Author/s:
Nesic, D;Teel, AR

Title:
On averaging and the ISS property

Date:
1999-12-01

Citation:
Nesic, D. & Teel, A. R. (1999). On averaging and the ISS property. Proceedings of the 38th IEEE Conference on Decision and Control, 4, pp.3346-3351. IEEE. https://doi.org/10.1109/cdc.1999.827789.

On averaging and the ISS property

D. Nešić[1]
Dept. of Elect. and Electronic Eng.
The University of Melbourne
Parkville, 3052, Victoria, Australia

A. R. Teel[2]
Dept. of Elect. and Comp. Eng.
University of California
Santa Barbara, CA 93106-9560

Abstract

Input-to-state stability (ISS) is investigated in the context of averaging. "Strong" and "weak" averages are introduced and their ISS properties are used to deduce ISS properties of the actual system. ISS of the strong average implies uniform semi-global practical ISS of the actual system. The weak average is a more general concept, but does not yield the same strong implication. On the other hand, ISS of the weak average implies uniform semi-global practical ISS of the actual system when the disturbances are uniformly Lipschitz.

1 Introduction

Averaging is an important tool in the stability analysis and synthesis of time-varying nonlinear control systems. Some classical results on averaging and their applications to adaptive control can be found in [3, 6, 16] and references therein. In classical averaging, one either proves the closeness of trajectories of the averaged and the actual system on finite time intervals or, under the assumption of local exponential stability (LES) of the origin of the averaged system, uniform LES of a small periodic trajectory of the actual system. The main proof technique in the classical approach is a change of coordinates in which it becomes apparent that the actual system can be regarded as a small perturbation of the averaged system. Another approach, that is well-suited to stability studies, is to compare the solutions of averaged and actual systems at sampling instances. A Lyapunov theorem expressed in terms of solutions at sampling instances was introduced in [1] and used to address global and semi-global stability for the cases of exponential stability of averaged system in [2] and homogeneous systems in [10]. In [15] it was shown that global asymptotic stability (GAS) of the averaged system implies uniform semi-global practical stability for the actual system.

Averaging results in the literature are concerned only with stability although very often one needs to analyze the performance with disturbances. The recently introduced ISS property [11] provides a nice framework for $\mathcal{L}_\infty$ stability analysis of nonlinear systems. It has proved to be very useful in the analysis and synthesis of nonlinear control systems (see, e.g., [7, 11, 12, 14]). We investigate it in the context of averaging.

We introduce two definitions of an average system for a time-varying system with disturbances ("strong average" and "weak average") and provide a connection between the ISS properties of averaged systems and the ISS properties of the actual system. Our first result is that *ISS of the strong average of a system implies uniform semi-global practical ISS of the actual system*, which generalizes the result obtained in [15]. We make use of the proof technique presented in [9] that relates discrete-time and sampled-data ISS $\mathcal{KL}$-estimates and which can be regarded as a generalization of the stability result in [1] to the ISS case. The limitation of our first result is that, even for periodic functions, a strong average may not exist in general. The weak average may exist when strong average does not but, unfortunately, it is not the case that ISS of the weak average of a system implies uniform semi-global practical ISS of the actual system. On the other hand, ISS of the weak average does imply that when the disturbances are uniformly Lipschitz (or equivalently the disturbances are absolutely continuous with $\|\dot{w}\|_\infty \in \mathcal{L}_\infty$) the actual system exhibits semi-global practical "ISS-like" behavior. This property is shown to be useful in the analysis of stability of time-varying cascaded systems, which motivates the use of weak averages. The result for the weak average is similar to the ISS result for singularly perturbed systems in [4].

The paper is organized as follows. In Section 2 we present definitions and preliminary results that are needed in the sequel. Our main results are stated in Section 3 and the proofs are given in Sections 4 and 5.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ ($\gamma \in \mathcal{G}$) if it is continuous, zero at zero and nondecreasing. It is of class-$\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \to 0^+} \beta(s,t) = 0$, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \to \infty} \beta(s,t) = 0$. Given a measurable function $w(t)$, we define $\|w\|_\infty := \operatorname{ess\,sup}_{t \geq 0} |w(t)|$. If $\|w\|_\infty < \infty$, then we write $w \in \mathcal{L}_\infty$. If $w(t)$ is absolutely continuous, its derivative is defined almost everywhere and we can write $w(t) - w(t_\circ) = \int_{t_\circ}^t \dot{w}(\tau) d\tau$. Consider the time-varying system:

$$\dot{x} = f(t, x, w) \tag{1}$$

with state $x \in \mathbb{R}^n$ and disturbance $w \in \mathbb{R}^m$.

Assumption 1 *The function $f(t, x, w)$ is measurable in t, locally Lipschitz in x, w uniformly in t, and $\exists c \geq 0$ s.t. $|f(t, 0, 0)| \leq c, \forall t \geq 0$.*

We investigate also the time-varying system that depends on a small parameter $\epsilon > 0$:

$$\dot{x}(t) = f\left(t/\epsilon, x(t), w(t)\right). \tag{2}$$

We are not aware of references in the literature on averaging of systems with disturbances and we introduce below two definitions of average for such systems. In the absence of disturbances both definitions reduce to the standard definition of an average.

[1] Work done while visiting UCSB. Research supported by NSF under grant ECS-9812346. Email: `d.nesic@ee.mu.oz.au`

[2] Supported in part by AFOSR grant F49620-98-1-0087 and in part by NSF grant ECS-9896140. Email: `teel@ece.ucsb.edu`

Definition 1 (strong average) *A locally Lipschitz function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the strong average of $f(t,x,w)$ if there exist $\beta \in \mathcal{KL}$ and $T^* > 0$ such that for all $T \geq T^*$, $t \geq 0$, $w \in \mathcal{L}_\infty, x \in \mathbb{R}^n$,*

$$\left| \frac{1}{T} \int_t^{t+T} (f_{sa}(x, w(s)) - f(s, x, w(s)))\, ds \right| \leq \beta(\max\{|x|, \|w\|_\infty, 1\}, T). \tag{3}$$

The strong average of system (1) is then defined as

$$\dot{x} = f_{sa}(x, w). \tag{4}$$

Definition 2 (weak average) *A locally Lipschitz function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the weak average of $f(t,x,w)$ if there exist $\beta \in \mathcal{KL}$ and $T^* > 0$ such that for all $T \geq T^*, t \geq 0$, $w \in \mathbb{R}^m, x \in \mathbb{R}^n$,*

$$\left| f_{wa}(x,w) - \frac{1}{T} \int_t^{t+T} f(s,x,w) ds \right| \leq \beta(\max\{|x|, |w|, 1\}, T).$$

The weak average of system (1) is then defined as

$$\dot{x} = f_{wa}(x, w). \tag{5}$$

The averages of (1) and (2) are the same. Any strong average is a weak average.

3 Main results

In this section we relate the ISS properties of the strong and weak average to the ISS properties of the actual system. We show that ISS of the strong average implies uniform semi-global practical ISS of the actual system (Theorem 1). The main result of [15] is a corollary of Theorem 1. On the other hand, ISS of the weak average does not, in general, imply uniform semi-global practical ISS of the actual system and we illustrate this by an example (Example 1). However, if the weak average is ISS and disturbances are uniformly Lipschitz then we can prove a semi-global practical "ISS like" property for the actual system (Theorem 2). This (semi-global practical) ISS under Lipschitz disturbances property is shown to be useful for the analysis of certain time-varying cascades.

We start by recalling a Lyapunov characterization of ISS for time-invariant systems.

Fact 1 ([11]) *If $\exists V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\gamma \in \mathcal{G}$ such that, for all (x, w),*

$$\begin{aligned} \alpha_1(|x|) \leq \quad & V(x) \quad \leq \alpha_2(|x|) \\ |x| \geq \gamma(|w|) \quad \Rightarrow \quad & \frac{\partial V}{\partial x} f_a(x,w) \leq -\alpha_3(|x|) \end{aligned} \tag{6}$$

then the system $\dot{x} = f_a(x,w)$ is ISS; specifically, there exists $\beta \in \mathcal{KL}$ such that, for each $w \in \mathcal{L}_\infty$ and each $x_\circ \in \mathbb{R}^n$, the solution $x(t)$ of $\dot{x} = f_a(x,w)$ starting at $(x_\circ, t_\circ)$ exists for all $t \geq t_\circ$ and satisfies, for all $t \geq t_\circ \geq 0$,

$$|x(t)| \leq \max\{\beta(|x_0|, t - t_\circ), \tilde{\gamma}(\|w\|_\infty)\} \tag{7}$$

where $\tilde{\gamma} = \alpha_1^{-1} \circ \alpha_2 \circ \gamma$.

Remark 1 *When (6) holds with α_3 only positive definite, it is always possible to modify V so that (6) holds with $\alpha_3 \in \mathcal{K}_\infty$ and $\alpha_1^{-1} \circ \alpha_2$ is unchanged.*

Remark 2 *While it is known that the condition (7) implies the existence of a smooth function $V(x)$ satisfying conditions like in (6) (see [13]), it is not known whether (7) implies the existence of $V, \alpha_1, \alpha_2, \alpha_3$ such that (6) holds with $\alpha_1^{-1} \circ \alpha_2 \circ \gamma = \tilde{\gamma}$.*

The main objective of this paper is to determine to what extent (6) for the averaged system implies (7) for the actual system. The main result for the strong average is stated below:

Theorem 1 *Under Assumption 1, if $f(t,x,w)$ has the strong average $f_{sa}(x,w)$ and there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty, \gamma \in \mathcal{G}$ such that, for all (x,w):*

$$\begin{aligned} \alpha_1(|x|) \leq \quad & V(x) \quad \leq \alpha_2(|x|) \\ |x| \geq \gamma(|w|) \quad \Rightarrow \quad & \frac{\partial V}{\partial x} f_{sa}(x,w) \leq -\alpha_3(|x|), \end{aligned} \tag{8}$$

then there exists $\beta \in \mathcal{KL}$ and, given any strictly positive real numbers $\Omega_x, \Omega_w, \delta$, there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ the solutions of (2) satisfy, for all $t \geq t_\circ \geq 0$, $|x(t_\circ)| \leq \Omega_x, \|w\|_\infty \leq \Omega_w$,

$$|x(t)| \leq \max\{\beta(|x(t_\circ)|, t - t_\circ), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + \delta\ .$$

In the next corollary, and in Corollary 2, we will need a precise definition of uniform semi-global practical asymptotic stability.

Definition 3 *The origin of the system $\dot{x} = f(t,x,\epsilon)$ is said to be uniformly semi-globally practically asymptotically stable in ϵ if there exists $\beta \in \mathcal{KL}$ and, for each pair of strictly positive real numbers δ, Δ, there exists ϵ^* such that for all $\epsilon \in (0, \epsilon^*)$, the solutions of $\dot{x} = f(t, x, \epsilon)$ satisfy, for all $t \geq t_\circ \geq 0$, $|x_\circ| \leq \Delta$,*

$$|x(t)| \leq \beta(|x_\circ|, t - t_\circ) + \delta\ . \tag{9}$$

A corollary of our main result is the result in [15]:

Corollary 1 *If the origin of the average of system $\dot{x} = F(\frac{t}{\epsilon}, x)$ is globally asymptotically stable then the origin of the actual system $\dot{x} = F(\frac{t}{\epsilon}, x)$ is uniformly semi-globally practically asymptotically stable in ϵ.*

The conclusion of Theorem 1 does not follow if strong average is replaced by weak average. Also, for some systems there may not exist a strong average whereas a weak average exists. The following example illustrates both of these points:

Example 1 *Consider the system:*

$$\dot{x} = -cx^3 + \cos(t/\epsilon)\, x^3 w \tag{10}$$

where $x, w \in \mathbb{R}$, $c \in (0, 0.5)$. Its weak average is $\dot{x} = -cx^3$, which is GAS but we will show that the system (10) is not uniformly semi-globally practically ISS. Consider the system (10) with the disturbance $w_\epsilon(t) = \cos(\frac{t}{\epsilon})$. Note that $\|w_\epsilon\|_\infty = 1, \forall \epsilon > 0$ and $\|\dot{w}_\epsilon\|_\infty = \frac{1}{\epsilon}$. We recall that $\int_t^{t+T} \cos^2(s) ds = 0.5T + 0.25(\sin(2t+2T) - \sin(2t))$. By direct integration of (10) with this disturbance:

$$\int_{x(t_\circ)}^{x(t)} \frac{d\chi}{\chi^3} = \int_{t_\circ}^{t} (\cos^2(\frac{s}{\epsilon}) - c) ds$$

we obtain

$$x^2(t) = \frac{x^2(t_\circ)}{1 - 2\psi(\epsilon, t, t_\circ) x^2(t_\circ)},$$

where $\psi(\epsilon, t, t_\circ)$ is given by

$$(0.5 - c)(t - t_\circ) + 0.25\epsilon \left(\sin\left(\frac{2t}{\epsilon}\right) - \sin\left(\frac{2t_\circ}{\epsilon}\right) \right).$$

Fix $t_\circ \geq 0, \epsilon > 0$ and let $x(t_\circ) = 1$. Since $0.5 - c > 0$, there exists $t^* \geq t_\circ$ such that $\psi(\epsilon, t^*, t_\circ) = 0.5$ and hence there are finite escape times. Hence, the actual system is not uniformly semi-globally practically ISS.

We now show that there does not exist a strong average for system (10). Pick an arbitrary $\tilde{x} \neq 0$ and note that, given any function $f_{sa}(x,w)$, we have two possibilities:

1. either $f_{sa}(\tilde{x}, w) + c\tilde{x}^3 = 0, \forall w$, or
2. $\exists \tilde{w}$ such that $f_{sa}(\tilde{x}, \tilde{w}) + c\tilde{x}^3 \neq 0$.

Suppose that $f_{sa}(x, w)$ is the strong average and the first case holds. Let $w(t) = \cos(t)$. We must have:

$$\left| \frac{\tilde{x}^3}{T} \int_t^{t+T} \cos^2(s) ds \right| \leq \beta_{av}(\max\{|\tilde{x}|, 1, 1\}, T)$$

for some $\beta_{av} \in \mathcal{KL}$ and for all T sufficiently large. However, if $T_k = k\pi, k \in \mathbb{N}$, the left hand side in the above expression is equal to $0.5\tilde{x}^3 \neq 0$ for all $k \in \mathbb{N}$ and it does not converge to zero as $k \to \infty$ (as $T_k \to \infty$). Suppose now that $f_{sa}(x, w)$ is the strong average and the second case holds. Pick $w(s) = \tilde{w}$ and then

$$\left| \tfrac{1}{T} \int_t^{t+T} (f_{sa}(\tilde{x}, \tilde{w}) + c\tilde{x}^3 - \tilde{w}\tilde{x}^3 \cos(s)) ds \right| \leq \beta_{av}(\max\{|\tilde{x}|, |\tilde{w}|, 1\}, T)$$

for some $\beta_{av} \in \mathcal{KL}$ and all T sufficiently large. However, for $T_k = 2k\pi, k \in \mathbb{N}$ the left-hand side is equal to $\left|c\tilde{x}^3 + f_{sa}(\tilde{x}, \tilde{w})\right| > 0$ for all $k \in \mathbb{N}$ and it does not converge to zero as $k \to \infty$ (as $T_k \to \infty$). Hence, there does not exist a strong average for (10).

Example 1 indicates that it is important to identify the class of systems for which the strong average exists. We address this issue below. Consider the system:

$$\dot{x} = f(t, x, w) = F(t, x) + g(x, w). \tag{11}$$

Suppose that there exists the (strong and weak) average for $F(t, x)$, denoted as $F_{av}(x)$. Then, it is easy to check that $F_{sa}(x, w) := F_{av}(x) + g(x, w)$ satisfies our definition of the strong average for $F(t, x) + g(x, w)$. It is natural to ask: *Does there exist a system which is not of the form (11) for which the strong average exists?* We state now (proof omitted for space reasons) that if $f(t, x, w)$ is periodic, then the systems of the form (11) are the only systems for which the strong average exists. For the general case one can argue similarly to show that systems of the form (11) cover "most of the cases" when the strong average exists. This emphasizes the importance of the structure (11) for the existence of the strong average.

Proposition 1 *Suppose that $f(t, x, w)$ is continuous and periodic in t. Suppose that there exists a strong average $f_{sa}(x, w)$ for $f(t, x, w)$. Then $f(t, x, w)$ can be written as the sum of a term that is independent of t plus a term that is independent of w and that has a well-defined average.*

If there does not exist a strong average there still may exist a weak average. However, in Example 1 we showed that ISS of the weak average of a system does not imply semi-global practical ISS of the actual system in general. The disturbance that we used to show this result is differentiable and it has arbitrarily large rates. A natural question is: *What (strongest) property can we can prove for the solutions of the actual system if the weak average is ISS?* In the definition of average for systems without disturbances we take constant x in the integral and then, assuming GAS of $\dot{x} = F_{av}(x)$, we can establish semi-global practical asymptotic stability for $\dot{x} = f(\frac{t}{\epsilon}, x)$. From the conclusion, it follows that the solutions $x(t)$ considered are uniformly Lipschitz. The disturbances w in the definition of weak average (Definition 2) are also constant and it is reasonable to expect that a semi-global result using the weak average can be proved under the assumption that the disturbances are uniformly Lipschitz. This is the next result.

Theorem 2 *Under Assumption 1, if $f(t, x, w)$ has the weak average $f_{wa}(x, w)$, there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty, \gamma \in \mathcal{G}$ such that:*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$|x| \geq \gamma(\|w\|_\infty) \Rightarrow \frac{\partial V}{\partial x} f_{wa}(x, w) \leq -\alpha_3(|x|), \tag{12}$$

and if only absolutely continuous disturbances w are acting on the system (2), then there exists $\beta \in \mathcal{KL}$ and, given any strictly positive real numbers $\Omega_x, \Omega_w, \Omega_{\dot{w}}, \nu$, there exists $\epsilon^* > 0$ such that, $\forall \epsilon \in (0, \epsilon^*)$, the solutions of (2) satisfy, for all $t \geq t_\circ \geq 0$,

$$|x(t)| \leq \max\{\beta(|x(t_\circ)|, t - t_\circ), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + \nu$$

whenever $|x(t_\circ)| \leq \Omega_x, \|w\|_\infty \leq \Omega_w, \|\dot{w}\|_\infty \leq \Omega_{\dot{w}}$.

Example 1 does not contradict Theorem 2 since the disturbance that we used in the example has $\|\dot{w}_\epsilon\|_\infty = \frac{1}{\epsilon}$ which becomes arbitrarily large for arbitrarily small $\epsilon > 0$. Hence, w_ϵ does not satisfy the condition of Theorem 2: $\|\dot{w}_\epsilon\|_\infty \leq \Omega_{\dot{w}}$ for some fixed $\Omega_{\dot{w}} > 0$.

The weak average (and the result of Theorem 2) is useful for analysis of stability of time varying cascade systems. For example, we can state the following:

Corollary 2 *Consider the system:*

$$\dot{\xi} = f_1(t/\epsilon, \xi, \eta) \ , \ \dot{\eta} = f_2(t, \eta, \epsilon) \tag{13}$$

where $\xi \in \mathbb{R}^{n_1}, \eta \in \mathbb{R}^{n_2}$ and suppose f_1 satisfies Assumption 1. If the weak average of the ξ subsystem exists and is ISS with respect to η and if the origin of the η-subsystem is uniformly semi-globally practically asymptotically stable in ϵ then the origin of the system (13) is uniformly semi-globally practically asymptotically stable in ϵ.

Remark 3 *A similar result can be stated when the assumptions of the ξ are unchanged but the η subsystem is allowed to depend on ξ and a disturbance w and the η subsystem is assumed to be semi-globally practically ISS with respect to (ξ, w) with a gain from ξ to η that satisfies a small gain condition relative to the ISS gain from η to ξ for the weak average of the ξ subsystem. Under this assumption, the overall system is semi-globally practically ISS with respect to the disturbance w. As a special case, we see that the definition of a weak average is also useful for concluding semi-global practical ISS for the case when disturbances are known to enter the system through an ISS filter.*

The next example emphasizes that the ξ system does not have to be (uniformly, semi-globally, practically) ISS with respect to η to conclude uniform semi-global practical asymptotic stability of the cascade:

Example 2 *Consider the cascade:*

$$\dot{\xi} = -c\xi^3 + \cos(\frac{t}{\epsilon}) \ \xi^3 \eta \ , \ \dot{\eta} = -(1 - 0.5\cos(t))\eta \tag{14}$$

where $c \in (0, 0.5)$. The weak average of the ξ system was considered in Example 1 and it was shown to be ISS with zero gain (GAS). On the other hand, since the η subsystem does not depend on the small parameter ϵ, we can not apply the results of [15] to show uniform semi-global practical stability of the cascade. The η system is also uniformly GAS and hence from Corollary 2 we conclude that the origin of the cascade (14) is uniformly semi-globally practically asymptotically stable in ϵ.

4 Proofs of main results

Our proofs parallel those in [1, 2, 10, 15]. Namely, we show that, even though the Lyapunov function associated with the average system is not necessarily monotonically decreasing along the trajectories of the time-varying system, it is decreasing at appropriate sampling instances, at least when x is big relative to w, for ϵ sufficiently small. (See Lemma 2 for Theorem 1 and Lemma 3 for Theorem 2.) We use this observation to construct, for the trajectories of the time-varying system, a "discrete-time" estimate that resembles the continuous-time estimate in the conclusion of each theorem (See Lemma 4). Finally, in Lemma 5, we combine the discrete-time estimate with a standard "intersample growth" result (Lemma 1), like in [9].

Since the proofs of Theorems 1 and 2 are similar, most of the details of the proof of Theorem 2 are omitted. We do, however, provide some of the details that establish the decrease of the Lyapunov function at sampling instances under the assumptions of Theorem 2.

In this section we state the main technical results needed to establish the theorems. The proofs of the technical results, unless short, are delayed until the next section. We start with a standard "intersample growth" result that we will need.

Lemma 1 *Under Assumption 1, given any strictly positive real numbers r^b and r_1^b, there exists $d>0$ and $M>0$ s.t. for each $\epsilon>0$ the following property holds:*

Property 00: *for each $t_\circ \geq 0$, if $|x(t_\circ)| \leq r^b$, $\|w\|_\infty \leq r_1^b$ then the solution $x(t)$ of (2) exists for all $t \in [t_\circ, t_\circ + d]$ and satisfies*

$$|x(t) - x(t_\circ)| \leq M(t - t_\circ) \ , \ \forall t \in [t_\circ, t_\circ + d] \ . \qquad (15)$$

For convenience, we state an obvious corollary:

Corollary 3 *Under Assumption 1, given any strictly positive real numbers r^b and r_1^b and ν_1, there exists $d>0$ s.t. for each $\epsilon>0$ the following property holds:*

Property P0: *for each $t_\circ \geq 0$, if $|x(t_\circ)| \leq r^b$, $\|w\|_\infty \leq r_1^b$ then the solution $x(t)$ of (2) exists for all $t \in [t_\circ, t_\circ + d]$ and satisfies*

$$|x(t)| \leq |x(t_\circ)| + \nu_1 \ , \ \forall t \in [t_\circ, t_\circ + d] \ . \qquad (16)$$

The next two results, on the decrease of the Lyapunov function at sampling instances in the setting of Theorems 1 and 2 respectively, are established in Sections 5.1 and 5.2, respectively.

Lemma 2 *Under the assumptions of Theorem 1, given any strictly positive real numbers Δ, δ, Δ_1, there exists $d^*>0$ and, for any $d \in (0, d^*)$, there exists $\epsilon^*>0$ such that for all $\epsilon \in (0,\epsilon^*)$ the following property holds:*

Property P1.1: *Given any $t_\circ \geq 0$, if $|x(t_\circ)| \leq \Delta, \|w\|_\infty \leq \Delta_1$ then the solution $x(t)$ of (2) exists for all $t \in [t_\circ, t_\circ + d]$ and satisfies*

$$|x(t_\circ)| \geq \gamma(\|w\|_\infty) + \delta \implies V(x(t_\circ + d)) - V(x(t_\circ)) \leq -\tfrac{d}{2}\alpha_3(0.5\,|x(t_\circ)|).$$

Lemma 3 *Under the assumptions of Theorem 2, given any strictly positive real numbers $\Delta, \delta, \Delta_1, \Delta_2$, there exists $d^*>0$ and, for any $d \in (0, d^*)$, there exists $\epsilon^*>0$ such that for all $\epsilon \in (0,\epsilon^*)$ the following property holds:*

Property P1.2: *Given any $t_\circ \geq 0$, if $|x(t_\circ)| \leq \Delta, \|w\|_\infty \leq \Delta_1, \|\dot{w}\| \leq \Delta_2$ then the solution $x(t)$ of (2) exists for all $t \in [t_\circ, t_\circ + d]$ and satisfies*

$$|x(t_\circ)| \geq \gamma(\|w\|_\infty) + \delta \implies V(x(t_\circ + d)) - V(x(t_\circ)) \leq -\tfrac{d}{2}\alpha_3(0.5\,|x(t_\circ)|).$$

A corollary of Lemmas 1 and 2 is the following.

Corollary 4 *Under the assumptions of Theorem 1, given any strictly positive real numbers $\Delta, \Delta_1, \mu, \mu_1$, there exists $d^*>0$ and, for any $d \in (0,d^*)$, $\exists \epsilon^* > 0$ s.t. for all $\epsilon \in (0,\epsilon^*)$ the following property holds:*

Property P2: *Given any $t_\circ \geq 0$, if $|x(t_\circ)| \leq \Delta, \|w\|_\infty \leq \Delta_1$ then the solution $x(t)$ of (2) exists for all $t \in [t_\circ, t_\circ + d]$ and satisfies $V(x(t_\circ + d)) \leq V(x(t_\circ)) + \mu_1$ and*

$$V(x(t_\circ)) \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu \implies V(x(t_\circ + d)) - V(x(t_\circ)) \leq -\tfrac{d}{2}\alpha_3(0.5\alpha_2^{-1}(V(x(t_\circ)))) \ .$$

Proof. Given Δ, Δ_1 and μ_1, it follows from Lemma 1 and the continuity of V that there exists $d^*>0$ such that for any $d \in (0,d^*)$ if $|x(t_\circ)| \leq \Delta, \|w\|_\infty \leq \Delta_1$ then the solution $x(t)$ of (2) exists for all $t \in [t_\circ, t_\circ + d]$ and satisfies $V(x(t_\circ + d)) \leq V(x(t_\circ)) + \mu_1$. Given $\Delta_1 > 0$ and $\mu > 0$, let $\delta > 0$ be sufficiently small so that

$$\max_{s \in [0, \gamma(\Delta_1)]} [\alpha_2(s+\delta) - \alpha_2(s)] \leq \mu \ . \qquad (17)$$

Such a δ exists from the continuity of $\alpha_2(\cdot)$. Then note that, for $\|w\|_\infty \leq \Delta_1$,

$$\alpha_2 \circ \gamma(\|w\|_\infty) + \mu \geq \alpha_2(\gamma(\|w\|_\infty) + \delta) \ . \qquad (18)$$

Using $V(x) \leq \alpha_2(|x|)$ we get

$$V(x(t_\circ)) \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu \implies |x(t_\circ)| \geq \gamma(\|w\|_\infty) + \delta \ .$$

Property P2 then follows from Property P1.1. Q.E.D.

In preparation for the next result, given functions α_1, α_2 and α_3 all in class-$\mathcal{K}_\infty$, we define $\beta_\alpha \in \mathcal{KL}$ as

$$\beta_\alpha(s,t) := \alpha_1^{-1}\left(\beta_V(\alpha_2(s), t)\right) \qquad (19)$$

where $\beta_V \in \mathcal{KL}$ satisfies

$$\dot{u} \leq -0.5\alpha_3(0.5\alpha_2^{-1}(u)) \implies u(t) \leq \beta_V(u(0), t) \ . \qquad (20)$$

Such a β_V exists from standard comparison results.

Now we state that Property P2 implies a particular discrete-time ISS estimate. (For proof, see Section 5.3.)

Lemma 4 *If Property P2 holds, $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$, and $\Delta > \alpha_1^{-1}(\alpha_2 \circ \gamma(\Delta_1) + \mu + \mu_1)$ then there exist $R>0, r^s>0, r_1^s>0$, (in particular we can take*

$$\begin{aligned} r^s &:= \alpha_2^{-1} \circ \alpha_1(\Delta) \qquad r_1^s := \Delta_1 \qquad (21) \\ R &:= \max_{s \in [0, \alpha_2 \circ \gamma(\Delta_1)]} (\alpha_1^{-1}(s + \mu + \mu_1) - \alpha_1^{-1}(s)) \ ), \end{aligned}$$

such that the following property holds:

Property P3: *for all $t_\circ \geq 0$, $|x(t_\circ)| \leq r^s, \|w\|_\infty \leq r_1^s$, the solution of (2) exists $\forall t \geq t_\circ$ and satisfies, $\forall k \geq 0$,*

$$|x(t_\circ + kd)| \leq \max\{\beta_\alpha(|x(t_\circ)|, kd), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + R$$

where β_α was defined by (19)-(20).

The last result we will need is the following:

Lemma 5 *If Property P0 and P3 hold and $R < r^b/2$ then then there exist strictly positive real numbers Δ_x, Δ_w, ν and $\bar{\beta} \in \mathcal{KL}$, (in particular we can take*

$$\begin{aligned} \bar{\beta}(s,t) &:= 2\max_{\eta \in [0,t]} 2^{-\eta}\beta_\alpha(s, t-\eta) \\ \nu &:= R + \nu_1 \\ \Delta_x &:= \min\left\{r^s, \beta_0^{-1}\left(\frac{r^b}{2}\right)\right\} \\ \Delta_w &:= \min\left\{\chi(r^b), r_1^b, r_1^s\right\} \qquad (22) \end{aligned}$$

where $\beta_0 \in \mathcal{K}_\infty$ satisfies $\beta_0(s) \geq \beta_\alpha(s,0)$ for all $s \geq 0$ and $\chi(r^b) := \sup\left\{s : \alpha_1^{-1} \circ \alpha_2 \circ \gamma(s) \leq \frac{r^b}{2}\right\}$) such that the following property holds:

Property P4: $|x(t_\circ)| \leq \Delta_x, \|w\|_\infty \leq \Delta_w \Rightarrow \forall t \geq t_\circ \geq 0$

$$|x(t)| \leq \max\{\bar{\beta}(|x(t_\circ)|, t - t_\circ), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + \nu \ .$$

Proof: (The same proof technique was used in [9].) Fix an arbitrary $t_\circ \geq 0$ let $t_k := t_\circ + kd$, and note that with the choice Δ_x, Δ_w given in (22) we have that if $|x(t_\circ)| \leq \Delta_x, \|w\|_\infty \leq \Delta_w$ then $\|w\|_\infty \leq r_1^b$ and $|x(t_k)| \leq r^b$ for all $k \geq 0$. Hence the trajectory exists for all $t \geq t_\circ$ and moreover we can write:

$$|x(t)| \leq |x(t_k)| + \nu_1, \forall t \in [t_k, t_{k+1}], \forall k \geq 0.$$

This implies that, for all $t \geq t_\circ \geq 0$,

$$|x(t)| \leq \max\{\beta_\alpha(|x(t_\circ)|, kd), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + R + \nu_1 \,. \tag{23}$$

It was shown in [9] that for any $\beta_\alpha \in \mathcal{KL}$ there exists $\bar{\beta} \in \mathcal{KL}$ s.t. $\beta_\alpha(s, kd) \leq \bar{\beta}(s, (k+1)d)$ for all $s \geq 0$, $k \geq 0$. Moreover, one possible choice is $\bar{\beta}(s,t) := 2\max_{\eta \in [0,t]} 2^{-\eta} \beta_\alpha(s, t-\eta)$. Since $t - t_\circ < (k+1)d, \forall t \in [t_k, t_{k+1}), k \geq 0$ we can write $\bar{\beta}(s, (k+1)d) \leq \bar{\beta}(s, t - t_\circ), \forall t \geq t_\circ \geq 0$, which completes the proof by letting $\nu = R + \nu_1$ in (23). Q.E.D.

Proof of Theorem 1. Let α_1, α_2 and α_3 generate the function $\beta_\alpha \in \mathcal{KL}$ according to the definition in (19)-(20). Let $\beta_0 \in \mathcal{K}_\infty$ satisfy $\beta_0(s) \geq \beta_\alpha(s, 0)$ for all $s \geq 0$. Let Ω_x, Ω_w and δ be arbitrary strictly positive real numbers. Define $\Delta_1 := \Omega_w$. Let μ and μ_1 be strictly positive real numbers such that

$$\max_{s \in [0, \alpha_2 \circ \gamma(\Delta_1)]} (\alpha_1^{-1}(s + \mu + \mu_1) - \alpha_1^{-1}(s)) \leq \frac{\delta}{2} \,.$$

Let Δ satisfy

$$\Delta \geq \max\{\alpha_1^{-1} \circ \alpha_2(\Omega_x), \alpha_1^{-1}(\alpha_2 \circ \gamma(\Delta_1) + \mu + \mu_1) + 1\} \,.$$

Define $\nu_1 := \frac{\delta}{2}$, $r_1^b := \Omega_w$ and let $r^b \geq \max\{2\beta_0(\Omega_x), 2\delta\}$ while also satisfying $\chi(r^b) \geq \Omega_w$. With these definitions, apply Corollaries 3 and 4 to generate a $d > 0$ and an ϵ^* such that Property P0 and Property P2 hold for all $\epsilon \in (0, \epsilon^*(d))$. Next, applying Lemma 4, we get that Property P3 holds with $r^s \geq \Omega_x$, $r_1^s = \Omega_w$ and $R \leq \frac{\delta}{2}$. Finally, we apply Lemma 5 to get that Property P4 holds with $\nu \leq \delta$, $\Delta_x \geq \Omega_x$ and $\Delta_w \geq \Omega_w$, i.e., the claim of the theorem is established. Q.E.D.

5 Proofs of Technical Lemmas

5.1 Proof of Lemma 2

Let Δ, Δ_1 and δ be given strictly positive real numbers. Let $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ be such that (3) holds for all $T \geq T^*$. Let $L > 0$ be a (uniform) Lipschitz constant for $\frac{\partial V}{\partial x}, f(\frac{t}{\epsilon}, x, w), f_{sa}(x, w)$ over the set where $|x| \leq 2\Delta, |w| \leq \Delta_1$. Let $T \geq T^*$ be such that

$$L\Delta\beta_{av}(\max\{\Delta, \Delta_1, 1\}, T) \leq 0.5\alpha_3\left(\frac{\delta}{4}\right) \,. \tag{24}$$

Define $r^b := \Delta$, $r_1^b := \Delta_1$ and $K := 4L\Delta + L\Delta_1 + c$. Throughout the rest of the proof we assume that $|x(t_\circ)| \leq \Delta$ and $\|w\|_\infty \leq \Delta_1$. Apply Lemma 1 and Corollary 3 to generate d_1^* and M such that Properties P00 and P0 hold for any $d \in (0, d_1^*)$. Use the continuity of α_3 and α_3^{-1} to find $d_2^* > 0$ such that

$$2Md_2^* + 2\alpha_3^{-1}\left(2KLMd_2^* + \alpha_3\left(\frac{\delta}{4}\right)\right) \leq \delta \,. \tag{25}$$

Define $d^* = \min\{d_1^*, d_2^*\}$. Fix $d \in (0, d^*)$ and define $\epsilon^*(d) := \frac{d}{T}$. Let $\epsilon \in (0, \epsilon^*)$. From Property P00 we have

$$|x(t_\circ)| \geq \gamma(\|w\|_\infty) + Md \Rightarrow |x(t)| \geq \gamma(\|w\|_\infty), \forall t \in [t_\circ, t_\circ + d] \,.$$

So $|x(t_\circ)| \geq \gamma(\|w\|_\infty) + Md$ implies $\forall t \in [t_\circ, t_\circ + d]$

$$\begin{aligned}
&\frac{\partial V}{\partial x}(x(t)) f\left(\frac{t}{\epsilon}, x(t), w(t)\right) \leq \underbrace{-\alpha_3(|x(t)|)}_{1} \\
&\underbrace{-\frac{\partial V}{\partial x}(x(t_\circ)) f_{sa}(x(t_\circ), w(t)) + \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{t}{\epsilon}, x(t_\circ), w(t)\right)}_{2} \\
&\underbrace{+\frac{\partial V}{\partial x}(x(t)) f\left(\frac{t}{\epsilon}, x(t), w(t)\right) - \frac{\partial V}{\partial x}(x(t_\circ)) f\left(\frac{t}{\epsilon}, x(t_\circ), w(t)\right)}_{3} \\
&\underbrace{-\frac{\partial V}{\partial x}(x(t)) f_{sa}(x(t), w(t)) + \frac{\partial V}{\partial x}(x(t_\circ)) f_{sa}(x(t_\circ), w(t))}_{4} \,.
\end{aligned} \tag{26}$$

We now integrate both sides of the preceding inequality along the solution $x(t)$ over the interval $[t_\circ, t_\circ + d]$ to get that $V(x(t_\circ + d)) - V(x(t_\circ))$ is bounded by the sum of the integral of the four terms on the right-hand side. The latter are bounded as follows:

1 : Using Property P00, we have that

$$-\int_{t_\circ}^{t_0 + d} \alpha_3(|x(s)|) ds \leq -d\alpha_3\left(\max\{|x(t_\circ)| - Md, 0\}\right)$$

and thus $|x(t_\circ)| \geq 2Md$ implies

$$-\int_{t_\circ}^{t_0 + d} \alpha_3(|x(s)|) ds \leq -d\alpha_3\left(0.5|x(t_\circ)|\right) \,.$$

2 : Since $\left|\frac{\partial V}{\partial x}(x(t_\circ))\right| \leq L|x(t_\circ)| \leq L\Delta$ and $d = \epsilon^* T$ we can over-bound the integral of **2** as $\epsilon L\Delta$ times

$$\left|\int_{t_\circ}^{t_0 + T\epsilon^*} \left(f_{sa}(x(t_\circ), w(s)) - f\left(\frac{s}{\epsilon}, x(t_\circ), w(s)\right)\right) d\left(\frac{s}{\epsilon}\right)\right| .$$

Introduce the change of variables $\tau = s/\epsilon$ in the above integral and introduce $w_1(\tau) := w(\epsilon\tau)$ (note that $\|w_1\|_\infty = \|w\|_\infty \leq \Delta_1$) and $T_1 := \epsilon^* T/\epsilon > T$. Then by the definition of strong average we have:

$$\begin{aligned}
&\left|\int_{\frac{t_\circ}{\epsilon}}^{\frac{t_0}{\epsilon} + T_1} \left(f_{sa}(x(t_\circ), w_1(\tau)) - f(\tau, x(t_\circ), w_1(\tau))\right) d\tau\right| \\
&\qquad \leq T_1\beta_{av}(\max\{|x(t_\circ)|, \|w_1\|_\infty, 1\}, T_1) \\
&\qquad \leq T_1\beta_{av}(\max\{\Delta, \Delta_1, 1\}, T)
\end{aligned}$$

and hence, using (24) and the fact that $\epsilon T_1 = d$, the integral of **2** can be over-bounded by $d\ 0.5\alpha_3\left(\delta/4\right)$.

3 and 4 : From the definition for $L > 0$, $\forall x, w$ s.t. $\max\{|x_1|, |x_2|\} \leq 2\Delta, |w| \leq \Delta_1$, we have

$$\begin{aligned}
&\left|\frac{\partial V}{\partial x}(x_1) f\left(\frac{s}{\epsilon}, x_1, w\right) - \frac{\partial V}{\partial x}(x_2) f\left(\frac{s}{\epsilon}, x_2, w\right)\right| \\
&\leq (4L\Delta + L\Delta_1 + c)L|x_1 - x_2| = KL|x_1 - x_2| \,.
\end{aligned}$$

The same bound holds if $f(\cdot,\cdot,\cdot)$ is replaced by $f_{sa}(\cdot,\cdot)$. Using Properties P00 and P0, we can overbound the sum of integrals of terms **3** and **4** by $KLMd^2$.

From these bounds, it follows that

$$|x(t_\circ)| \geq \gamma(\|w\|_\infty) + 2Md + 2\alpha_3^{-1}\left(2KLMd + \alpha_3\left(\frac{\delta}{4}\right)\right)$$

implies

$$V(x(t_\circ + d)) - V(x(t_\circ)) \leq -\frac{d}{2}\alpha_3(0.5|x(t_\circ)|) \,.$$

From (25) and $d < d_2^*$, the result follows. Q.E.D.

5.2 Proof of Lemma 3

The main difference in the proof of Lemma 3 (compared to Lemma 2) is that the right-hand side of (26) becomes

$$\underbrace{-\alpha_3(|x(t)|)}_{1}$$
$$\underbrace{-\frac{\partial V}{\partial x}(x(t_\circ))f_{wa}(x(t_\circ),w(t_\circ))+\frac{\partial V}{\partial x}(x(t_\circ))f(\frac{t}{\epsilon},x(t_\circ),w(t_\circ))}_{2}$$
$$\underbrace{+\frac{\partial V}{\partial x}(x(t))f(\frac{t}{\epsilon},x(t),w(t))-\frac{\partial V}{\partial x}(x(t_\circ))f(\frac{t}{\epsilon},x(t_\circ),w(t_\circ))}_{3}$$
$$\underbrace{-\frac{\partial V}{\partial x}(x(t))f_{wa}(x(t),w(t))+\frac{\partial V}{\partial x}(x(t_\circ))f_{wa}(x(t_\circ),w(t_\circ))}_{4}.$$

Term **1** is the same as before. Term **2** is in a form such that the definition of the weak average can be exploited in the same way that the definition of the strong average was exploited in the proof of Lemma 2. Terms **3** and **4** have been modified so that $w(t_\circ)$ replaces $w(s)$ in some places. Nevertheless, these terms can be over-bounded like in the proof of Lemma 2. The key observations are that

$$\left|\frac{\partial V}{\partial x}(x_1)f(\frac{s}{\epsilon},x_1,w_1)-\frac{\partial V}{\partial x}(x_2)f(\frac{s}{\epsilon},x_2,w_2)\right| \le KL\left(|x_1-x_2|+|w_1-w_2|\right)$$

(similarly for $f_{wa}(\cdot,\cdot)$ in place of $f(\cdot,\cdot,\cdot)$) and, since w is absolutely continuous and $\|\dot{w}\|_\infty \le \Delta_2$, $\int_{t_\circ}^{t_\circ+d}|w(s)-w(t_\circ)|\,ds \le 0.5\Delta_2 d^2$. Q.E.D.

5.3 Proof of Lemma 4

Introduce the set $\mathcal{V} := \{x : V(x) \le \alpha_1(\Delta)\}$. Define r^s, r_1^s and R as in (21). Consider any $t_\circ, x(t_\circ), w(t)$ with $t_\circ \ge 0, |x(t_\circ)| \le r^s \le \Delta, \|w\|_\infty \le r_1^s$. All the calculations below are given for this (fixed) triple $(t_\circ, x(t_\circ), w(t))$. Define $t_k := t_\circ + kd$ and, for those k such that $x(t_k)$ is defined, $V_k := V(x(t_k))$. We will use:

Fact 2 *If $V_0 \le \alpha_1(\Delta)$, then $V_k \le \alpha_1(\Delta), \forall k \in \mathbb{N}$.*
Proof: Let $j \ge 0$ be s.t. V_j exists and $V_j \le \alpha_1(\Delta)$. Either $V_j \ge \alpha_2\circ\gamma(\Delta_1)+\mu$ in which case it follows from the second part of Property P2 that $V_{j+1} \le V_j \le \alpha_1(\Delta)$, or $V_j \le \alpha_2\circ\gamma(\Delta_1)+\mu$ in which case it follows from the first part of Property P2 and the assumption that $\Delta > \alpha_1^{-1}(\gamma(\Delta_1)+\mu+\mu_1)$ that $V_{j+1} \le \gamma(\Delta_1)+\mu+\mu_1 \le \alpha_1(\Delta)$. The proof follows by induction. Q.E.D.

For $s \in [kd,(k+1)d), k \ge 0$, introduce the variable:

$$u(s) := V_k + (s/d-k)(V_{k+1}-V_k)\ . \qquad (27)$$

From Fact 2 it follows that $u(s) \le \alpha_1(\Delta), \forall s \ge 0$. The function $u(s)$ is a continuous, piecewise linear function of s and hence it is absolutely continuous. Thus, $u(s)$ is differentiable for almost all $s \ge 0$. We also have

$$0 \le u(s) \le \max\{V_k, V_{k+1}\}, \forall s \in [kd,(k+1)d), k \in \mathbb{N}.$$

Hence using the first part of Property P2 we conclude that whenever $|x(t_\circ)| \le \Delta, \|w\|_\infty \le \Delta_1$ we have that

$$u(s) \le V_k + \mu_1, \forall s \in [kd,(k+1)d), k \ge 0. \qquad (28)$$

Using (28) and the second part of Property P2, the following implications hold $\forall s \in [kd,(k+1)d),\ k \in \mathbb{N}$:

$$\begin{aligned} u(s) \ge \alpha_2\circ\gamma(\|w\|_\infty)+\mu+\mu_1 \Rightarrow \\ V_k \ge \alpha_2\circ\gamma(\|w\|_\infty)+\mu \Rightarrow \\ V_{k+1}-V_k \le -\tfrac{d}{2}\alpha_3(0.5\alpha_2^{-1}(V_k)). \end{aligned} \qquad (29)$$

From (27) and (29) we have $\forall s \in [kd,(k+1)d),\ k \in \mathbb{N}$:

$$\begin{aligned} &u(s) \ge \alpha_2\circ\gamma(\|w\|_\infty)+\mu+\mu_1 \ \Rightarrow \\ &\tfrac{d}{ds}u(s) = \tfrac{1}{d}(V_{k+1}-V_k) \le -0.5\alpha_3(0.5\alpha_2^{-1}(V_k)). \end{aligned} \qquad (30)$$

Finally, since whenever $V_k \ge \alpha_2\circ\gamma(\|w\|_\infty)+\mu$ we have $V_{k+1} \le V_k$ and hence $u(s) \le V_k, s \in [kd,(k+1)d), k \in \mathbb{N}$, we can rewrite (30) as:

$$\begin{aligned} &u(s) \ge \alpha_2\circ\gamma(\|w\|_\infty)+\mu+\mu_1 \ \Rightarrow \\ &\tfrac{d}{ds}u(s) \le -0.5\alpha_3(0.5\alpha_2^{-1}(u(s))), \forall s \ge 0. \end{aligned}$$

Using, e.g., [6, Theorem 5.1], we have, $\forall s \ge 0$,

$$|u(s)| \le \max\{\beta_V(|u_0|, s), \alpha_2\circ\gamma(\|w\|_\infty)+\mu+\mu_1\}$$

with β_V defined by (20). Since $u(kd) = V_k \ge \alpha_1(|x(t_k)|)$, $u_0 = V_0 \le \alpha_2(|x(t_0)|)$, for $s = kd$ we get

$$|x(t_k)| \le \max\{\beta_\alpha(|x(t_\circ)|), kd), \alpha_1^{-1}(\alpha_2\circ\gamma(\|w\|_\infty)+\mu+\mu_1)\}$$

where β_α was defined in (19). Using the definition of R in (21) the result follows. Q.E.D.

References

[1] D. Aeyels and J. Peuteman, *A new asymptotic stability criterion for nonlinear time-variant differential equations*, IEEE TAC, 43 (1998), pp. 968-971.

[2] D. Aeyels and J. Peuteman, *On exponential stability of nonlinear time-variant differential equations*, Automatica, 1999.

[3] B.D.O. Anderson, R.R. Bitmead, C.R. Johnson Jr., P.V. Kokotović, R. L. Kosut, I.M.Y. Mareels, L. Praly and B.D. Riedle, *Stability of adaptive systems: passivity and averaging analysis.* MIT Press, 1986.

[4] P.D.Christofides and A.R.Teel, *Singular perturbations and input-to-state stability*, IEEE Trans. Automat. Contr., 41 (1996), pp. 1645-1650.

[5] Z.P.Jiang, Y.Wang and E.D.Sontag, *Input-to-state stability properties for discrete-time nonlinear systems*, in Proc. 14th IFAC World Congress, Beijing, China, 5-9 July 1999, vol. E, pp. 277-282.

[6] H.K. Khalil, *Nonlinear systems.* Prentice-Hall, 1996.

[7] M.Krstić, I.Kanellakopoulos and P.Kokotović, *Nonlinear and adaptive control design.* J. Wiley & Sons, 1995.

[8] V. Lakshmikantham and S. Leela, *Differential and integral inequalities: Theory and applications, Part 1.* Academic Press: New York, 1969.

[9] D. Nešić, A.R. Teel and E.D. Sontag, *On stability and input-to-state stability $\mathcal{KL}$ estimates of discrete-time and sampled-data nonlinear systems*, to appear in Systems & Control Letters, 1999.

[10] J. Peuteman and D. Aeyels, *Averaging results and the study of uniform asymptotic stability of homogeneous differential equations that are not fast time varying*, to appear in SIAM J. Contr. Optimiz., 1998.

[11] E. D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE TAC, 34 (1989), pp. 435-443.

[12] E. D. Sontag, *On input-to-state stability property*, Europ. J. Contr., 1 (1995), pp. 24-36.

[13] E.D. Sontag and Y. Wang, *On characterizations of the input-to-state stability property*, Systems & Control Letters, 24 (1995), pp. 351-359.

[14] E.D.Sontag and Y.Wang, *New characterizations of the input-to-state stability property*, IEEE Trans. Automat. Contr., 41 (1996), pp. 1283-1294.

[15] A.R Teel, J. Peuteman and D.Aeyels, *Global asymptotic stability for the averaged implies semi-global practical asymptotic stability for the actual*, Proc. Conf. Decis. Control, Tampa, Florida, pp. 1458-1463, 1998.

[16] V.M.Volosov, *Averaging in systems of ordinary differential equations*, Russian Math. Surveys, 17, 1962.