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Model Reduction of Automotive Engines using Perturbation Theory

R. Sharma, D. Nesic and C. Manzie

***Abstract*— In this paper, a new constructive and versatile procedure to systematically reduce the order of control oriented engine models is presented. The technique is governed by the identification of time scale separation within the dynamics of various engine state variables and hence makes extensive use of the perturbation theory. On the basis of the dynamic characteristics and the geometry of engines, two methods for model reduction are proposed. Method 1 involves collective use of the regular and singular perturbation theories to eliminate temperature dynamics and approximate them with their quasi-steady state values, while Method 2 deals with the elimination of fast pressures. The result is a library of engine models which are associated with each other on a sound theoretical basis and simultaneously allow sufficient flexibility in terms of the reduced order modeling of a variety of engines. Different assumptions under which this model reduction is justified are presented and their implications are discussed.**

I. INTRODUCTION

For the development of engine control, a critical initial step is to acquire the system characteristics in a set of mathematical equations referred to as a *control oriented model*. Among the existing approaches for control oriented engine modeling, a major trend has been to use the measured engine data to obtain steady state engine maps. However, this empirical, quasi-static modeling approach, though widely used, lacks the generalization capability for usage on different engines and leads to lengthy and costly calibrations [2], [3].

An alternative is to formulate models based on physical principles and conduct the few experiments necessary to identify key parameters. An important class of engine models established on physical laws, which in recent decades have proven to be quite effective in performing studies on engine dynamics, supervision and control, is the mean value engine models (MVEM) [2]-[5]. MVEMs describe the average engine behavior over several engine event cycles.

A critical survey of the existing literature reveals that a range of mean value models of different order and complexity have so far been reported. Whereas, the high order engine models are typically a closer description of the actual system (e.g. [10]-[12]), often the task of controller development can be substantially alleviated if the model is in a sufficiently simplified form. In other words, there exits a trade-off between the model accuracy and ease with which the controller development problem can be solved. While considering this trade-off several reduced order models have been proposed and subsequently used for control (see for example [6]-[9]).

However, the existing reduced order models are entirely based on empirical findings and *ad-hoc* assumptions and thus correspond only to specific engines. More specifically, none of the existing works utilize the dynamic and physical attributes of the engine to develop a systematic procedure for model reduction. The knowledge of such a rigorous procedure will not only ease the task of model reduction but will also improve the portability of the same reduced order engine model to different engines leading to reduced engine calibration times as prior work in controller development on one system can be utilized on multiple systems.

The aim of this paper is twofold. Firstly, the paper presents a systematic and rigorous approach for model reduction, applicable to a wide range of internal combustion engines (ICEs). Secondly, noting that the controllers based on the existing reduced order models often perform well, the procedure provides valuable theoretical insight into the validity of the existing reduced order models so as to enhance their portability to other engines and operating regimes. The procedure utilizes approximation techniques, based on *regular and singular perturbation theory*, which, on the one hand, can be theoretically justified and, on the other hand, the errors introduced by these approximations characterized through simulations. Different sets of assumptions under which these model reductions are justified are presented and discussed. The eventual outcome is a library of engine models, which are shown to evince close characteristics under a wide range of operating conditions provided certain sets of assumptions hold. Nevertheless, these assumptions should to be checked on a case-by-case basis for each engine. We verified their validity quantitatively for a 13th order model of a turbocharged engine. In addition, the techniques developed are applicable to other systems such as gas turbines, where model reduction is a necessary step in control design.

The structure of this paper is as follows. In section II, a generic description of the mean value control oriented models of ICEs is introduced. Then, in section III, the procedure of model order reduction is devised. This section is partitioned into two section to discuss the two methods of model reduction. In order to demonstrate the effectiveness and flexibility of the new model reduction technique, a case study is presented in section IV.

II. SYSTEM DESCRIPTION

In this section, we present a generic description of mean value engine models (including both spark ignition and com-

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pression ignition engines). The common strategy in engine modeling is to incorporate the physics due to individual constituent components and their interactions through pipes and/or manifolds (referred to as *control volumes/reservoirs*). Then, the modeling approach utilized is to place *components* (like air filter, compressor, intercooler, throttle, engine, turbine and turbo-shaft) between the control volumes. Next, we present the dynamic description of an individual control volume as a basic building block of a complete engine model.

A. Control Volume (CV) Modeling

The pressure and temperature within the control volumes are determined by mass flows into and out of the volume. On the other hand, mass flows and the temperatures of the flows at the inlet of control volumes are determined by the components on the basis of the pressure and temperature in the control volumes before and after them. In other words, the behavior of the gas within the control volumes is dictated by filling and emptying dynamics of temperatures and pressures.

Therefore, control volume of an ICE can be modeled as a dynamic element with two states, namely temperature T_{cv} and pressure P_{cv}. The dynamic equation for temperature is based on the law of conservation of energy, while that of pressure dynamics originates from the ideal gas law [3]:

$$\begin{aligned}
\dot{P}_{cv} &= \left(\frac{\gamma R}{V}\right)\left[\dot{m}_{cv_{in}}\left(P,T,u(t)\right)T_{cv_{in}}\left(P,T\right)\right. \\
&\qquad \left. -\dot{m}_{cv_{out}}\left(P,T,u(t)\right)T_{cv}\right] \qquad (1) \\
\dot{T}_{cv} &= \left(\frac{T_{cv}}{P_{cv}}\right)\left(\frac{\gamma R}{V_{cv}}\right)\left[\dot{m}_{cv_{in}}\left(P,T,u(t)\right)T_{cv_{in}}\left(P,T\right)\right. \\
&\qquad -\dot{m}_{cv_{out}}\left(P,T,u(t)\right)T_{cv} \\
&\qquad \left. -\frac{T_{cv}}{\gamma}\left(\dot{m}_{cv_{in}}\left(P,T,u(t)\right)-\dot{m}_{cv_{out}}\left(P,T,u(t)\right)\right)\right] \\
&\qquad (2)
\end{aligned}$$

where, P and T are the vectors of pressures and temperatures, respectively, corresponding to all engine control volumes. $u(t)$ denotes the vector of control inputs. $\dot{m}_{cv_{in}}$ and $\dot{m}_{cv_{out}}$ represent the mass flows into and out of the CV and $T_{cv_{in}}$ symbolizes the temperature of the gas at the CV inlet. The expressions of $\dot{m}_{cv_{in}}$, $\dot{m}_{cv_{out}}$ and $T_{cv_{in}}$ in terms of P, T and $u(t)$ for various ICEs can be found in [12]. The ratio of the specific heats, γ is taken to be 1.4.

***Remark** 1:* It may be noted that mean value modeling of any ICE can be carried out as an interconnection of components and control volumes with subtle differences with respect to the components involved. However, the equations governing the dynamics of pressure and the temperatures within the control volumes remain the same as (1)-(2).

B. Engine model

Let us consider an engine with N control volumes. Then, based on (1)-(2), the overall state-space representation of such an engine system is given by

$$\Sigma : \left\{ \begin{array}{l} \frac{dP}{dt} = f\left(P,T,u(t)\right) \\ \frac{dT}{dt} = \varepsilon_T(P,T)G\left(P,T,u(t)\right) \end{array} \right. \qquad (3)$$

In (3), $P = [P_1,\ldots,P_N]^T$; $T = [T_1,\ldots,T_N]^T$; $f = [f_1,\ldots,f_N]^T$ where for each $cv \in \{1,\ldots,N\}$ f_{cv} and $\left(\frac{T_{cv}}{P_{cv}}\right) g_{cv}$ represent the right hand sides of (1) and (2), respectively; $\varepsilon_{cv}(P_{cv},T_{cv}) = \left(\frac{T_{cv}}{P_{cv}}\right)$;

$$\varepsilon_T(P,T) \quad = \quad \underset{cv\in\{1,\ldots,N\}}{min}\, \varepsilon_{cv}(P_{cv},T_{cv}) \qquad (4)$$

and

$$G\left(P,T,u(t)\right) = \left[\begin{array}{l} \left(\frac{\varepsilon_1(P_1,T_2)}{\varepsilon_T(P,T)}\right) g_1\left(P,T,u(t)\right) \\ \left(\frac{\varepsilon_2(P_2,T_2)}{\varepsilon_T(P,T)}\right) g_2\left(P,T,u(t)\right) \\ \vdots \\ \left(\frac{\varepsilon_N(P_N,T_N)}{\varepsilon_T(P,T)}\right) g_N\left(P,T,u(t)\right) \end{array}\right].$$

The functions f, G and ε_T in (3) are continuously differentiable in their arguments for $(P,T) \in D_P \times D_T \;\forall\, t \in [t_0,t_1]$, where $D_P \subset \mathcal{R}_+^N$ and $D_T \subset \mathcal{R}_+^N$. It may be noted that ε_T, due to its dependence on P and T, is dependent on time and initial conditions.

***Remark** 2:* In the context of the mean value modeling of a turbocharged engine, in addition to temperatures and pressures, an additional state of turbo speed, to account for the turbo-shaft dynamics, should also be included. Generally, the dynamics of turbo speed are of the same magnitude as those of pressures and can be assumed to correspond to the same time scale as pressure when considered.

III. MODEL REDUCTION PROCEDURE

This section presents the new model reduction procedure and the conditions under which these model reductions are most applicable. The procedure comprises of two methods and is based on the use of perturbation theory [1]. In the first method (as elaborated in subsection III-A), dynamic characteristics of pressure and temperature of mass in the control volumes are considered. In this direction, first *regular perturbation theory* is applied on Σ to obtain a preliminary system denoted by Σ_1. Under certain assumptions, Σ_1 is identified to demonstrate two time scale separation. The corresponding fast and the slow systems, after the application of *singular perturbation theory*, are expressed in the form of Σ_2^{fast} and Σ_2^{slow}, respectively. On the other hand, Method 2 of model reduction, as demonstrated in subsection III-B, is based on relative sizes of the various engine control volumes. It is shown that, under some conditions on the engine geometry, Σ_2^{fast} can be decoupled into two separate time scale subsystems, Σ_3^{fast} and Σ_3^{slow}. Interestingly, the procedure allows for a great deal of modeling flexibility as the two methods can be implemented either in conjunction or completely independent of one another depending upon the satisfaction of certain conditions.

A. Model reduction: Method 1

This model simplification is based on the observation that in all engine control volumes the magnitude of the derivative of pressure is significantly larger in comparison to the magnitude of derivative of temperature. This difference in magnitudes can be attributed to the $\varepsilon_T\left(P,T\right)$ term in

(3). Typically, $\left(\frac{T_{cv}}{P_{cv}}\right)$ and hence $\varepsilon_T\left(P,T\right)$ in the context of engine CVs is a very small positive quantity. This time scale separation allows for the use of perturbation theory to obtain reduced order models of Σ, as demonstrated next.

If $\varepsilon_{T_{max}}$ and $\varepsilon_{T_{min}}$ denote the maximum and minimum values of $\varepsilon_T(P,T)$ over $D_P \times D_T$, respectively, then its average value, ε_{av}, in this domain of operation becomes

$$\varepsilon_{av} = \frac{\varepsilon_{T_{max}} + \varepsilon_{T_{min}}}{2} \tag{5}$$

From (5), we deduce that $\exists\ \Delta\varepsilon_T(P,T)$, which signifies the variation of ε_{av} from its average value, such that

$$\varepsilon_T(P,T) = \varepsilon_{av} + \Delta\varepsilon_T(P,T) \tag{6}$$

By substituting (6) in (3) we obtain

$$\begin{aligned}\frac{dT}{dt} \quad &= \quad \varepsilon_{av} G\left(P,T,u(t)\right) + \Delta\varepsilon_T(P,T) G\left(P,T,u(t)\right), \\ &\qquad\qquad\qquad\qquad\qquad\qquad T\left(t_0\right) = T_0 \quad (7)\end{aligned}$$

***Assumption** 1: The magnitude of $\Delta\varepsilon_T(P,T)$ is sufficiently small in comparison to ε_{av} for $(P,T) \in D_P \times D_T$, $\forall\ t \in [t_0, t_1]$.*

***Assumption** 2: The average value $\varepsilon_{av} << 1$.*

***Remark** 3:* The conditions on $\Delta\varepsilon_T(P,T)$ and ε_{av}, as per Assumptions 1 and 2 above, is a qualitative requirement. In practice, their smallness can be easily ascertained quantitatively by carrying out comprehensive simulations and experimental studies. Nonetheless, the magnitudes of ε_{av} and $\Delta\varepsilon_T(P,T)$ depend on initial conditions and the inputs. Therefore, we require Assumptions 1-2 to hold for all reasonable initial conditions and inputs.

Due to the smallness of $\Delta\varepsilon_T(P,T)$, solving equation (7) can be seen as a *regular perturbation* problem. By setting $\Delta\varepsilon_T(P,T) = 0$, the following nominal or unperturbed system is obtained:

$$\frac{d\bar{T}}{dt} \quad = \quad \varepsilon_{av} G\left(P,\bar{T},u(t)\right), \quad \bar{T}\left(t_0\right) = \bar{T}_0 \tag{8}$$

where, $\forall\ t \in [t_0, t_1]$, $\bar{T} \in D_T \subset \mathcal{R}_+^N$.

For a given continuous control input $u(t)$, the closeness of solutions of dynamic equations (7) and (8) can be ensured by following Theorem 3.4 of [1]. In the context of the engine we have the following.

***Corollary** 1: Let $\varepsilon_{av} G(P,T,u(t))$ be continuous in t and Lipschitz in $(P,T) \in D_P \times D_T\ \forall\ t \in [t_0, t_1]$ with Lipschitz constant L. Let $T(t)$ and $\bar{T}(t)$ be the solutions of (7) and (8), respectively, such that $T(t), \bar{T}(t) \in D_T\ \forall\ t \in [t_0, t_1]$. Suppose that $\|\Delta\varepsilon_T(P,T) G\left(P,T,u(t)\right)\| \leq \mu$ for some $\mu >$ 0. Then,*

$$\|T(t) - \bar{T}(t)\| \leq \|T_0 - \bar{T}_0\| exp[L(t-t_0)] + \frac{\mu}{L}\{exp[L(t-t_0)] - 1\}$$

$\forall\ t \in [t_0, t_1]$.

***Remark** 4:* The continuous differentiability of $G\left(P,T,u(t)\right)$ is ensured only for continuous inputs. Here, we refer to such an input as *admissible*. The time continuity of the control inputs can be assumed because inputs to the engine (like throttle and waste-gate) vary smoothly with time.

Thus (7), which governs dynamics of the temperatures, can be approximated by equations (8). Accordingly, the original system Σ can be approximated by the following system

$$\Sigma_1 : \begin{cases} \frac{dP}{dt} = f\left(P,\bar{T},u(t)\right), & P(t_0) = P_0 \\ \frac{d\bar{T}}{dt} = \varepsilon_{av} G\left(P,\bar{T},u(t)\right), & \bar{T}(t_0) = \bar{T}_0 \end{cases} \tag{9}$$

By introducing a change of time variable as $\varepsilon_{av} t = \tau$, Σ_1 can be rewritten as

$$\begin{aligned}\varepsilon_{av}\frac{dP}{d\tau} \quad &= \quad F(P,\bar{T},u(\tau/\varepsilon_{av})), \qquad P(t_0) = P_0 \quad (10) \\ \frac{d\bar{T}}{d\tau} \quad &= \quad G(P,\bar{T},u(\tau/\varepsilon_{av})), \qquad \bar{T}(t_0) = \bar{T}_0 \quad (11)\end{aligned}$$

The system description (10)-(11) is in standard *singularly perturbed form* [1] and demonstrates the time scale separation between the dynamics of pressure and temperature. Specifically, the dynamics of pressures are much faster than those of temperatures. The smallness of ε_{av} permits the application of singular perturbation theory to interpret Σ_1 in two separate time scales. For that, we set $\varepsilon_{av} = 0$ to obtain the following reduced order system

$$\begin{aligned}0 \quad &= \quad F(P,\bar{T},u(\tau/\varepsilon_{av})) \qquad\qquad (12) \\ \frac{d\bar{T}}{d\tau} \quad &= \quad G(h(\bar{T},u),\bar{T},u(\tau/\varepsilon_{av})) \qquad (13)\end{aligned}$$

where, $h = \left[h_1, \ldots, h_N\right]^T$ is the solution of (12) for P in terms of $\bar{T}$ and $u(\tau/\varepsilon_{av})$. One way to analytically evaluate function $h(\bar{T}, u(\tau/\varepsilon_{av}))$ is to expand the right hand side of (12) in its Taylor series expansion and solve for P by equating it to zero.

The more explicit representation of slow time scale subsystem is given by following equations:

$$\Sigma_2^{slow} : \begin{cases} \frac{d\bar{T}_1}{d\tau} = \left(\frac{\varepsilon_1(h_1(\bar{T},u(\tau/\varepsilon_{av})),\bar{T}_1)}{\varepsilon_T(h(\bar{T},u(\tau/\varepsilon_{av})),\bar{T})}\right) g_1\left(h(\bar{T},u(\tau/\varepsilon_{av})),\bar{T}\right) \\ \frac{d\bar{T}_2}{d\tau} = \left(\frac{\varepsilon_2(h_2(\bar{T},u(\tau/\varepsilon_{av})),\bar{T}_2)}{\varepsilon_T(h(\bar{T},u(\tau/\varepsilon_{av})),\bar{T})}\right) g_2\left(h(\bar{T},u(\tau/\varepsilon_{av})),\bar{T}\right) \\ \vdots \\ \frac{d\bar{T}_N}{d\tau} = \left(\frac{\varepsilon_N(h_N(\bar{T}),\bar{T}_N)}{\varepsilon_T(h(\bar{T},u(\tau/\varepsilon_{av})),\bar{T})}\right) g_N\left(h(\bar{T},u(\tau/\varepsilon_{av})),\bar{T}\right) \end{cases}$$

For a given admissible control input $u(\tau/\varepsilon_{av})$, the closeness of solutions of systems (10)-(11) and (12)-(13) is ascertained by the application of Theorem 11.1 (page 434) of [1] (commonly known as *Tikhonov's theorem*) whose natural consequence for the case at hand can be expressed as follows.

***Corollary** 2: Consider the singularly perturbed system (10)-(11) and let $P = h\left(\bar{T}, u(\tau/\varepsilon_{av})\right)$ be an isolated root of (12). Assume that the following conditions are satisfied for all*

$$\left[\tau, \bar{T}, P - h\left(\bar{T}, u(\tau/\varepsilon_{av})\right), \varepsilon_{av}\right] \in [0, \tau_1] \times D_T \times D_{y_T} \times (0, \bar{\varepsilon}_{av}]$$

for the domains $D_T \subset \mathcal{R}_+^N$ and $D_{y_T} \subset \mathcal{R}^N$:

- *The functions F, G, their partial derivatives with respect to $(\bar{T}, P)$, and first partial derivative of F with respect to τ are continuous for admissible control inputs; the function $h\left(\bar{T}, u(\tau/\varepsilon_{av})\right)$ and the Jacobian $[\partial F/\partial P]$ have continuous first partial derivatives with respect to their arguments.*
- *The reduced order system (13) has a unique solution $\tilde{T} \in S$, where S is a compact subset of D_T.*

- *The boundary layer system,*

$$\frac{dy_T}{dt} = F\left(\bar{T}_0, y_T + h\left(\bar{T}_0, u(t)\right), u(t)\right) \tag{14}$$

where $y_T = P - h\left(\bar{T}, u(\tau/\varepsilon_{av})\right)$, is exponentially stable at the origin, uniformly in $\left(\tau, \bar{T}\right)$; let $M_T \subset D_{y_T}$ be the region of attraction of (14) and Ω_{y_T} be a compact subset of M_T.
Then, there exists a physical constant ε^* such that $\forall\ P_0 - h(\bar{T}_0, u(\tau_0/\varepsilon_{av})) \in \Omega_{y_T}$ and $0 < \varepsilon_{av} < \varepsilon^*$, the singular perturbation problem of (10)-(11) has a unique solution of $\bar{T}\left(\tau, \varepsilon_{av}\right)$ and $P\left(\tau, \varepsilon_{av}\right)$ on $[\tau_0, \tau_1]$, and

$$\begin{aligned} \bar{T}\left(\tau, \varepsilon_{av}\right) - \tilde{T}\left(\tau\right) &= O(\varepsilon_{av}) \\ P\left(\tau, \varepsilon_{av}\right) - h(\tilde{T}\left(\tau\right), u(\tau/\varepsilon_{av})) - \hat{y}_T\left(\tau/\varepsilon_{av}\right) &= O(\varepsilon_{av}) \end{aligned}$$

holds uniformly for $\tau \in [\tau_0, \tau_1]$, $\tilde{T}$ and $\hat{y}_T$ are the solutions of (13) and (14), respectively. Moreover, given any $\tau_b > \tau_0$, there exists $\varepsilon^{**} \leq \varepsilon^*$ such that

$$P\left(\tau, \varepsilon_{av}\right) - h(\tilde{T}(\tau), u(\tau/\varepsilon_{av})) = O(\varepsilon_{av})$$

holds uniformly for $\tau \in [\tau_b, \tau_1]$, whenever $\varepsilon_{av} < \varepsilon^{**}$.

The usual practice in singular perturbation theory is to approximate the fast dynamic with its quasi steady state value and reduce the order of the system by considering only slow dynamic. For the purpose of control oriented modeling, we pursue the alternative mode by focusing on the fast dynamic (pressure) and approximating the slow dynamic (temperature) by a fixed value. This course of action is motivated by the following:

- Most engine control algorithms are related to torque, which in turn requires scheduling of pressures.
- The transient fluctuations in temperatures are much smaller than those in pressures. This makes elimination of the dynamics of temperatures a more viable choice.

Thus, following (14), the *fast time scale subsystem* can be expressed as

$$\frac{d\tilde{P}}{dt} = F\left(\tilde{P}, \bar{T}_0, u(t)\right) \tag{15}$$

More explicitly, the following reduced order control oriented model is obtained:

$$\Sigma_2^{fast} : \begin{cases} \frac{dP_1}{dt} = \left(\frac{\gamma R}{V_1}\right)\left[\dot{m}_{1_{in}}\left(P, \bar{T}_0, u(t)\right) T_{1_{in}} \right. \\ \qquad\qquad \left. -\dot{m}_{1_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{1_0}\right] \\ \frac{dP_2}{dt} = \left(\frac{\gamma R}{V_2}\right)\left[\dot{m}_{2_{in}}\left(P, \bar{T}_0, u(t)\right) T_{2_{in}}\left(P, \bar{T}_0, u(t)\right) \right. \\ \qquad\qquad \left. -\dot{m}_{2_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{2_0}\right] \\ \vdots \\ \frac{dP_N}{dt} = \left(\frac{\gamma R}{V_N}\right)\left[\dot{m}_{N_{in}}\left(P, \bar{T}_0, u(t)\right) T_{N_{in}}\left(P, \bar{T}_0\right) \right. \\ \qquad\qquad \left. - \dot{m}_{N_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{N_0}\right] \end{cases}$$

B. Model reduction: Method 2

While the Method 1 of model reduction is based on the dynamic characteristics of the engine control volumes, Method 2 is governed by relative sizes of the engine control volumes and hence on the engine geometry. The motivation for this follows from the fact that the magnitude of the pressure and temperature dynamics, as shown in (1) and (2), is inversely proportional to volume V_{cv}. Thus, smaller V_{cv} will lead to faster transients and hence there may exist a time scale separation within the dynamics of control volume states. In order to demonstrate Method 2, we consider the problem of further model reduction of reduced order model Σ_2^{fast} obtained after the application of Method 1 on Σ. This entails the classification of control volume pressures as slow and fast on the basis of relative values of V_{cv}.

***Assumption** 3: First r of the N control volumes are of sufficiently larger sizes than those of the rest (with appropriate reordering if necessary). That is $\{V_1, \ldots, V_r\} >> \{V_{r+1}, \ldots, V_N\}$.*

***Remark** 5:* Assumption 3 is justified as sizes of the engine control volumes are sufficiently different. For instance, sizes of intake manifold and exhaust manifold are typically much larger than rest of the control volumes.

***Remark** 6:* It is easy to see that the Method 2 can also be implemented to further reduce the order of Σ_2^{slow}. However, the main use of Σ_2^{slow} is in conjunction with Σ_2^{fast} for performing simulation studies, while countering numerical stiffness issues that may be encountered when high order model Σ is used for simulations. Nevertheless, model reduction is usually needed to aid the control development and, hence, implementation of Method 2 on Σ_2^{slow} may not be required.

Therefore, the control volume pressures can be separated into two time scales. That is, $(P_{r+1}, \ldots, P_N)$ are much faster in comparison with $(P_1, \ldots, P_r)$. In order to conceptualize this time scale separation, let us express the reduced order model Σ_2^{fast} in the following form:

$$\begin{aligned} \frac{dP_s}{dt} &= f_s\big(P, \bar{T}_0, u(t)\big), \qquad P_s(t_0) = P_{s_0} \quad (16) \\ \varepsilon_{P_{min}}\frac{dP_f}{dt} &= f_f\big(P, \bar{T}_0, u(t)\big), \qquad P_f(t_0) = P_{f_0} \quad (17) \end{aligned}$$

where, $P = \left[P_s^T, P_f^T\right]^T$; $P_s = [P_1, \ldots, P_r]^T$; $P_f = [P_{r+1}, \ldots, P_N]^T$;

$$f_s\left(P, \bar{T}_0, u(t)\right) = \begin{bmatrix} \frac{\gamma R}{V_1}\left(\dot{m}_{1_{in}}\left(P, \bar{T}_0, u(t)\right) T_{1_{in}}\left(P, \bar{T}_0\right)\right. \\ \qquad \left. -\dot{m}_{1_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{1_0}\right) \\ \frac{\gamma R}{V_2}\left(\dot{m}_{2_{in}}\left(P, \bar{T}_0, u(t)\right) T_{2_{in}}\left(P, \bar{T}_0\right)\right. \\ \qquad \left. -\dot{m}_{2_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{2_0}\right) \\ \vdots \\ \frac{\gamma R}{V_r}\left(\dot{m}_{r_{in}}\left(P, \bar{T}_0, u(t)\right) T_{r_{in}}\left(P, \bar{T}_0\right)\right. \\ \qquad \left. -\dot{m}_{r_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{r_0}\right) \end{bmatrix};$$

$$f_f\left(P, \bar{T}_0, u(t)\right) = \begin{bmatrix} \left(\frac{1}{\alpha_{r+1}}\right)\left(\frac{\gamma R}{V_{max}}\right)\left(\dot{m}_{r+1_{in}}\left(P, \bar{T}_0, u(t)\right) T_{amb}\right. \\ \qquad \left. -\dot{m}_{r+1_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{r+1_0}\right) \\ \vdots \\ \left(\frac{1}{\alpha_i}\right)\left(\frac{\gamma R}{V_{max}}\right)\left(\dot{m}_{i_{in}}\left(P, \bar{T}_0, u(t)\right) T_{i_{in}}\left(P, \bar{T}_0\right)\right. \\ \qquad \left. -\dot{m}_{i_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{i_0}\right) \\ \vdots \\ \left(\frac{1}{\alpha_N}\right)\left(\frac{\gamma R}{V_{max}}\right)\left(\dot{m}_{N_{in}}\left(P, \bar{T}_0, u(t)\right) T_{N_{in}}\left(P, \bar{T}_0\right)\right. \\ \qquad \left. -\dot{m}_{N_{out}}\left(P, \bar{T}_0, u(t)\right) \bar{T}_{N_0}\right) \end{bmatrix};$$

$V_{max} = max(V_1, \ldots, V_r)$; for each $i = [r+1, \ldots, N]$; $\varepsilon_{P_i} = \left(\frac{V_i}{V_{max}}\right)$; $\varepsilon_{P_{min}} = min\{\varepsilon_{P_{r+1}}, \ldots, \varepsilon_{P_N}\}$; $\alpha_i = \left(\frac{\varepsilon_{P_i}}{\varepsilon_{P_{min}}}\right)$.

Due to Assumption 3, we have $\varepsilon_{P_{min}}$ to be sufficiently small. By setting $\varepsilon_{P_{min}} = 0$, from (16)-(17) the following reduced order system is obtained

$$\frac{dP_s}{dt} = f_s\big(P_s, \psi(P_s, \bar{T}_0, u(t)), \bar{T}_0, u(t)\big) \quad (18)$$

$$0 = f_f\big(P_s, P_f, \bar{T}_0, u(t)\big) \quad (19)$$

where, $\psi(P_s, \bar{T}_0, u(t)) = [\psi_{r+1}, \dots, \psi_N]^T$ is the solution of (19) for P_f in terms of P_s, expressed in the following form:

$$P_i = \psi_i\left(P_s, \bar{T}_0, u(t)\right), \quad i \in \{r+1, \dots, N\} \quad (20)$$

The functions ψ_i can be approximated by expanding $f_f\big(P_s, P_f, \bar{T}_0, u(t)\big)$ in its Taylor series and solving for P_f by equating it to zero.

For a given admissible control input $u(t)$, the closeness of the solutions of (16)-(17) and (18)-(19) is ascertained by the application of *Tikhonov's Theorem* (Theorem 11.1, [1]) whose validity in this case is elaborated in the Corollary 3.

***Corollary** 3: Consider the singularly perturbed system (16)-(17) and let* $P_f = \psi\left(P_s, \bar{T}_0, u(t)\right)$ *be an isolated root of (19). Assume that the following conditions are satisfied for all*

$$\left[t, P_s, P_f - \psi\left(P_s, \bar{T}_0, u(t)\right), \varepsilon_{P_{min}}\right] \in [0, t_1] \times D_{P_s} \times D_{y_P} \times (0, \bar{\varepsilon}_{P_{min}}]$$

for the domains $D_{P_s} \subset \mathcal{R}_+^r$ *and* $D_{y_P} \subset \mathcal{R}^{N-r}$*:*

- *The functions* f_s*,* f_f*, their partial derivatives with respect to* (P_s, P_f)*, and first partial derivative of* f_f *with respect to* t *are continuous; the function* $\psi\left(P_s, \bar{T}_0, u(t)\right)$ *and the Jacobian* $[\partial f_f / \partial P_f]$ *have continuous first partial derivatives with respect to their arguments.*
- *The reduced order system (18) has a unique solution* $\bar{P}_s \in S_1$*, where* $S_1 \subset D_{P_s}$*.*
- *The boundary layer system (fast time scale subsystem),*

$$\Sigma_3^{fast} : \frac{dy_P}{d\tau_P} = f_f\left(P_{s_0}, y_P + \psi(P_{s_0}, \bar{T}_0, u(t_0)), \bar{T}_0, u(t_0)\right)$$

where, $\tau_P = \left(\frac{t - t_0}{\varepsilon_{P_{min}}}\right)$ *and* $y_P = P_f - \psi(P_s, \bar{T}_0, u(t))$*, is exponentially stable at the origin, uniformly in* (t, P_s)*; let* $M_P \subset D_{y_P}$ *be the region of attraction of (21) and* Ω_{y_P} *be a compact subset of* M_P*.*

Then, there exists a physical constant $\varepsilon^*_{P_{min}}$ *such that* $\forall\ P_{f_0} - \psi(P_{s_0}, \bar{T}_0, u(t_0)) \in \Omega_{y_P}$ *and* $0 \le \varepsilon_{P_{min}} \le \varepsilon^*_{P_{min}}$*, the singular perturbation problem (16)-(17) has a unique solution of* $P_s\left(t, \varepsilon_{P_{min}}\right)$ *and* $P_f\left(t\right)$ *on* $[t_0, t_1]$ *and*

$$P_s\left(t, \varepsilon_{P_{min}}\right) - \bar{P}_s(t) = O(\varepsilon_{P_{min}})$$

$$P_f\left(t, \varepsilon_{P_{min}}\right) - \psi(\bar{P}_s(t), \bar{T}_0, u(t)) - \hat{y}_P\left(t / \varepsilon_{P_{min}}\right) = O(\varepsilon_{P_{min}})$$

holds uniformly for $t \in [t_0, t_1]$*,* $\bar{P}_s(t)$ *and* $\hat{y}_P\left(\tau_P\right)$ *are the solutions of (18) and (21), respectively. Moreover, given any* $\tau_b > 0$*, there exists* $\varepsilon^{**}_{P_{min}} \le \varepsilon^*_{P_{min}}$ *such that*

$$P_f\left(t, \varepsilon_{P_{min}}\right) - \psi(\bar{P}_s(t), \bar{T}_0, u(t)) = O(\varepsilon_{P_{min}})$$

holds uniformly for $t \in [t_b, t_1]$*, whenever* $\varepsilon_{P_{min}} < \varepsilon^{**}_{P_{min}}$*.*

***Remark** 7:* It is worth highlighting that the procedure allows for a sufficient flexibility in the manner in which functions ψ_i can be obtained. Expanding in the Taylor series and solving, as demonstrated in this paper, is but one of the possible ways.

The reduced order model thus obtained after the exclusion of fast pressures from Σ_2^{slow} and replacing them with their quasi-steady state values is given by the following r^{th} order system, Σ_3^{slow}:

$$\Sigma_3^{slow} : \begin{cases} \dot{P}_1 = \left(\frac{\gamma R}{V_1}\right)\big[\dot{m}_{1_{in}}\left(P_s, \psi\left(P_s, \bar{T}_0, u(t)\right), \bar{T}_0, u(t)\right) T_{1_{in}}\left(P, \bar{T}_0\right) \\ \qquad\qquad -\dot{m}_{1_{out}}\left(P_s, \psi\left(P_s, \bar{T}_0, u(t)\right), \bar{T}_0, u(t)\right) \bar{T}_{1_0}\big] \\ \dot{P}_2 = \left(\frac{\gamma R}{V_2}\right)\big[\dot{m}_{2_{in}}\left(P_s, \psi\left(P_s, \bar{T}_0, u(t)\right), \bar{T}_0, u(t)\right) T_{2_{in}}\left(P, \bar{T}_0\right) \\ \qquad\qquad -\dot{m}_{2_{out}}\left(P_s, \psi\left(P_s, \bar{T}_0, u(t)\right), \bar{T}_0, u(t)\right) \bar{T}_{2_0}\big] \\ \vdots \\ \dot{P}_r = \left(\frac{\gamma R}{V_r}\right)\big[\dot{m}_{r_{in}}\left(P_s, \psi\left(P_s, \bar{T}_0, u(t)\right), \bar{T}_0\right) T_{r_{in}}\left(P, \bar{T}_0\right) \\ \qquad\qquad -\dot{m}_{r_{out}}\left(P_s, \psi\left(P_s, \bar{T}_0, u(t)\right), \bar{T}_0, u(t)\right) \bar{T}_{r_0}\big] \end{cases}$$

IV. CASE STUDY

The model reduction procedure developed in this paper is now implemented on a 13^{th} order model of a turbocharged (TC) spark ignition (SI) engine as developed in [10], [11]. The model comprises of six control volumes (cv), namely, air-filter (af), compressor (c), intercooler (ic), intake manifold (im), exhaust manifold (em) and turbine (tb). Each of the control volumes is modeled as a dynamic element with two states, namely, temperature (T_{cv}) and pressure (P_{cv}). The 13^{th} state corresponds to the speed of the turbocharger. The model order reduction is accomplished in two stages which involve the sequential application of Method 1 and Method 2. In Method 1, the original 13^{th} order model Σ is decoupled into two time scale systems, namely, Σ_2^{fast} (pressures and the speed of the turbocharger) and Σ_2^{slow} (temperatures). This is followed by the elimination of fast pressures in Method 2. In the subsequent subsections, we examine the errors introduced by the reduced order models (obtained by the successive application of Method 1 and Method 2 on 13^{th} order model) with respect to higher order engine model.

A. Method 1: Comparison of Σ_2^{fast} *and* Σ_2^{slow} *with* Σ

In this subsection, we examine the errors in the pressures and temperatures introduced by the reduced order engine model Σ_2^{fast} (7^{th} order in this case) and Σ_2^{slow} (6^{th} order in this case) with respect to 13^{th} order model of TC SI engine. Figures 1 and 2 show the *percentage* deviations in the responses of pressures and temperatures under changing operating conditions. In the simulation, we assume that at time $t = 4s$ a change in the throttle takes place followed by opening of the wastegate at time $t = 8s$. It is clear from Figure 1 that the errors introduced in the pressures by the control oriented model Σ_2^{fast} under changing operating conditions are quite small. On the other hand, in case of the temperatures, Σ_2^{slow} (based on second order of Taylor series approximations) incorporates errors of up to 18% with respect to Σ temperature responses (elaborated in Remark 8).

***Remark** 8:* The percentage errors in temperatures in Figure 2 are based on the 2^{nd} order Taylor series approximation of $h\left(\bar{T}, u(t)\right)$. With a higher order approximation, the errors can be reduced further. However, due to the complexity

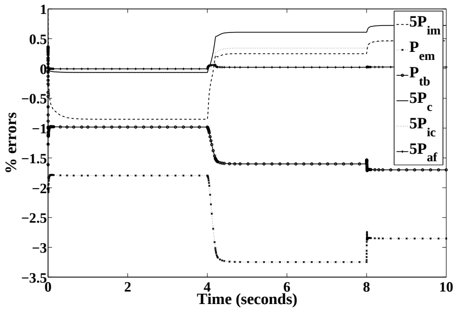

Fig. 1. Percentage errors introduced by Σ_2^{fast} w.r.t Σ pressures

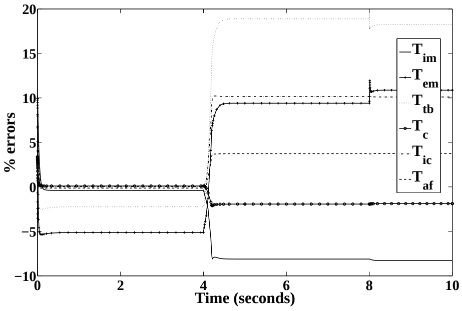

Fig. 2. Percentage errors introduced by Σ_2^{slow} w.r.t Σ temperatures

and order of $F(.)$ (as per (12)), obtaining higher order approximations poses significant numerical challenges and increases the complexity of solutions.

B. Method 2: Comparison of ψ with corresponding Σ_2^{fast} responses

In order to apply Method 2 model reduction on Σ_2^{fast}, we assume that the V_{cv} for intake manifold, exhaust manifold and intercooler have a much higher values than those for air-filter, compressor and turbine. Accordingly, it is easy to say that $\{P_{af}, P_c, P_{tb}\}$ have much faster transients than $\{P_{im}, P_{em}, P_{ic}\}$.

The key to the accuracy of reduced order engine models obtained after Method 2 lies in the accuracy with which the fast pressures, $\{P_{af}, P_c, P_{tb}\}$, can be approximated by ψ (as per (20)). Figures 3-5 show the solution of (19) for fast pressures. It is easy to see that the approximated fast pressures (as obtained in the form of ψ) approach the corresponding responses of Σ_2^{fast} as the order of approximation is increased.

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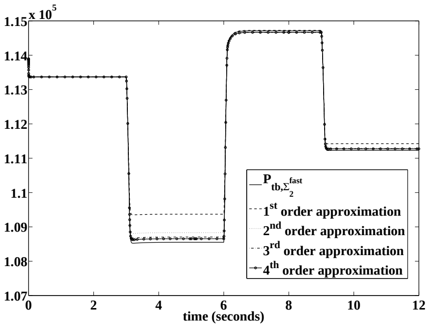

Fig. 3. Approximation of P_{tb} for change in throttle at 3, 6 and 9 seconds

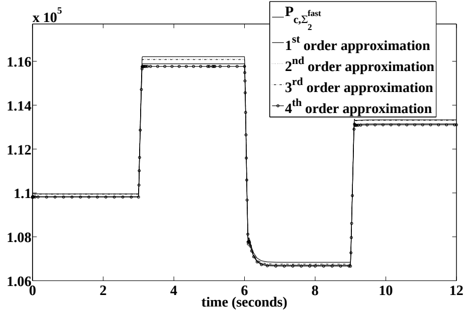

Fig. 4. Approximation of P_c for change in throttle at 3, 6 and 9 seconds

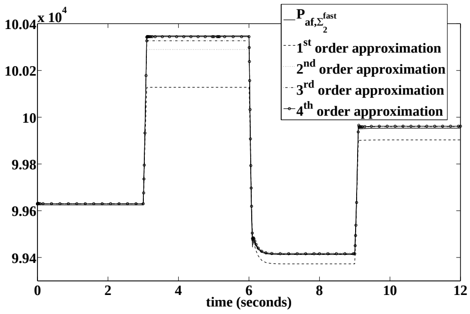

Fig. 5. Approximation of P_{af} for change in throttle at 3, 6 and 9 seconds

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