λ_m^* and $T_{\mathrm{MASP},m}$ v.s. λ_m, $m \in \{1,2\}$.

For any $c_m \in (0, a_m)$ and $m \in \{1,2\}$, we can calculate $T_{\mathrm{dwell,m}}$ and λ_m^* according to (20) and (22). For each $\lambda_m \in (0, \lambda_m^*)$, we calculate $T_{\mathrm{MASP},m}$ according to (23). See the calculation results in Figure 2, it is clear that λ_m^* decreases when $T_{\mathrm{dwell,m}}$ is reduced, which in general means more transmissions are needed when control modes switch frequently. Moreover, there exists a tradeoff between the bound of the sampling period $T_{\mathrm{MASP},m}$ with the triggering parameter λ_m. The simulation results show that states of the closed loop system converge to the origin when we choose $T_{\mathrm{dwell,m}}$, λ_m and T_m properly, as suggested in Theorem 1.

VI. Conclusions

We have considered the design of periodic event-triggered supervisory control for nonlinear systems. We have followed an emulation approach for this purpose, in the sense that we started with a supervisory control scheme which is designed to robustly stabilize the origin of the plant in the absence of network. The supervisory control scheme consists of two state-feedback candidate controllers and a control-mode switching logic. We then implemented it on the network and studied conditions on the control-mode dwell time and the network to preserve stability properties of the system. We provided the minimum control-mode dwell time for respectively two candidate controllers, and for each controller, we designed criterion to generate transmission events and provided explicit bound on the maximum sampling period with which the triggering rules are evaluated. We showed that the stability of the whole system is ensured when the bounds on control-model dwell time and sampling period are satisfied. The analysis reveals relationships among the control-mode dwell time, a parameter used to define the transmission event-triggering condition and the bound of the sampling period. An example is given to illustrate how to apply the results.

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