A note on the robustness of input-to-state stability

A.R. Teel[1] and L. Moreau[2] and D. Nešić[3]

Abstract

This paper develops a unified framework for studying robustness of the input-to-state stability (ISS) property and presents new results on robustness of ISS to slowly varying parameters, to rapidly varying signals, and to generalized singular perturbations.

1 Introduction

1.1 Background

One of the strongest threads in the fabric of nonlinear stability theory is the inherent robustness of uniform asymptotic stability to regular perturbations, slowly varying parameters, rapidly varying signals (averaging) and fast unmodeled dynamics (singular perturbations). See this paper's extended version [16] for references.

Averaging and singular perturbation techniques have been combined, especially in the identification and adaptive control literature, under the title "two time-scale averaging". In this setting the boundary layer system, obtained in the singular perturbation approach by setting the derivative of the slow state variables to zero, is time-varying and possesses a time-varying integral manifold on which the derivative of the slow state variables can be averaged. See [16] for references.

Averaging can be useful in singular perturbation studies even when the boundary layer system is not time-varying. Instead of insisting that the trajectories of the boundary layer system converge to an equilibrium manifold, as in the classical singular perturbation theory of Tikhonov, or to a time-varying integral manifold like in certain adaptive control problems, the "steady-state" behavior of the boundary layer may be complex. For example, the trajectories of the boundary layer may converge to a family of limit cycles parameterized by the slow state variables. The steady-state behavior can then be used to average the derivative of the slow state variables. This idea can be found in the work of Anosov [1], more recently in [7], [8], and in the elegant, pioneering formulation of Artstein and coauthors (e.g., [3], [2]) where the averaging is done using invariant measures and the reduced system is typically a differential inclusion. Except for the results on near asymptotic stability of the origin in [2], these papers focus on compact time interval approximations.

Many recent robustness studies have focused on systems with exogenous disturbances where uniform asymptotic stability is replaced by the input-to-state stability (ISS) property, introduced by Sontag [13]. Robustness of ISS has been established for systems with small time delays [15], for singularly perturbed systems [5] and for systems having an average [11], [10]. All of these results rely on the existence of a converse Lyapunov function for ISS, which was established in [14].

1.2 Contribution

We develop a unified framework for studying robustness of general ISS properties and present new results on robustness of ISS to slowly varying parameters, rapidly varying signals, and generalized singular perturbations. The common feature in these problems is a time-scale separation between slow and fast variables which permits the definition of a boundary layer system like in classical singular perturbation theory. To address the various robustness problems simultaneously, the asymptotic behavior of the boundary layer is allowed to be complex, and it generates an average for the derivative of the slow state variables, as in the work of Artstein [2]. The main results establish that if the boundary layer and averaged systems, also called "limiting systems", are ISS then the ISS bounds also hold for the actual system with an offset that converges to zero with the parameter that characterizes the separation of time-scales. The set of initial conditions from which this ISS behavior holds approaches the set from which the behavior holds for the limiting systems.

Instead of relying on converse Lyapunov functions, our results are derived directly from the closeness of the actual solutions to the limiting systems' solutions on compact time intervals. We use the proof technique we introduced in [9] which enables capturing the behavior of the system on the infinite time interval from information about the behavior on compact time intervals. This is possible even though the actual system's solutions are typically not close to the limiting systems' solutions on the infinite interval due to exogenous disturbances and the general ISS properties considered.

Our results extend to the general ISS setting well-known facts from the literature on averaging, regular perturbations, and singular perturbations. Due to space limitations, we only briefly comment on these applications; a detailed exposition is given in [16].

We present these results as follows: In Section 2 we give an example that illustrates the main concept we will be developing in more generality in the core of the paper.

[1]ECE Dept., UCSB, teel@ece.ucsb.edu, Supported in part by AFOSR F49620-00-1-0106,NSF ECS-9988813 and the Australian Research Council under the small ARC grants scheme.

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[3]Electrical and Electronic Eng. Dept., The Univ. of Melbourne, Parkville, 3010, Victoria, Australia. Supported by the Australian Research Council under the small ARC grants scheme.

We study the robustness of generalized ISS to generalized singular perturbations in Section 3 where our main assumptions and result are given. In Section 4 we present sufficient conditions for the main assumptions of Section 3 to hold. In Section 5 we briefly discuss applications of our results. The applications are discussed in much more detail in [16].

1.3 Notation

- We use $\dot{x} := \frac{dx}{dt}$ and $x' := \frac{dx}{d\tau}$ where, typically, $\tau = \varepsilon t$.
- We will often write (x_s, x_f) in place of $(x_s^T \ x_f^T)^T$.
- A function defined on $\mathbb{R}_{\geq 0}$ taking values in $\mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing.
- A function defined on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ taking values in $\mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is class-$\mathcal{K}$ in its first argument and decreasing to zero in its second argument.
- $\mathcal{B}$ is a closed unit ball, $\rho\mathcal{B}$ a closed ball of radius $\rho > 0$, and $\mathcal{X} + \rho\mathcal{B}$ the union of all sets obtained by taking a closed ball of radius ρ around each point in $\mathcal{X}$.
- For a function $d(\cdot)$ belonging to a set of functions $\mathcal{D}$ that take values in $\mathcal{V}$, we will use $d \in \mathcal{D}$ for the function and $d \in \mathcal{V}$ for a value of the function.
- For $d : \mathbb{R}_{\geq 0} \to \mathcal{V}$, define $||d||_\infty := \text{ess sup}_{t \geq 0} |d(t)|$.

2 A motivational example

We start with an example where, as in a wide variety of industrial applications, the control action is due not to the instantaneous motion of the actuator but rather to some average effect of this motion. In our example, the van der Pol oscillator is a prototype vibrating actuator and an RL circuit is an elementary linear plant. The voltage and current in the van der Pol circuit vary rapidly and the average of the motion can be adjusted by varying the circuit's capacitance. The control algorithm will adjust the capacitance based on the error between the voltage across the resistor in the RL plant and its reference value.

We combine the RL circuit input-output equations

$$\begin{aligned} v_s' &= -v_s + u \\ y &= v_s \ , \end{aligned} \tag{1}$$

where v_s is the voltage across the resistance, with those for the van der Pol circuit

$$\begin{aligned} \varepsilon\, v_f' &= \exp(u_c)\left(-\tfrac{1}{3}v_f^3 + v_f - I_f\right) \\ \varepsilon\, I_f' &= v_f \\ y_c &= K|v_f| - v_{dc} \end{aligned} \tag{2}$$

where v_f is the voltage across the capacitor, I_f is the current through the inductor, and v_{dc} is a constant bias voltage. Equations (1) and (2) are in normalized units. The value u_c is related to the van der Pol circuit's capacitance C, which is the sum of a nominal value $C_\circ$ and an adjustable value $\widetilde{C}$, by

$$\exp(-u_c) = \frac{C_\circ + \widetilde{C}}{C_\circ} \ . \tag{3}$$

We have normalized $\sqrt{L/C_\circ}$ to one and defined $\varepsilon := \sqrt{LC_\circ}$. Central to our control algorithm derivation will be the assumption that ε is small. The output equation for the van der Pol circuit is realized with the combination of a rectifier and an operational amplifier.

One interconnection condition that we impose is

$$y_c = u \tag{4}$$

which indicates that the output of the van der Pol circuit is the input voltage to the RL circuit.

The second interconnection condition will include the control law to be inserted between the measurement of the RL circuit voltage, v_s, and the adjustable capacitor, $\widetilde{C}$ or u_c, in the van der Pol circuit. To derive a control algorithm we exploit our assumption that ε is small which will make the van der Pol circuit oscillations fast compared to the dynamics in the RL circuit. Because of this time-scale separation property, we will approximate the effect of the oscillating actuator by the static mapping

$$u_c \mapsto f(u_c) := \lim_{T\to\infty} \frac{1}{T}\int_0^T K|v_f(t, v_f(0), I_f(0), u_c, \varepsilon)|dt - v_{dc} \tag{5}$$

where the right-hand side turns out to be independent of $\varepsilon > 0$ and $(v_f(0), I_f(0)) \in \mathbb{R}^2 \setminus \{0\}$. This mapping is well-defined and locally Lipschitz continuous. A numerical approximation of this function, with $K = 29.63$ and $v_{dc} = 38.926$, for u_c in the range $[-1, 1]$ is shown in Figure 1. Replacing the vibrating actuator with the static,

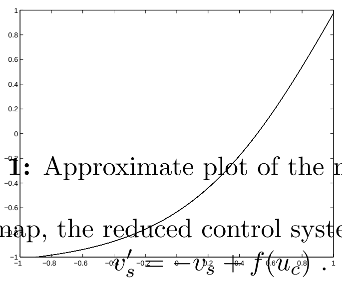

Figure 1: Approximate plot of the map $u_c \mapsto f(u_c)$.

average map, the reduced control system becomes

$$v_s' = -v_s + f(u_c) \ . \tag{6}$$

To allow the values of the nonlinear mapping (5) to be uncertain, we employ an integral controller

$$\begin{aligned} \xi' &= r - v_s \\ u_c &= \text{sat}(\xi) \end{aligned} \tag{7}$$

where r is the reference for the voltage v_s and where the saturation is to keep the capacitor values in a reasonable range. Our reduced closed-loop system is then

$$\begin{aligned} v_s' &= -v_s + f(\text{sat}(\xi)) \\ \xi' &= r - v_s \ . \end{aligned} \tag{8}$$

We wish to derive information about the behavior of the system (1)-(4), (7) from the behavior of the system (8). Assuming the nonlinearity is monotone on $[-, 1, 1]$ and r is strictly in the range of $f([-1, 1])$ so that $f^{-1}(r)$ exists, the point $(v_s, \xi) = (r, f^{-1}(r))$ is globally asymptotically stable for the simplified closed-loop system (8). (This is shown via Lyapunov/LaSalle techniques in [16].) The following result then comes as a special case of our upcoming main results: Let $\mathcal{A}_f(u_c)$ be the set of points on the limit cycle of the van der Pol equation corresponding to the constant input u_c. Then, for the complete closed-loop system (1)-(4), (7), by choosing $\varepsilon > 0$ sufficiently small we can make a set arbitrarily close to the set $\{(v_s, \xi, v_f, I_f) : (v_s, \xi) = (r, f^{-1}(r)), (v_f, I_f) \in \mathcal{A}_f(\xi)\}$ asymptotically stable with basin of attraction arbitrarily close to the set $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})$.

In fact, a similar statement can be made even if f is not assumed to be monotone. See [16] for more details.

3 Unified Framework

3.1 Assumptions

We consider nonlinear dynamical systems with state x decomposed into a "slow" state $x_s \in \mathbb{R}^{n_s}$ and a "fast" state $x_f \in \mathbb{R}^{n_f}$, and driven by a "slow" disturbance d_s and a "fast" disturbance d_f. The governing differential equation has the form

$$\begin{array}{rcl} \dot{x}_s & = & F_s(x_s, x_f, d_s(t), d_f(t), \varepsilon) \\ \dot{x}_f & = & F_f(x_s, x_f, d_s(t), d_f(t), \varepsilon) \end{array} \quad (9)$$

where ε is a small, positive parameter. To fit time-varying systems into the form (9), we augment the state-space with the equations $\dot{t} = 1$ and/or $\dot{t} = \varepsilon$. The equation (9) is assumed to have at least one solution, locally in time, for each initial condition and disturbance of interest. The functions d_s and d_f belong to the sets $\mathcal{D}_{s,\varepsilon}$ and $\mathcal{D}_f$ of measurable functions taking values in subsets of Euclidean space $\mathcal{V}_s$ and $\mathcal{V}_f$, respectively. We assume that the disturbance sets are such that, for each $\varepsilon > 0$, the solution set to (9) does not depend on the starting time, for convenience. This is guaranteed by:

Assumption 1 *The sets $\mathcal{D}_{s,\varepsilon}$ and $\mathcal{D}_f$ are shift invariant, i.e., if $t \mapsto d_s(t)$ belongs to $\mathcal{D}_{s,\varepsilon}$ then $t \mapsto d_s(t + t_\circ)$ belongs to $\mathcal{D}_{s,\varepsilon}$ for all $t_\circ$, and similarly for $\mathcal{D}_f$.*

The "slow" (and, by extension, "fast") terminology we are using is justified by the following assumption which is geared toward ensuring that, when ε is small, the rates of change of $t \mapsto x_s(t)$ and $t \mapsto d_s(t)$ are small.

Assumption 2 *The following conditions hold:*

1. *$F_s(x_s, x_f, d_s, d_f, 0) = 0$ for all $(x_s, x_f, d_s, d_f) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f} \times \mathcal{V}_s \times \mathcal{V}_f$.*
2. *For each $T > 0$ and $\rho > 0$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,*
$$d_s \in \mathcal{D}_{s,\varepsilon} \implies |d_s(t) - d_s(0)| \le \rho \quad \forall t \in [0, T] \ .$$

Remark 3.1 *An example of a class of functions $\mathcal{D}_{s,\varepsilon}$ that satisfies Assumption 2 is the class of functions given by $t \to d_s(\varepsilon t)$ where $d_s(\cdot)$ belongs to a shift invariant class $\mathcal{D}_s$ of uniformly equi-continuous functions. In particular, notice that item 2 of Assumption 2 is satisfied for this class of functions if for each $\rho > 0$ there exists $\mu > 0$ such that, for all $d_s \in \mathcal{D}_s$, $|d_s(t) - d_s(0)| \le \rho$ for all $t \in [0, \mu]$. In general, the condition on the set $\mathcal{D}_{s,\varepsilon}$ does not impose that its elements are continuous, let alone absolutely continuous or possessing a uniformly small derivative. For example, in the special case mentioned in the preceding paragraph, equi-continuity of $d_s(\cdot)$ does not necessarily imply that the derivative of $t \mapsto d_s(\varepsilon t)$ exists or is small when ε is small.*

Remark 3.2 *In our motivational example, the states of the plant and controller, v_s and ξ, correspond to the slow state variables while the state of the oscillating van der Pol circuit, (v_f, I_f), corresponds to the fast state variables. The motivational example can be put into a form where Assumption 2 holds by changing from the original time-scale τ to a new time-scale $t = \tau/\varepsilon$.*

Remark 3.3 *The fast disturbance d_f can represent environmental influences and/or can be used to realize the behavior of a differential inclusion with a nonempty, compact, convex right-hand side. There are several ways to express a set-valued map in terms of a function with a parameter ranging over a unit ball, in this case d_f ranging over the unit ball in $\mathbb{R}^{n_s} \times \mathbb{R}^{n_f} =: \mathcal{V}_f$. The parameterization based on the Steiner selection is one the most appealing because the resulting function inherits the continuity properties of the set-valued map it is parameterizing. In particular, the parameterization theorem [4, Theorem 9.7.2] or [6, Proposition 2.22] is useful for some of the applications discussed in [16].*

For the system (9), we are interested in the infinite time interval input-to-state behavior resulting from stability assumptions on two limiting systems that arise from (9). The first limiting system we consider, obtained from (9) by setting $\varepsilon = 0$ and using Assumption 2, is

$$\begin{array}{rcl} \dot{x}_s & = & 0 \\ \dot{x}_f & = & F_f(x_s, x_f, d_s, d_f(t), 0) \\ \dot{d}_s & = & 0 \ . \end{array} \quad (10)$$

We will refer to this system as the *boundary layer* system so that we are consistent with the classical singular perturbation literature. We will use z_{bl} to denote the composite state for this system, i.e., $z_{bl} := (x_s, x_f, d_s)$. We express our stability assumption on the boundary layer system, which is to hold from a set $\mathcal{H}_f$ of initial conditions, in terms of an output measuring function $z_{bl} \mapsto \omega_{f,o}(z_{bl})$ and an input measuring function $d_f \mapsto \omega_{f,i}(d_f)$. Measuring functions take values in $\mathbb{R}_{\ge 0} \bigcup \{\infty\}$ and hence are not required to be continuous, *a priori*. When the boundary layer has a globally asymptotically stable origin, respectively closed set, we will use the output measuring function corresponding to the norm, i.e., $z_{bl} \mapsto \omega_{f,o}(z_{bl}) := |z_{bl}|$, respectively the distance to a closed set $\mathcal{A}$, i.e., $z_{bl} \mapsto \omega_{f,o}(z_{bl}) := |z_{bl}|_{\mathcal{A}} := \inf_{\zeta \in \mathcal{A}} |\zeta - z_{bl}|$. For an appropriate output measuring function for the boundary layer of the motivational example, see Remark 3.4. A common example of an input measuring function is a class-$\mathcal{K}$ function γ of the norm, i.e., $d_f \mapsto \gamma(|d_f|)$. Another common input measuring function, relevant for the case where d_f is used to realize the behavior of a differential inclusion, is the identically zero function.

Assumption 3 *There exists a class-$\mathcal{KL}$ function β_f such that, for all initial conditions $z_{bl}(0) \in \mathcal{H}_f$ and all disturbances $d_f \in \mathcal{D}_f$, the solutions of (10) exist and satisfy, for all $t \ge 0$,*

$$\omega_{f,o}(z_{bl}(t)) \le \max \{ \ \beta_f(\omega_{f,o}(z_{bl}(0)), t) \ , \ ||\omega_{f,i}(d_f)||_\infty \ \} \quad (11)$$

Remark 3.4 *For our motivational example, we can take $\omega_{f,i}$ identically zero (we considered no disturbances), and $\mathcal{H}_f = \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})$. We can also construct a continuous function $\omega_{f,o}$ of the composite state (v_s, ξ, v_f, I_f) for which Assumption 3 holds. This function vanishes if (v_f, I_f) belongs to the limit cycle associated with $\mathrm{sat}(\xi)$, is positive elsewhere and is unbounded as $|(v_f, I_f)| \to 0$ or $\to \infty$. See [16] for more details.*

In the next assumption, we define the right-hand side of a *reduced* system using the average effect of the boundary layer's asymptotic behavior on the right-hand side of the slow dynamics, F_s. In what follows $\mathcal{S}_{bl}(z_{bl}, d_f)$ represents all solutions of the boundary layer system (10) starting at z_{bl} under the influence of the function d_f. The set $\mathcal{R}_s$ represents a set over which we expect the x_s component of the solution to (9) to range. The set $\mathcal{K}_f$ is a set that we expect to be recurrent (see Assumption 6) for (x_s, x_f, d_s). $\omega_{av,i}$ is a measuring function. The role of $e(\cdot)$ is to allow the possibility of an ensemble of solutions for the average system corresponding to multiple steady-state solutions of the boundary layer system.

Assumption 4 *There exist an integer m and a function $F_{av} : \mathbb{R}^{n_s} \times \mathcal{V}_s \times \mathcal{V}_f \times \mathbb{R}^m \mapsto \mathbb{R}^{n_s}$ so that for each $\rho > 0$, there exist $T^* > 0$ and $\varepsilon^* > 0$ such that for each*

$$\left.\begin{aligned} T &\geq T^* \\ \varepsilon &\in (0, \varepsilon^*] \\ (x_s, d_s, d_f) &\in \mathcal{R}_s \times \mathcal{V}_s \times \mathcal{D}_f \\ z_{bl} = (x_s, x_f, d_s) &\in \mathcal{K}_f \\ \omega_{f,o}(z_{bl}) &\leq ||\omega_{f,i}(d_f)||_\infty \\ \phi_{bl} &\in \mathcal{S}_{bl}(z_{bl}, d_f) \end{aligned}\right\} \tag{12}$$

$\exists$ a measurable function $e : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ satisfying

$$||e||_\infty \leq ||\omega_{av,i}(d_f)||_\infty \tag{13}$$

such that

$$\left| \int_0^T [F_s(x_s, \phi_{f_{bl}}(t), d_s, d_f(t), \varepsilon) - \varepsilon F_{av}(x_s, d_s, d_f(t), e(t))]\, dt \right| \leq T\rho\varepsilon . \tag{14}$$

Remark 3.5 *For our example, take $m = 0$, F_{av} to be the right-hand side of the reduced system (8), $\omega_{av,i}$ identically zero, $\mathcal{R}_s = \mathbb{R}^2$ and $\mathcal{K}_f = \mathbb{R}^4$. See [16].*

Remark 3.6 *We restrict our attention to asymptotic trajectories of the boundary layer by considering only those initial conditions $z_{bl} \in \mathcal{K}_f$ of the boundary layer (10) satisfying $\omega_{f,o}(z_{bl}) \leq ||\omega_{f,i}(d_f)||_\infty$ (compare with (11) letting $t \to \infty$). For the motivational example, this corresponds to only considering initial conditions of the boundary layer on a limit cycle, since $\omega_{f,i} \equiv 0$ and $\omega_{f,o}$ is zero only on a limit cycle.*

Remark 3.7 *As mentioned above Assumption 4, the role of $e(\cdot)$ in the definition of the average is to allow the possibility of an ensemble of solutions for the average system corresponding to multiple steady-state solutions of the boundary layer system. The average system proposed in [2], expressed in terms of invariant measures and pertaining to the case where d_s and d_f are not present, is typically a differential inclusion with a nonempty, compact, convex right-hand side because the steady-state behavior of the boundary layer system is often different for different initial conditions. A differential inclusion can be recovered through e by taking the measuring function $\omega_{av,i}$ to be identically equal to one and using the idea in Remark 3.3. For more details, see [16].*

Assumption 4 is used to generate the *reduced* system

$$\dot{x}_s = \varepsilon F_{av}(x_s, d_s(t), d_f(t), e(t)) . \tag{15}$$

We make the following stability assumption for (15) from initial conditions in some set $\mathcal{H}_s$:

Assumption 5 *There exists a class-$\mathcal{KL}$ function β_s such that, for all $\varepsilon > 0$, all initial conditions $x_s(0) \in \mathcal{H}_s$, all disturbances $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$, and all e satisfying $||e||_\infty \leq ||\omega_{av,i}(d_f)||_\infty$, the solutions of (15) exist and satisfy, for all $t \geq 0$,*

$$\omega_{s,o}(x_s(t)) \leq \max\{\beta_s(\omega_{s,o}(x_s(0)), \varepsilon t), ||\omega_{s,i}(d_s, d_f)||_\infty\} \tag{16}$$

Now we want to pass from the stability bounds on the limiting systems (see Assumptions 3 and 5) to the corresponding bounds on the original system (9). Assumptions 6 and 7, which follow, will make this possible. We will later guarantee Assumptions 6 and 7 by joining Assumptions 2-5 with regularity assumptions on the functions characterizing the problem.
In Assumptions 6 and 7, $\mathcal{K}_s$ and $\mathcal{K}_f$ are sets of initial conditions from which we want the stability bounds to apply. The set $\mathcal{K}_f$ is the same one considered in Assumption 4. The first of the final two assumptions asks that the solutions of (9) be close to the solutions of the boundary layer on compact time intervals.

Assumption 6 *The following hold:*

1. $\sup_{z \in \mathcal{K}_f} \omega_{f,o}(z) < \infty$*;*
2. *There exists $T^* > 0$ so that for each $T \geq T^*$ and $\delta > 0$ there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, each $(x_s(0), x_s(0), x_f(0), d_s(0)) \in \mathcal{K}_s \times \mathcal{K}_f$, each $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$, and each solution $(x_s(\cdot), x_f(\cdot))$ of (9), there exists $z_{bl}(0) \in \mathcal{H}_f$ and a solution $z_{bl}(\cdot)$ of (10) such that, with $z(t) := (x_s(t), x_f(t), d_s(t))$,*
$$|\omega_{f,o}(z(t)) - \omega_{f,o}(z_{bl}(t))| \leq \delta \qquad \forall t \in [0, T] \tag{17}$$
and
$$z(t) \in \mathcal{K}_f \qquad \forall t \in [T^*, T] . \tag{18}$$

The final assumption before our main result asks that the x_s component of the solutions to (9) be close, in an appropriate sense, to the solutions of the reduced system, on compact time intervals of length proportional to $1/\varepsilon$.

Assumption 7 *The following hold:*

1. $\sup_{x_s \in \mathcal{K}_s} \omega_{s,o}(x_s) < \infty$*;*
2. *There exists $T^* > 0$ so that for each $T \geq T^*$ and $\delta > 0$ there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in$*

$(0,\varepsilon^*]$, each $(x_s(0), x_s(0), x_f(0), d_s(0)) \in \mathcal{K}_s \times \mathcal{K}_f$, each $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$, and each solution $x_s(\cdot)$ of (9), there exists $x_{s,av}(0) \in \mathcal{H}_s$, $e(\cdot)$ such that $||e||_\infty \le ||\omega_{av,i}(d_f)||_\infty$ and a solution $x_{s,av}(\cdot)$ of (15) such that

$$|\omega_{s,o}(x_s(t)) - \omega_{s,o}(x_{s,av}(t))| \le \delta \qquad \forall t \in [0, T/\varepsilon] \tag{19}$$

and

$$x_s(t) \in \mathcal{K}_s \qquad \forall t \in [T^*/\varepsilon, T/\varepsilon] \ . \tag{20}$$

3.2 General Result

We are now prepared to state our main result. The proof can be found in [16]. In the next section, we will give sufficient conditions for the functions defining the problem to guarantee that Assumptions 6 and 7 hold. After that, we will discuss the meaning of our assumptions for several applications.

Theorem 1 *If Assumptions 1, 3, and 5-7 hold then for each $\delta > 0$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0,\varepsilon^*]$, all $(d_s, d_f) \in \mathcal{D}_{s,\varepsilon} \times \mathcal{D}_f$ and all initial conditions such that $x_s(0) \in \mathcal{K}_s$ and $z(0) := (x_s(0), x_f(0), d_s(0)) \in \mathcal{K}_f$, the solutions of (9) exist and satisfy*

$$\omega_{s,o}(x_s(t)) \le \max\{\ \beta_s(\omega_{s,o}(x_s(0)), \varepsilon t)\ ,\ ||\omega_{s,i}(d_s, d_f)||_\infty\ \} + \delta \tag{21}$$

and, with $z(t) := (x_s(t), x_f(t), d_s(t))$,

$$\omega_{f,o}(z(t)) \le \max\{\ \beta_f(\omega_{f,o}(z(0)), t)\ ,\ ||\omega_{f,i}(d_f)||_\infty\ \} + \delta \tag{22}$$

for all $t \ge 0$.

4 Conditions for close solutions

In this section we present sufficient conditions on the measuring functions in Assumptions 3 and 5 and the functions on right-hand side of the actual system (9) and the average system (15) that guarantee Assumptions 6 and 7 hold. Our first assumption concerns the measuring functions. It is made to guarantee the recurrence of the sets $\mathcal{K}_f$ and $\mathcal{K}_s$ as in Assumptions 6 and 7.

Assumption 8 *The following conditions hold:*

1. $\mathcal{K}_s \subseteq \mathcal{H}_s$,
2. $\mathcal{K}_f \subseteq \mathcal{H}_f$,
3. $\sup_{z\in\mathcal{K}_f} \omega_{f,o}(z) =: c_{f,o} < \infty$
4. $\sup_{d_f\in\mathcal{V}_f} \omega_{f,i}(d_f) =: c_{f,i} < \infty$
5. $\sup_{x_s\in\mathcal{K}_s} \omega_{s,o}(x_s) =: c_{s,o} < \infty$
6. $\sup_{(d_s,d_f)\in\mathcal{V}_s\times\mathcal{V}_f} \omega_{s,i}(d_s, d_f) =: c_{s,i} < \infty$
7. *There exists $\alpha > 0$ such that*
$$\{x_s : \omega_{s,o}(x_s) \le c_{s,i} + \alpha\} \subseteq \mathcal{K}_s$$
and
$$\{z = (x_s, x_f, d_s) : d_s \in \mathcal{V}_s\ ,\ \omega_{f,o}(z) \le c_{f,i} + \alpha\ ,\ \omega_{s,o}(x_s) \le \max\{\beta_s(c_{s,o}, 0)\ ,\ c_{s,i}\} + \alpha\} \subseteq \mathcal{K}_f \quad .$$

Remark 4.1 *For our motivational example, we can take $\mathcal{K}_s$ to be an arbitrary compact subset of $\mathbb{R}^2$ that contains a neighborhood of the set where $\omega_{s,o}$ is zero, since $c_{s,i} = 0$, and we can take $\mathcal{K}_f$ to be $\mathbb{R}^2 \times \Omega$ where Ω is an arbitrary compact subset of $(\mathbb{R}^2 \backslash \{0\})$ containing a neighborhood of the set $\bigcup_{u\in[-1,1]} \mathcal{A}_f(u)$, since $c_{f,i} = 0$. It is quite common to be able to pick $\mathcal{K}_s$ and $\mathcal{K}_f$ arbitrarily close to $\mathcal{H}_s$ and $\mathcal{H}_f$, respectively, in order to satisfy all of the conditions of Assumption 8. In this case, according to our main results, the set of initial conditions from which the ISS behavior holds for the actual system approaches the set from which the behavior holds for the limiting systems.*

The last assumption we make is on the continuity of the functions that define the problem. According to the statement of our main results and using Assumption 8, we expect the variable $z = (x_s, x_f, d_s)$ for the system (9) to evolve so that x_s remains in a neighborhood of the set

$$\mathcal{X}_s := \{x_s : \omega_{s,o}(x_s) \le \max\{\beta_s(c_{s,o}, 0)\ ,\ c_{s,i}\}\ \}\ , \tag{23}$$

z remains in a neighborhood of the set

$$\mathcal{Z}_f := \{z :\ \omega_{f,o}(z) \le \max\{\beta_f(c_{f,o}, 0)\ ,\ c_{f,i}\}\ \}\ , \tag{24}$$

and, by definition, (d_s, d_f) evolves in $\mathcal{V}_s \times \mathcal{V}_f$. On these sets, we assume Lipschitz and/or uniform continuity:

Assumption 9 *$\exists\ L > 0$, $M > 0$ and $\sigma > 0$ such that:*

1. *$\omega_{s,o}$ is uniformly continuous on $\mathcal{X}_s + \sigma\mathcal{B}$,*
2. *$\mathcal{X}_s + \sigma\mathcal{B} \subseteq \mathcal{R}_s$,*[1]
3. *$\omega_{f,o}$ is uniformly continuous on*
$$\mathcal{U}_f(\sigma) := \{z = (x_s, x_f, d_s) : x_s \in \mathcal{X}_s + \sigma\mathcal{B}\ ,\ z \in \mathcal{Z}_f + \sigma\mathcal{B}\ , \quad d_s \in \mathcal{V}_s\ \}$$
4. *for each $\rho > 0$ there exists $\varepsilon^* > 0$ such that*
   (a) *for each $c \in [0, c_{f,i}]$, if $\omega_{f,o}(z) \le c + \varepsilon^*$ then there exists z_c such that $\omega_{f,o}(z_c) \le c$ and $|z - z_c| \le \rho$,*
   (b) *for all $d_f \in \mathcal{V}_f$, $(x_s, x_f, d_s), (y_s, y_f, w_s) \in \mathcal{U}_f(\sigma)$, $|d_s - w_s| \le \varepsilon^*$, $|e| \le \sup_{d_f\in\mathcal{V}_f} \omega_{av,i}(d_f)$, $\varepsilon \in (0,\varepsilon^*]$,*
$$\begin{aligned} |F_s(x_s, x_f, d_s, d_f, \varepsilon)| &\le \varepsilon M \\ |F_{av}(x_s, d_s, d_f, e)| &\le M \end{aligned}$$
$$\max\{|x_s - y_s|\ ,\ |x_f - y_f|\} \le \varepsilon^* \Longrightarrow$$
$$|F_s(x_s, x_f, d_s, d_f, \varepsilon) - F_s(y_s, y_f, w_s, d_f, \varepsilon)| \le \varepsilon\rho$$
$$|x_s - y_s| \le \varepsilon^* \Longrightarrow$$
$$|F_f(x_s, x_f, d_s, d_f, \varepsilon) - F_f(y_s, y_f, w_s, d_f, 0)| \le L|x_f - y_f| + \rho$$
$$|F_{av}(x_s, d_s, d_f, e) - F_{av}(y_s, w_s, d_f, e)| \le L|x_s - y_s| + \rho\ .$$

Remark 4.2 *The purpose of item 4a is to guarantee that if $\omega_{f,o}(z)$ is close to c then z is close to the set $\{\zeta : \omega_{f,o}(\zeta) \le c\}$. Consider the special case where $c_{f,i} = 0$. In this case, if it is possible to find $\eta > 0$ and a class-$\mathcal{K}$ function κ such that, with $\mathcal{A}_f := \{z : \omega_{f,o}(z) = 0\}$,*

$$\omega_{f,o}(z) \le \eta \qquad \Longrightarrow \qquad \kappa(|z|_{\mathcal{A}_f}) \le \omega_{f,o}(z) \tag{25}$$

then item 4a is satisfied for any $\varepsilon^* \le \min\{\eta, \kappa(\rho)\}$). The condition (25) is satisfied for our motivational example. For more details, see [16].

Remark 4.3 *For our example, $\mathcal{U}_f(\sigma)$ is a compact subset of $\mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{0\})$ for σ sufficiently small, and $\mathcal{X}_s$ compact. The right-hand side of our example is locally*

[1] Recall that the set $\mathcal{R}_s$ comes from Assumption 4 and characterizes a region where the integral of F_s is approximately equal to the integral of εF_{av}.

Lipschitz and our measuring functions are continuous on these sets. Combining this observation with the previous remark establishes that our motivational example, in the appropriate time-scale, satisfies Assumption 9.

The main result of this section (see [16] for the proof) is that, under Assumptions 2-5, we can guarantee Assumptions 6 and 7, which are assumptions about trajectories, by replacing them with Assumptions 8 and 9 which are assumptions about functions.

Proposition 1 *If Assumptions 2 - 5, 8 and 9 hold then Assumptions 6 and 7 hold.*

5 Applications

We now briefly discuss how our general result applies to several situations where F_s and/or F_f have special structure that corresponds to classical robustness problems. More details can be found in [16].

Our results can easily be applied to what [12, Chapter 5] refers to as "averaging over spatial variables". This type of averaging arises, for example, when casting into polar coordinates the weakly nonlinear oscillator equation $\ddot{y} + y = \varepsilon g(y, \dot{y})$. We recover asymptotic stability results for weakly nonlinear oscillators by considering the magnitude of $(y, \dot{y})$ as the slow state variable and the phase angle as the fast state variable.

When the fast state variable corresponds to a fast time state, i.e., $\dot{x}_f = 1$, and the slow state variables include a slow time state, we recover classical and partial averaging results as well as recent results on averaging with fast and slow disturbances [11].

When the fast state variables include a fast time state and the boundary layer system has a uniformly globally asymptotically stable time-varying invariant manifold we obtain "two time-scale averaging" resutls from adaptive control and identification.

Results on robustness with respect to slowly varying parameters are obtained when the slow state variables are not present. It is noteworthy that our results, when specialized to equilibrium manifolds parameterized by d_s, impose no differentiability requirements on the equilibrium manifold.

Our framework also addresses standard singular perturbation problems, leading to extensions of the results in [5], as well as unconventional situations like when the boundary layer has an unstable equilibrium manifold parameterized by the slow state variables [2, Remark 5.1].

6 Conclusion

We have developed a unified framework for studying robustness of the input-to-state stability (ISS) property and have presented new results on robustness of ISS to slowly-varying parameters, to rapidly varying signals, and to generalized singular perturbations. The framework assumes a time-scale separation between slow and fast variables which permits the definition of a boundary layer system like in classical singular perturbation theory. To address various robustness problems simultaneously, we have allowed the asymptotic behavior of the boundary layer to be complex and have required it to generate an average for the derivative of the slow state variables. Our main result has shown that if the boundary layer and averaged systems are ISS then the ISS bounds also hold for the actual system with an offset that converges to zero with the parameter that characterizes the separation of time-scales.

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