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Singularly Perturbed Networked Control Systems [*]

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Abstract: We study networked control systems (NCSs) where the controller is given by a state-feedback law and the plant is modeled by a dynamical system evolving on two time-scales, representing a characterization by some slow and fast dynamics. When using the stability analysis frameworks for NCSs from the literature, this time-scale separation is ignored and, as a result, the slow dynamics are in general updated at the same rate as the fast dynamics, leading to many redundant transmissions of the slow dynamics. Therefore, we assume in this paper that the slow dynamics and fast dynamics can be transmitted separately over the network, allowing us to use techniques inspired by singular perturbation methods in the stability analysis. That is, we show by means of a Lyapunov-based proof how to obtain conditions on the transmission rates (expressed in maximal allowable transmission intervals (MATIs)) for the slow and fast dynamics separately such that stability of the NCS is guaranteed, based only on approximated models of the slow and the fast dynamics.

Keywords: Networked control systems, singular perturbed systems, hybrid dynamical systems

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1. INTRODUCTION

Many systems are characterized by both slow and fast dynamics, and operate, consequently, on multiple time-scales. As mentioned in Kokotović et al. (1999), examples of such systems often occur when mechanical and electrical components are combined. For instance, an electrically driven robot manipulator can have slower mechanical dynamics and faster electrical dynamics. Other examples arise when there is a need to implement a feedback control algorithm through a fast actuator, see Sanfelice and Teel (2011), or in power systems, see, for instance, Chow (1982).

For these multi time-scale systems, the high-frequency phenomena are often neglected to simplify the model for analysis. However, a controller design based on a simplified model might result in a system far from its desired performance (or even an unstable system). Therefore, one needs an extra step in the design procedure that takes into account the disregarded (fast) phenomena. Because most control systems are dynamic, one way of modeling and analyzing systems according to this two step design procedure is by exploiting the multi time-scales, i.e., the decomposition in stages is dictated by a separation of time-scales, which happens to be the fundamental characteristic of the singular perturbation method, see Kokotović et al. (1999) or Khalil (2002). Using approximated models for both the slow and the fast dynamics, called the reduced model and the boundary layer model, respectively, stability of the overall system may be readily addressed by assessing the stability of these approximated models.

Unfortunately, when considering systems that exploit (wireless) packet-based networks to communicate sensor and actuator data to and from the plant/controller, this time-scale separation appears to be ignored in the design and therefore also in the existing methodologies for the stability analysis in the literature. That is, for these so-called networked control systems (NCSs), it is always assumed that the networked values (i.e., the most recently received values) corresponding to the slow and the fast dynamic states are updated at the same rate; there is only one maximal allowable transmission interval (MATI) for the entire system, see, e.g., Nešić and Teel (2004), Carnevale et al. (2007), Heemels et al. (2010), and the references therein. However, one can imagine that maintaining such a communication rate for both the fast *and* the slow dynamics leads to many redundant transmissions of the slow dynamics since they will not change (much) between updates.

Therefore, we consider in this paper the scenario in which the plant is a two time-scale nonlinear system for which its slow and fast states are transmitted through *separate* digital communication channels to the controller. As a result of this separation, the slow dynamics can be updated independently of the fast dynamics and therefore do not have to be updated at the same rate. Consequently, similar to the continuous-time case as described in Kokotović et al. (1999) and Khalil (2002), we can again analyze the overall system by analyzing its slow and fast dynamics separately.

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To this end, following Nešić and Teel (2004) and Carnevale et al. (2007), we use the emulation-based approach to design the NCS, meaning that we assume that the controller, being here a state-feedback law, is such that it stabilizes the plant in the absence of a network (i.e., under perfect communication), combined with hybrid systems analysis tools. Doing so allows us to rewrite the overall NCS as a *singularly perturbed hybrid system*. As such, using a Lyapunov-based proof along the lines of Sanfelice and Teel (2011) and Abdelrahim et al. (2015), stability of the NCS can be addressed by assessing the stability of the boundary layer and reduced systems corresponding to this singularly perturbed hybrid system, expressed in terms of individual MATIs for the slow *and* fast dynamics. Here, in contrary to Sanfelice and Teel (2011) and Wang et al. (2012), we model *both* the reduced system and the boundary layer system as a hybrid system.

Notation: The set of real numbers is denoted by $\mathbb{R}$ and the sets of non-negative real numbers and integers by $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$, respectively. For vectors $v_i \in \mathbb{R}^{n_i}$, $i \in \{1, 2, \ldots, N\}$, we denote by $(v_1, v_2, \ldots, v_N)$ the vector $\begin{bmatrix} v_1^\top & v_2^\top & \cdots & v_N^\top \end{bmatrix}^\top$, and by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the usual inner product, respectively. We use the notation $r^+(t) = r(t^+) = \lim_{\tau \downarrow t} r(\tau)$ for $r : \mathbb{R} \to \mathbb{R}^n$, provided the limit exists.

2. SYSTEM DESCRIPTION

In this section, we introduce the NCS setup and a hybrid model describing the overall dynamics.

2.1 System Setup

We consider the NCS setup as depicted in Fig. 1 where the plant $\mathcal{P}$ is controlled by the controller $\mathcal{C}$ by means of communicating the plant states (x, z) via the network $\mathcal{N}$. To complete the description, we consider the plant, the controller, and the network individually in more detail.

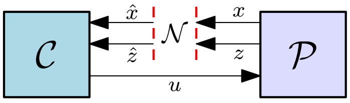

Fig. 1. The considered NCS setup.

The plant $\mathcal{P}$
The plant $\mathcal{P}$ is for some small constant $0 < \varepsilon \ll 1$ given by the nonlinear singular perturbed system

$$\mathcal{P} : \begin{cases} \dot{x} = f(x, z, u) \\ \varepsilon \dot{z} = g(x, z, u) \end{cases} \tag{1}$$

with $(x, z) \in \mathcal{D}_p \times \mathcal{D}_z$ the state of the system (where $\mathcal{D}_x \subset \mathbb{R}^{m_x}$ and $\mathcal{D}_z \subset \mathbb{R}^{m_z}$ are open connected sets that contain the origin $(x, z) = (0, 0)$) and $u \in \mathcal{D}_u \subseteq \mathbb{R}^{m_u}$ the control input. We assume that the functions f and g are locally Lipschitz in their first two arguments.

The controller $\mathcal{C}$
As mentioned in the introduction, following the emulation-based approach we design a state-feedback controller $\mathcal{C}$ for the plant $\mathcal{P}$ assuming perfect communication. In particular, inspired by the singular perturbed method it is assumed that there exists a composite control law

$$\mathcal{C} : \; u = \Gamma_s(\hat{x}) + \Gamma_f(\hat{x}, \hat{z}) \tag{2}$$

with $(\hat{x}, \hat{z})$ the networked versions of (x, z), such that the origin of (1) is an asymptotically stable equilibrium when $(\hat{x}(t), \hat{z}(t)) = (x(t), z(t))$ for all times $t \in \mathbb{R}_{\geq 0}$. See, e.g., (Kokotović et al., 1999, Section 7.6) for a design procedure for such a composite controller for the plant $\mathcal{P}$ in the case that $(\hat{x}, \hat{z}) = (x, z)$ (i.e., under perfect communication).

The network $\mathcal{N}$
Observe that, based on the dynamics of the closed-loop system (1)-(2), we have a separation of time scales. In particular, the plant $\mathcal{P}$ is characterized by slow dynamics (related to) x and fast dynamics (related to) z. As a result, one can imagine that updating the slow dynamics x at the same rate as the fast dynamics z results in many redundant transmissions (since the value for x will not change (much) between updates in this case), which would be the case when using the analysis framework of Nešić and Teel (2004), Carnevale et al. (2007) or Heijmans et al. (2017). Therefore, as can be seen from Fig. 1 we assume in this work that the network $\mathcal{N}$ has two separate communication channels that are respectively dedicated to transmissions of x and z. Such a NCS setup can be realized for many practical network implementations, including, e.g., a WirelessHART network, see Maass et al. (2017). As a result of this separation, the slow dynamics x can be updated independently of the fast dynamics z and therefore do not have to be updated at the same rate, potentially preventing many redundant communications.

To be more precise, we assume that the network has two collections of transmission times, t_j^s, $j \in \mathbb{N}$, and t_j^f, $j \in \mathbb{N}$, corresponding to transmissions of the slow and fast dynamics, respectively. For these collections we assume that the transmission intervals are bounded by

$$\tau^s_{miati} \leq t^s_{j+1} - t^s_j \leq \tau^s_{mati} \;\text{ and }\; \tau^f_{miati} \leq t^f_{j+1} - t^f_j \leq \tau^f_{mati} \tag{3}$$

where $0 < \tau^s_{miati} \leq \tau^s_{mati}$ denote the *minimal allowable transmission interval* (MIATI)[1] and the *maximal allowable transmission interval* (MATI), respectively, between two consecutive transmission instants at which the slow dynamics x are updated and $0 < \tau^f_{miati} \leq \tau^f_{mati}$ the MIATI/MATI for the fast dynamics z. The two MATIs have to be selected appropriately to guarantee stability properties of the NCS, see also Section 3.2 below.

At each of those transmission times, parts of the plant state (x, z) are sampled and transmitted to the controller $\mathcal{C}$, which results in an update of the networked values $(\hat{x}, \hat{z})$. That is, for all t_j^s, $j \in \mathbb{N}$, we have a transmission of the slow dynamics x, meaning that an update of the networked values occurs according to

$$\begin{aligned} \hat{x}\left((t_j^s)^+\right) &= x\left(t_j^s\right) + h_x\left(j, e_x(t_j^s)\right) \\ \hat{z}\left((t_j^s)^+\right) &= \hat{z}\left(t_j^s\right), \end{aligned} \tag{4}$$

where the function $h_x : \mathbb{N} \times \mathbb{R}^{m_x} \to \mathbb{R}^{m_x}$ models the scheduling protocol for the slow dynamics, while for all t_j^f, $j \in \mathbb{N}$, we have a transmission of the fast dynamics z, leading to an update of the networked values according to

$$\begin{aligned} \hat{x}\left(\left(t_j^f\right)^+\right) &= \hat{x}\left(t_j^f\right) \\ \hat{z}\left(\left(t_j^f\right)^+\right) &= z\left(t_j^f\right) + h_z\left(j, e_z\left(t_j^f\right)\right), \end{aligned} \tag{5}$$

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[1] The MIATI represents physical hardware limitations and is employed to rule out Zeno behavior, see, e.g., Nešić and Teel (2004).

where the function $h_z : \mathbb{N} \times \mathbb{R}^{m_z} \to \mathbb{R}^{m_z}$ models the scheduling protocol for the fast dynamics. Here, $e = (e_x, e_z)$ denotes the network induced error defined as

$$e := (e_x, e_z) = (\hat{x} - x, \hat{z} - z) \in \mathbb{R}^{m_x} \times \mathbb{R}^{m_z},$$

which is a result of the sampling/transmitting behavior, i.e., we have that in general $\hat{x} \neq x$ and $\hat{z} \neq z$.

Finally, it is assumed that $\hat{x}$ and $\hat{z}$ are constant in between two successive transmissions (zero-order-hold (ZOH)), i.e., $\dot{\hat{x}} = 0$ and $\dot{\hat{z}} = 0$. However, this can easily be modified, if desired, see, e.g., Nešić and Teel (2004).

2.2 A Hybrid Modeling Framework

Following the works of Nešić and Teel (2004), Carnevale et al. (2007), and Heijmans et al. (2017), the above NCS setup can be rewritten in the hybrid system formalism of Goebel et al. (2012). To do so, in contrast to the mentioned works, we need to be able to keep track of the time between two consecutive transmissions of the slow dynamics x *and* we need to be able to keep track of the time between two consecutive transmission of the fast dynamics z. Therefore, we introduce two separate timer variables $\tau_s, \tau_f \in \mathbb{R}_{\geq 0}$, modeled by

$$\mathcal{T}_s : \begin{cases} \dot{\tau}_s = 1, & \tau_s \in [0, \tau^s_{mati}] \\ \tau_s^+ = 0, & \tau_s \in [\tau^s_{miati}, \tau^s_{mati}] \end{cases} \tag{6a}$$

and

$$\mathcal{T}_f : \begin{cases} \varepsilon\dot{\tau}_f = 1, & \varepsilon\tau_f \in [0, \tau^f_{mati}] \\ \tau_f^+ = 0, & \varepsilon\tau_f \in [\tau^f_{miati}, \tau^f_{mati}]. \end{cases} \tag{6b}$$

Note that we thus model the timer τ_f to be evolving in the fast time scale of z, i.e., its time-derivative depends on ε, which is needed to model the boundary layer system, as we will see later on in Section 3.1.

In addition to the timers, we also introduce the two counters $\kappa_s, \kappa_f \in \mathbb{N}$, which keep track of the number of transmissions for, respectively, the slow dynamics x and the fast dynamics z. Using now these auxiliary variables, the NCS consisting of the plant model (1), the control law (2), and the network $\mathcal{N}$ with (3)-(5) can be expressed as the hybrid model

$$\mathcal{H} : \begin{cases} \left.\begin{aligned} \dot{x} &= \hat{f}(x, z, e) \\ \varepsilon\dot{z} &= \hat{g}(x, z, e) \\ \dot{e}_x &= -\hat{f}(x, z, e) \\ \varepsilon\dot{e}_z &= -\hat{g}(x, z, e) \\ \dot{\tau}_s &= 1, \ \dot{\kappa}_s = 0 \\ \varepsilon\dot{\tau}_f &= 1, \ \dot{\kappa}_f = 0 \end{aligned}\right\} & \begin{aligned} &\text{when } \tau_s \in [0, \tau^s_{mati}] \\ &\text{and } \varepsilon\tau_f \in [0, \tau^f_{mati}] \end{aligned} \\ \left.\begin{aligned} &x^+ = x, \ z^+ = z, \ e_z^+ = e_z, \\ &\tau_f^+ = \tau_f, \ \kappa_f^+ = \kappa_f \\ &e_x^+ = h_x(\kappa_s, e_x) \\ &\tau_s^+ = 0 \\ &\kappa_s^+ = \kappa_s + 1 \end{aligned}\right\} & \begin{aligned} &\text{when} \\ &\tau_s \in [\tau^s_{miati}, \tau^s_{mati}] \end{aligned} \\ \left.\begin{aligned} &x^+ = x, \ z^+ = z, \ e_x^+ = e_x, \\ &\tau_s^+ = \tau_s, \ \kappa_s^+ = \kappa_s \\ &e_z^+ = h_z(\kappa_f, e_z) \\ &\tau_f^+ = 0 \\ &\kappa_f^+ = \kappa_f + 1 \end{aligned}\right\} & \begin{aligned} &\text{when} \\ &\varepsilon\tau_f \in [\tau^f_{miati}, \tau^f_{mati}] \end{aligned} \end{cases} \tag{7}$$

where

$$\begin{aligned} \hat{f}(x, z, e) &:= f(x, z, \Gamma_s(e_x + x) + \Gamma_f(e_x + x, e_z + z)) \\ \hat{g}(x, z, e) &:= g(x, z, \Gamma_s(e_x + x) + \Gamma_f(e_x + x, e_z + z)) \end{aligned}$$

and with the full state of the hybrid system

$$\xi := ((x, z), e, (\tau_s, \tau_f), (\kappa_s, \kappa_f)) \in \mathbb{X},$$

with $\mathbb{X} := \mathcal{D}_x \times \mathcal{D}_z \times \mathbb{R}^{m_x} \times \mathbb{R}^{m_z} \times \mathbb{R}^2_{\geq 0} \times \mathbb{N}^2$, which can be separated in a set of "slow" and a set of "fast" dynamical sates, given, respectively, by

$$\begin{aligned} \xi_s &:= (x, e_x, \tau_s, \kappa_s) \in \mathbb{X}_s := \mathcal{D}_x \times \mathbb{R}^{m_x} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \\ \xi_f &:= (z, e_z, \tau_f, \kappa_f) \in \mathbb{X}_f := \mathcal{D}_z \times \mathbb{R}^{m_z} \times \mathbb{R}_{\geq 0} \times \mathbb{N}. \end{aligned}$$

We are interested in the stability of this hybrid model (7).

Definition 1. For the system $\mathcal{H}$ given by (7), the set

$$\mathcal{E} = \{\xi \in \mathbb{X} \mid x = 0 \wedge z = 0 \wedge e = 0\} \tag{9}$$

is said to be *uniformly globally asymptotically stable* (UGAS) if there exists a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\beta \in \mathcal{KL}$ such that for any initial condition $\xi(0,0) \in \mathbb{X}$, all corresponding maximal solutions ξ to $\mathcal{H}$ are complete [2] and satisfy $|(x(t,j), z(t,j), e(t,j))| \leq \beta\big(|(x(0,0), z(0,0), e(0,0))|, t + j\big)$ for all $(t,j) \in \operatorname{dom} \xi$. Moreover, if β is of the form $\beta(r,s) = Kr\exp(-cs)$ for some $K, c > 0$, then the set $\mathcal{E}$ is uniformly globally exponentially stable (UGES).

3. STABILITY ANALYSIS

Observe that the hybrid system $\mathcal{H}$ in (7) is of a similar class of singularly perturbed hybrid systems as discussed in Sanfelice and Teel (2011) and Wang et al. (2012). However, the results from Sanfelice and Teel (2011) and Wang et al. (2012) are not applicable as important differences will arise in the stability analysis, as we will see below. In any case, similar to the continuous-time case from Kokotović et al. (1999), its stability can thus be analyzed by means of analyzing the stability of the reduced and boundary layer systems corresponding to (7), which we will first derive.

3.1 Reduced and Boundary Layer Models

For the hybrid system $\mathcal{H}$, we first define its quasi-steady-state equilibrium manifold, which in this case appears as the set-valued mapping $H : \mathbb{R}^{m_x} \times \mathbb{R}^{m_x} \rightrightarrows \mathbb{R}^{m_z} \times \mathbb{R}^{m_z}$, see also Sanfelice and Teel (2011). Since both z and e_z evolve with respect to the fast time scale, we need to define the quasi-steady-state equilibrium for the fast state dynamics z as well as the quasi-steady-state equilibrium for its network-induced error e_z. Observe that for a sufficient amount of communications (i.e., updates of $\hat{z}$), e_z will converge to zero (i.e., its quasi-steady-state equilibrium). As such, let, for all $(\bar{x}, \bar{e}_x) \in \mathcal{D}_x \times \mathbb{R}^{m_x}$, $\bar{z} = H_z(\bar{x}, \bar{e}_x)$ with $H_z(0,0) = 0$ be the unique root of

$$0 = g(\bar{x}, \bar{z}, \Gamma_s(\bar{e}_x + \bar{x}) + \Gamma_f(\bar{e}_x + \bar{x}, \bar{z})),$$

then, we can define the quasi-steady-state equilibrium manifold for the overall hybrid system $\mathcal{H}$ as

$$H(\bar{x}, \bar{e}_x) = \begin{cases} (H_z(\bar{x}, \bar{e}_x), 0), & \text{for all } (\bar{x}, \bar{e}_x) \in \mathcal{D}_x \times \mathbb{R}^{m_x} \\ 0, & \text{for all } (\bar{x}, \bar{e}_x) \notin \mathcal{D}_x \times \mathbb{R}^{m_x}. \end{cases} \tag{10}$$

Similar to the classical continuous-time case, see, e.g., Khalil (2002) or Kokotović et al. (1999), we can now define the boundary layer system for (7) by setting $\varepsilon = 0$ and

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[2] For details and terminology on hybrid systems of the form (7), see Goebel et al. (2012).

using the stretched time scale $\sigma = {}^{t}\!/\!_{\varepsilon}$, see, e.g., Sanfelice and Teel (2011) or Wang et al. (2012). However, in contrast to Sanfelice and Teel (2011) and Wang et al. (2012), we cannot ignore the jump map for the boundary layer system corresponding to (7) since its stability with respect to the error dynamics e_z depends on the property of persistently updating the networked value $\hat{z}$. Therefore, we define the boundary layer system *with* its jump map. Moreover, we will use the change of coordinates $y = z - H_z(x, e_x) \in \mathcal{D}_y \subset \mathbb{R}^{m_z}$ to express the boundary layer system as this shifts its equilibrium towards to origin. As a result, we define the *boundary layer system* to be given by the hybrid system

$$\mathcal{H}_{bl} : \begin{cases} \left.\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}\sigma} = 0, \frac{\mathrm{d}e_x}{\mathrm{d}\sigma} = 0, \frac{\mathrm{d}\tau_s}{\mathrm{d}\sigma} = 0 \\ &\frac{\mathrm{d}y}{\mathrm{d}\sigma} = \hat{g}(x, y + H_z(x, e_x), e) \\ &\frac{\mathrm{d}e_z}{\mathrm{d}\sigma} = -\hat{g}(x, y + H_z(x, e_x), e) \\ &\frac{\mathrm{d}\tau_f}{\mathrm{d}\sigma} = 1, \frac{\mathrm{d}\kappa_f}{\mathrm{d}\sigma} = 0, \frac{\mathrm{d}\kappa_s}{\mathrm{d}\sigma} = 0 \end{aligned}\right\} \begin{array}{l} \text{when} \\ \varepsilon\tau_f \in [0, \tau^f_{mati}] \end{array} \\ \left.\begin{aligned} &x^+ = x,\ y^+ = y,\ e_x^+ = e_x, \\ &\tau_s^+ = \tau_s,\ \kappa_s^+ = \kappa_s \\ &e_z^+ = h_z(\kappa_f, e_z) \\ &\tau_f^+ = 0 \\ &\kappa_f^+ = \kappa_f + 1 \end{aligned}\right\} \begin{array}{l} \text{when} \\ \varepsilon\tau_f \in [\tau^f_{miati}, \tau^f_{mati}]. \end{array} \end{cases} \tag{11}$$

For notational convenience, let now $\xi_f^y := (y, e_z, \tau_f, \kappa_f) \in \mathbb{X}_f^y := \mathcal{D}_y \times \mathcal{D}_{e_z} \times \mathbb{R}_{\geq 0} \times \mathbb{N}$ represent the fast dynamical states expressed using the change of coordinates $y = z - H_z(x, e_x)$. Hence, the full state of (11) is given by $\xi^y := (\xi_s, \xi_f^y)$.

Remark 2. When the network-induced error e_z is absent, i.e., in the case of perfect communication, implying $\hat{z} = z$, (11) simplifies to continuous-time case boundary layer system corresponding to (1) with $\hat{z} = z$ (without considering the jump map), as expressed in, for instance, Khalil (2002) or Kokotović et al. (1999).

Using (10), we can now also obtain the *reduced system* associated with (7), which is given by

$$\mathcal{H}_r : \begin{cases} \left.\begin{aligned} &\dot{x} = \hat{f}(x, H_z(x, e_x), (e_x, 0)) \\ &\dot{e}_x = -\hat{f}(x, H_z(x, e_x), (e_x, 0)) \\ &\dot{\tau}_s = 1 \\ &\dot{\kappa}_s = 0 \end{aligned}\right\} \text{when } \tau_s \in [0, \tau^s_{mati}] \\ \left.\begin{aligned} &x^+ = x \\ &e_x^+ = h(\kappa_s, e_x) \\ &\tau_s^+ = 0 \\ &\kappa_s^+ = \kappa_s + 1 \end{aligned}\right\} \begin{array}{l} \text{when} \\ \tau_s \in [\tau^s_{miati}, \tau^s_{mati}]. \end{array} \end{cases} \tag{12}$$

Note that the jump map of (12) does not depend on $z = H_z(x, e_x)$, so the reduced system ignores the fast dynamics z when determining jumps.

3.2 Lyapunov Conditions for UGAS (or UGES)

Based on the results from Goebel et al. (2012) for hybrid systems, it can be shown that UGAS for the system (7) is guaranteed when there exists a hybrid Lyapunov function $U : \mathbb{X}_s \times \mathbb{X}_f^y \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its arguments corresponding to (x, y, e) and satisfies for some functions $\underline{\alpha}_U, \overline{\alpha}_U \in \mathcal{K}_\infty$ and some positive definite function ϱ:

- For all $(\xi_s, \xi_f^y) \in \mathbb{X}_s \times \mathbb{X}_f^y$
$$\underline{\alpha}_U(|(x, y, e)|) \leq U(\xi_s, \xi_f^y) \leq \overline{\alpha}_U(|(x, y, e)|). \tag{13a}$$
- For almost all $(\xi_s, \xi_f^y) \in \mathbb{X}_s \times \mathbb{X}_f^y$
$$\left\langle \nabla U(\xi_s, \xi_f^y), F^y(\xi_s, \xi_f^y) \right\rangle \leq -\varrho(|(x, y, e)|), \tag{13b}$$
when $\tau_s \in [0, \tau^s_{mati}]$ and $\varepsilon\tau_f \in [0, \tau^f_{mati}]$ and where $F^y(\xi_s, \xi_f^y) := (F_s^y(\xi_s, \xi_f^y), F_f^y(\xi_s, \xi_f^y))$ with $F_s^y(\xi_s, \xi_f^y) := (\hat{f}(x, y + H_z(x, e_x), e), -\hat{f}(x, y + H_z(x, e_x), e), 1, 0)$ and $F_f^y(\xi_s, \xi_f^y) := (\hat{g}(x, y + H_z(x, e_x), e), -\hat{g}(x, y + H_z(x, e_x), e), 1, 0)$.
- For all $(\xi_s, \xi_f^y) \in \mathbb{X}_s \times \mathbb{X}_f^y$ when $\tau_s \in [\tau^s_{miati}, \tau^s_{mati}]$
$$U(G_s(\xi_s), \xi_f^y) - U(\xi_s, \xi_f^y) \leq 0, \tag{13c}$$
where $G_s(\xi_s) := (x, h(\kappa_s, e_x), 0, \kappa_s + 1)$.
- For all $(\xi_s, \xi_f^y) \in \mathbb{X}_s \times \mathbb{X}_f^y$ when $\varepsilon\tau_f \in [\tau^f_{miati}, \tau^f_{mati}]$
$$U(\xi_s, G_f^y(\xi_f^y)) - U(\xi_s, \xi_f^y) \leq 0, \tag{13d}$$
where $G_f^y(\xi_f^y) = (y, h_z(\kappa_f, e_z), 0, \kappa_f + 1)$.

To obtain such a Lyapunov function, similar to the continuous case (see, e.g., (Khalil, 2002, Sec. 11.5)) and the analysis in Sanfelice and Teel (2011), we aim to compose a so-called composite Lyapunov function given by

$$U(\xi_s, \xi_f^y) = (1 - d)V_s(\xi_s) + dV_f(\xi_s, \xi_f^y), \quad 0 < d < 1, \tag{14}$$

where $V_f : \mathbb{X}_s \times \mathbb{X}_f^y \to \mathbb{R}_{\geq 0}$ and $V_s : \mathbb{X}_s \to \mathbb{R}_{\geq 0}$ are hybrid Lyapunov functions for the boundary layer system (11) and the reduced system (12), respectively. Hence, we aim to analyze under which conditions the overall NCS is stable by means of analyzing the stability of the boundary layer and the reduced system.

To this end, observe that both the boundary layer system (11) and the reduced system (12) are of the standard hybrid form for NCSs as introduced/described in Nešić and Teel (2004) and Carnevale et al. (2007). Hence, their stability (including the construction of a hybrid Lyapunov function) can be determined by means of, e.g., (Carnevale et al., 2007, Th. 1) or (Heijmans et al., 2017, Th. 1). Moreover, when we assume that for the boundary layer system $\mathcal{H}_{bl}$ the MATI τ^f_{mati} is such that the set $\mathcal{E}_f := \{(\xi_s, \xi_f^y) \in \mathbb{X}_s \times \mathbb{X}_f^y \mid y = 0 \wedge e_z = 0\}$ is UGAS uniformly[3] in ξ_s, if follows from the result in Cai et al. (2008) that there exists a (smooth) Lyapunov function $V_f : \mathbb{X}_s \times \mathbb{X}_f^y \to \mathbb{R}_{\geq 0}$ such that for (almost) all $(\xi_s, \xi_f^y) \in \mathbb{X}_s \times \mathbb{X}_f^y$

$$\underline{\alpha}_f(|(y, e_z)|) \leq V_f(\xi_s, \xi_f^y) \leq \overline{\alpha}_f(|(y, e_z)|), \tag{15a}$$

$$\left\langle \nabla_{\xi_f^y} V_f(\xi_s, \xi_f^y), F_f^y(\xi_s, \xi_f^y) \right\rangle \leq -\eta_f \beta_f^2(|(y, e_z)|), \quad \text{when } \varepsilon\tau_f \in [0, \tau^f_{mati}] \tag{15b}$$

$$V_f(\xi_s, G_f^y(\xi_f^y)) - V_f(\xi_s, \xi_f^y) \leq 0, \quad \text{when } \varepsilon\tau_f \in [\tau^f_{miati}, \tau^f_{mati}], \tag{15c}$$

and when we assume that for the reduced system $\mathcal{H}_r$ the MATI τ^s_{mati} is such that the set $\mathcal{E}_s := \{\xi_s \in \mathbb{X}_s \mid x = 0 \wedge e_x = 0\}$ is UGAS, there also exists a (smooth) Lyapunov function $V_s : \mathbb{X}_s \to \mathbb{R}_{\geq 0}$ such that for (almost) all $\xi_s \in \mathbb{X}_s$

$$\underline{\alpha}_s(|(x, e_x)|) \leq V_s(\xi_s) \leq \overline{\alpha}_s(|(x, e_x)|), \tag{16a}$$

$$\langle \nabla V_s(\xi_s), F_s(\xi_s) \rangle \leq -\eta_s \beta_s^2(|(x, e_x)|), \quad \text{when } \tau_s \in [0, \tau^s_{mati}] \tag{16b}$$

$$V_s(G_s(\xi_s)) - V_s(\xi_s) \leq 0, \text{ when } \tau_s \in [\tau^s_{miati}, \tau^s_{mati}] \tag{16c}$$

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[3] With uniformly we mean here that the UGAS property does not depend on the value for $\xi_s \in \mathbb{X}_s$. See for more details Khalil (2002).

with $F_s(\xi_s) := (\hat{f}(x, H_z(x, e_x), (e_x, 0)), -\hat{f}(x, H_z(x, e_x), (e_x, 0)), 1, 0)$ and where $\underline{\alpha}_f, \overline{\alpha}_f, \underline{\alpha}_s, \overline{\alpha}_s \in \mathcal{K}_\infty$, β_f and β_s are continuous positive definite functions, and $\eta_f, \eta_s > 0$.

Using (15) and (16) we can now verify under which conditions the inequalities (13) for the composite Lyapunov function (14) hold. Obviously, it follows directly from (15a) and (16a) that (13a) is satisfied. This leaves us with analyzing the conditions (13c) and (13d) during jumps of the hybrid system (cf. updates of the NCS) and the flow condition (13b).

During Jumps

When we have an update of the fast dynamics z, i.e., when $\varepsilon\tau_f \in [\tau^f_{miati}, \tau^f_{mati}]$, we have that

$$\begin{aligned} U(\xi_s, G^y_f(\xi^y_f)) &= (1-d)V_s(\xi_s) + dV_f(\xi_s, G^y_f(\xi^y_f)) \\ &\le (1-d)V_s(\xi_s) + dV_f(\xi_s, \xi^y_f) \qquad (17a) \\ &= U(\xi_s, \xi^y_f). \end{aligned}$$

Hence, (13d) also holds under the assumption that the boundary layer systems as well as the reduced system are asymptotically stable. On the other hand, when we have an update of the slow dynamics x, i.e., when $\tau_s \in [\tau^s_{miati}, \tau^s_{mati}]$, it follows that

$$\begin{aligned} U(G_s(\xi_s), \xi^y_f) &= (1-d)V_s(G_s(\xi_s)) + dV_f(G_s(\xi_s), \xi^y_f) \\ &\le (1-d)V_s(\xi_s) + dV_f(G_s(\xi_s), \xi^y_f). \end{aligned} \qquad (17b)$$

Hence, for (13c) to be satisfied, it is sufficient to require the additional condition that for all $(\xi_s, \xi^y_f) \in \mathbb{X}_s \times \mathbb{X}^y_f$

$$V_f(G_s(\xi_s), \xi^y_f) \le V_f(\xi_s, \xi^y_f). \qquad (18)$$

Remark 3. Condition (18) captures the "neglected" effect of the slow dynamics in the boundary layer system during jumps. That is, no matter how slow the slow dynamics are (or even "frozen" in $\mathcal{H}_{bl}$), if they exhibit a jump, their change is instantaneous and faster than any fast continuous dynamics and this effect should be taken into account.

During Flows

In between updates of the networked values (during flows of the hybrid system $\mathcal{H}$), for almost all $(\xi_s, \xi^y_f) \in \mathbb{X}_s \times \mathbb{X}^y_f$ when $\tau_s \in [0, \tau^s_{mati}]$ and $\varepsilon\tau_f \in [0, \tau^f_{mati}]$ it holds that

$$\begin{aligned} &\left\langle \nabla U\left(\xi_s, \xi^y_f\right), F^y(\xi_s, \xi^y_f)\right\rangle \\ &= (1-d)\left\langle \nabla_{\xi_s} V_s(\xi_s), F^y_s(\xi_s, \xi^y_f)\right\rangle \\ &+ d\left\langle \nabla_{\xi_s} V_f(\xi_s, \xi^y_f), F^y_s(\xi_s, \xi^y_f)\right\rangle + \frac{d}{\varepsilon}\left\langle \nabla_{\xi^y_f} V_f(\xi_s, \xi^y_f), F^y_f(\xi_s, \xi^y_f)\right\rangle \\ &- d\frac{\partial V_f(\xi_s, \xi^y_f)}{\partial y}\left\langle \nabla_{\xi_s} H_z(x, e_x), F^y_s(\xi_s, \xi^y_f)\right\rangle \\ &= (1-d)\left\langle \nabla_{\xi_s} V_s(\xi_s), F_s(\xi_s)\right\rangle + \frac{d}{\varepsilon}\left\langle \nabla_{\xi^y_f} V_f(\xi_s, \xi^y_f), F^y_f(\xi_s, \xi^y_f)\right\rangle \\ &+ (1-d)\left\langle \nabla_{\xi_s} V_s(\xi_s), F^y_s(\xi_s, \xi^y_f) - F_s(\xi_s)\right\rangle \\ &+ d\left\langle \nabla_{\xi_s} V_f(\xi_s, \xi^y_f) - \frac{\partial V_f(\xi_s, \xi^y_f)}{\partial y}\nabla_{\xi_s} H_z(x, e_x), F^y_s(\xi_s, \xi^y_f)\right\rangle \\ &\le -(1-d)\eta_s\beta^2_s(|(x, e_x)|) - \frac{d}{\varepsilon}\eta_f\beta^2_f(|(y, e_z)|) \\ &+ (1-d)\left\langle \nabla_{\xi_s} V_s(\xi_s), F^y_s(\xi_s, \xi^y_f) - F_s(\xi_s)\right\rangle \\ &+ d\left\langle \nabla_{\xi_s} V_f(\xi_s, \xi^y_f) - \frac{\partial V_f(\xi_s, \xi^y_f)}{\partial y}\nabla_{\xi_s} H_z(x, e_x), F^y_s(\xi_s, \xi^y_f)\right\rangle. \end{aligned}$$

As such, similar to the continuous-time case as described in Kokotović et al. (1999) and Khalil (2002), it is sufficient to have that the *interconnection conditions*

$$\begin{aligned} &\left\langle \nabla_{\xi_s} V_s(\xi_s), F^y_s(\xi_s, \xi^y_f) - F_s(\xi_s)\right\rangle \\ &\qquad \le \eta_{fs}\beta_s\left(|(x, e_x)|\right)\beta_f\left(|(y, e_z)|\right) \end{aligned} \qquad (19a)$$

$$\begin{aligned} &\left\langle \nabla_{\xi_s} V_f(\xi_s, \xi^y_f) - \frac{\partial V_f(\xi_s, \xi^y_f)}{\partial y}\nabla_{\xi_s} H_z(x, e_x), F^y_s(\xi_s, \xi^y_f)\right\rangle \\ &\quad \le \eta_{ff}\beta^2_f\left(|(y, e_z)|\right) + \eta_{sf}\beta_s\left(|(x, e_x)|\right)\beta_f\left(|(y, e_z)|\right) \end{aligned} \qquad (19b)$$

hold for some nonnegative constants η_{ff}, η_{fs}, and η_{sf} in order for (13b) to be satisfied. In particular, when the conditions (19) hold, along the same lines as in (Khalil, 2002, Sec. 11.5) it can be shown that (13b) is satisfied for $d = \frac{\eta_{sf}}{\eta_{sf}+\eta_{fs}}$ and for all $0 < \varepsilon < \varepsilon^*$ with

$$\varepsilon^* = \frac{\eta_s\eta_f}{\eta_s\eta_{ff} + \eta_{sf}\eta_{fs}}. \qquad (20)$$

3.3 Main Result

Based on all the above, we can now state our main result.

Theorem 4. For given values of $\tau^s_{miati}, \tau^f_{miati} > 0$, if the MATIs $\tau^s_{mati} > \tau^s_{miati}$ and $\tau^f_{mati} > \tau^f_{miati}$ are such that for the boundary layer system (11) and for the reduced system (12) there exist Lyapunov functions $V_f : \mathbb{X}_s \times \mathbb{X}^y_f \to \mathbb{R}_{\ge 0}$ and $V_s : \mathbb{X}_s \to \mathbb{R}_{\ge 0}$ satisfying (15) and (16), respectively, and if (18) and (19) are satisfied, then there exists an $\varepsilon^* > 0$ given by (20) such that for all $0 < \varepsilon < \varepsilon^*$ the set $\mathcal{E}$ given by (9) for the hybrid system (7) is UGAS.

As mentioned above, stability (and, therefore, the existence of a Lyapunov function) of the boundary layer system and the reduced system can be readily assessed by means of the results in Carnevale et al. (2007) or Heijmans et al. (2017). Moreover, when the boundary layer system itself is uniform in ξ_s (i.e., the dynamics of y and e_z are independent of x and e_x) and V_f exists, then (18) also readily holds, which is, for instance, the case for the illustrative example considered in the next section. Finally, we have that the interconnection conditions (19) can be simplified when the Lyapunov functions V_f and V_s are so-called quadratic-type Lyapunov functions, see also Saberi and Khalil (1984). Moreover, following (Khalil, 2002, Sec. 11.5) we have also the following.

Corollary 5. In the case that both the boundary layer system and the reduced system are UGES and (18) holds, it can be directly obtained that there always exists an $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$ the system (7) is UGES.

4. ILLUSTRATIVE EXAMPLE

In this section, we provide an illustrative example to show how the quasi-steady-state equilibrium manifold H, the boundary layer system $\mathcal{H}_{bl}$ and the reduced system $\mathcal{H}_r$ can be computed, and how stability of the overall NCS can be determined by means of Theorem 4.

Consider the plant and composite controller given by

$$\mathcal{P} : \begin{cases} \dot{x} = xz^3 \\ \varepsilon\dot{z} = z + u \end{cases} \quad \text{and} \quad \mathcal{C} : \; u = -3\hat{z} - 2\hat{x}^{4/3}, \qquad (21)$$

for $\mathcal{D}_x = [-1, 1]$ and $\mathcal{D}_z = [-\frac{1}{2}, \frac{1}{2}]$. Observe that in the case of perfect communication (i.e., $\hat{x} = x$ and $\hat{z} = z$), the controller $\mathcal{C}$ indeed stabilizes the origin $x = 0$, $z = 0$ for $\varepsilon < \frac{3}{7}$, see also (Kokotović et al., 1999, Chap. 7, Example 6.1).

Combining the plant and controller dynamics, we obtain that the flow dynamics for the hybrid system (7) are

$$\begin{cases} \dot{x} = xz^3 \\ \varepsilon\dot{z} = -3e_z - 2z - 2(e_x + x)^{4/3} \\ \dot{e}_x = -xz^3 \\ \varepsilon\dot{e}_z = 3e_z + 2z + 2(e_x + x)^{3/4} \\ \dot{\tau}_s = 1,\ \dot{\kappa}_s = 0 \\ \varepsilon\dot{\tau}_f = 1,\ \dot{\kappa}_f = 0 \end{cases} \tag{22}$$

and that we have for the quasi-steady-state manifold that

$$H_z(\bar{x}, \bar{e}_x) = -(\bar{x} + \bar{e}_x)^{4/3}. \tag{23}$$

Moreover, observe that, since both x and z have dimension 1, an update of the networked values $\hat{x}$ and/or $\hat{z}$ always results in the errors e_x and/or e_z to be reset to zero.

Combining (22) and (23), it follows that the flow dynamics of the boundary layer system are uniform in (x, e_x), i.e., the dynamics of y and e_z do not depend on (x, e_x). As such, determining UGAS (or UGES) uniformly in ξ_s for the boundary layer system is now equivalent of determining UGAS (or UGES) of the "boundary layer" system given by

$$\mathcal{H}_{bl}^* : \begin{cases} \left.\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}\sigma} &= -2y - 3e_z \\ \frac{\mathrm{d}e_z}{\mathrm{d}\sigma} &= 2y + 3e_z \\ \frac{\mathrm{d}\tau_f}{\mathrm{d}\sigma} &= 1,\ \frac{\mathrm{d}\kappa_f}{\mathrm{d}\sigma} = 0 \end{aligned}\right\} \text{ when } \varepsilon\tau_f \in [0, \tau_{mati}^f] \\ \left.\begin{aligned} y^+ &= y,\ e_z^+ = 0 \\ \tau_f^+ &= 0,\ \kappa_f^+ = \kappa_f + 1 \end{aligned}\right\} \text{ when } \varepsilon\tau_f \in [\tau_{miati}^f, \tau_{mati}^f]. \end{cases} \tag{24}$$

Note that we have not included the dynamics for the state ξ_s in this model as they do not influence the dynamics of y nor e_z (and can therefore be left out). As a result, when we can construct a Lyapunov function for (24), then we can also construct V_f such that (18) is satisfied for this NCS. Additionally, the reduced system is in this case given by

$$\mathcal{H}_r : \begin{cases} \left.\begin{aligned} \dot{x} &= -x(x + e_x)^4 \\ \dot{e}_x &= x(x + e_x)^4 \\ \dot{\tau}_s &= 1,\ \dot{\kappa}_s = 0 \end{aligned}\right\} \text{ when } \tau_s \in [0, \tau_{mati}^s] \\ \left.\begin{aligned} x^+ &= x,\ e_x^+ = 0 \\ \tau_s^+ &= 0,\ \kappa_s^+ = \kappa_s + 1 \end{aligned}\right\} \text{ when } \tau_s \in [\tau_{miati}^s, \tau_{mati}^s]. \end{cases} \tag{25}$$

Observe now that, in the case of perfect communication (i.e., $(e_x, e_z) = (0, 0)$), we have the dynamics $\frac{\mathrm{d}y}{\mathrm{d}\sigma} = -2y$ and $\dot{x} = -x^5$, which are both asymptotically stable. As such, we know from the results of Nešić and Teel (2004) and Carnevale et al. (2007) that there always exist $\tau_{mati}^s > 0$ and $\tau_{mati}^f > 0$ small enough for which both (24) and (25) are UGAS. In other words, the UGAS property for the boundary layer system and reduced system is maintained when we update "fast enough". Hence, for small enough values for τ_{mati}^s and τ_{mati}^f, which do *not* have to be necessarily the same, and using the results from Carnevale et al. (2007) or Heijmans et al. (2017), we can construct the Lyapunov functions V_s and V_f, verify for those whether or not (19) is satisfied, and use Theorem 4 to conclude if the set $\mathcal{E}$ given by (9) is UGAS for the NCS.

5. CONCLUSION

In this paper we considered NCSs that exhibit both some slow and fast dynamics. By modeling the overall system as a singularly perturbed hybrid system, we were able to use a singular perturbed method to derive the boundary layer and reduced systems and address the stability of the overall NCS by assessing the stability of these boundary layer and reduced systems. We foresee that this work opens up new insights and can possible inspire to obtain new analyzing techniques for NCSs with multiple time-scales.

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