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Title:
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Date:
2014

Citation:
Taringoo, F., Nesic, D., Tan, Y. & DOWER, P. M. (2014). Extremum seeking control for nonlinear systems on compact Riemannian manifolds. Proceedings of the IEEE Conference on Decision and Control, 2015-February, (February), pp.2667-2672. IEEE Press. https://doi.org/10.1109/CDC.2014.7039797.

Persistent Link:
https://hdl.handle.net/11343/299848

53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

# Extremum Seeking Control for Nonlinear Systems on Compact Riemannian Manifolds

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*Abstract*— **This paper formulates the extremum seeking control problem for nonlinear dynamical systems which evolve on Riemannian manifolds and presents stability results for a class of numerical algorithms defined in this context. The results are obtained based upon an extension of extremum seeking algorithms in Euclidean spaces and a generalization of Lyapunov stability theory for dynamical systems defined on Rimannian manifolds. We employ local properties of Lyapunov functions to extend the singular perturbation analysis on Riemannian manifolds. Consequently, the results of the singular perturbation on manifolds are used to obtain the convergence of extremum seeking algorithms for dynamical systems on Riemannian manifolds.**

## I. INTRODUCTION

Extremum seeking is a class of on-line or real-time optimization methods for optimization of the steady-state behavior of dynamical systems. In this setting, the plant model is assumed to be uncertain and unknown, see [6], [7], [12], [14], [24]. This topic has been considered within two different classes of algorithms referred to as black box algorithms and gray box algorithms, see [18]–[20], [27]. Black box algorithms do not require apriori knowledge of the dynamical equations. However, the gray box approach requires partial information about the dynamical equations, see [18]. A recent geometric class of extremum seeking algorithms which is for optimization of static cost functions is presented in [4].

The core mathematical tools for the stability analysis of extremum seeking algorithms are averaging techniques and singular perturbation methods, see [14]. As shown in [14], these techniques can be used to address the stability of extremum seeking algorithms via analyzing dynamical systems with several time scales where the closed loop of a dynamical system and a numerical algorithm gives rise to a system with slow and fast variables. Hence, singular perturbation methods provide a suitable framework within which the stability results of the closed loop system for extremum seeking control systems can be obtained, see [14].

Traditionally, Euclidean spaces have been considered as the underlying state spaces of dynamical systems, see [11], [22]. Most of the research work related to nonlinear systems and their analyses has been carried out for dynamical systems on Euclidean spaces. The vector space properties of Euclidean spaces simplify the stability analysis and enable us to analyze the stability of a generic equilibrium at the origin, see [11]. In a more general framework, the underlying Euclidean spaces can be replaced by Riemannian manifolds as configuration spaces for many dynamical systems. Two link manipulators, rotating bodies, rolling disks, etc. are examples of such systems found in many mechanical settings, see [3].

In this paper we consider dynamical systems with underlying Riemannian manifolds as their state configuration spaces and extend the extremum seeking framework of dynamical system in $\mathbb{R}^n$ to systems evolve on Riemannian manifolds. This necessitates a generalization of the Lyapunov stability theory for systems defined on Riemannian manifolds and consequently a generalization of the singular perturbation analysis on Riemannian manifolds. To this end, we present the converse Lyapunov results for the existence of Lyapunov functions for locally exponentially dynamical systems on Riemannian manifolds in a normal neighborhood of an equilibrium as given in [25]. We employ the converse Lyapunov results to obtain the boundedness of solutions of slowly parameter varying vector fields on Riemannian manifolds. Then a version of singular perturbation analysis for dynamical systems on compact Riemanian manifolds is presented. Finally we address the extremum seeking problem on connected compact Riemannian manifolds by employing the obtained singular perturbation analysis.

In terms of exposition, Section II presents some mathematical preliminaries needed for the analyses of the paper. Section III presents the extremum seeking problem for nonlinear dynamical systems on compact Riemannian manifolds. In the appendix, we present the mathematical results needed for the stability analysis of the extremum seeking control problem defined in Section III. The results concerning the singular perturbation analysis on compact Riemannian manifolds are obtained by employing the stability analysis on Riemannian manifolds and applying the local properties of Lyapunov functions in normal neighborhoods of equilibriums.

## II. PRELIMINARIES AND PROBLEM FORMULATION

In this section we provide the differential geometric material which is necessary for the analyses presented in the rest of the paper. In this notation, we define some of the frequently used symbols of this paper in Table I.

*Definition 1:* (see [16], Chapter 3) A Riemannian manifold $(M, g)$ is a differentiable manifold $M$ together with a Riemannian metric $g$, where $g$ is defined for each $x \in M$ via an inner product $g_x : T_xM \times T_xM \to \mathbb{R}$ on the tangent space $T_xM$ (to $M$ at $x$), such that the function defined

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This work was supported by the Australian Research Council Discovery Project DP120101144.

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TABLE I

SYMBOLS AND THEIR DESCRIPTIONS

| Symbol | Description |
|---|---|
| $M$ | Riemannian manifold |
| $\mathfrak{X}(M \times \mathbb{R})$ | space of smooth time varying vector fields on $M$ |
| $T_xM$ | tangent space at $x \in M$ |
| $TM$ | tangent bundle of $M$ |
| $\frac{\partial}{\partial x_i}$ | basis tangent vectors at $x \in M$ |
| $f(x,t)$ | vector fields on $M$ |
| $\|\| \cdot \|\|_g$ | Riemannian norm |
| $\|\| \cdot \|\|_e$ | Euclidean norm |
| $g_x(.,.)$ | Riemannian metric on $M$ |
| $d(.,.)$ | Riemannian distance on $M$ |
| $\nabla$ | Levi-Civita Connection on $M$ |
| $\Phi_f$ | flow associated with $f$ |
| $TF$ | push-forward of $F$ |
| $C^\infty(M, \mathbb{R})$ | Space of smooth functions on $M \times \mathbb{R}$ |

by $x \mapsto g_x(X(x), Y(x))$ is smooth for any vector fields $X, Y \in \mathfrak{X}(M)$. In addition,

(i) $(M, g)$ is $n$-dimensional if $M$ is $n$-dimensional;

(ii) $(M, g)$ is connected if for any $x, y \in M$, there exists a piecewise smooth curve connecting them.

■

(Note that in the special case where $M \doteq \mathbb{R}^n$, the Riemannian metric $g$ is defined everywhere by $g_x = \sum_{i=1}^n dx_i \otimes dx_i$, where $\otimes$ is the tensor product on $T_x^*M \times T_x^*M$, see [16].) In this paper $||\cdot||_g$ ($||\cdot||_e$) denotes the Riemannian (Euclidean) norm, i.e. $||X||_g = \sqrt{g(X,X)}$. As formalized in Definition 1, connected Riemannian manifolds possess the property that any pair of points $x, y \in M$ can be connected via a path $\gamma \in \mathscr{P}(x,y)$, where

$$\mathscr{P}(x,y) \doteq \left\{ \gamma : [a,b] \to M \;\middle|\; \begin{array}{l} \gamma \text{ piecewise smooth,} \\ \gamma(a) = x\,,\ \gamma(b) = y \end{array} \right\} \tag{1}$$

*Theorem 1:* ([15], Page 94) Suppose $(M, g)$ is an $n$-dimensional connected Riemannian manifold. Then, for any $x, y \in M$, there exists a piecewise smooth path $\gamma \in \mathscr{P}(x,y)$ that connects $x$ to $y$. ■

The existence of connecting paths (via Theorem 1) between pairs of elements of an $n$-dimensional connected Riemannian manifold $(M, g)$ facilitates the definition of a corresponding Riemannian distance. In particular, the Riemannian distance $d : M \times M \to \mathbb{R}$ is defined as the infimal path length between any two elements of $M$, with

$$d(x,y) \doteq \inf_{\gamma \in \mathscr{P}(x,y)} \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt\,. \tag{2}$$

(Note that in the special case where $M \doteq \mathbb{R}^n$, the Riemannian distance (2) simplifies to $d(x,y) = \|x - y\|$.)

Using the definition of Riemannian distance $d$ of (2), it may be shown that $(M, d)$ defines a metric space.

*Theorem 2:* ([15], Page 94) Any $n$-dimensional connected Riemannian manifold $(M, g)$ defines a metric space $(M, d)$ via the Riemannian distance $d$ of (2). Furthermore, the induced topology of $(M, d)$ is the same as the manifold topology of $(M, g)$. ■

*Definition 2:* For a given smooth mapping $F : M \to N$ from manifold $M$ to manifold $N$ the pushforward $TF$ is defined as the following linear map:

$$T_xF : T_xM \to T_{F(x)}N. \tag{3}$$

### *A. Dynamical systems on Riemannian manifolds*

This paper focuses on dynamical systems governed by differential equations on a connected $n$ dimensional Riemannian manifold $M$. Locally these differential equations are expressed by (see [16])

$$\begin{aligned} &\dot{x}(t) = f(x(t), t), \\ &f(x(t), t) \in T_{x(t)}M, \;\; x(0) = x_0 \in M, t \in [t_0, t_f] \end{aligned} \tag{4}$$

The time dependent flow associated with a differentiable time dependent vector field $f$ is a map $\Phi_f$ satisfying :

$$\begin{aligned} &\Phi_f : [t_0, t_f] \times [t_0, t_f] \times M \to M, \\ &(t_0, s, x) \hookrightarrow \Phi_f(s, t_0, x) \in M, \end{aligned} \tag{5}$$

and

$$\frac{d\Phi_f(s, t_0, x)}{ds}|_{s=t} = f(\Phi_f(t, t_0, x), t). \tag{6}$$

One may show, for a smooth vector field $f$, the integral flow $\Phi_f(s, t_0, .) : M \to M$ is a local diffeomorphism , see [16]. Here we assume that the vector field $f$ is smooth and *complete*, i.e. $\Phi_f$ exists for all $t \in (t_0, \infty)$.

### *B. Geodesic Curves*

As known (see [10]), geodesics are defined as length minimizing curves on Riemannian manifolds which satisfy

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0, \tag{7}$$

where $\gamma(\cdot)$ is a geodesic curve on $(M, g)$. The solution of the Euler-Lagrange variational problem associated with the length minimizing problem shows that all the geodesics on an $n$ dimensional Riemannian manifold $(M, g)$ must satisfy the following system of ordinary differential equations:

$$\ddot{\gamma}_i(s) + \sum_{j,k=1}^{n} \Gamma^i_{j,k} \dot{\gamma}_j(s) \dot{\gamma}_k(s) = 0, \quad i = 1, ..., n, \tag{8}$$

where $\Gamma^i_{j,k} = \frac{1}{2} \sum_{l=1}^n g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l})$, $g_{jl,k} = \frac{\partial g_{jl}}{\partial x_k}$, and all the indices $i, j, k, l$ run from 1 up to $n = dim(M)$ and $[g^{ij}] \doteq [g_{ij}]^{-1}$. Note that $g_{ij}$ is the $(i, j)$ entity of the matrix $g$.

*Definition 3:* ([15]) The restricted exponential map is defined by

$$\exp_x : T_xM \to M, \;\; \exp_x(v) = \gamma_v(1), v \in T_xM, \tag{9}$$

where $\gamma_v(1)$ is the geodesic initiating from $x$ with the velocity $v$ up to $t = 1$. ■

For brevity, in this paper we refer the restricted exponential maps as exponential maps. For $x \in M$, consider a $\delta$ ball in $T_xM$ such that $B_\delta(0) \doteq \{v \in T_xM |\ ||v||_g < \delta\}$. Then the geodesic ball is defined as follows.

*Definition 4:* ([15]) In a neighborhood of $x \in M$ where $\exp_x$ is a local diffeomorphism (this neighborhood always exists by Lemma 1 below), a geodesic ball of radius $\delta > 0$ is denoted by $\exp_x(B_\delta(0)) \subset M$. Also we call $\exp_x(\overline{B}_\delta(0))$ a closed geodesic ball of radius $\delta$. ■

*Lemma 1:* ([15]) For any $x \in M$ there exists a neighborhood $B_\delta(0)$ in $T_xM$ on which $\exp_x$ is a diffeomorphism onto $\exp_x(B_\delta(0)) \subset M$. ■

*Definition 5:* For a vector space $V$, a *star-shaped neighborhood* of $0 \in V$ is any open set $U$ such that if $u \in U$ then $\alpha u \in U, \alpha \in [0, 1]$. ■

*Definition 6:* ([15]) A normal neighborhood around $x \in M$ is any open neighborhood of $x$ which is a diffeomorphic image of a star shaped neighborhood of $0 \in T_xM$ under $\exp_x$ map. ■

*Definition 7:* The injectivity radius of $M$ is

$$i(M) \doteq \inf_{x \in M} i(x), \tag{10}$$

where

$$i(x) \doteq \sup\{r \in \mathbb{R}_{\geq 0} \,|\, \exp_x \text{is differmorphic onto} \exp_x B_r(0)\}. \tag{11}$$

■

*Definition 8:* The metric ball with respect to $d$ on $(M, g)$ is defined by

$$B(x, r) \doteq \{y \in M \;\; s.t. \;\; d(x, y) < r\}. \tag{12}$$

■

The following lemma displays a relationship between normal neighborhoods and metric balls defined before on $M$.

*Lemma 2:* ([21]) If $\exp_x(\cdot), \;\; x \in M$ is a diffeomorphism on $B_\epsilon(0) \subset T_xM, \;\; \epsilon \in \mathbb{R}_{>0}$, and $B(x, r) \subset \exp_x B_\epsilon(0)$, then

$$\exp_x B_r(0) = B(x, r). \tag{13}$$

■

We note that $B_\epsilon(0)$ is the metric ball of radius $\epsilon$ with respect to the Riemannian metric $g$ in $T_xM$.

The following lemma bounds the injectivity radius of compact Riemannian manifolds, see Definition 7.

*Lemma 3:* ([13]) The injectivity radius $i(x), x \in M$ is continuous with respect to $x$ and is bounded from below for compact Riemannian manifolds. ■

By the results of [21], Corollary 5.3, and Lemma 3, in the case $i(M) > 0$, for any $r \leq i(M)$, such that $B(x, r) \subset \exp_x B_{i(M)}(0)$, we have

$$B(x, r) = \exp_x B_r(0). \tag{14}$$

## III. EXTREMUM SEEKING ON RIEMANNIAN MANIFOLDS

In this section we extend the results presented in [14] to extremum seeking control of dynamical systems on compact Riemanian manifolds. In order to obtain the stability of the extremum seeking closed loop, we need to extend the Lyapunov stability results for dynamical systems on Riemannian manifolds and characterize the local properties of Lyapunov functions with respect to the Riemannian distance function. These stability results will be employed to address the closeness of solutions for a class of singular perturbation problems on Riemannian manifolds, see Appendix A. We use the obtained results to conclude closeness of solutions for the extremum seeking problem defined on compact Riemannian manifolds. The main proof is an extension of the results presented in [14].

Consider a nonlinear system on a Riemannian manifold $M$ together with an extremum seeking control loop, see Figure 1. In this setting, the controlled dynamical equations are given by

$$\begin{aligned} &\dot{x} = f(x, u), \;\; x(t) \in M, u(t) \in \mathbb{R}^m, \\ &y(t) = h(x(t)) \in \mathbb{R}, \end{aligned} \tag{15}$$

where $y$ is the output of the dynamical equations and $h$ and $f$ are both smooth. The goal is to find an input $u$ which maximizes or minimizes the output $y$ at the equilibrium corresponding to $u$. We use the following assumptions borrowed from [14] for our analysis.

*Assumption 1*

- $u$ has a feedback characterization $u = \alpha(x, \theta)$, where $\theta$ is a parameter determined by the controller, see Figure 1.
- The equilibrium of $f(x, \alpha(x, \theta))$ is given by an unknown function $\bar{x} = l(\theta)$.
- $\alpha$ and $l$ are both smooth i.e. $\alpha \in C^\infty(M, \mathbb{R})$ and $l \in C^\infty(\mathbb{R})$.
- The equilibrium $l(\theta)$ is locally exponentially stable uniformly with respect to $\theta$ (in the sense of Riemannian distance function). ■

The dynamical equations for the extremum seeking problem on manifolds are locally given as follows, see [14] and Figure 1:

$$\begin{cases} \dot{x} = f(x, \alpha(x, \hat{\theta} + a\sin(\omega t))) \\ \dot{\hat{\theta}} = k\zeta \\ \dot{\zeta} = -\omega_l\zeta + \omega_h(h(x) - \eta)a\sin(\omega t) \\ \dot{\eta} = -\omega_h\eta + \omega_h h(x) \end{cases} \tag{16}$$

Following [14], we change the coordinates as follows:

$$\tau = \omega t, \;\; \tilde{\theta} = \hat{\theta} - \theta^*, \tag{17}$$

where $\theta^*$ is the optimal parameter. Then (16) is written as

$$\begin{cases} \omega\frac{dx}{d\tau} = f(x, \alpha(x, \theta^* + \tilde{\theta} + a\sin(\tau))) \\ \frac{d\tilde{\theta}}{d\tau} = \frac{k}{\omega}\zeta \\ \frac{d\zeta}{d\tau} = \frac{-\omega_l\zeta + \omega_h(h(x) - \eta)a\sin(\tau)}{\omega} \\ \frac{d\eta}{d\tau} = \frac{-\omega_h\eta + \omega_h h(x)}{\omega} \end{cases} \tag{18}$$

Following the standard assumptions in [14] for selecting the controller parameters to achieve the closed loop stability, we assume

$$\begin{aligned} \omega_h &= \omega\omega_H = \omega\delta\acute{\omega}_H = O(\omega\delta), \\ \omega_l &= \omega\omega_L = \omega\delta\acute{\omega}_L = O(\omega\delta), \\ k &= \omega K = \omega\delta\acute{K} = O(\omega\delta), \end{aligned} \tag{19}$$

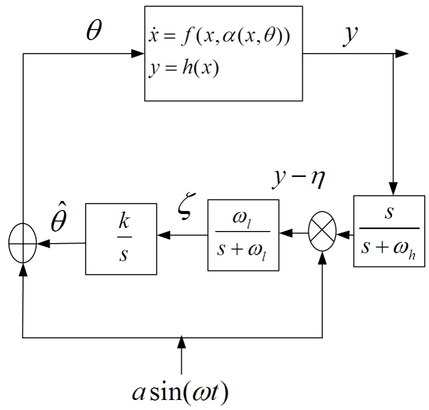

Fig. 1. Extremum seeking closed loop system (16), see also [14]

where $\omega$ and $\delta$ are small positive constants and $\acute{\omega}_H, \acute{\omega}_L$ and $\acute{K}$ are positive constants bounded from below. Then (18) is written as

$$\begin{cases} \omega\frac{dx}{d\tau} = f\left(x, \alpha(x, \theta^* + \tilde{\theta} + a\sin(\tau))\right), \\ \frac{d\tilde{\theta}}{d\tau} = \delta\acute{K}\zeta, \\ \frac{d\zeta}{d\tau} = \delta\left(-\acute{\omega}_L\zeta + \acute{\omega}_H(h(x) - \eta)a\sin(\tau)\right), \\ \frac{d\eta}{d\tau} = \delta\left(-\acute{\omega}_H\eta + \acute{\omega}_H h(x)\right). \end{cases} \tag{20}$$

Note that $x \in M$ and $(\theta, \zeta, \eta) \in \mathbb{R}^3$. Hence, (20) is in a form of singular perturbation systems with different time scales, see [11].

By freezing the fast variable $x$ on $(M, g)$ to its equlibrium $l(\theta)$, we obtain the reduced dynamical system on $\mathbb{R}^3$ as

$$\begin{cases} \frac{d\tilde{\theta}}{d\tau} = \delta\acute{K}\zeta, \\ \frac{d\zeta}{d\tau} = \delta\Big( - \acute{\omega}_L\zeta + \acute{\omega}_H(h\left(l(\theta^* + \tilde{\theta} + a\sin(\tau))\right) - \\ \qquad \eta)a\sin(\tau)\Big), \\ \frac{d\eta}{d\tau} = \delta\left(-\acute{\omega}_H\eta + \acute{\omega}_H h\left(l(\theta^* + \tilde{\theta} + a\sin(\tau))\right)\right). \end{cases} \tag{21}$$

The averaged dynamical system is then given as

$$\begin{cases} \frac{d\tilde{\theta}}{d\tau} = \delta\acute{K}\zeta, \\ \frac{d\zeta}{d\tau} = \delta\Big( - \acute{\omega}_L\zeta + \frac{\acute{\omega}_H}{2\pi}\int_0^{2\pi}(h\left(l(\theta^* + \tilde{\theta} + a\sin(\tau))\right) \\ \qquad -\eta)a\sin(\tau)d\tau\Big), \\ \frac{d\eta}{d\tau} = \delta\left(-\acute{\omega}_H\eta + \frac{\acute{\omega}_H}{2\pi}\int_0^{2\pi} h\left(l(\theta^* + \tilde{\theta} + \right.\right. \\ \qquad a\sin(\tau)))\,d\tau). \end{cases}, \tag{22}$$

where for a generic $T$ periodic time varying vector field $f$, the averaged vector field is given by

$$\hat{f}(x) = \frac{1}{T}\int_0^T f(x, \tau)du. \tag{23}$$

Since $z \doteq (\tilde{\theta}, \zeta, \eta) \in \mathbb{R}^3$, we employ the results of [11] to show the existence of an exponentially stable periodic trajectory of (21) in a sufficiently small neighborhood of the equilibrium of the averaged system of (22). A general averaging problem for nonlinear dynamical systems evolving on Riemannian manifolds is addressed in [26].

*Lemma 4:* ([11], [14]) Consider the dynamical system (21) and assume the output equilibrium map $h(l(\theta))$ has a maximizer (minimizer) $\theta^*$ such that $\frac{dh(l(\theta))}{d\theta}|_{\theta=\theta^*} = 0$ and $\frac{d^2h(l(\theta))}{d\theta^2}|_{\theta=\theta^*} < 0$. Then there exists $(\delta^*, a^*)$ such that for all $0 < \delta < \delta^*$ and $0 < a < a^*$, (21) has an exponentially stable periodic solution $z^{2\pi}(\tau) \doteq (\tilde{\theta}^{2\pi}(\tau), \zeta^{2\pi}(\tau), \eta^{2\pi}(\tau))$ which satisfies

$$||z^{2\pi}(\tau) - z^*||_e = O(\delta + a^3), \ \ \forall\tau \in \mathbb{R}_{>0}, \tag{24}$$

where $z^* \doteq (\tilde{\theta}^*, \zeta^*, \eta^*)$ is an equilibrium of (22). ■

Now let us consider an extremum seeking control problem on a compact Riemannian manifold $(M, g)$, i.e., consider (16) where $x \in M, f(x, \alpha(x, \hat{\theta} + a\sin(\omega t))) \in T_xM$. Following the singularly perturbed dynamical systems introduced in Appendix A, the equilibrium map $x = l(\theta^* + \tilde{\theta} + a\sin(\omega t))$ gives the solution $x = h(z, t)$ in (38). Since $z \in \mathbb{R}^3$ then we employ the coordinate changes $(\acute{\theta}, \acute{\zeta}, \acute{\eta}) \doteq \acute{z} = z - z^{2\pi}$, then we have

$$\begin{cases} \omega\frac{dx}{d\tau} = F(x, \acute{z}, \tau), \\ \frac{d\acute{z}}{d\tau} = \delta\, G(x, \acute{z}, \tau), \end{cases} \tag{25}$$

where

$$F(x, \acute{z}, \tau) = f\left(x, \alpha(x, \theta^* + \acute{\theta}(\tau) + \theta^{2\pi}(\tau) + a\sin(\tau))\right),$$

and

$$G(x, \acute{z}, \tau) = \\ \begin{pmatrix} \acute{K}\acute{\zeta} \\ -\acute{\omega}_L\acute{\zeta} + \acute{\omega}_H\Big(h(x) - \hat{\eta}\Big)a\sin(\tau) \\ -\acute{\omega}_H\hat{\eta} + \acute{\omega}_H h(x) \end{pmatrix} - \\ \begin{pmatrix} 0 \\ \acute{\omega}_H\Big(h\left(l(\theta^* + \acute{\theta} + \theta^{2\pi} + a\sin(\tau))\right)\Big)a\sin(\tau) \\ \acute{\omega}_H h\left(l(\theta^* + \acute{\theta} + \theta^{2\pi} + a\sin(\tau))\right) \end{pmatrix}. \tag{26}$$

By the results of Lemma 4, variable $\acute{z}$ exponentially converges to the origin in the reduced system $\frac{d\hat{z}}{d\tau} = \delta G(x, \hat{z}, \tau)$. By the construction above, $G$ is a smooth function of $x, \acute{z}$ and $\tau$ and since $M$ is compact, on any compact set $\hat{D}_{cmp} \subset \mathbb{R}^3$, we have $||G(x, \acute{z}, \tau)|| \le c_0, \ x \in M, \acute{z} \in \hat{D}_{cmp}, \tau \in \mathbb{R}_{\ge 0}$, for some $c_0 > 0$. Note that $G$ depends upon $\tau$ via $\sin(\tau)$, hence, its norm remains bounded for all $\tau \in \mathbb{R}_{\ge 0}$. The boundary layer model for the singular perturbation analysis of this problem is given in the time scale $t$ as follows:

$$\frac{dx_b}{dt} = F(x_b, \acute{z}, \tau) = f\left(x_b, \alpha(x_b, \theta)\right), \tag{27}$$

where by Assumption 1, $x_b$ converges to $l(\theta)$ exponentially fast. We note that for the boundary layer model above, $\acute{z}$ and $\tau$ are considered as parameters. The following theorem gives the convergence results for the extremum seeking problem defined above on compact Riemannian manifolds.

*Theorem 3:* Consider the extremum seeking dynamics (25), satisfying Assumption 1, on a compact $n$ dimensional Riemannian manifold $M$, where $F$ and $G$ are locally given by (26) and (26) respectively. Then for any $\tau_1 > \tau_0$, if $d\left(x(\tau_0), l(\theta^* + \tilde{\theta}(\tau_0) + a\sin\tau_0)\right) \leq \delta^*$ for a sufficiently small $\delta^*$, then there exist small enough $\omega_1 > 0$, $\delta_1 > 0$ and $a_1 > 0$, such that

$$\begin{aligned} & d\left(\Phi_{F(x,\acute{z},\tau)}(\tau,\tau_0,x_0), l(\acute{z}(\tau))\right) = O(\omega), \tau \in [\tau_1,\infty) \\ & \limsup_{\tau\to\infty} ||\acute{z}(\tau)||_e = O(\omega), \;\; \omega \in (0,\omega_1), \\ & \delta \in (0,\delta_1), a \in (0,a_1). \end{aligned} \tag{28}$$

*Sketch of the proof*: The proof is long and technical and we do not include it in this version of the paper. It follows from the results of Lemma 4 and Theorem 4 where the variable $\hat{z}$ lies on a compact subset of $\mathbb{R}^3$ containing the origin, see Appendix and A. ■

The results of Theorem 3 and Lemma 4 together imply that for sufficiently small $\delta, a$ and $\omega$, we have

$$\limsup_{t\to\infty} ||z(t) - z^*||_e = O(\delta + a + \omega). \tag{29}$$

## IV. CONCLUSION

In this paper we presented the stability results for the extremum seeking control of nonlinear dynamical systems on compact Riemannian manifolds. The results are based upon the construction of Lyapunov functions (see [25]) for dynamical systems on Riemannian manifolds and extending the singular perturbation analysis on Riemannian manifolds, see Appendix A. The vector space properties of tangent spaces enabled us to employ the standard results of Converse Lyapunov Theorem to construct Lyapunov functions on Riemannian manifolds. These results are used to obtain the closeness of solutions for a class of singularly perturbed dynamical systems where fast variables evolve on Riemannian manifolds and slow variables are elements of Euclidean spaces.

## APPENDIX

### A. Singularly Perturbed Dynamical Systems on Compact Riemannian Manifolds

Let us introduce the following system of differential equations in the form of singularly perturbed systems on differentiable manifolds.

$$\begin{aligned} \epsilon\dot{x}(t) &= f(x,z,t), && x \in M_1, f(x,z,t) \in T_xM_1 \\ \dot{z}(t) &= e(x,z,t), && z \in M_2, e(x,z,t) \in T_zM_2, \end{aligned} \tag{30}$$

where $M_1$ and $M_2$ are $n$ and $m$ dimensional Riemannian manifolds respectively and $\epsilon \in [0,\epsilon_1],\ \epsilon_1 \in \mathbb{R}_{>0}$. Since we need to apply the stability results of singularly perturbed systems to the framework of extremum seeking control, we restrict our analysis to the case where $M_2 = \mathbb{R}^m$. In order to analyze the singular perturbation, defined above, on $M$, we need to extend the notion of stability for dynamical systems evolving on Riemannian manifolds. This problem has been addressed in [1], [3], [17] in a geometric framework.

*Definition 9:* For the time varying dynamical system $\dot{x} = f(x,t),\ \ f \in \mathfrak{X}(M\times\mathbb{R}),\ \bar{x} \in M$ is an equilibrium if

$$\Phi_f(t,t_0,\bar{x}) = \bar{x}, \;\; t \in [t_0,\infty), \tag{31}$$

where $\Phi$ is the integral flow of $f$ defined by (5). ■

*Definition 10:* ([2], [3], [5], [11]) For the dynamical system $\dot{x} = f(x,t),\ \ f \in \mathfrak{X}(M\times\mathbb{R})$, an equilibrium $\bar{x} \in M$ is

(i): *Lyapunov stable* if for any neighborhood $\mathcal{U}_{\bar{x}}$ of $\bar{x}$ and any time $t_0$, there exits a neighborhood $\mathcal{W}_{\bar{x}}(t_0)$ of $\bar{x}$, such that

$$x_0 \in \mathcal{W}_{\bar{x}}(t_0) \Rightarrow \Phi_f(t,t_0,x_0) \in \mathcal{U}_{\bar{x}}, \;\; t \in [t_0,\infty). \tag{32}$$

(ii): locally asymptotically stable if it is Lyapunov stable and for all $t_0 \in \mathbb{R}$, there exits $\mathcal{U}_{\bar{x}}(t_0)$, such that

$$\begin{aligned} & x_0 \in \mathcal{U}_{\bar{x}}(t_0) \Rightarrow \lim_{t\to\infty} \Phi_f(t,t_0,x_0) = \bar{x}, \;\; \text{i.e.} \\ & \lim_{t\to\infty} d(\Phi_f(t,t_0,x_0),\bar{x}) = 0. \end{aligned} \tag{33}$$

(iii): globally asymptotically stable if it is Lyapunov stable and for all $t_0 \in \mathbb{R}$,

$$\forall x_0 \in M, \;\; \lim_{t\to\infty} \Phi_f(t,t_0,x_0) = \bar{x}. \tag{34}$$

(iv): locally exponentially stable if it is locally asymptotically stable and for all $t_0 \in \mathbb{R}$, there exist $\mathcal{U}_{\bar{x}}(t_0)$ and $K,\lambda \in \mathbb{R}_{>0}$, such that

$$\begin{aligned} d(\Phi_f(t,t_0,x_0),\bar{x}) &\leq K d(x,\bar{x}) \exp(-\lambda(t-t_0)), \\ & K,\lambda \in \mathbb{R}_{>0}, x_0 \in \mathcal{U}_{\bar{x}}(t_0). \end{aligned} \tag{35}$$

(v): globally exponentially stable if it is globally asymptotically stable and for all $t_0 \in \mathbb{R}$, there exist $K,\lambda \in \mathbb{R}_{>0}$, such that

$$\begin{aligned} d(\Phi_f(t,t_0,x_0),\bar{x}) &\leq K d(x,\bar{x}) \exp(-\lambda(t-t_0)), \\ & K,\lambda \in \mathbb{R}_{>0}, x_0 \in M. \end{aligned} \tag{36}$$

■

Let us consider the following dynamical systems on an $n$ dimensional compact Riemannian manifold $M$:

$$\begin{aligned} \epsilon\dot{x}(t) &= f(x,z,t), && x \in M, \\ \dot{z}(t) &= e(x,z,t), && z \in \mathbb{R}^m, \end{aligned} \tag{37}$$

where $f$ and $e$ are smooth with respect to their arguments. Since $M$ is compact it is covered by a finite number of charts $(\mathcal{U}_i,\psi_i)$, where $M = \bigcup_{i=1,..,N} \mathcal{U}_i$. For $\epsilon = 0$, we assume the solution of $f(x,z,t) = 0 \in T_xM$ is given by a smooth algebraic equation

$$x = h_i(z,t), \;\; i = 1,...,N, \tag{38}$$

where $h_i : \mathbb{R}^m \times \mathbb{R} \to M$. For the new time variable $\tau = \frac{t-t_0}{\epsilon}$, the dynamical equations in (37) are written as

$$\begin{aligned} \frac{dx}{d\tau} &= f(x,z,t), && x \in M, \\ \frac{dz}{d\tau} &= \epsilon e(x,z,t), && z \in \mathbb{R}^m. \end{aligned} \tag{39}$$

By treating $z$ and $t$ as fixed parameters, the *boundary layer model* is then defined by

$$\frac{d\hat{x}}{d\tau} = f(\hat{x}, z, t), \;\; \hat{x} \in M, z \in \mathbb{R}^m, \tau \in [0, \infty). \tag{40}$$

Note that $\frac{dt}{d\tau} = \epsilon, \frac{dz}{d\tau} = \epsilon e(x, z, t)$, hence for sufficiently small $\epsilon$, parameters $t$ and $z$ are slow with respect to $\tau$.

The *reduced model* for $z$ is then defined by

$$\dot{z} = e(h_i(z, t), z, t), \;\; z \in \mathbb{R}^m, i = 1, ..., N. \tag{41}$$

The following theorem is a version of Tikhonov theorem (see [8], [9], [11]) for dynamical systems where the fast variable $x$ evolves on a compact Riemannian manifold $M$ and the slow variable $z$ lies on $\mathbb{R}^m$.

*Theorem 4:* Consider the singular perturbation system defined by (37) on a compact $n$ dimensional Riemannian manifold $(M, g)$. Assume $||e(x, z, t)||_e \leq c_0, \;\; x \in M, z \in \mathbb{R}^m, t \in [t_0, \infty)$. Also assume all derivatives of $h$ with respect to $x$ and $t$ are bounded and there exists $l_1, l_2 \in \mathbb{R}_{>0}$, such that

$$\begin{aligned} &||T_{(x,(z,t))} f(x, z, t)|| \doteq ||T_x T_{(z,t)} f(x, z, t)|| \leq l_1, \\ &||T_x f(x, z, t)|| \leq l_2, \;\; x \in M, z \in \mathbb{R}^m, t \in [t_0, \infty), \end{aligned} \tag{42}$$

where $|| \cdot ||$ is the linear operator norm. Let $h_i(z, t)$ be given by (38) for all the local charts $(\mathcal{U}_i, \psi_i), \;\; i = 1, ..., N$, where $N$ is the number of charts needed to cover $M$. Assume the equilibrium $\bar{x}(t) = h_i(z, t)$ of the boundary layer model (40) and $z_i$ of the reduced system (41) are exponentially stable, uniformly with respect to $(t, z)$ for the boundary system and uniformly with respect to $t$ for the reduced system, with regard to Definition 10 and $\bar{z} \doteq z_i = z_j, i, j = 1, ..., N$. Then for any $t_1 > t_0$, there exists $\epsilon_1 > 0$ and a neighborhood $\mathcal{U}_{\bar{x}(t_0)}$, such that

$$\begin{aligned} &d(\Phi_f(t, t_0, x_0), \bar{x}(t)) = O(\epsilon), \\ &\epsilon \in (0, \epsilon_1], \forall x_0 \in \mathcal{U}_{\bar{x}(t_0)}, t \in [t_1, \infty), \end{aligned} \tag{43}$$

and

$$\limsup_{t \to \infty} ||z(t) - \bar{z}||_e = O(\epsilon), \;\; \epsilon \in (0, \epsilon_1]. \tag{44}$$

*Sketch of the proof:* The proof is detailed and technical and is not included for the economy of the space. The main idea is employing the results of [25] and applying the results of Taylor expansion of functions defined on Riemannian manifolds, see [23]. In this case we need to study the variation of Lyapunov functions for boundary layer model on $TM$ and the reduced model on $\mathbb{R}^m$. ∎

*Remark 1:* Note that the closeness of solutions for trajectories on $(M, g)$ are given for any time $t_1 > t_0$. Also the closeness of solutions for $z$ trajectory is only provided in the limit point.

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