Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems

Dina Shona Laila and Dragan Nešić[1]
Electrical and Electronic Engineering Department,
The University of Melbourne,
Parkville, 3010, Victoria, Australia

Abstract

Input-to-state stability (ISS) of a feedback interconnection of two discrete-time ISS systems satisfying an appropriate small gain condition is investigated via the Lyapunov method. In particular, an ISS Lyapunov function for the overall system is constructed from the ISS Lyapunov functions of the two subsystems. We consider parameterized families of discrete-time systems that naturally arise when an approximate discrete-time model is used to design a controller for a sampled-data system.

Keywords: Discrete-time; Small gain; Input-to-state stability; Lyapunov method; Nonlinear.

1 Introduction

The small gain theorem is one of the most important tools in robustness analysis and controller design for nonlinear control systems. A particularly useful version of the small gain theorem for nonlinear continuous-time systems was proved in [2], based on the input-to-state stability (ISS) property introduced in [10]. A range of related result for continuous-time systems can be found in [11, 12] and for nonlinear discrete-time systems in [4]. All of the above results rely on trajectory based proofs. Explicit construction of Lyapunov function for interconnected systems was presented for instance in [9], while the first partial construction of a Lyapunov function for the feedback connection of two continuous-time ISS systems satisfying a small-gain condition was proposed in [3].

It is the main purpose of this paper to present a discrete-time version of the results in [3]. Indeed, we present a partial construction of an ISS Lyapunov function from the ISS Lyapunov functions of two interconnected discrete-time ISS systems satisfying a small-gain condition. While the constructed Lyapunov function in the discrete-time case has the same form as the one constructed in [3] for continuous-time systems, the proofs of the two results are significantly different. The main result of this paper is closely related to results on changes of supply rates [7, 8], and it can be regarded as an appropriate generalization of the results in [8].

We believe that the result of this paper is a useful tool for a range of nonlinear discrete-time control problems. In particular, the constructed Lyapunov function can be used together with results in [5, 6] to design ISS controllers for nonlinear sampled-data systems via their approximate discrete-time plant models.

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2 Preliminaries

The set of real numbers is denoted by $\mathbb{R}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and zero at zero; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Functions of class $\mathcal{K}_\infty$ are invertible. We say that a function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is *positive* if it is continuous and $q(s) > 0$ for all $s \geq 0$. A function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *positive definite* if it is continuous, $q(0) = 0$ and $q(s) > 0$ for all $s > 0$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is of class $\mathcal{L}$ if it is positive and $\alpha(s)$ is strictly decreasing to zero as $s \to \infty$. Given two functions $\alpha(\cdot)$ and $\gamma(\cdot)$, we denote their composition and multiplication respectively as $\alpha \circ \gamma(\cdot)$ and $\alpha(\cdot) \cdot \gamma(\cdot)$. Identity function is denoted by Id, that is $\mathrm{Id}(s) := s$.

Consider a family of parameterized discrete-time systems

$$x(k+1) = F_T(x(k), u(k)) \ . \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and input of the system. It is assumed that F_T is well defined on arbitrarily large compact sets for sufficiently small T, where $T > 0$ is the sampling period, which parameterizes the system and can be arbitrarily assigned. Parameterized discrete-time systems (1) commonly arise when an approximate discrete-time model is used for designing a digital controller for a nonlinear sampled-data system (see [5, 6]). For instance, if we use the Euler model of $\dot{x} = f(x, u)$ for controller design then we have $F_T(x, u) := x + Tf(x, u)$. Non-parameterized discrete-time systems are a special case of (1) when T is constant (for instance $T = 1$). We use the following definition.

Definition 2.1 *The system (1) is semi-globally practically input-to-state stable (SP-ISS) w.r.t. input u if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, a positive definite function α and a function $\gamma \in \mathcal{K}$, and for any strictly positive real numbers Δ_x, Δ_u, ν, $\tilde{\nu}$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$ there exists a continuous function $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ and $T \in (0, T^*)$ the following holds:*

$$\underline{\alpha}(|x|) \leq V_T(x) \leq \overline{\alpha}(|x|) \ , \tag{2}$$

$$V_T(x) \geq \gamma(|u|) + \nu \Rightarrow V_T(F_T) - V_T(x) \leq -T\alpha(V_T(x)) \ , \tag{3}$$

$$V_T(F_T) \leq V_T(x) + \tilde{\nu} \ . \tag{4}$$

The function V_T is called a SP-ISS Lyapunov function for the system (1). ■

Definition 2.2 (Lipschitz uniform in small T) *A family of functions $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is Lipschitz uniformly in small T if given any $\Delta_x > 0$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and $\max\{|x|, |y|\} \leq \Delta_x$ it satisfies $|V_T(x) - V_T(y)| \leq L\,|x - y|$.* ■

Remark 2.1 *If instead of (3), we used the following Lyapunov condition in Definition 2.1*

$$\Delta V_T \leq -Ta(|x|) + T\gamma(|u|) + T\nu \ , \quad a, \gamma \in \mathcal{K}_\infty \ , \tag{5}$$

then we would not need (4). However, formulation (5) leads to a more complicated statement and proof of our main result and hence we have opted to use the conditions as stated in Definition 2.1. We emphasize that (3) and (4) are equivalent to (5) if an appropriate condition holds. Indeed, it is trivial to see that (5) implies both (3) and (4). The opposite holds if there exists $\sigma \in \mathcal{K}_\infty$ such that for any strictly positive r, ν there exists $T^* > 0$ such that the following holds

$$\max_{T \in (0,T^*), |x| \leq \gamma(r), |u| \leq r} \left| \frac{\Delta V_T}{T} \right| \leq \sigma(r) + \nu \ , \tag{6}$$

and then we can write that for any $(\Delta_x, \Delta_u, \nu)$ there exists $T^* > 0$ such that the following holds:

$$\Delta V_T \leq -T\alpha(V_T) + T\sigma(|u|) + T\nu \ .$$

The condition (6) is slightly stronger than (4) but it often holds.

We note that the condition (5) was used in [6] to provide a framework for design of input-to-state stabilizing controllers for sampled-data systems via their approximate discrete-time models. Hence, results of this paper in cases when the condition (6) holds provide a tool for ISS controller design within the framework of [6]. ■

The following are used in proving the main result.

Lemma 2.1 *[3] Let $\sigma_1 \in \mathcal{K}$ and $\sigma_2 \in \mathcal{K}_\infty$ satisfy $\sigma_1(r) < \sigma_2(r)$ for all $r > 0$. Then there exists $\sigma \in \mathcal{K}_\infty$ such that*

- *$\sigma_1(r) < \sigma(r) < \sigma_2(r)$ for all $r > 0$;*
- *$\sigma(r)$ is C^1 on $(0, \infty)$; $\sigma'(r) =: \tilde{q}(r)$ is positive.* ■

Lemma 2.2 *[1] Let $\tilde{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be a positive function. Then there exist a positive definite function q and functions $q_1 \in \mathcal{K}_\infty$ and $q_2 \in \mathcal{L}$ such that $\tilde{q}(r) \geq q(r) \geq q_1(r) \cdot q_2(r)$, $\forall r \geq 0$.* ■

Remark 2.2 *For any function $\rho \in \mathcal{K}_\infty$, $r_1 \geq 0$, $r_2 \geq 0$ we have $\frac{1}{2}r_1 + \frac{1}{2}\rho(r_2) \leq \max\{r_1, \rho(r_2)\} \leq r_1 + \rho(r_2)$.* ■

Remark 2.3 *Since for any $\alpha \in \mathcal{K}$ we have $\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2)$ for all $s_1 \geq 0, s_2 \geq 0$, then for any $\alpha_1, \alpha_2 \in \mathcal{K}$, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}$ such that:*

$$\underline{\alpha}(s_1 + s_2) \leq \alpha_1(s_1) + \alpha_2(s_2) \leq \overline{\alpha}(s_1 + s_2), \forall s_1, s_2 \geq 0,$$

In particular, we can take $\underline{\alpha}(s) := \min\{\alpha_1(\frac{s}{2}), \alpha_2(\frac{s}{2})\}$ and $\overline{\alpha}(s) := \max\{2\alpha_1(s), 2\alpha_2(s)\}$. ■

3 Main result

In this section we state and prove Theorem 3.1, the main result of this paper. Theorem 3.1 is a discrete-time version of the continuous-time result [3]. The statements of both results are similar but the proofs are notably different, since we need to use the Main Value Theorem in the proof of Theorem 3.1.

The focus of this paper is a family of parameterized discrete-time interconnected systems

$$\begin{array}{l} \Sigma_1 : \ x_1(k+1) = F_{1T}(x_1(k), x_2(k), u(k)) \\ \Sigma_2 : \ x_2(k+1) = F_{2T}(x_1(k), x_2(k), u(k)) \ . \end{array} \tag{7}$$

In the sequel we will assume that the subsystem Σ_1 is SP-ISS with respect to inputs x_2 and u and the subsystem Σ_2 is SP-ISS with respect to inputs x_1 and u. More precisely, we suppose that for $i, j \in \{1, 2\}, i \neq j$, there exist functions $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, positive definite functions α_i, functions $\gamma_{x_i}, \gamma_{u_i} \in \mathcal{K}$, and for any strictly positive real numbers $(\Delta_{x_i}, \Delta_{x_j}, \Delta_{u_i}, \nu_i, \tilde{\nu}_i)$ there exist $T_i^* > 0$ and for any $T \in (0, T_i^*)$ there exist $V_{iT} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following hold for all $T \in (0, T_i^*)$, $|x_i| \leq \Delta_{x_i}$, $|x_j| \leq \Delta_{x_j}$ and $|u| \leq \Delta_{u_i}$:

$$\underline{\alpha}_i(|x_i|) \leq V_{iT}(x_i) \leq \overline{\alpha}_i(|x_i|) \ , \tag{8}$$

$$V_{iT}(x_i) \geq \max\{\gamma_{x_i}(V_{jT}(x_j)), \gamma_{u_i}(|u|) + \nu_i\} \Longrightarrow V_{iT}(F_{iT}) - V_{iT}(x_i) \leq -T\alpha_i(V_{iT}(x_i)) \ , \tag{9}$$

$$V_{iT}(F_{iT}) \leq V_{iT}(x_i) + \tilde{\nu}_i \ . \tag{10}$$

Under the above given conditions and an appropriate small gain condition, we show that the overall system (7) is SP-ISS with respect to the input u. Moreover, we construct a SP-ISS Lyapunov function V_T for the overall system (7) using the SP-ISS Lyapunov functions V_{1T} and V_{2T} of the subsystems Σ_1 and Σ_2. More precisely, we can state the following result.

Theorem 3.1 *Consider the parameterized discrete-time interconnected system (7). Suppose that the following conditions hold:*
A1. The subsystem Σ_1 is SP-ISS with inputs x_2 and u and SP-ISS Lyapunov function V_{1T}.
A2. The subsystem Σ_2 is SP-ISS with inputs x_1 and u and SP-ISS Lyapunov function V_{2T}.
A3. There exist $\tau_1, \tau_2 \in \mathcal{K}_\infty$ such that $(\mathrm{Id}+\tau_1)\circ\gamma_{x_1}\circ(\mathrm{Id}+\tau_2)\circ\gamma_{x_2}(s) < s, \quad \forall s > 0$.
Then, the system (7) is SP-ISS with respect to the input u and moreover there exists $\rho \in \mathcal{K}_\infty$ such that

$$V_T(x_1,x_2) := \max\{V_{1T}(x_1), \rho(V_{2T}(x_2))\} \ , \qquad (11)$$

is SP-ISS Lyapunov function for system (7). Moreover, if V_{1T}, V_{2T} are locally Lipschitz uniformly in small T, then V_T is locally Lipschitz uniformly in small T. ∎

Proof of Theorem 3.1: Suppose that all conditions of Theorem 3.1 are satisfied. Let $\underline{\alpha}_1$, $\overline{\alpha}_1$, α_1, γ_{x_1}, γ_{u_1} come from conditions A1, and let $\underline{\alpha}_2$, $\overline{\alpha}_2$, α_2, γ_{x_2}, γ_{u_2} come from condition A2. Let $\tau_1, \tau_2 \in \mathcal{K}_\infty$ come from the small gain condition A3. Note that without loss of generality we can assume that $(\mathrm{Id}-\alpha_i)$, $i=1,2$ are positive definite. For simplicity of notation we introduce $\tilde\gamma_{x_1}(s) := (\mathrm{Id}+\tau_1)\circ\gamma_{x_1}$, $\tilde\gamma_{x_2}(s) := (\mathrm{Id}+\tau_2)\circ\gamma_{x_2}$. Similar to [3] we denote $b := \lim_{r\to\infty}\tilde\gamma_{x_2}(r)$ and since $\tilde\gamma_{x_2} \in \mathcal{K}$, then $\tilde\gamma_{x_2}^{-1}$ is defined on $[0,b)$, $\tilde\gamma_{x_2}^{-1}(r) \to \infty$ as $r \to b^-$ and from A3 we have that

$$\tilde\gamma_{x_1}(r) < \tilde\gamma_{x_2}^{-1}(r), \ \forall r \in (0,b) \ . \qquad (12)$$

Let $\hat\gamma_x \in \mathcal{K}_\infty$ be such that
- $\hat\gamma_x(r) \le \tilde\gamma_{x_2}^{-1}(r)$ for all $r \in [0,b)$;
- $\tilde\gamma_{x_1}(r) < \hat\gamma_x(r)$ for all $r > 0$.

(if $\tilde\gamma_{x_2} \in \mathcal{K}_\infty$, then we can take $\hat\gamma_x(r) = \tilde\gamma_{x_2}^{-1}(r)$). Let $\rho \in \mathcal{K}_\infty$ come from Lemma 2.1 such that

$$\tilde\gamma_{x_1}(r) < \rho(r) < \hat\gamma_x(r) \ , \quad \forall r > 0 \ . \qquad (13)$$

Denote $\tilde q(r) := \frac{d\rho}{dr}(r)$, where $\tilde q$ is a positive function. Let V_T be defined as:

$$V_T(x_1,x_2) := \max\{V_{1T}(x_1), \rho(V_{2T}(x_2))\} \ . \qquad (14)$$

We use the notation $x := (x_1^T\ x_2^T)^T$, $F_T := (F_{1T}^T\ F_{2T}^T)^T$ and the norm $|x| := |x_1| + |x_2|$. We show that the interconnected system (7) is SP-ISS with input u by proving that V_T is a SP-ISS Lyapunov function for (7). Let arbitrary strictly positive real numbers $(\Delta_x, \Delta_u, \nu, \tilde\nu)$ be given. Let $\Delta_{x_1} = \Delta_{x_2} = \Delta_x$, $\Delta_{u_1} = \Delta_{u_2} = \Delta_u$. Let $\varepsilon_1, \varepsilon_2 \in \mathcal{K}_\infty$ be arbitrary functions such that $(\mathrm{Id}-\varepsilon_i)$ are positive definite for $i=1,2$. Let ν_1 be such that

$$\max\left\{\nu_1, \max_{s\in[0,\Delta_u]}[\varepsilon_1^{-1}(\gamma_{u_1}(s)+\nu_1) - \varepsilon_1^{-1}\circ\gamma_{u_1}(s)]\right\} \le \nu$$

and let ν_2 be such that

$$\max\left\{\max_{s\in[0,\Delta_u]}[\varepsilon_2^{-1}\circ\rho(\gamma_{u_2}(s)+\nu_2) - \varepsilon_2^{-1}\circ\rho\circ\gamma_{u_2}(s)], \max_{s\in[0,\Delta_u]}[\rho(\gamma_{u_2}(s)+\nu_2) - \rho\circ\gamma_{u_2}(s)]\right\} \le \nu \qquad (15)$$

Let $\tilde\nu_1 > 0$ and $\tilde\nu_2 > 0$ be such that

$$\max\{\tilde\nu_1, \max_{s\in[0,\overline{\alpha}_2(\Delta_{x_2})]}[\rho(s+\tilde\nu_2) - \rho(s)]\} \le \tilde\nu \ , \qquad (16)$$

$$(\mathrm{Id}+\tau_1^{-1})(\tilde\nu_1) \le \frac{\nu_1}{2} \ , \quad (\mathrm{Id}+\tau_2^{-1})(\tilde\nu_2) \le \frac{\nu_2}{2} \ . \qquad (17)$$

Let $(\Delta_{x_1}, \Delta_{x_2}, \Delta_{u_1}, \frac{\nu_1}{2}, \tilde\nu_1)$ determine $T_1^* > 0$ via the condition A1. Let $(\Delta_{x_1}, \Delta_{x_2}, \Delta_{u_2}, \frac{\nu_2}{2}, \tilde\nu_2)$ determine $T_2^* > 0$ via the condition A2. Let $T^* := \min\{1, T_1^*, T_2^*\}$. In the rest of the proof we assume that $|x| \le \Delta_x$, $|u| \le \Delta_u$ and $T \in (0, T^*)$.

Note that $\tilde\gamma_{x_1}(s) \ge \gamma_{x_1}(s)$, $\tilde\gamma_{x_2}(s) \ge \gamma_{x_2}(s)$ for all $s \ge 0$. Conditions A1 and A2 imply that the following hold:

$$\begin{aligned} V_{1T}(x_1) &\ge \max\{\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|)+\nu_1/2\} \\ &\Rightarrow V_{1T}(F_{1T}) - V_{1T}(x_1) \le -T\alpha_1(V_{1T}(x_1)) \qquad (18) \\ V_{2T}(x_2) &\ge \max\{\gamma_{x_2}(V_{1T}(x_1)), \gamma_{u_2}(|u|)+\nu_2/2\} \\ &\Rightarrow V_{2T}(F_{2T}) - V_{2T}(x_2) \le -T\alpha_2(V_{2T}(x_2)) \ , \qquad (19) \\ V_{1T}(F_{1T}) &\le V_{1T}(x_1) + \tilde\nu_1 \ , \qquad (20) \\ V_{2T}(F_{2T}) &\le V_{2T}(x_2) + \tilde\nu_2 \ . \qquad (21) \end{aligned}$$

Moreover, using respectively A1 and A2 and our choice of $\nu_i, \tilde\nu_i$, $i=1,2$ we can write respectively

$$\begin{aligned} V_{1T}(F_{1T}) &\le \max\{(\mathrm{Id}-T\alpha_1)(V_{1T}(x_1)), \\ &\qquad \tilde\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|)+\nu_1\} \qquad (22) \\ V_{2T}(F_{2T}) &\le \max\{(\mathrm{Id}-T\alpha_2)(V_{2T}(x_2)), \\ &\qquad \tilde\gamma_{x_2}(V_{1T}(x_1)), \gamma_{u_2}(|u|)+\nu_2\} \ . \qquad (23) \end{aligned}$$

We only prove (22) and the proof of (23) is omitted since it follows the same steps. Note that if $V_{1T}(x_1) \ge \max\{\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|)+\nu_1/2\}$, then from (18) we can write that:

$$V_{1T}(F_{1T}) \le (\mathrm{Id}-T\alpha_1)(V_{1T}(x_1)) \ . \qquad (24)$$

On the other hand, if $V_{1T}(x_1) \le \max\{\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|)+\nu_1/2\}$, then from (20) we have

$$\begin{aligned} V_{1T}(F_{1T}) &\le V_{1T}(x_1) + \tilde\nu_1 \\ &\le \max\{\gamma_{x_1}(V_{2T}(x_2)) + \tilde\nu_1, \gamma_{u_1}(|u|)+\nu_1/2+\tilde\nu_1\} \ . \end{aligned}$$

By considering two sub-cases $\tau_1\circ\gamma_{x_1}(V_{2T}(x_2)) \ge \tilde\nu_1$ and $\tau_1\circ\gamma_{x_1}(V_{2T}(x_2)) \le \tilde\nu_1$, and from definition of ν_1 and $\tilde\nu_1$ we can write that

$$\begin{aligned} V_{1T}(F_{1T}) &\le \max\{(\mathrm{Id}+\tau_1)\circ\gamma_{x_1}(V_{2T}(x_2)), \\ &\qquad (\mathrm{Id}+\tau_1^{-1})(\tilde\nu_1), \gamma_{u_1}(|u|)+\nu_1/2+\tilde\nu_1\} \\ &\le \max\{\tilde\gamma_{x_1}(V_{2T}(x_2)), (\mathrm{Id}+\tau_1^{-1})(\tilde\nu_1), \\ &\qquad \gamma_{u_1}(|u|)+\nu_1/2+\nu_1/2\} \qquad (25) \\ &\le \max\{\tilde\gamma_{x_1}(V_{2T}(x_2)), \nu_1/2, \gamma_{u_1}(|u|)+\nu_1\} \\ &= \max\{\tilde\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|)+\nu_1\} \ , \end{aligned}$$

and (24), (25) complete the proof of (22). We assume in the sequel that the equations (18)-(23) hold.

Using the definition of V_T, Remark 2.2 and Remark 2.3, we have that V_T satisfies (2) with $\underline{\alpha}(s) := 1/2\min\{\underline{\alpha}_1(s/2), \rho\circ\underline{\alpha}_2(s/2)\}$ and $\overline{\alpha}(|x|) := 2\max\{\overline{\alpha}_1(s), \rho\circ\overline{\alpha}_2(s)\}$.

Moreover, from the definition of V_T and (16) we have

$$\begin{aligned} V_T(F_T) &= \max\{V_{1T}(F_{1T}), \rho(V_{2T}(F_{2T})\} \\ &\le \max\{V_{1T}(x_1)+\tilde{\nu}_1, \rho(V_{2T}(x_2)+\tilde{\nu}_2)\} \\ &\le \max\{V_{1T}(x_1), \rho(V_{2T}(x_2))\}+\tilde{\nu} \\ &= V_T(x)+\tilde{\nu}\ , \end{aligned} \tag{26}$$

which proves that (4) holds. To show that V_T satisfies (3), we consider four cases:

1: $V_{1T}(x_1) \ge \rho(V_{2T}(x_2))$ and $V_{1T}(F_{1T}) \ge \rho(V_{2T}(F_{2T}))$ It holds that

$$\Delta V_T := V_T(F_T) - V_T(x) = V_{1T}(F_{1T}) - V_{1T}(x_1)$$

Conditions $V_{1T}(x_1) \ge \rho(V_{2T}(x_2))$ and $\gamma_{x_1}(r) \le \tilde{\gamma}_{x_1}(r) < \rho(r), \forall r>0$ imply $V_{1T}(x_1) > \gamma_{x_1}(V_{2T}(x_2))$. Hence, from (18) it holds that if $V_{1T}(x_1) \ge \gamma_{u_1}(|u|)+\nu_1$ then we have

$$V_{1T}(F_{1T}) - V_{1T}(x_1) \le -T\alpha_1(V_{1T}(x_1))$$

Since $V_T(x) = V_{1T}(x_1)$ and $\nu \ge \nu_1$, we have

$$V_T(x) \ge \gamma_{u_1}(|u|)+\nu \Rightarrow \Delta V_T \le -T\alpha_1(V_T(x)). \tag{27}$$

2: $V_{1T}(x_1) < \rho(V_{2T}(x_2))$ and $V_{1T}(F_{1T}) < \rho(V_{2T}(F_{2T}))$ It holds that

$$\Delta V_T = \rho(V_{2T}(F_{2T})) - \rho(V_{2T}(x_2))\ . \tag{28}$$

Conditions $V_{1T}(x_1) < \rho(V_{2T}(x_2))$ and $\rho^{-1}(r) > \tilde{\gamma}_{x_2}(r) \ge \gamma_{x_2}(r), \forall r>0$ imply $V_{2T}(x_2) > \gamma_{x_2}(V_{1T}(x_1))$. Hence, from (19) it holds that if $V_{2T}(x_2) \ge \gamma_{u_2}(|u|)+\nu_2$, we have that

$$\begin{aligned} \Delta V_{2T} = V_{2T}(F_{2T}) - V_{2T}(x_2) &\le -T\alpha_2(V_{2T}(x_2)) \\ \Rightarrow\ V_{2T}(F_{2T}) &\le (\mathrm{Id}-T\alpha_2)(V_{2T}(x_2)) \end{aligned} \tag{29}$$

Then using the Mean Value Theorem and the construction of ρ via Lemma 2.1, we have that

$$\begin{aligned} \Delta V_T &= \rho(V_{2T}(F_{2T})) - \rho(V_{2T}(x_2)) \\ &\le \rho\circ(\mathrm{Id}-T\alpha_2)(V_{2T}(x_2)) - \rho(V_{2T}(x_2)) \\ &= -T\tilde{q}(V_{2T}^\star)\cdot\alpha_2(V_{2T}(x_2)) \end{aligned} \tag{30}$$

with $V_{2T}^\star \in [(\mathrm{Id}-\alpha_2)(V_{2T}(x_2)), V_{2T}(x_2)]$ (since $T<1$) and $\tilde{q}$ is a positive function. Via Lemma 2.2, let $\tilde{q}$ generate $q_1 \in \mathcal{K}_\infty$, $q_2 \in \mathcal{L}$. Using $V_T(x) = \rho(V_{2T}(x_2))$, we write

$$\begin{aligned} \Delta V_T &\le -T\tilde{q}(V_{2T}^\star)\cdot\alpha_2(V_{2T}(x_2)) \\ &\le -Tq_1(V_{2T}^\star)\cdot q_2(V_{2T}^\star)\cdot\alpha_2(V_{2T}(x_2)) \\ &\le -Tq_1\circ(\mathrm{Id}-\alpha_2)(V_{2T}(x_2))\cdot q_2(V_{2T}(x_2)) \\ &\quad \cdot\alpha_2(V_{2T}(x_2)) \\ &=: -Ta_{2a}(V_{2T}(x_2)) \\ &= -Ta_{2a}\circ\rho^{-1}(V_T(x)) = -Ta_2(V_T(x))\ . \end{aligned} \tag{31}$$

Since $V_T(x) = \rho(V_{2T}(x_2))$ and by (15), we have

$$\begin{aligned} V_T(x) &\ge \rho\circ\gamma_{u_2}(|u|)+\nu \\ &\qquad \Rightarrow \Delta V_T \le -Ta_2(V_T(x)) \end{aligned} \tag{32}$$

3: $V_{1T}(x_1) < \rho(V_{2T}(x_2))$ and $V_{1T}(F_{1T}) \ge \rho(V_{2T}(F_{2T}))$ Using (22) it holds that

$$\begin{aligned} \Delta V_T &= V_{1T}(F_{1T}) - \rho(V_{2T}(x_2)) \\ &\le \max\{(\mathrm{Id}-T\alpha_1)(V_{1T}(x_1)), \tilde{\gamma}_{x_1}(V_{2T}(x_2)), \\ &\quad \gamma_{u_1}(|u|)+\nu_1\} - \rho(V_{2T}(x_2))\ . \end{aligned} \tag{33}$$

3.1: If $(\mathrm{Id}-T\alpha_1)(V_{1T}(x_1))$ is the maximum in (33):

$$\Delta V_T \le (\mathrm{Id}-T\alpha_1)(V_{1T}(x_1)) - \rho(V_{2T}(x_2)) \tag{34}$$

We then consider two sub-cases:

3.1.1: If $V_{1T}(x_1) \le \frac{1}{2}\rho(V_{2T}(x_2))$, then from (34):

$$\begin{aligned} \Delta V_T &\le (\mathrm{Id}-T\alpha_1)(V_{1T}(x_1)) - \rho(V_{2T}(x_2)) \\ &\le V_{1T}(x_1) - \rho(V_{2T}(x_2)) \\ &\le -\frac{1}{2}\rho(V_{2T}(x_2)) \le -\frac{T}{2}V_T(x)\ . \end{aligned} \tag{35}$$

3.1.2: Let $V_{1T}(x_1) \in [\frac{1}{2}\rho(V_{2T}(x_2)), \rho(V_{2T}(x_2))]$ and let α_1 generate $q_1 \in \mathcal{K}_\infty$ and $q_2 \in \mathcal{L}$. Then we can write:

$$\begin{aligned} \Delta V_T &\le (\mathrm{Id}-T\alpha_1)(V_{1T}(x_1)) - \rho(V_{2T}(x_2)) \\ &\le -T\alpha_1(V_{1T}(x_1)) \\ &\le -Tq_1(V_{1T}(x_1))\cdot q_2(V_{1T}(x_1)) \\ &\le -Tq_1(\frac{1}{2}\rho(V_{2T}(x_2)))\cdot q_2\circ\rho(V_{2T}(x_2)) \\ &= -Ta_{3a}\circ\rho(V_{2T}(x_2)) = -Ta_{3a}(V_T(x))\ . \end{aligned} \tag{36}$$

3.2: If $\tilde{\gamma}_{x_1}(V_{2T}(x_2))$ is the maximum in (33) then, since $\tilde{\gamma}_{x_1}(r) < \rho(r), \forall r>0$, the function $a_{3b}(r) := \rho(r) - \tilde{\gamma}_{x_1}(r)$ is positive definite and since $T<1$ we have that

$$\begin{aligned} \Delta V_T &\le \tilde{\gamma}_{x_1}(V_{2T}(x_2)) - \rho(V_{2T}(x_2)) \\ &= -a_{3b}(V_{2T}(x_2)) \le -Ta_{3b}\circ\rho^{-1}(V_T(x)) \end{aligned} \tag{37}$$

3.3: If $\gamma_{u_1}(|u|)+\nu_1$ is the maximum in (33) then we have that

$$\Delta V_T \le \gamma_{u_1}(|u|)+\nu_1 - \rho(V_{2T}(x_2)) \tag{38}$$

Let $\varepsilon_1 \in \mathcal{K}_\infty$ be such that $\mathrm{Id}-\varepsilon_1$ is a positive definite function. If $\rho(V_{2T}(x_2)) > \varepsilon_1^{-1}\circ(\gamma_{u_1}(|u|)+\nu_1)$ then it holds that

$$\Delta V_T \le -(\mathrm{Id}-\varepsilon_1)\circ\rho(V_{2T}(x_2)) \tag{39}$$

Since $V_T(x) = \rho(V_{2T}(x_2))$ and using the definition of ν_1 and $T<1$, we can write:

$$\begin{aligned} V_T(x) &> \varepsilon_1^{-1}\circ\gamma_{u_1}(|u|)+\nu \\ &\Rightarrow \rho(V_{2T}(x_2)) > \varepsilon_1^{-1}\circ(\gamma_{u_1}(|u|)+\nu_1) \\ &\quad \Rightarrow \Delta V_T \le -(\mathrm{Id}-\varepsilon_1)(V_T(x)) \\ &\qquad =: -a_{3c}(V_T(x)) \le -Ta_{3c}(V_T(x))\ . \end{aligned} \tag{40}$$

Combining 3.1, 3.2 and 3.3, with $a_3(r) := \min\{r/2, a_{3a}(r), a_{3b} \circ \rho^{-1}(r), a_{3c}(r)\}$, we have that

$$V_T(x)) > \varepsilon_1^{-1} \circ \gamma_{u_1}(|u|) + \nu \Rightarrow \Delta V_T \leq -T a_3(V_T(x)) \quad (41)$$

4: $V_{1T}(x_1) \geq \rho(V_{2T}(x_2))$ and $V_{1T}(F_{1T}) < \rho(V_{2T}(F_{2T}))$
Using condition (23) we have

$$\begin{aligned}\Delta V_T &= \rho(V_{2T}(F_{2T})) - V_{1T}(x_1) \\ &\leq \max\{\rho \circ (\mathrm{Id} - T\alpha_2)(V_{2T}(x_2)), \\ &\quad \rho \circ \tilde{\gamma}_{x_2}(V_{1T}(x_1)), \rho(\gamma_{u_2}(|u|) + \nu_2)\} - V_{1T}(x_1)\end{aligned} \quad (42)$$

4.1: If $\rho \circ (\mathrm{Id} - T\alpha_2)(V_{2T}(x_2))$ is the maximum in (42), then we can write:

$$\Delta V_T \leq \rho \circ (\mathrm{Id} - T\alpha_2)(V_{2T}(x_2)) - V_{1T}(x_1) \quad (43)$$

and we consider two sub-cases:

4.1.1: If $\frac{1}{2}V_{1T}(x_1) \geq \rho(V_{2T}(x_2))$, since $T < 1$ then we have from (43):

$$\begin{aligned}\Delta V_T &\leq \rho \circ (\mathrm{Id} - T\alpha_2)(V_{2T}(x_2)) - V_{1T}(x_1) \\ &\leq \rho(V_{2T}(x_2)) - V_{1T}(x_1) \\ &\leq -\frac{1}{2}V_{1T}(x_1) \leq -\frac{T}{2}V_T(x) \;.\end{aligned} \quad (44)$$

4.1.2: If $\rho(V_{2T}(x_2)) \in [V_{1T}(x_1)/2, V_{1T}(x_1)]$, then using (43) and the Mean Value Theorem we have that:

$$\begin{aligned}\Delta V_T &\leq \rho \circ (\mathrm{Id} - T\alpha_2)(V_{2T}(x_2)) - V_{1T}(x_1) \\ &\leq \rho \circ (\mathrm{Id} - T\alpha_2)(V_{2T}(x_2)) - \rho(V_{2T}(x_2)) \\ &= \tilde{q}(V_{2T}^\star)[-T\alpha_2(V_{2T}(x_2))]\end{aligned} \quad (45)$$

where $V_{2T}^\star \in [(\mathrm{Id} - \alpha_2)(V_{2T}(x_2)), V_{2T}(x_2)]$ (since $T < 1$) and $\tilde{q}$ is a positive function. Let $\tilde{q}$ generate via Lemma 2.2 the functions $q_1 \in \mathcal{K}_\infty$ and $q_2 \in \mathcal{L}$. Since $\mathrm{Id} - \alpha_2$ is a positive definite function, we can write using (45)

$$\begin{aligned}\Delta V_T &\leq -T\tilde{q}(V_{2T}^\star) \cdot \alpha_2(V_{2T}(x_2)) \\ &\leq -Tq_1(V_{2T}^\star) \cdot q_2(V_{2T}^\star) \cdot \alpha_2(V_{2T}(x_2)) \\ &\leq -Tq_1 \circ (\mathrm{Id} - \alpha_2)(V_{2T}(x_2)) \\ &\quad \cdot q_2(V_{2T}(x_2)) \cdot \alpha_2(V_{2T}(x_2)) \\ &=: -T\alpha_2^*(V_{2T}(x_2))\end{aligned} \quad (46)$$

Let α_2^* generate via Lemma 2.2 functions $q_1^* \in \mathcal{K}_\infty$ and $q_2^* \in \mathcal{L}$ and let a_{4a} is positive definite, then we can write:

$$\begin{aligned}\Delta V_T &\leq -T\alpha_2^*(V_{2T}(x_2)) \\ &\leq -Tq_1^*(V_{2T}(x_2)) \cdot q_2^*(V_{2T}(x_2)) \\ &\leq -Tq_1^* \circ \rho^{-1}(\frac{1}{2}V_{1T}(x_1)) \cdot q_2^* \circ \rho^{-1}(V_{1T}(x_1)) \\ &=: -Ta_{4a}(V_{1T}(x_1)) = -Ta_{4a}(V_T(x)).\end{aligned} \quad (47)$$

4.2: If $\rho \circ \tilde{\gamma}_{x_2}(V_{1T}(x_1))$ is the maximum in (42), then since $a_{4b}(r) := (\mathrm{Id} - \rho \circ \tilde{\gamma}_{x_2})(r)$ is positive definite, $T < 1$ and $V_T(x) = V_{1T}(x_1)$, we have that

$$\begin{aligned}\Delta V_T &\leq \rho \circ \tilde{\gamma}_{x_2}(V_{1T}(x_1)) - V_{1T}(x_1) \\ &\leq (\rho \circ \tilde{\gamma}_{x_2} - \mathrm{Id})(V_{1T}(x_1)) \\ &=: -a_{4b}(V_{1T}(x_1)) \leq -Ta_{4b}(V_T(x)) \;.\end{aligned} \quad (48)$$

4.3: If $\rho(\gamma_{u_2}(|u|) + \nu_2)$ is the maximum in (42), then we have that

$$\Delta V_T \leq \rho(\gamma_{u_2}(|u|) + \nu_2) - V_{1T}(x_1) \quad (49)$$

Let $\varepsilon_2 \in \mathcal{K}_\infty$ is such that $a_{4c} := \mathrm{Id} - \varepsilon_2$ is a positive definite function. Using the definition of ν_2, $V_T(x) = V_{1T}(x_1)$ and the fact that $T < 1$, we can write that

$$\begin{aligned}&V_T(x) \geq \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(|u|) + \nu \\ &\Rightarrow V_{1T}(x) \geq \varepsilon_2^{-1} \circ \rho(\gamma_{u_2}(|u|) + \nu_2) \\ &\Rightarrow \Delta V_T \leq -(\mathrm{Id} - \varepsilon_2)(V_T(x)) \\ &\qquad =: -a_{4c}(V_T(x)) \leq -Ta_{4c}(V_T(x)) \;.\end{aligned} \quad (50)$$

Combining 4.1, 4.2 and 4.3, with $a_4(s) := \min\{s/2, a_{4a}(s), a_{4b}(s), a_{4c}(s)\}$, we can write that

$$V_T(x) \geq \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(|u|) + \nu \Rightarrow \Delta V_T \leq -Ta_4(V_T(x)) \;, \quad (51)$$

By combining (27), (32), (41) and (51), we have shown that (3) holds with

$$\alpha(r) := \min\{\alpha_1(r), a_2(r), a_3(r), a_4(r)\} \quad (52)$$
$$\gamma(r) := \max\{\varepsilon_1^{-1} \circ \gamma_{u_1}(r), \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(r)\} \quad (53)$$

where α is a positive definite function and and $\gamma \in \mathcal{K}$. Hence, the system (7) is SP-ISS.

The last thing left to prove is that if V_{1T} and V_{2T} are Lipschitz, uniformly in small T then V_T is Lipschitz, uniformly in small T. Let $\Delta_x > 0$ be given. Let L_1, T_1^* and L_2, T_2^* come respectively from the Lipschitz properties of V_{1T} and V_{2T} for the set $|x_i| \leq \Delta_x, i = 1, 2$. Note that since $\rho \in C^1$, it is locally Lipschitz and let L_ρ be its Lipschitz constant for the set $V_{2T}(x_2) \leq \overline{\alpha}_2(\Delta_x)$. Denote $x := (x_1^T \; x_2^T)^T$ and $y := (y_1^T \; y_2^T)^T$. Let $T^* := \min\{1, T_1^*, T_2^*\}$ and consider arbitrary $T \in (0, T^*)$ and $\max\{|x|, |y|\} \leq \Delta_x$. Introduce the sets:

$$\begin{aligned}A &:= \{x : V_{1T}(x_1) > \rho(V_{2T}(x_2))\} \\ B &:= \{x : V_{1T}(x_1) = \rho(V_{2T}(x_2))\} \\ C &:= \{x : V_{1T}(x_1) < \rho(V_{2T}(x_2))\} \;.\end{aligned}$$

We consider the following cases, to prove our claim:

Case 1: $(x, y \in A)$ or ($x \in A$ and $y \in B$) or ($x \in B$ and $y \in A$) or $(x, y \in B)$

$$|V_T(x) - V_T(y)| = |V_{1T}(x_1) - V_{1T}(y_1)| \leq L_1 |x_1 - y_1|$$

Case 2: $(x, y \in C)$ or $(x \in C$ and $y \in B)$ or $(x \in B$ and $y \in C)$.

$$\begin{aligned} &|V_T(x) - V_T(y)| \\ &\quad = |\rho(V_{2T}(x_2)) - \rho(V_{2T}(y_2))| \leq L_\rho L_2 |x_2 - y_2| \end{aligned}$$

Case 3: $x \in A$ and $y \in C$

$$|V_T(x) - V_T(y)| = |V_{1T}(x_1) - \rho(V_{2T}(y_2))| \qquad (54)$$

Since $x \in A$ implies $V_{1T}(x_1) > \rho(V_{2T}(x_2))$ and $y \in C$ implies $V_{1T}(y_1) < \rho(V_{2T}(y_2))$, we have that:

1. If $V_{1T}(x_1) > \rho(V_{2T}(y_2))$ then

$$\begin{aligned} |V_{1T}(x_1) - \rho(V_{2T}(y_2))| &= V_{1T}(x_1) - \rho(V_{2T}(y_2)) \\ &\leq V_{1T}(x_1) - V_{1T}(y_1) \leq L_1 |x_1 - y_1| \end{aligned} \qquad (55)$$

2. If $V_{1T}(x_1) \leq \rho(V_{2T}(y_2))$ then

$$\begin{aligned} &|V_{1T}(x_1) - \rho(V_{2T}(y_2))| = -V_{1T}(x_1) + \rho(V_{2T}(y_2)) \\ &\quad \leq \rho(V_{2T}(y_2)) - \rho(V_{2T}(x_2)) \leq L_\rho L_2 |x_2 - y_2| \end{aligned} \qquad (56)$$

Case 4: $x \in C$ and $y \in A$, follows Case 3 by symmetry.

Hence, we can conclude that

$$|V_T(x) - V_T(y)| \leq L(|x_1 - y_1| + |x_2 - y_2|), \qquad (57)$$

where $L := \max\{L_1, L_\rho L_2\}$. Therefore, V_T is Lipschitz uniformly in small T. ∎

Example 3.1 *The following example illustrates that it may happen an approximate discrete-time model satisfies a small gain condition but the gains depend on T (hence, the subsystems are not SP-ISS by Definition 2.1), then the approximate discrete-time model may be stable for all small values of T but the exact discrete-time model is unstable for all small values of T.*

Consider a continuous-time plant $\dot{x}_1 = x_1 + u$, which is between a sampler and zero order hold. Suppose that we carry out the controller design using the Euler discrete-time approximate model of the plant

$$x_1(k+1) = (1+T)x_1(k) + Tu(k) \ . \qquad (58)$$

Suppose that we use the family of dynamic controllers

$$x_2(k+1) = -0.5x_2(k) - T^2 x_1(k) \qquad (59)$$

$$u(k) = -\frac{1}{T}x_2(k) - \frac{2}{T}x_1(k) \ . \qquad (60)$$

Note that the approximate closed-loop system (58), (59), (60) can be regarded as a feedback interconnection of two scalar systems (58) with (60) and (59). Moreover, using Lyapunov functions $V_{1T}(x_1) = |x_1|$ and $V_{2T}(x_2) = |x_2|$ and suppose that $T < 1$, we can write the following:

$$|x_1| \geq \frac{2}{T}|x_2| \quad \Rightarrow \quad \Delta V_{1T} \leq -\frac{T}{2}|x_1| \qquad (61)$$

$$|x_2| \geq 4T^2 |x_1| \quad \Rightarrow \quad \Delta V_{2T} \leq -\frac{T}{4}|x_2| \qquad (62)$$

In this case the gains are $\gamma_{x_1}(s) = \frac{2}{T}s$ and $\gamma_{x_2}(s) = 4T^2 s$. Let $\tau_1(s) = \tau_2(s) = \mathrm{Id}(s)$. Then we have that the small gain condition $(\mathrm{Id}+\tau_1)\circ\gamma_{x_1}\circ(\mathrm{Id}+\tau_2)\circ\gamma_{x_2} = 32Ts \leq s$ is satisfied for all $T \in (0, \frac{1}{32})$. Hence, the approximate model is stable for all $T \in (0, \frac{1}{32})$. Using Maple Software Package we have computed the eigenvalues of the approximate closed-loop system matrix and we have obtained that $\lambda_1^a = -\frac{1}{2} + 2T^2 + O(T^3)$ and $\lambda_2^a = -1 + T - 2T^2 + O(T^3)$, which indicates that indeed the approximate closed-loop model is stable for sufficiently small T. However, if we consider the exact closed-loop system consisting of the exact discrete-time plant model

$$x_1(k+1) = e^T x_1(k) + (e^T - 1)u(k) \qquad (63)$$

and (59), (60), we obtain that the eigenvalues of the system matrix are $\lambda_1^e = -\frac{1}{2} + 2T^2 + O(T^3)$ and $\lambda_2^e = -1 - \frac{11}{6}T^2 + O(T^3)$ and obviously we have that $|\lambda_2^e| > 1$ for all sufficiently small T. ∎

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