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Piecewise quadratic Lyapunov functions for linear control systems with First Order Reset Elements

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Abstract: For a class of reset control systems, we reconsider the piecewise quadratic Lyapunov function construction proposed in Zaccarian et al. [2005]. In particular, we propose an alternative construction that addresses an analysis subtlety overlooked in that preliminary result. An example illustrates the potential pitfalls of ignoring this analysis subtlety. Another example is used to show that the new statement leads to numerical results that are similar to the ones that come from Zaccarian et al. [2005].

Keywords: First Order Reset Element, piecewise quadratic Lyapunov function, reset system

1. INTRODUCTION

Perhaps the first work on reset control systems dated back to Clegg [1958] where a nonlinear modification of the analog integrator circuit (called "Clegg integrator") was proposed to reset its output when the input crossed the zero value. Later work (Horowitz and Rosenbaum [1975] and references therein) extended the concept to a much more general class of reset systems, called First Order Reset Elements (FORE), where rather than an integrator, the system to be reset was a first order linear filter.

In recent years, the potential of this class of reset systems and its ability to overcome fundamental limitations of linear control was emphasized Beker et al. [2001] and the stability analysis of FORE control systems was cast as a Lyapunov type of problem (see Beker et al. [2004] and references therein). In this work, the original characterization of FOREs in terms of zero crossings of its input was retained, leading to a Lyapunov characterization imposing a flow condition on almost all the input/output space of the FORE, by only removing a suitably modified version of the zero measure set where the input is zero. A new characterization of the reset rule for FORE was then proposed in Zaccarian et al. [2005] by showing that with this new reset characterization, the closed-loop trajectories are substantially the same but weaker Lyapunov conditions can be checked to establish stability properties of the reset control system. This extension allowed to establish exponential and $\mathcal{L}_2$ stability of reset control systems whose underlying linear dynamics (before resets) is exponentially unstable. This extension also led to a number of subsequent results hinging upon this new reset characterization (see Loquen et al. [2008], Witvoet et al. [2007], Zaccarian et al. [2007], Nešić et al. [2008] and references therein).

Already in Horowitz and Rosenbaum [1975] it was noted that the problem of optimizing the nonlinear response "will be solved more satisfactorily when stability criteria are developed for feedback loops containing non-linear elements such as FORE." The goal of developing such stability criteria was pursued in Zaccarian et al. [2005] and the later work Nešić et al. [2008] and references therein. In this paper we further develop toward the achievement of this goal by revisiting the piecewise quadratic Lyapunov construction of Zaccarian et al. [2005] pointing out a subtlety in the Lyapunov conditions and the corresponding analysis, which calls into question the result in [Zaccarian

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et al., 2005, Theorem 3]. The potential pitfalls of ignoring this subtlety are illustrated here by a simple example for which the piecewise quadratic construction of Zaccarian et al. [2005] leads to a function satisfying all the required conditions, even though the closed-loop system generates diverging trajectories. This example is however only illustrative and cannot be taken as a counterexample because it does not completely fit into the framework used in Zaccarian et al. [2005] and also used here. We also propose in this paper a new set of Lyapunov conditions, solving the subtlety of Zaccarian et al. [2005] while still being capable to establish exponential and $\mathcal{L}_2$ stability. In particular, we show on another example that there are cases when the construction proposed here leads to substantially the same exponential stability conclusions and $\mathcal{L}_2$ gain bounds.

The paper is organized as follows. In Section 2, we give a description of the problem data. In Section 3 we describe the Lyapunov construction and formulate our main result. Finally, in Section 4 we discuss two simulation examples. All the proofs are omitted due to space constraints.

2. FORE SISO CONTROL LOOPS

Consider a strictly proper SISO linear plant whose dynamics is described by

$$\mathcal{P} \begin{cases} \dot{x}_p = A_p x_p + B_{pu} u + B_{pd} d, \\ y = C_p x_p, \end{cases} \tag{1}$$

where $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}^{n_d}$ is an $\mathcal{L}_2$ bounded disturbance input, $y \in \mathbb{R}$ is the measured plant output and A_p, B_{pu}, B_{pd} and C_p are matrices of appropriate dimensions.

For the plant (1), assume that a control system is designed with a FORE element described by the following dynamics:

$$\text{FORE} \begin{cases} \dot{x}_r = \lambda_r x_r + v, & \text{if } v x_r \geq 0 \\ x_r^+ = 0, & \text{if } v x_r \leq 0, \end{cases} \tag{2}$$

$$\text{Interc'n} \begin{cases} u = k x_r, \\ v = -y \end{cases} \tag{3}$$

where k denotes the loop gain and $\lambda_r \in \mathbb{R}$ denotes the pole of the FORE. Note that λ_r can be any real number (including positive ones) while k should be positive. For example, choosing $k = 1$ and $\lambda_r = 0$ corresponds to implementing in the FORE the well known Clegg integrator introduced in Clegg [1958] and recently discussed in Zaccarian et al. [2005], Nešić et al. [2009, submitted] using the notation adopted here. The closed-loop system (1), (2), (3) is of interest because it falls into the category of linear reset control systems that received much attention in recent years (see Beker et al. [2004], Nešić et al. [2008] and references therein). For example, in Beker et al. [2001] it was shown that this type of control architecture can overcome intrinsic limitations of linear control.

To avoid Zeno behavior and ensure that solutions are defined for all positive times (see, Goebel et al. [2004], Nešić et al. [2008]), the overall closed-loop system is augmented with temporal regularization namely with a timer τ ensuring that between two consecutive resets (or jumps) there is a minimum dwell time ρ. Then the closed-loop system is described by the following equations, where $x = [x_p^T \; x_r]^T \in \mathbb{R}^n$:

$$\left.\begin{array}{l} \dot{\tau} = 1, \\ \dot{x} = Ax + B_d d, \end{array}\right\} \text{ if } x^T M x \geq 0 \text{ or } \tau \leq \rho,$$
$$\left.\begin{array}{l} \tau^+ = 0, \\ x^+ = A_r x, \end{array}\right\} \text{ if } x^T M x \leq 0 \text{ and } \tau \geq \rho, \tag{4}$$
$$y = Cx$$

where A denotes the flow matrix, A_r denotes the reset matrix and M characterizes the flow and the jump sets (note that these two sets have their boundaries in common). Based on the values in (1), (2) and (3), the matrices in (4) are

$$\left[\begin{array}{c|c} A & B_d \\ \hline C & \end{array}\right] = \left[\begin{array}{cc|c} A_p & B_{pu}k & B_{pd} \\ -C_p & \lambda_r & 0 \\ \hline C_p & 0 & \end{array}\right] \tag{5a}$$

$$\left[\begin{array}{c|c} A_r & M \end{array}\right] = \left[\begin{array}{cc|cc} I & 0 & 0 & -C_p^T \\ 0 & 0 & -C_p & 0 \end{array}\right] \tag{5b}$$

In this paper we provide a piecewise quadratic Lyapunov construction to analyze the exponential and $\mathcal{L}_2$ stability from d to y of (4), (5).

3. PIECEWISE QUADRATIC LYAPUNOV CONSTRUCTION

Following the preliminary results in Zaccarian et al. [2005], we use here piecewise quadratic Lyapunov functions arising from patching together several quadratic functions, selected in different cones of the state space, to form a unique piecewise quadratic function. The arising patched function is continuous as long as the quadratic functions coincide at the patching surfaces and is homogeneous of degree two because the patching surfaces are cone boundaries.

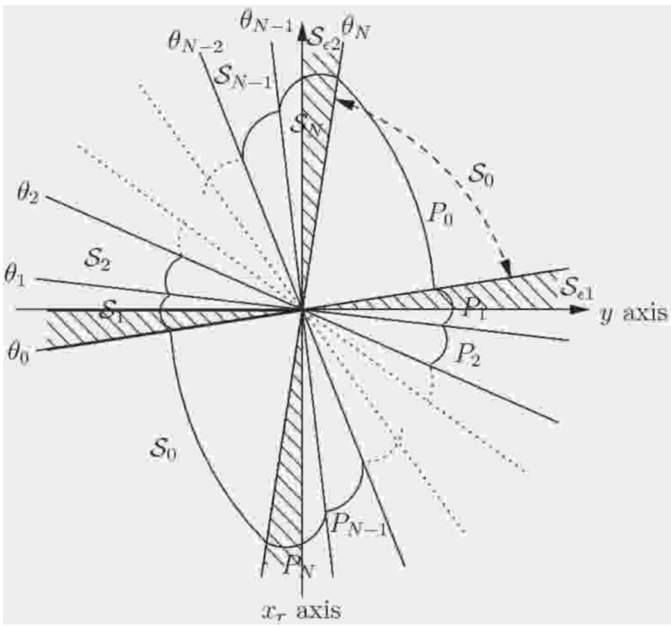

Fig. 1. The type of piecewise quadratic Lyapunov functions constructed in Theorem 1.

As shown in Figure 1, the proposed Lyapunov function is patched on suitable sectors of the (x_r, y) plane. More specifically, given $N \geq 2$ specifying the total number of sectors in the partition and a small value $\theta_\epsilon > 0$, denote the patching angles $-\theta_\epsilon = \theta_0 < 0 < \theta_1 < \cdots < \frac{\pi}{2} < \theta_N = \frac{\pi}{2} + \theta_\epsilon$ (for example, in our case studies we select $\theta_i = \frac{i}{N}\frac{\pi}{2}$, for all $i \in \{1, \ldots, N-1\}$). Then, for each $i \in \{0, \ldots, N\}$, define the patching hyperplanes generated by those angles as

$$\Theta_i = \begin{bmatrix} 0_{1\times(n-2)} & \sin(\theta_i) & \cos(\theta_i) \end{bmatrix}^T, \tag{6a}$$

and their orthogonal matrices $\Theta_{i\perp}$ (so that $\Theta_{i\perp}^T \Theta_i = 0$) as

$$\Theta_{i\perp} := \begin{bmatrix} I & 0 & 0 \\ 0 & \cos(\theta_i) & -\sin(\theta_i) \end{bmatrix}^T. \tag{6b}$$

Based on the patching hyperplanes, also define the sign indefinite matrices characterizing each sector between two patching surfaces as

$$\begin{aligned} S_0 &:= \Theta_0 \Theta_N^T + \Theta_N \Theta_0^T \\ S_i &:= -(\Theta_i \Theta_{i-1}^T + \Theta_{i-1}\Theta_i^T), \quad i = 1, \ldots, N, \\ S_{\epsilon 1} &:= \begin{bmatrix} 0_{(n-2)\times(n-2)} & 0 & 0 \\ 0 & 0 & \sin(\theta_\epsilon) \\ 0 & \sin(\theta_\epsilon) & -2\cos(\theta_\epsilon) \end{bmatrix} \\ S_{\epsilon 2} &:= \begin{bmatrix} 0_{(n-2)\times(n-2)} & 0 & 0 \\ 0 & -2\cos(\theta_\epsilon) & \sin(\theta_\epsilon) \\ 0 & \sin(\theta_\epsilon) & 0 \end{bmatrix}. \end{aligned} \tag{6c}$$

so that, with reference to Figure 1, $x \in \mathcal{S}_i$, namely in the section of state-space spanned by $\theta \in [\theta_i, \theta_{i+1}]$ whenever $x^T S_i x \geq 0$. Similarly, S_0 and $S_{\epsilon 1}$, $S_{\epsilon 2}$ characterize the cones of Figure 1 corresponding to the strict subset of the jump set and to the two shaded regions close to the horizontal and vertical axes, respectively. Based on this parametrization of the partition by way of θ_i, $i = 0, \ldots, N$, the following theorem can be stated.

Theorem 1. *Consider the reset control system (4) with the matrix selection (5). Assume (without loss of generality) that the plant (1) is in observability canonical form (so that $C_p = [0 \ \cdots \ 0 \ 1]$). Choose any $N \geq 2$ and any $\theta_\epsilon > 0$. For any selection of θ_i, $i = 0, \ldots, N$ such that $-\theta_\epsilon = \theta_0 < 0 < \theta_1 < \cdots < \frac{\pi}{2} < \theta_N = \frac{\pi}{2} + \theta_\epsilon$, define the matrices in (6) and $Z = [I_{n-2} \ 0_{(n-2)\times 2}]$.*

If the following linear matrix inequalities in the variables $P_i = P_i^T > 0$, $\tau_{Fi} \geq 0$, $i = 1, \ldots, N$, $P_0 = P_0^T > 0$, $\tau_J, \tau_{\epsilon 1}, \tau_{\epsilon 2} \geq 0$, $\gamma > 0$ are feasible:

$$\begin{bmatrix} A^T P_i + P_i A + \tau_{Fi} S_i & P_i B_d & C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{bmatrix} < 0, \quad i = 1, \ldots, N, \tag{7a}$$

$$\begin{bmatrix} Z(A^T P_0 + P_0 A)Z^T & Z P_0 B_d & Z C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{bmatrix} < 0, \tag{7b}$$

$$A_r^T P_1 A_r - P_0 + \tau_J S_0 \leq 0 \tag{7c}$$

$$A_r^T P_1 A_r - P_1 + \tau_{\epsilon 1} S_{\epsilon 1} \leq 0 \tag{7d}$$

$$A_r^T P_1 A_r - P_N + \tau_{\epsilon 2} S_{\epsilon 2} \leq 0 \tag{7e}$$

$$\Theta_{i\perp}^T (P_i - P_{i+1}) \Theta_{i\perp} = 0, \quad i = 0, \ldots, N-1, \tag{7f}$$

$$\Theta_{N\perp}^T (P_N - P_0) \Theta_{N\perp} = 0 \tag{7g}$$

then there exists a small enough $\rho^* > 0$ such that for any fixed $\rho \in (0, \rho^*)$, the FORE control system (4) is exponentially stable and has a finite $\mathcal{L}_2$ gain from d to y which is smaller than γ.

Remark 1. The piecewise quadratic Lyapunov function arising from Theorem 1 is obtained by patching together N quadratic functions (characterized by the matrices $P_1, \ldots, P_N$) defined in the (inflated) flow set and one quadratic function (characterized by the matrix P_0) in the jump set, as represented in Figure 1. Note that in most of the state space either a flow or a jump condition needs to be satisfied, except for the dashed sectors in the figure, where both the jump and flow conditions are enforced by (7). The level set sketched in Figure 1 represents a possible solution arising from the LMI constraints. In particular, conditions (7f), (7g) ensure that the Lyapunov function is continuous on the patching surfaces. Moreover, condition (7c) enforces the jump condition from the set $\mathcal{S}_0$, condition (7d) enforces it from the set $\mathcal{S}_{\epsilon 1}$ corresponding to the portion of the set $\mathcal{S}_1$ overlapping with the jump set, and similarly for condition (7e) and $\mathcal{S}_{\epsilon 2}$. Conditions (7a) ensure that the proposed Lyapunov function is a disturbance attenuation Lyapunov function everywhere except for $\mathcal{S}_0$. Finally, condition (7b) ensures that the function $x^T P_0 x$ used in the jump set is a disturbance attenuation Lyapunov function at the boundary with the flow set consisting in the origin of Figure 1, namely the set where $(x_r, y) = (0, 0)$. ◦

Remark 2. For implementation purposes, the LMI constraints (7) can be solved via an auxiliary LMI problem consisting of only strict linear matrix inequalities. In particular, it is first useful to impose $P_i > \zeta I$, $i = 0, \ldots, N$, where ζ is a positive number small enough not to make the $\mathcal{L}_2$ gain estimate too conservative (we selected $\zeta = 0.001$ for our example studies). Then a very small tolerance ε can be fixed (for our examples we used $\varepsilon = 10^{-10}$) and the non strict LMIs (7c), (7e) can be replaced by the strict LMIs

$$\begin{aligned} A_r^T P_1 A_r - P_0 + \tau_J S_0 &< 0 \\ A_r^T P_1 A_r - P_N + \tau_{\epsilon 2} S_{\epsilon 2} &< 0 \end{aligned} \tag{8}$$

while the nonstrict LMI (7d) should be broken in the following two conditions

$$-\begin{bmatrix} 0_{1\times(n-1)} & 1 \end{bmatrix} P_1 \begin{bmatrix} 0_{(n-2)\times 1} \\ 1 \\ 0 \end{bmatrix} + \tau_{\epsilon 1}\sin(\theta_\epsilon) < 0 \tag{9}$$

$$\begin{bmatrix} -\varepsilon I & \begin{bmatrix} I_{n-2} & 0_{(n-2)\times 2} \end{bmatrix} P_1 \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix} \\ \star & -\varepsilon \end{bmatrix} < 0. \tag{10}$$

Finally, the equality constraints (7f) can be replaced by the LMIs

$$\begin{bmatrix} -\varepsilon I & \Theta_{i\perp}^T (P_i - P_{i+1}) \Theta_{i\perp} \\ \star & -\varepsilon I \end{bmatrix} < 0,$$

(and similarly for (7g)). The arising solutions will satisfy the LMIs (7) up to a very small tolerance (proportional to the size of ε). This can be shown by slightly twisting the arising matrices to enforce the required continuity constraints using the tools given in Appendix A. ◦

Remark 3. Note that, different from the preliminary results in Zaccarian et al. [2005], in the numerical construction of Theorem 1, we explicitly impose that there is a (generally small) sector where the flow and the jump conditions are both satisfied (this corresponds to the dashed areas in Figure 1). Moreover, the new condition (7b) enforces a good property for $x^T P_0 x$ at the boundary between the flow and jump sets where $(x_r, y) = (0, 0)$. This extra constraint allows to guarantee good flow conditions in an inflated version of the flow set, so that the results of Nešić et al. [2008] apply. These extra conditions were not enforced in Zaccarian et al. [2005], thus calling into question the result of [Zaccarian et al., 2005, Theorem 3]. A counterexample seems difficult to find because the structure of the FORE control loops in (5) helps ruling out defective cases. Nevertheless, we illustrate in Example 1 (in the next section) the possible issues arising from omitting the overlap conditions (7d), (7e). This example does not fit into the class of reset systems described by (4), (5) but is still indicative of the potential problems with [Zaccarian et al., 2005, Theorem 3]. ◦

Remark 4. Unlike most of the existing work on FORE control systems, in the stability conditions considered here we are not incorporating any reference signal. However, by following the same reasonings as in Beker et al. [2004] it is straightforward to generalize the approach to set point regulation whenever the plant contains an integrator (namely an internal model of the reference signal). For more general results with reference signals, the reader is referred to the developments in Zaccarian et al. [2007], Loquen et al. [2008], Witvoet et al. [2007]. ◦

Remark 5. In Theorem 1, the LMI conditions involve: $\mathcal{D} = N + 4 + \frac{n(n+1)}{2}(N+1)$ decision variables and $\mathcal{L} = n(2N+5) + n_d(N+1) - 2$ lines. Using the same notation, the number of decision variables and lines corresponding to [Zaccarian et al., 2005, Theorem 3] are, respectively, $\mathcal{D} = \frac{n(n+1)}{2} + 3$ and $\mathcal{L} = 2n + n_d + 1$. When the complexity analysis of the numerical method is addressed, the numbers of lines and of variables of the considered LMIs play a crucial role. Indeed, the computation gets more complex when these numbers become larger: see, for example the complexity analysis of the interior-point method in Gahinet et al. [1995], or the complexity analysis for a primal method in [de Klerk, 2002, Theorem 5.1] or for a dual method in [de Klerk, 2002, Theorem 5.2]. LMI conditions can be solved in polynomial time for instance by specialized algorithms as in Gahinet et al. [1995], with complexity proportional to $\mathcal{D}^3\mathcal{L}$. It clearly appears that the numerical complexity grows with the number N (for given n and n_d). Of course, other LMI solvers may perform differently. ◦

4. SIMULATION EXAMPLES

Example 1. Consider the planar reset system characterized by $d = 0$, $A = \begin{bmatrix} 0.5 & -1 \\ 1 & -1 \end{bmatrix}$, and $M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. For this system, the conditions in [Zaccarian et al., 2005, Theorem 3] are feasible and the corresponding construction gives the function with the level set shown in Figure 2. However, despite the fact that the underlying linear system without resets is exponentially stable (it leads to the thin solid trajectory in the figure), when adding resets, the system's state response starting from the initial condition $x_0 = \begin{bmatrix} 10 & -10 \end{bmatrix}^T$ goes to infinity sliding along the boundary between the flow and the jump set. The corresponding trajectory is represented by the bold curve in the figure. The trajectory, after crossing the horizontal flow set boundary, exhibits the sawtoothed shape arising from flowing in the jump set until $\tau < \rho$ and then jumping back to the horizontal axis in some sort of chattering-like behavior. Note that this example does not satisfy the conditions of Theorem 1 because the diverging chattering is not allowed based on the overlapping flow and jump conditions. Note also that this example does not fit into the framework (5) which would require $A_{(2,1)} = -1$ because $C_p = 1$ by the observability canonical form of the plant. In other words, it requires changing the sign of v in the second interconnection equation of (3) and, consequently, swapping the jump and flow set inequalities in (2). This results in a quite unnatural FORE implementation where the state is reset when the FORE input and state have the same sign, rather than opposite signs as in (2). However, since the proof of [Zaccarian et al., 2005, Theorem 3] does not use any information on the structure of A in (5) (nor the proof of Theorem 1 of this paper does), this example is illustrative of the potential problems with [Zaccarian et al., 2005, Theorem 3]. ⋆

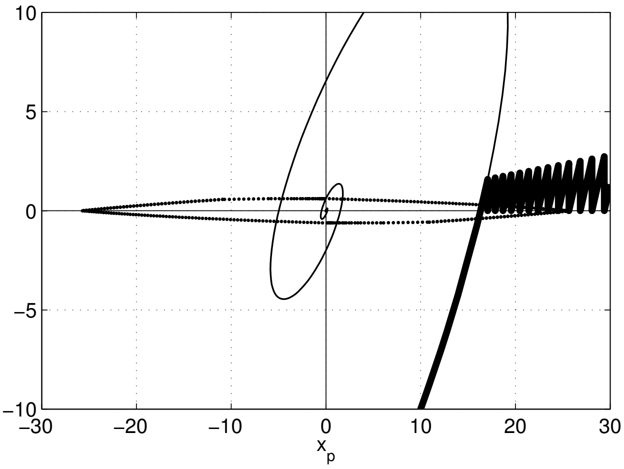

Fig. 2. Trajectories (solid) and a level set (dotted) for Example 1.

Example 2. Consider a two dimensional linear plant controlled by a FORE discussed in Nešić et al. [2005], where [5] the (incorrect) tools of Zaccarian et al. [2005] were used to establish closed-loop stability and finite $\mathcal{L}_2$ gain for different values of λ_r. Here we revisit the same example and employ Theorem 1. For this specific example, the missing conditions in [Zaccarian et al., 2005, Theorem 3] do not cause any evident difference on the stability and $\mathcal{L}_2$ gain results so the curves arising from applying [Zaccarian

[5] Note that in Nešić et al. [2005] the $\mathcal{L}_2$ gain estimate is different from here because following [Zaccarian et al., 2005, Remark 5], the extra constraints $P_i > I$, $i = 0, \ldots, N$ were added to the LMI constraints, thus making the estimate more conservative.

et al., 2005, Theorem 3] and Theorem 1 of this paper are substantially coincident (see Figure 3).

In this example, originally discussed in Hollot et al. [2001], a FORE element whose linear part is characterized by the transfer function $\frac{1}{s+1}$ (namely with $\lambda_r = -1$) controls via a negative unitary feedback a SISO plant whose transfer function is $\mathcal{P} = \frac{s+1}{s(s+0.2)}$. For this example, the control system involving the FORE is shown in Hollot et al. [2001] to behave more desirably than the linear control system. It was shown in Hollot et al. [2001] that the reset system had only about 40% overshoot of the linear closed-loop system while retaining the rise time of the linear design. This example can be further interpreted using the results of this paper. Indeed, when computing the $\mathcal{L}_2$ gain from a disturbance d acting at the plant input to the plant output y, the linear closed-loop system has an H_∞ norm around 5, while using the piecewise quadratic construction of Theorem 1 we obtain that the $\mathcal{L}_2$ gain of the reset system is smaller than 2.7.

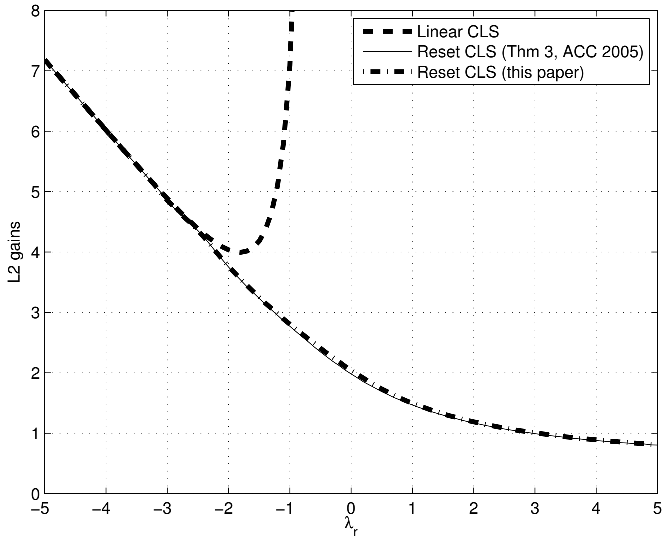

Fig. 3. $\mathcal{L}_2$ gains of linear and reset closed loops for Example 2, as a function of the pole of the FORE. The thin solid curve corresponds to the one reported in Nešić et al. [2005].

Figure 3 shows the $\mathcal{L}_2$ gains for the linear closed-loop and the reset closed-loop as a function of the pole of the FORE ($N = 20$ is chosen in Theorem 1). Similar to the previous example, the introduction of resets does not cause any increase of the $\mathcal{L}_2$ gain estimate, thus suggesting that the closed-loop performance is improved by the resets for any value of λ_r. Moreover, for positive values of λ_r (unstable FOREs) the linear closed-loop is unstable, while the reset closed-loop guarantees smaller gains. The case studied in Hollot et al. [2001] corresponds to the horizontal coordinate $\lambda_r = -1$ in Figure 3.

According to Remark 5, a growing value of N could result into unfeasible conditions due to numerical problems. This is however not the case in this example since a solution is obtained although N is increased until 10 (with associated complexity quantity $\mathcal{D}^3\mathcal{L} = 43008\ 10^3$). ⋆

5. CONCLUSIONS

In this paper we proposed a piecewise quadratic Lyapunov construction in terms of LMIs for exponential stability and $\mathcal{L}_2$ stability analysis of linear control systems with First Order Reset Elements. In particular, we revisited the previous results in Zaccarian et al. [2005] providing new conditions which address a subtle issue overlooked in those preliminary results. The new conditions have been tested on two examples, one illustrating the problems associated with the results in Zaccarian et al. [2005] and the second one illustrating that in some cases the conditions given here and those of Zaccarian et al. [2005] lead to the same results.

Appendix A. PATCHING DISCONTINUOUS LEVEL SETS

In this appendix we give some general results on how to modify the level sets of two quadratic Lyapunov functions to allow for a continuous patched Lyapunov function (which is piecewise quadratic), where the patching zone is a hyperplane. In particular, we show that the modification that needs to be enforced on one of the two functions is bounded in size by the mismatch between the two functions on the patching hyperplane.

Proposition 1. *(Patching on one hyperplane) Consider two functions* $V_1(x) = x^T P_1 x$ *and* $V_2(x) = x^T P_2 x$*, with* $x \in \mathbb{R}^n$*, and a patching hyperplane* $\mathcal{H}_\theta := \{x : x_1 \sin\theta - x_2 \cos\theta = 0\}$*, for* $\theta \in [0, \pi)$*. This hyperplane can be also defined as* $\mathcal{H}_\theta = \{x : \; h^T x = 0\}$*, where* $h = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & \cdots & 0 \end{bmatrix}^T$*. Define the following worst case mismatch between* V_1 *and* V_2 *at the patching hyperplane:*

$$\delta := \sup_{x \in \mathcal{H}_\theta \setminus \{0\}} \frac{x^T (P_1 - P_2) x}{x^T x}. \tag{A.1}$$

Then there exists a matrix ΔP *such that*

$$|\Delta P| \le \delta \tag{A.2}$$

$$x^T (P_1 - P_2 + \Delta P) x = 0, \; \forall x \in \mathcal{H}_\theta, \tag{A.3}$$

where $|\Delta P|$ *denotes the induced Euclidean norm of the matrix* ΔP*. Equivalently, the patching between* $V_1 = x^T P_1 x$ *and* $\widetilde{V}_2 = x^T (P_2 - \Delta P) x$ *(alternatively,* $V_2 = x^T P_2 x$ *and* $\widetilde{V}_1 = x^T (P_1 + \Delta P) x$*) is continuous on the hyperplane* $\mathcal{H}_\theta$*.*

Example 3. In this example we patch together the two functions with $P_1 = \left[\begin{smallmatrix} 2.5 & 2.5 \\ 2.5 & 7.5 \end{smallmatrix}\right]$ and $P_2 = \left[\begin{smallmatrix} 4 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ on the hyperplane with $\theta = \pi/4$. We get $\delta = 3.5$ and $\Delta P = -\left[\begin{smallmatrix} 1.75 & 1.75 \\ 1.75 & 1.75 \end{smallmatrix}\right]$, with $|\Delta P| = 3.5$.

The level sets of the patched functions are represented in Figure A.1, where the first column represents the functions before patching, the second column represents the functions after modifying P_1 and the third column represents the functions after modifying P_2. ⋆

Proposition 2. *(Patching on two hyperplanes) Consider two functions* $V_1(x) = x^T P_1 x$ *and* $V_2(x) = x^T P_2 x$*, with* $x \in \mathbb{R}^n$*, and two patching hyperplanes* $\mathcal{H}_{\theta_1} := \{x : x_1 \sin\theta_1 - x_2 \cos\theta_1 = 0\}$*, for* $\theta_1 \in [0, \pi)$ *and* $\mathcal{H}_{\theta_2} := \{x : x_1 \sin\theta_2 - x_2 \cos\theta_2 = 0\}$*, for* $\theta_2 \in [0, \pi)$*,* $\theta_2 \neq \theta_1$ *(namely, the hyperplanes are not coincident). These hyperplanes*

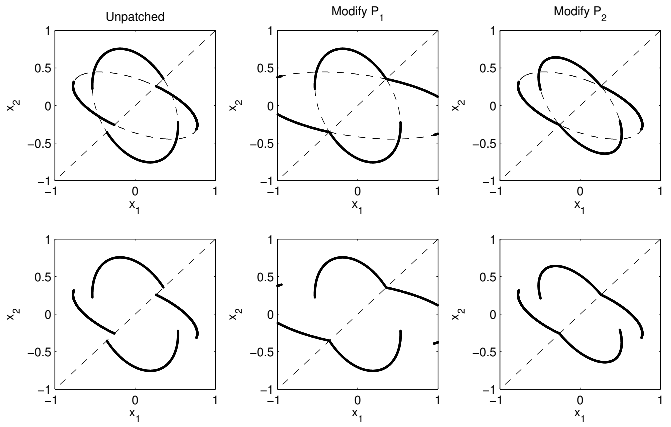

Fig. A.1. The level sets for Example 3. The patching hyperplane is the dashed line.

can be also defined as $\mathcal{H}_{\theta_1} = \{x : h_1^T x = 0\}$, *where* $h_1 = \begin{bmatrix}\sin\theta_1 & -\cos\theta_1 & 0 & \cdots & 0\end{bmatrix}^T$, $\mathcal{H}_{\theta_2} = \{x : h_2^T x = 0\}$, *where* $h_2 = \begin{bmatrix}\sin\theta_2 & -\cos\theta_2 & 0 & \cdots & 0\end{bmatrix}^T$. *Define the following worst case mismatch between* V_1 *and* V_2 *at the patching hyperplanes:*

$$\delta := \sup_{x\in\mathcal{H}_{\theta_1}\cup\mathcal{H}_{\theta_2}\setminus\{0\}} \frac{x^T(P_1-P_2)x}{x^Tx}. \tag{A.4}$$

Then there exists matrix ΔP *such that*

$$|\Delta P| \le 3\delta\,(1-|\cos(\theta_1-\theta_2)|)^{-1} \tag{A.5}$$

$$x^T(P_1-P_2+\Delta P)x = 0,\ \forall x\in\mathcal{H}_{\theta_1}\cup\mathcal{H}_{\theta_2}, \tag{A.6}$$

where $|\Delta P|$ *denotes the induced Euclidean norm of the matrix* ΔP. *Equivalently, the patching between* $V_1 = x^TP_1x$ *and* $\widetilde{V}_2 = x^T(P_2-\Delta P)x$ *(alternatively,* $V_2 = x^TP_2x$ *and* $\widetilde{V}_1 = x^T(P_1+\Delta P)x$*) is continuous on the hyperplanes* $\mathcal{H}_{\theta_1}$ *and* $\mathcal{H}_{\theta_2}$. *(As a consequence, the corresponding piecewise quadratic function is Lipschitz).*

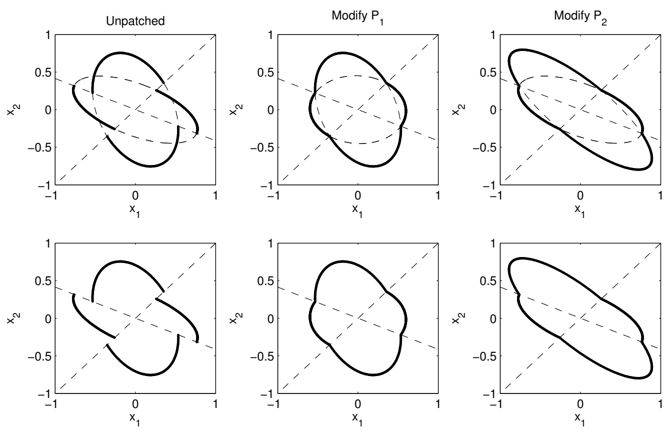

Fig. A.2. The level sets for Example 4. The patching hyperplanes are shown by dashed lines.

Example 4. In this example we patch together again the two functions with $P_1 = \left[\begin{smallmatrix}2.5 & 2.5\\ 2.5 & 7.5\end{smallmatrix}\right]$ and $P_2 = \left[\begin{smallmatrix}4 & 1\\ 1 & 2\end{smallmatrix}\right]$ on the hyperplanes with $\theta_1 = \pi/4$ and $\theta_2 = \frac{3.5}{4}\pi$. We get $\delta = 3.5$, so that the bound in (A.5) is $3\delta(1-|\cos(\theta_1-\theta_2)|)^{-1} \approx 17$. We get for this example $\Delta P = \left[\begin{smallmatrix}0.299 & -2.3492\\ -2.3492 & -2.6005\end{smallmatrix}\right]$, with $|\Delta P| = 3.9113$.

The level sets of the patched functions are represented in Figure A.2 where the first column represents the functions before patching, the second column represents the functions after modifying P_1 and the third column represents the functions after modifying P_2. ⋆

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