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Author/s:

Ljesnjanin, M;Quevedo, DE;Nesic, D

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Packetized MPC with Dynamic Scheduling Constraints and Bounded Packet Dropouts[☆]

Merid Lješnjani^a, Daniel E. Quevedo^b, Dragan Nešić^a

^a*Department of Electrical and Electronic Engineering, The University of Melbourne, Carlton, VIC 3010, Australia*

^b*School of Electrical Engineering & Computer Science, The University of Newcastle, Callaghan, NSW 2308, Australia*

Abstract

We study a Networked Control System architecture which uses a communication network in the controller-actuator links. The network is affected by packet dropouts and allows access to only one plant input node at each time instant. This limits control performance significantly. To mitigate these limitations we propose a control and network protocol co-design method. Succinctly, the underlying features of the proposed method are as follows: a sequence of predicted optimal control values over a finite horizon, for an optimally chosen input node, is obtained using Model Predictive Control ideas; the *entire* resulting sequence is sent to the chosen input node; a smart actuator is used to store the predictions received and apply them accordingly. We show that if the number of consecutive packet dropouts is uniformly bounded, then partial nonlinear gain ℓ_2 stability and also a more traditional linear gain ℓ_2 stability can be ensured via appropriate choice of design parameters and the right assumptions. Whilst our results apply to general nonlinear discrete-time multiple input plants affected by exogenous disturbances, for a disturbance-free case we prove that Global Asymptotic Stability follows from our main result. Moreover, we show that by imposing stronger assumptions, Input-to-State Stability is achievable as well. Finally we demonstrate the potential of the proposed method via simulations.

Keywords: Networked Control System, Model Predictive Control, Packet dropouts, Dynamic Scheduling, Nonlinear gain ℓ_2 stability, Input-to-State Stability

1. Introduction

A Networked Control System (NCS) is a system which uses a communication network as a communication medium in at least one of its links. Usage of a network instead of traditional point-to-point connections is motivated by the cost reduction, increased flexibility and reliability, easier interoperability of devices, and simpler maintenance and troubleshooting (Moyne and Tilbury, 2007). Moreover, due to the rapid advancements in communication technology and affordability of networks in general, NCSs are considered to be one of the future directions in the field of control (Murray et al., 2002).

A communication network itself is a dynamical system with intrinsic properties such as delays, scheduling and packet dropouts. These properties impose serious challenges in the corresponding NCS design and analysis. Moreover, they often constitute a significant bottleneck in the achievable performance (Bushnell, 2001; Antsaklis and Baillieul, 2004, 2007; Hespanha et al., 2007).

We note that essentially there are two types of communication networks, namely, control-like and data-like

networks (Lian et al., 2005). Unlike control-like networks which are used in some specialized areas (e.g., the automotive industry), data-like networks are more widespread. In fact, they are being increasingly adopted for low level control technology (Moyne and Tilbury, 2007). Unfortunately, modern communication data-like networks usually are not designed for time critical applications. On the other hand, a common feature of all data-like networks is that data packets, used for transmitting data, contain fairly large data fields. Hence, bit-rates are high and quantization issues will often play only a minor role. Moreover, via appropriate control strategy, these large data fields can be exploited to mitigate negative effects of delays, scheduling and packet dropouts; for instance, see Bemporad (1998); Quevedo et al. (2008b); Polushin et al. (2008); Grüne et al. (2009); Findeisen et al. (2011); Quevedo and Nešić (2011); Greco et al. (2012) and references therein. More precisely, one can use the Model Predictive Control (MPC) framework to design a corresponding controller that predicts control values over a finite horizon. Then, due to the size of a data field readily available, rather than sending only the first member of the corresponding sequence of predicted control values, the whole sequence is sent to the receiving end, e.g., a smart actuator. Finally, depending on transmission outcomes and via appropriate buffering and processing at the receiving end, some predicted control values are applied to the plant.

[☆]A preliminary version of parts of this work was presented as Quevedo et al. (2008b).

Email addresses: meridl@student.unimelb.edu.au (Merid Lješnjani), dquevedo@ieee.org (Daniel E. Quevedo), dnesic@unimelb.edu.au (Dragan Nešić)

The latter control strategy, introduced in Bemporad (1998), underpins recent NCS architectures concerned with delays, scheduling and packet dropouts. For instance, the corresponding analysis for NCS configurations with *one input* LTI plants is documented in Casavola et al. (2006); Tang and de Silva (2006, 2007); Liu et al. (2006); Zhao et al. (2008), while for *one input* NCS configurations with general nonlinear plants in Polushin et al. (2008); Grüne et al. (2009); Findeisen and Varutti (2009); Pin and Parisini (2009); Findeisen et al. (2011); Quevedo and Nešić (2011); Nagahara and Quevedo (2011); Quevedo and Nešić (2012); Greco et al. (2012); Reble et al. (2012). As one can imagine, due to the sharing of resources of a network, scheduling access to input and/or output plant nodes can become unavoidable. To the best of the authors' knowledge, scheduling issues (together with packet dropouts) in controller-actuator links within the control strategy presented above for general discrete-time nonlinear systems was only considered in our conference contribution Quevedo et al. (2008b). In the present manuscript, which expands upon the latter work, we investigate *robustness properties* of an NCS configuration which uses a data-like network as a communication medium in controller-actuator links, see Fig.1. The network is affected by packet dropouts, d , and allows access to only one smart actuator node at each time instant. We conduct our investigation within the packetized predictive control strategy described above. Thus, we assume that smart actuators have a memory (a buffer) and a simple processing unit, so that they are capable to store and process a sequence of predicted control values. The access constraint imposed by the network necessitates scheduling, and we address it by proposing a network protocol and controller co-design method with the focus on closed loop performance. When compared to Quevedo et al. (2008b), a distinguishing feature here is that a general nonlinear discrete-time multiple input plant is affected by exogenous disturbances w . As will become apparent, this complicates the analysis significantly.

We employ the concept of nonlinear gains introduced in Kameneva and Nešić (2008) to capture robustness of the system at hand. We show that, if the number of consecutive packet dropouts is uniformly bounded, then partial nonlinear gain ℓ_2 stability can be achieved via similar assumptions used for showing stability in networked MPC (Quevedo and Nešić, 2011). We also present additional assumptions which lead to, more traditional, linear gain ℓ_2 stability. For the one node case, we present nonlinear gain ℓ_2 stability which represent alternative robustness characterisation of the NCS configuration considered in Quevedo and Nešić (2011). We also show that by assuming stronger assumptions, which mimic those used in the latter work, Input-to-State Stability (ISS) can be achieved in the similar way for the aggregated state of plant and buffer state. Finally, we illustrate that dynamic scheduling using the proposed method outperforms static scheduling.

The contributions of the manuscript are as follows: Thorough and rigorous *robustness* analysis of the setup

considered in Quevedo et al. (2008b); Recovery of the main result from the latter work with relaxed assumption (see Remark 2); Alternative robustness characterisation of NCS configuration considered in Quevedo and Nešić (2011); ISS result for the aggregated state of plant and buffer state.

The remainder of this manuscript is organized as follows: In Sections 2 and 3 we present the NCS under study. Robustness properties are established in Section 4. Simulation results are presented in Section 5 while Section 6 draws conclusions.

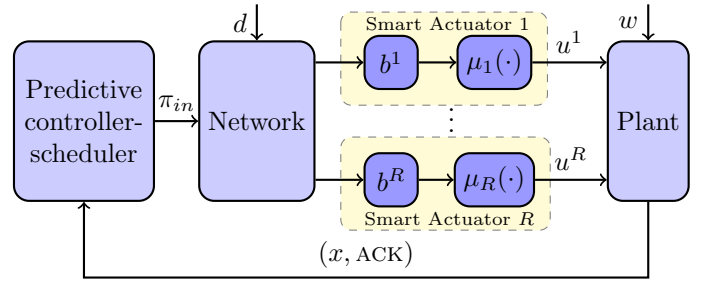


Figure 1: NCS configuration

Notation and preliminaries

Let $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Z}$ the set of integers. For any $p \in \mathbb{R}$, we use the notation $\mathbb{R}_{\geq p}$ ($\mathbb{R}_{>p}$) to refer to the subset $\{v \in \mathbb{R} : v \geq p\}$ ($\{v \in \mathbb{R} : v > p\}$); the similar notation applies to the set of integer numbers as well. Euclidean norm is denoted with $|\cdot|$, while the superscript $\top$ stands for transposition. Often we use tuple notation to denote a column vector. The zero element of $\mathbb{R}^m$ is denoted by $\bar{0}_m$; the symbol I_m stands for the $m \times m$ identity matrix, while $0_m = 0 \cdot I_m$. We denote any sequence $\{a_i\}_{i=1}^j$ by a_i^j for $i \leq j$ while $a_i^j \triangleq \{\}$ for $i > j$. Further, a sequence consisting only of zero values of appropriate length is denoted by $\mathbf{0}$. Moreover we define the set consisting of infinite sequences starting at time zero whose elements take values in some set $\mathcal{V}$ as $\mathcal{S}^{\mathcal{V}} \triangleq \{a_0^\infty : a_i \in \mathcal{V}, \forall i \in \mathbb{Z}_{\geq 0}\}$.

The identity function is defined as $\text{Id}(v) \triangleq v$ for all $v \in \mathbb{R}_{\geq 0}$. Further, a function $\alpha: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a class $\mathcal{K}$ function ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing; it is said to be a class $\mathcal{K}_\infty$ function ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and it is unbounded. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a class $\mathcal{KL}$ function ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t and $\beta(v, \cdot)$ is decreasing to zero for each $v > 0$.

Often we use the fact that for any class $\mathcal{K}_\infty$ ($\mathcal{K}$) function α and any $a, b \in \mathbb{R}_{\geq 0}$ it holds that

$$\frac{1}{2}\alpha(a) + \frac{1}{2}\alpha(b) \leq \alpha(a+b) \leq \alpha(2a) + \alpha(2b) \quad (1)$$

which is the consequence of non-decreasing character of α .

2. Networked Control System Architecture

We consider the NCS configuration depicted in Fig.1. A communication network is located in controller-actuator links. The network introduces packet dropouts and allows access to only one input node at each time instant. These intrinsic properties of a network may lead to poor performance or even instability of the NCS.

Next, we precisely describe each part of the NCS configuration depicted in Fig. 1.

2.1. Plant

We consider a plant model of the form

$$x^+ = f(x, u, w) \quad (2)$$

with state $x \in \mathcal{X} \subseteq \mathbb{R}^n$, control input $u \in \mathcal{U} \subseteq \mathbb{R}^p$ and disturbance $w \in \mathcal{W} \subseteq \mathbb{R}^m$ where $n, p, m \geq 1$. The function $f: \mathcal{X} \times \mathcal{U} \times \mathcal{W} \rightarrow \mathcal{X}$ is assumed to be a general nonlinear function which satisfies the following assumption, which will be used for robustness analysis in Section 4.

Assumption 1 (Continuity). There exist class $\mathcal{K}$ functions χ and δ such that

$$|f(x, u, w) - f(y, u, 0)| \leq \chi(|x - y|) + \delta(|w|) \quad (3)$$

for all $x, y \in \mathcal{X}$, all $u \in \mathcal{U}$ and all $w \in \mathcal{W}$. $\blacksquare$

Note that Assumption 1 may preclude some systems with cross terms. However, if the sets $\mathcal{U}$ and $\mathcal{W}$ are compact (which is often the case) then Assumption 1 does not preclude the latter type of systems.

The input vector u is connected at the actuator side via $R \leq p$ nodes according to the partition

$$u \triangleq (u^1, \dots, u^R) \quad (4)$$

where $u^r \in \mathcal{U}_r \subseteq \mathbb{R}^{p_r}$ for all $r \in \mathcal{N}$ and $\sum_{r=1}^R p_r = p$, where

$$\mathcal{N} \triangleq \{1, 2, \dots, R\} \quad (5)$$

is the set of nodes.

Throughout the paper we introduce additional notational convention to simplify the presentation. The first one is related to finite control and disturbance sequences. Namely, for a finite horizon $N \in \mathbb{Z}_{\geq 1}$ we write $u_0^{N-1} \triangleq \mathbf{u}$ and $w_0^{N-1} \triangleq \mathbf{w}$. Note that, using our notation, both $\mathbf{u}$ and $\mathbf{w}$ are sub-sequences of some infinite sequences $u_0^\infty \in \mathcal{S}^{\mathcal{U}}$ and $w_0^\infty \in \mathcal{S}^{\mathcal{W}}$, respectively. Further, we denote the solution of (2), k steps into the future, starting at initial condition x , under the influence of a control input sequence $\mathbf{u}$ and the disturbance sequence $\mathbf{w}$ by $\phi_x(k, x, \mathbf{u}, \mathbf{w})$; note that $\phi_x(0, x, \mathbf{u}, \mathbf{w}) = x$ (e.g., see Chapter 2 in Sontag (1998)).

2.2. Network

We consider a data-like communication network, such as Ethernet, located between predictive controller-scheduler and smart actuators. The network is prone to packet dropouts and we address this issue explicitly. Additionally, we consider the scenario where the network allows access to only one actuator node at each time instant. Consequently, we model the network as a collection of R erasure channels (Imer et al., 2006; Wu and Chen, 2007) with a protocol that allows dynamic scheduling. Further, at *each* time instant, the transmission effect is modeled via the discrete dropout process

$$d = \begin{cases} 0, & \text{if packet dropout occurred,} \\ 1, & \text{if packet dropout did not occur.} \end{cases} \quad (6)$$

Additionally, we define the set of values from which d takes values as $\mathcal{D} \triangleq \{0, 1\}$.

Notice that, when longer input sequences are sent and buffered at the actuator side, in case of a packet dropout, the NCS is operated in open-loop. Hence, as one can expect, to obtain certain performance of the system the number of consecutive packet dropouts should be bounded. In fact, we restrict ourselves to data-like networks which satisfy the following assumption.

Assumption 2 (Packet dropouts). There exists a finite horizon $N \in \mathbb{Z}_{\geq 1}$, such that the number of consecutive packet dropouts is uniformly bounded by $N - 1$. $\blacksquare$

In order to precisely define the set consisting of all infinite sequences of dropout outcomes which satisfy Assumption 2 we introduce set of (relabelled) dropout indices which satisfy Assumption 2 as

$$K_N \triangleq \{k_i : k_i \in \mathbb{Z}_{\geq 0}, d_{k_i} = 1, k_{i+1} > k_i, m_i \leq N - 1, k_0 \leq N, \forall i \in \mathbb{Z}_{\geq 0}\},$$

where

$$m_i \triangleq k_{i+1} - k_i - 1, \forall k_i \in \mathbb{Z}_{\geq 0}, \forall i \in \mathbb{Z}_{\geq 0}, \quad (7)$$

is the number of consecutive packet dropouts. Now, the set consisting of all infinite sequences of dropout outcomes which satisfy Assumption 2 is defined as

$$\mathcal{S}_N^{\mathcal{D}} \triangleq \{d_0^\infty : d_0^\infty \in \mathcal{S}^{\mathcal{D}}, k_i \in K_N, \forall k_i \in \mathbb{Z}_{\geq 0}, \forall i \in \mathbb{Z}_{\geq 0}\}. \quad (8)$$

Note that $\mathcal{S}_N^{\mathcal{D}} \subset \mathcal{S}^{\mathcal{D}}$. In the sequel, if not stated otherwise, every time when we refer to a horizon, we mean a horizon that satisfies Assumption 2.

For the purpose of clearer analysis, we regard the network as a dynamical system, but we omit some details in its description to simplify the presentation; see Fig.2. So, the input is considered to be a packet π_{in} generated by the predictive controller-scheduler. It consists of an *address* field which we denote with lowercase letter $a \in \mathcal{N}$ (see (5)) and a *data* field which is denoted by Δ or more precisely

$$\pi_{in} \triangleq (a, \Delta) \in \Pi \subseteq \mathcal{N} \times \mathbb{R}^{p_m N}, p_m \triangleq \max p_r. \quad (9)$$

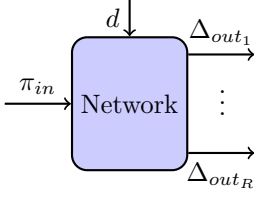


Figure 2: Block diagram of a network

The network has R outputs denoted as $\Delta_{out_1}, \dots, \Delta_{out_R}$. However, since the network allows access to only one channel at each time instant, only one network output will be *active*, i.e., $\Delta_{out_a} = d\Delta$. Finally, packet dropouts act like an exogenous disturbance.

2.3. Smart Actuators

We use R devices to which we refer to as smart actuators. These will be used to store sequences of predicted control values and apply the corresponding predicted control values accordingly. More precisely, each device has a memory and a simple processing unit. The memory unit is regarded as a buffer which is denoted as $b^r \in (\mathcal{U}_r)^N$ for all $r \in \mathcal{N}$. The corresponding processing units are denoted as $\mu_r(b^r) \in \mathcal{U}_r$ for all $r \in \mathcal{N}$ and their purpose is to pick the first value from the buffer and apply it to the corresponding plant input.

Next we describe the buffering mechanism. Recall that at each time instant a data packet as in (9) is sent. Let us first consider the case of a packet dropout. Also, recall that due to access constraint, only one of the network outputs is active, i.e., $\Delta_{out_a} = d\Delta$. However, due to packet dropout (i.e., $d = 0$), effectively, all outputs are inactive. In that case all buffers shift their existing contents. On the other hand, if the transmission was successful (i.e., $d = 1$), the contents of the addressed buffer is overwritten with the data Δ , i.e., $b^a = \Delta \in (\mathcal{U}_a)^N, a \in \mathcal{N}$, while other buffers shift their existing contents. The buffering mechanisms amount to *parallel-in serial-out shift registers*. These act as safeguards against packet dropouts and also offer advantages in the presence of the dynamic scheduling constraints considered here. More concretely, the buffering mechanism can be described as follows

$$\begin{cases} b^{a+} \triangleq d\Delta \\ b^{r+} \triangleq S_{p_r} b^r, \forall r \in \mathcal{N} \setminus \{a\}, \end{cases} \quad (10)$$

where the shift matrices $S_{p_r}, \forall r \in \mathcal{N}$ are defined as¹

$$S_{p_r} \triangleq \begin{bmatrix} 0_{p_r} & I_{p_r} & 0_{p_r} & \cdots & 0_{p_r} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{p_r} & \cdots & 0_{p_r} & I_{p_r} & 0_{p_r} \\ 0_{p_r} & \cdots & \cdots & 0_{p_r} & I_{p_r} \\ 0_{p_r} & \cdots & \cdots & \cdots & 0_{p_r} \end{bmatrix} \in \mathbb{R}^{p_r N \times p_r N}. \quad (11)$$

Note that, in the case of a packet dropout, the second equation of (10) is used for all $r \in \mathcal{N}$, since all network

¹The choice in (11) corresponds to setting a buffer state to zero if no data is received over N consecutive time instances. Alternatively, if one wants to hold the last value one could set the element of S_{p_r} which belongs to last column and last row to be equal to I_{p_r} .

outputs are inactive. The smart actuator outputs are given by

$$\mu_r(b^r) \triangleq [I_{p_r} 0_{p_r} \cdots 0_{p_r}] b^r \in \mathcal{U}_r, \forall r \in \mathcal{N}. \quad (12)$$

For the purpose of clearer analysis we introduce the overall network-buffer model which encapsulates all equations from (10) as

$$\beta^+ \triangleq \Psi(a, d) S \beta + \Upsilon(\Delta, a, d) \quad (13)$$

where

$$\beta \triangleq (b^1, \dots, b^R) \in \mathcal{U}^N; \quad (14)$$

$$\Psi(a, d) \triangleq \begin{cases} \text{diag}(I_{p_1 N}, \dots, I_{p_{a-1} N}, \\ 0_{p_a N}, I_{p_{a+1} N}, \dots, I_{p_R N}) & \text{for } d = 1, \\ I_{p_N \times p_N} & \text{for } d = 0; \end{cases} \quad (15)$$

$$S \triangleq \text{diag}(S_{p_1}, S_{p_2}, \dots, S_{p_R}); \quad (16)$$

$$\Upsilon(\Delta, a, d) \triangleq \begin{cases} [\bar{0}_{p_1 N}^\top \cdots \bar{0}_{p_{a-1} N}^\top \\ \Delta^\top \bar{0}_{p_{a+1} N}^\top \cdots \bar{0}_{p_R N}^\top]^\top & \text{for } d = 1, \\ 0_{p_N} & \text{for } d = 0. \end{cases} \quad (17)$$

Note that $\Delta \in \mathbb{R}^{p_a N}, \forall a \in \mathcal{N}$ in (17). Further, the vector of smart actuator outputs $v \in \mathcal{U}$ is defined now as

$$v \triangleq W(\Psi(a, d) S \beta + \Upsilon(\Delta, a, d)), \quad (18)$$

where

$$W \triangleq [V_{p_1} V_{p_2} \cdots V_{p_R}], \quad (19)$$

and $\forall r \in \mathcal{N}$

$$V_{p_r} \triangleq \begin{bmatrix} 0_{p_1} & 0_{p_1} & \cdots & 0_{p_1} \\ \vdots & \vdots & \cdots & \vdots \\ 0_{p_{r-1}} & 0_{p_{r-1}} & \cdots & 0_{p_{r-1}} \\ I_{p_r} & 0_{p_r} & \cdots & 0_{p_r} \\ 0_{p_{r+1}} & 0_{p_{r+1}} & \cdots & 0_{p_{r+1}} \\ \vdots & \vdots & \cdots & \vdots \\ 0_{p_R} & 0_{p_R} & \cdots & 0_{p_R} \end{bmatrix} \in \mathbb{R}^{p \times p_r N}. \quad (20)$$

We finish this section by stating the interconnection $v = u$, yielding

$$v \triangleq W(\Psi(a, d) S \beta + \Upsilon(\Delta, a, d)) = u. \quad (21)$$

3. Design of Control Sequences and Actuator Schedules

As foreshadowed in the introduction, we adopt the strategy introduced in Bemporad (1998) to address the issue of packet dropouts. The framework there considered only one node, hence, for our purposes, a refinement is necessary to account for access constraints. For that, we use ideas from Quevedo et al. (2008b). Succinctly, scheduling of input nodes will be assumed dynamic. Now, recall that the MPC based controller underpins the control strategy of interest and the necessary modifications have to be made

in the model used for generating predictions. Hence, we incorporate models of both network issues in the overall nominal model used to generate predicted plant trajectories. This model consists of R nominal models due to the fact that we have R nodes. However, it suffices to present only one model. So, for any node $r \in \mathcal{N}$ the corresponding nominal model used to generate predicted plant trajectories is given as

$$\bar{x}^+ \triangleq f(\bar{x}, u^{\beta,r}, \bar{0}_m), \quad \bar{x} = x \quad (22)$$

where $u^{\beta,r}$ is obtained according to

$$u^{\beta,r} \triangleq \begin{bmatrix} W_{p_1} \left(\Psi(r, \bar{d}) S \bar{\beta} + \Upsilon(\mathbf{u}^r, r, \bar{d}) \right) \\ \vdots \\ W_{p_{r-1}} \left(\Psi(r, \bar{d}) S \bar{\beta} + \Upsilon(\mathbf{u}^r, r, \bar{d}) \right) \\ u^r \\ W_{p_{r+1}} \left(\Psi(r, \bar{d}) S \bar{\beta} + \Upsilon(\mathbf{u}^r, r, \bar{d}) \right) \\ \vdots \\ W_{p_R} \left(\Psi(r, \bar{d}) S \bar{\beta} + \Upsilon(\mathbf{u}^r, r, \bar{d}) \right) \end{bmatrix}, \quad (23)$$

where

$$\begin{cases} \bar{\beta}^+ \triangleq \Psi(r, \bar{d}) S \bar{\beta} + \Upsilon(\mathbf{u}^r, r, \bar{d}), \quad \bar{\beta} = \beta, \\ \bar{d} \in \bar{\mathbf{d}} \triangleq \bar{\mathbf{d}}_0^{N-1} \triangleq \{1, 0, \dots, 0\}, \\ u^r \in \mathbf{u}^r \triangleq \{u_0^r, u_1^r, \dots, u_{N-1}^r\}, \end{cases} \quad (24)$$

and where $W_{p_r}, \forall r \in \mathcal{N}$ are rows of matrix W , i.e., $[W_{p_1}^\top \dots W_{p_R}^\top]^\top = W$.

Now, a few points related to a nominal model (22)-(24) should be discussed. First, note that our method assumes perfect knowledge of the plant dynamics, up to the unknown disturbances w (see (22)). Second, the control predictions (23) depend on buffer contents. This is the effect of having access to only one input node at each time instant. It is important to notice that we assume that buffer contents are known to the controller via reliable acknowledgments of receipt as indicated in Fig. 1 by ACK. However, if such acknowledgments are not available, then one could either adopt a stochastic control framework, e.g., as in Hekler et al. (2012), or use a control which accounts for all transmission scenarios, e.g., as in Quevedo et al. (2008a). Third, note that for each node the corresponding model (22)-(24) addresses the issues of packet dropouts and access constraint via a model of the buffer and the interconnection, see (13) and (21). Finally, note that the horizon N is provided by Assumption 2.

For the purposes of defining the cost function, first, we denote a sequence of predicted control values (23) over a finite horizon N by $\mathbf{u}^{\beta,r} \triangleq \{u_0^{\beta,r}, u_1^{\beta,r}, \dots, u_{N-1}^{\beta,r}\}$; next, we denote the solution of the nominal model for node $r \in \mathcal{N}$, i.e., (22), k steps into the future starting at the initial state x under the influence of $\mathbf{u}^{\beta,r}$ by $\phi_{\bar{x}}(k, x, \mathbf{u}^{\beta,r})$; again, it follows $\phi_{\bar{x}}(0, x, \mathbf{u}^{\beta,r}) = x$. Recall that for each nominal model, prediction of a control input depends on contents of

other buffers. Hence, it is convenient to define the overall NCS state as

$$\xi \triangleq (x, \beta) \in \bar{\mathcal{X}} \subseteq \mathbb{R}^{n+pN}. \quad (25)$$

In order to achieve good control performance despite network artifacts, the control is optimization based. We define the cost function as

$$J(\xi, r, \mathbf{u}^r) \triangleq g(\phi_{\bar{x}}(N, x, \mathbf{u}^{\beta,r})) + \sum_{k=0}^{N-1} l(\phi_{\bar{x}}(k, x, \mathbf{u}^{\beta,r}), u_k^{\beta,r}) \quad (26)$$

where $g : \mathcal{X}_f \rightarrow \mathbb{R}_{\geq 0}$ is the *terminal cost* function and $l : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$ is the *stage cost* function. The set $\mathcal{X}_f \subseteq \mathcal{X}$ is a given set which contains $\bar{0}_n \in \mathbb{R}^n$ and we denote a set of all feasible initial plant states as $\mathcal{X}_N \subseteq \mathcal{X}$; i.e., plant states for which (26) is bounded. Moreover, the weighting functions g and l satisfy the following assumption.

Assumption 3. There exist a class $\mathcal{K}_\infty$ function α such that

$$\begin{aligned} g(x) &\geq 0, \forall x \in \mathcal{X}_f, g(\bar{0}_n) = 0, \\ l(x, u) &\geq \alpha(|x|), \forall x \in \mathcal{X}_N, \forall u \in \mathcal{U}, l(\bar{0}_n, \bar{0}_p) = 0, \end{aligned} \quad (27)$$

is satisfied. Also, for *some* $r \in \mathcal{N}$ there exist a *terminal control law* $\kappa^r : \mathcal{X}_f \rightarrow \mathcal{V}_r$ such that

$$\begin{aligned} g(f(x, \kappa^r(x), \bar{0}_m)) - g(x) + l(x, \kappa^r(x)) &\leq 0, \\ f(x, \kappa^r(x), \bar{0}_m) &\in \mathcal{X}_f, \\ \kappa^r(x) &\in \mathcal{V}_r, \end{aligned} \quad (28)$$

holds for all $x \in \mathcal{X}_f$ where for any $r \in \mathcal{N}$ the set $\mathcal{V}_r \in \mathbb{R}^p$ is defined as

$$\mathcal{V}_r \triangleq \bar{0}_{p_1} \times \dots \times \bar{0}_{p_{r-1}} \times \mathcal{U}_r \times \bar{0}_{p_{r+1}} \times \dots \times \bar{0}_{p_R}. \quad (29)$$

The set (29) is related to the fact that we can access only one input node at each time instant. ■

Note that the terminal control law κ^r constitutes a locally stabilizing control law for the corresponding nominal model (22)-(24). In fact, the property (28) establishes that the *terminal cost* function g is a local control Lyapunov function and that the set $\mathcal{X}_f$ is controlled invariant. Important to note also, is that κ^r is not necessarily applied to the plant; it is just a construct needed to establish stability results, e.g., see (Quevedo et al., 2008b; Quevedo and Nešić, 2011) for networked case and (Mayne et al., 2000; Rawlings and Mayne, 2009) for non-networked case. ■

Minimization of the cost function (26) is carried out under constraints on control predictions (23) and resulting predicted plant solutions $\phi_{\bar{x}}$. These constraints are similar to those distilled in (Mayne et al., 2000) for non-networked cases. Namely, the control predictions (23) are constrained via

$$u_k^{\beta,r} \in \mathcal{U}, \quad (30)$$

for all $k \in \{0, \dots, N-1\}$. The solutions of the model (22) are constrained with

$$\phi_{\bar{x}}(k, x, \mathbf{u}^{\beta,r}) \in \mathcal{X}, \quad (31)$$

for all $k \in \{0, \dots, N-1\}$. The *final solution* has to satisfy the *terminal state*-like constraint

$$\phi_{\bar{x}}(N, x, \mathbf{u}^{\beta, r}) \in \mathcal{X}_f. \quad (32)$$

Recall that there are R models of (22)-(24) with the corresponding cost function and constraints (30)-(32). To obtain the sequence of optimal control predictions over a finite horizon N for the optimal node chosen, we first minimize each corresponding cost function with respect to the corresponding control sequence $\mathbf{u}^r$, i.e., we seek

$$V(\xi, r) \triangleq \min_{\mathbf{u}^r} J(\xi, r, \mathbf{u}^r) \text{ s.t. (22)-(25)\&(30)-(32)} \quad (33)$$

which results in the set $\{V(\xi, r)\}_{r \in \mathcal{N}}$. Then we take the minimum of the latter set, i.e.,

$$V(\xi) \triangleq \min_r V(\xi, r), \quad (34)$$

which represents the optimal value function. Moreover, assume that the optimal value function (34) satisfies the following assumption.

Assumption 4. There exist a class $\mathcal{K}$ function $\bar{\chi}_V$ and a class $\mathcal{K}_\infty$ function χ_V such that²

$$|V(\begin{bmatrix} x \\ \beta \end{bmatrix}) - V(\begin{bmatrix} y \\ \beta \end{bmatrix})| \leq \chi_V(|x - y|), \quad (35a)$$

$$V(\xi) \leq \bar{\chi}_V(|\xi|). \quad (35b)$$

holds $\forall x, y \in \mathcal{X}_N$, $\forall \xi \in \bar{\mathcal{X}}_N$ and $\forall \beta \in (\mathcal{U})^N$, where $\bar{\mathcal{X}}_N \subseteq \bar{\mathcal{X}}$ is the set of all feasible initial plant and buffer states such that corresponding cost function (26) is bounded.

When investigating robustness of a system governed by an MPC based controller it is useful to assume continuity of the optimal value function, e.g. see (Grimm et al., 2004; Magni et al., 2006; Rawlings and Mayne, 2009) for non-networked systems and (Quevedo and Nešić, 2011) for a networked case. Since our interest lies only with plant trajectories we assume (35a). Note that, if the set $\bar{\mathcal{X}}_N$ is compact, (35a) is not a strong assumption. However, if the set $\bar{\mathcal{X}}_N$ is not compact, finding general conditions which relax (35a) is an open question. The bound (35b) comes from the fact that we assume that the NCS is asymptotically controllable to the origin, e.g. see Section III in (Grimm et al., 2005) and Section II in (Kreisselmeier and Birkholzer, 1994).

Further, the optimal node is obtained via

$$r^* \triangleq \underset{r}{\operatorname{argmin}} V(\xi, r), \quad (36)$$

while the corresponding optimal sequence via

$$(\mathbf{u}^r)^* \triangleq \underset{\mathbf{u}^{r^*}}{\operatorname{argmin}} J(\xi, r^*, \mathbf{u}^{r^*}) \text{ s.t. (22)-(25)\&(30)-(32)}. \quad (37)$$

Note that in equation (37) we abuse notation and denote $\mathbf{u}^{r^*}$ as $(\mathbf{u}^r)^*$ which is also now in *tuple* form. Also, note that $V(\xi) = J(\xi, r^*, (\mathbf{u}^r)^*)$.

The solutions (36) and (37) form an input packet π_{in} , i.e.,

$$(a, \Delta) := (r^*, (\mathbf{u}^r)^*). \quad (38)$$

Further, it follows from (21) that

$$u = W(\Psi(r^*, d)S\beta + \Upsilon((\mathbf{u}^r)^*, r^*, d)). \quad (39)$$

Finally, the overall NCS closed-loop system (2), (6)-(20), (22)-(26), (30)-(36) with the interconnections above can be described via

$$\begin{aligned} \xi^+ &= \begin{bmatrix} f\left(x, W(\Psi(r^*, d)S\beta + \Upsilon((\mathbf{u}^r)^*, r^*, d)), w\right) \\ \Psi(r^*, d)S\beta + \Upsilon((\mathbf{u}^r)^*, r^*, d) \end{bmatrix} \\ &\triangleq F(\xi, d, w). \end{aligned} \quad (40)$$

Note that from an optimisation viewpoint, at every discrete time-step the controller needs to solve R deterministic finite horizon optimisation problems. A key feature here is that those R optimisations can be carried out in parallel. Hence the computational burden scales linearly with the number of actuator nodes. If implemented on hardware with parallel processors, computation times will be comparable to those of regular MPC with horizon N .

Another key feature of our method is that, due to the optimization, at each instant the controller will address the buffer most in need. Therefore, it may occur that some buffers will run out of data more often than others. However, as we shall see in our subsequent results, if appropriate assumptions are satisfied, then the proposed method guarantees stability in the presence of dropouts, scheduling constraints and exogenous disturbances.

We finish this section with remark on the absence of a network in sensor-controller links.

Remark 1. Throughout this work we assume that the controller has direct access to measurements of plant states at each time instant. Recall that these are crucial in making predictions in MPC framework (e.g., see Rawlings and Mayne (2009)). Furthermore, due to lack of network imperfections in sensor-controller links the algorithms for acknowledgments of receipt will be simpler and reliable (e.g., see Polushin et al. (2008); Grüne et al. (2009) for complications that arise when a network is included in sensor-controller links). A practical application is control of robots in confined space which communicate with the controller wirelessly while their location is measured with sensors directly connected to the controller. However, if one would take into account sensor-controller link imperfections within the scheme proposed one could use state observers with intermittent observations. If state observation errors are bounded (e.g. due to bounded dropouts), then the associated robustness problem could be dealt with

²Recall that ξ is augmented plant and the corresponding buffer state, see (25).

tools akin to those developed in the present work. Alternatively, one could also restrict the controller to only calculate control sequences at instances of successful sensor-data receptions. ■

4. Stability of the NCS

In this section we present several stability results. First, we establish sufficient conditions for a partial (plant state only) nonlinear gain ℓ_2 stability as introduced in Kameneva and Nešić (2008). Second, we establish conditions for the more familiar notion of partial linear gain ℓ_2 stability which is followed by considerations of two special cases, namely, the disturbance-free case and the one node case. We finish this section with establishing sufficient conditions for ISS of the overall NCS state. ■

4.1. Preliminaries

We assume that up until the first successful transmission the mapping $F(\cdot, 0, \cdot)$ satisfies the following assumption.

Assumption 5. There exist the class $\mathcal{K}$ functions $\bar{\chi}$ and $\bar{\delta}$ such that function F defined in (40) satisfies

$$|F(\xi, 0, w)| \leq \bar{\chi}(|\xi|) + \bar{\delta}(|w|)$$

for all $\xi \in \bar{\mathcal{X}}$ and all $w \in \mathcal{W}$. ■

This assumption ensures that before the first successful transmission trajectories will be within a set whose boundaries can be determined via the latter inequality.

Further, we denote the solution of the system (40) k steps into the future starting from the initial overall NCS state ξ (see (25)) under the influence of a sequence of dropout outcomes $\mathbf{d}$ and the sequence of disturbances $\mathbf{w}$ by $\phi_\xi(k, \xi, \mathbf{d}, \mathbf{w})$; recall that $\phi_\xi(0, \xi, \mathbf{d}, \mathbf{w}) = \xi$. Sometimes we abuse notation and write only ϕ_ξ . Note that the system (40) is time invariant, hence, we often use the following property of the solutions of the time invariant difference equations

$$\begin{aligned} \phi_\xi(k, \xi_0, d_0^{k-1}, w_0^{k-1}) &= \phi_\xi(k - \tilde{k}, \xi_{\tilde{k}}, d_{\tilde{k}}^{k-1}, w_{\tilde{k}}^{k-1}), \\ \xi_{\tilde{k}} &= \phi_\xi(\tilde{k}, \xi_0, d_0^{\tilde{k}-1}, w_0^{\tilde{k}-1}), \quad \forall k \geq \tilde{k} \geq 0. \end{aligned} \quad (41)$$

Note that $\phi_\xi = (\phi_x, \phi_\beta)$. Hence, in the sequel we use the property (41) for the mapping ϕ_x as well. Finally, we state the last assumption which is related to positive invariance of the set $\bar{\mathcal{X}}_N$ under the mapping ϕ_ξ , e.g. see (Quevedo and Nešić, 2011).

Assumption 6. For any $w_0^\infty \in \mathcal{S}^\mathcal{W}$ the solution of the system (40) satisfies

$$\phi_\xi(k, \xi_0, \mathbf{0}, \mathbf{w}) \in \bar{\mathcal{X}}_N \subseteq \bar{\mathcal{X}}, \quad \forall \xi_0 \in \bar{\mathcal{X}}_0 \subseteq \bar{\mathcal{X}}, \quad (42)$$

for all $k \in \{0, \dots, N\}$ where $\mathbf{w} \subset w_0^\infty \in \mathcal{S}^\mathcal{W}$ and $\bar{\mathcal{X}}_0$ is the set of initial plant and buffer states. Moreover, for the sequence of dropout outcomes $\bar{\mathbf{d}} \triangleq \{1, 0, \dots, 0\}$ the solutions of system (40) satisfy

$$\phi_\xi(k - \tilde{k}, \xi_{\tilde{k}}, \bar{\mathbf{d}}, \mathbf{w}) \in \bar{\mathcal{X}}_N, \quad \forall \xi_{\tilde{k}} \in \bar{\mathcal{X}}_N, \quad (43)$$

for all $k \in \{\tilde{k}, \dots, \tilde{k} + N\}$ and all $\tilde{k} > 0$, where, again, $\mathbf{w} \subset w_0^\infty \in \mathcal{S}^\mathcal{W}$. ■

The first part of Assumption 6 ensures that starting from any initial state $\xi_0 \in \bar{\mathcal{X}}_0$, the trajectories will end up in $\bar{\mathcal{X}}_N$. The second part mimics Assumption 3 from Quevedo and Nešić (2011), more precisely, the set $\bar{\mathcal{X}}_N$ is robust positive invariant for the mapping (43) for up to N steps.

4.2. Partial nonlinear gain ℓ_2 stability of the plant model

First, we state a precise definition of partial nonlinear gain ℓ_2 stability with respect to the system (40). Then, we state our main result and four lemmas which are used in the proof of main result. Next, we investigate partial linear gain ℓ_2 stability. We finish this section by considering two special cases.

Definition 1 (Partial nonlinear gain ℓ_2 stability).

The system (40) is said to be partially nonlinear gain ℓ_2 stable with respect to sets $\mathcal{S}^\mathcal{D}$ and $\mathcal{S}^\mathcal{W}$ if for any finite horizon $N \in \mathbb{Z}_{\geq 1}$ there exist class $\mathcal{K}$ functions γ_1 and γ_3 and class $\mathcal{K}_\infty$ functions γ and γ_2 such that

$$\sum_{l=0}^k \gamma(|\phi_x(l, \xi_0, d_0^{l-1}, w_0^{l-1})|) \leq \gamma_1(|\xi_0|) + \gamma_2 \left(\sum_{l=0}^{k-1} \gamma_3(|w_l|) \right)$$

holds for any $\xi_0 \in \bar{\mathcal{X}}_0$, all $k \in \mathbb{Z}_{\geq 0}$, all sub-sequences d_0^{k-1} of $d_0^\infty \in \mathcal{S}^\mathcal{D}$ and all sub-sequences w_0^{k-1} of $w_0^\infty \in \mathcal{S}^\mathcal{W}$. ■

Next, we state our main result.

Theorem 1. Let the Assumptions 1-6 be satisfied. Then, for a finite horizon N from Assumption 2, the system (40) is partially nonlinear gain ℓ_2 stable with respect to sets $\mathcal{S}_N^\mathcal{D}$ and $\mathcal{S}^\mathcal{W}$. ■

To prove Theorem 1 we use four lemmas whose proofs are given in the appendix. The first uses Assumptions 2 and 6 to prove invariance of a set $\bar{\mathcal{X}}_N$.

Lemma 1. Consider any $d_0^\infty \in \mathcal{S}_N^\mathcal{D}$ and any $w_0^\infty \in \mathcal{S}^\mathcal{W}$. Let the Assumptions 2 and 6 be satisfied. Then, for any $\xi_0 \in \bar{\mathcal{X}}_0$ it follows that $\xi_{k_i} \in \bar{\mathcal{X}}_N$ for all $k_i \in K_N$. ■

The next result considers values of the optimal value function (34) at two consecutive successful transmission instants.

Lemma 2. Consider any $d_0^\infty \in \mathcal{S}_N^\mathcal{D}$ and any $w_0^\infty \in \mathcal{S}^\mathcal{W}$. Let the Assumptions 1-4 and 6 be satisfied. Then, there exists a class $\mathcal{K}_\infty$ function α_1 and a class $\mathcal{K}$ function α_2 such that

$$\begin{aligned} V(\xi_{k_{i+1}}) - V(\xi_{k_i}) &\leq - \sum_{l=k_i}^{k_{i+1}-1} \alpha_1(|\phi_x(l - k_i, \xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|) \\ &\quad + \sum_{l=k_i}^{k_{i+1}-1} \alpha_2(|w_l|), \end{aligned} \quad (44)$$

holds $\forall k_i, k_{i+1} \in K_N$ and $\forall \xi_{k_i}, \xi_{k_{i+1}} \in \bar{\mathcal{X}}_N$. ■

Further, we use Lemma 2 to state the result that considers the time interval $\{k_0, \dots, k\}$.

Lemma 3. *Let the assumptions of Lemma 2 be satisfied. Then for the class $\mathcal{K}_\infty$ function α_1 (from Lemma 2) and the class $\mathcal{K}$ function $\bar{\chi}_V$ (from Assumption 4) there exists a class $\mathcal{K}$ function α_3 such that*

$$\sum_{l=k_0}^{k-1} \alpha_1(|\phi_x(l-k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) \leq \bar{\chi}_V(|\xi_{k_0}|) + \sum_{l=k_0}^{k-1} \alpha_3(|w_l|) \quad (45)$$

holds for any $\xi_{k_0} \in \bar{\mathcal{X}}_N$. ■

Finally, we state the result that will enable us to consider the time interval $\{0, \dots, k_0\}$.

Lemma 4. *Let Assumptions 2 and 5 be satisfied. Then, for the class $\mathcal{K}$ function $\bar{\chi}_V$ there exist class $\mathcal{K}$ functions γ_1 and α_4 such that*

$$\bar{\chi}_V(|\xi_{k_0}|) \leq \gamma_1(|\xi_0|) + \sum_{l=0}^{k_0-1} \alpha_4(|w_l|) - \sum_{l=0}^{k_0-1} \bar{\chi}_V(|\phi_x(l, \xi_0, d_0^{l-1}, w_0^{l-1})|) \quad (46)$$

holds for any $\xi_0 \in \bar{\mathcal{X}}$, where $\bar{\chi}_V$ is as in Assumption 4. ■

PROOF (THEOREM 1). Assumptions 1-6 are satisfied, hence, the conclusions of all preceding lemmas hold. Consider some arbitrary $\xi_0 \in \bar{\mathcal{X}}_0$, $d_0^\infty \in \mathcal{S}_N^\mathcal{D}$ and $w_0^\infty \in \mathcal{S}^\mathcal{W}$. Then, according to Assumption 6 and the conclusion of Lemma 1 the corresponding $\xi_k = \phi_\xi(k, \xi_0, d_0^{k-1}, w_0^{k-1}) \in \bar{\mathcal{X}}_N$ for all $k > 0$. Note that if $0 < k < k_0 \leq N$ from Assumption 6 (see (42)) it follows $\xi_k \in \bar{\mathcal{X}}_N$; otherwise, $k \in \{k_i, \dots, k_{i+1}\}$ for some $k_i, k_{i+1} \in K_N$, and then, from the Assumption 6 (see (43)) and the conclusion of Lemma 1 it follows $\xi_k \in \bar{\mathcal{X}}_N$. Next, we use (46) to bound the term $\bar{\chi}_V(|\xi_{k_0}|)$ in (45), i.e., in 6 (see (43)) and the conclusion of Lemma 1 it follows $\xi_k \in \bar{\mathcal{X}}_N$. Next, we use (46) to bound the term $\bar{\chi}_V(|\xi_{k_0}|)$ in (45), yielding

$$\sum_{l=0}^{k_0-1} \bar{\chi}_V(|\phi_x(l, \xi_0, d_0^{l-1}, w_0^{l-1})|) + \sum_{l=k_0}^{k-1} \alpha_1(|\phi_x(l-k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) \leq \gamma_1(|\xi_0|) + \sum_{l=0}^{k_0-1} \alpha_4(|w_l|) + \sum_{l=k_0}^{k-1} \alpha_3(|w_l|).$$

Finally, by defining $\gamma \triangleq \min\{\bar{\chi}_V, \alpha_1\} \in \mathcal{K}$, $\gamma_2 \triangleq \text{Id} \in \mathcal{K}_\infty$ and $\gamma_3 \triangleq \max\{\alpha_3, \alpha_4\} \in \mathcal{K}$ we obtain the desired bound in the sense of Definition 1, namely

$$\sum_{l=0}^k \gamma(|\phi_x(l, \xi_0, d_0^{l-1}, w_0^{l-1})|) \leq \gamma_1(|\xi_0|) + \gamma_2 \left(\sum_{l=0}^{k-1} \gamma_3(|w_l|) \right) \quad (47)$$

for all $k \in \mathbb{Z}_{\geq 0}$, any $\xi_0 \in \bar{\mathcal{X}}_0$ and with respect to $\mathcal{S}_N^\mathcal{D}$ and $\mathcal{S}^\mathcal{W}$. ■

4.2.1. Partial linear gain ℓ_2 stability

Next we establish necessary conditions to obtain a more familiar partial linear gain ℓ_2 stability property, defined as follows.

Definition 2 (Linear gain ℓ_2 stability). The system (40) is said to be linear gain ℓ_2 stable with respect to the sets $\mathcal{S}^\mathcal{D}$ and $\mathcal{S}^\mathcal{W}$ if for any finite horizon $N \in \mathbb{Z}_{\geq 1}$ there exist functions $\gamma(s) = gs^2$, $\gamma_1(s) = g_1s^2$, $\gamma_2(s) = g_2s$ and $\gamma_3(s) = g_3s^2$ for some $g, g_1, g_2, g_3 > 0$, such that inequality in Assumption 1 holds for any $\xi_0 \in \bar{\mathcal{X}}_0$, all $k \in \mathbb{Z}_{\geq 0}$, all sub-sequences d_0^{k-1} of $d_0^\infty \in \mathcal{S}^\mathcal{D}$ and all sub-sequences w_0^{k-1} of $w_0^\infty \in \mathcal{S}^\mathcal{W}$. ■

Corollary 1. *Let the cost function (26) be quadratic and let the following be satisfied: 1) Assumption 1, where $\chi(s) = L_\chi s$ and $\delta(s) = L_\delta s$, $L_\chi > 0$, $L_\delta > 0$; 2) Assumption 2; 3) Assumption 3; 4) Assumption 4, where $\chi_V(s) = L_{\chi_V} s^2$ and $\bar{\chi}_V(s) = L_{\bar{\chi}_V} s^2$, $L_{\chi_V} > 0$, $L_{\bar{\chi}_V} > 0$; 5) Assumption 5, where $\bar{\chi}(s) = L_{\bar{\chi}} s$ and $\bar{\delta}(s) = L_{\bar{\delta}} s$, $L_{\bar{\chi}} > 0$, $L_{\bar{\delta}} > 0$; 6) Assumption 6. Then, for the finite horizon N from Assumption 2, the system (40) is linear gain ℓ_2 stable with respect to sets $\mathcal{S}_N^\mathcal{D}$ and $\mathcal{S}^\mathcal{W}$. Moreover, the numbers that define gains of functions $\gamma(s)$, $\gamma_1(s)$, $\gamma_2(s)$ and $\gamma_3(s)$ are given as follows: $g = \min\{L_{\bar{\chi}_V}, 2^{-1}\}$; $g_1 = (N+1)L_{\bar{\chi}_V}(2L_{\bar{\chi}})^{2(N-1)}L_{\bar{\chi}}^2$; $g_2 = 1$; $g_3 = 2^{4(N-1)}\max\{L_{\chi}^{2(N-1)}L_{\bar{\delta}}^2, \max\{N2^{-1}, L_{\chi_V}\}, NL_{\bar{\chi}_V}L_{\bar{\chi}}^{2(N-1)}L_{\bar{\delta}}^2\}$. ■*

The proof is given in the appendix.

Next we consider two special cases. The first one is related to the nominal case when there are no disturbances affecting the plant while the second one considers the scenario when there is only one plant input.

4.2.2. Disturbance-free case

When there are no disturbances the considered NCS configuration becomes essentially the same as the one investigated in our previous work Quevedo et al. (2008b). In the following corollary we strengthen the result presented there.

Corollary 2. *Let the system (40) be nonlinear gain ℓ_2 stable with respect to sets $\mathcal{S}_N^\mathcal{D}$ and $\mathcal{S}^\mathcal{W}$ and let $\mathcal{W} \equiv \{0\}$. Then,*

$$\lim_{k \rightarrow \infty} |\phi_x(k, \xi_0, d_0^{k-1}, \mathbf{0})| = 0$$

for all $\xi_0 \in \bar{\mathcal{X}}_0$, all $d_0^\infty \in \mathcal{S}_N^\mathcal{D}$, where $\mathbf{0} \in \mathcal{S}^{\{0\}}$. ■

PROOF (COROLLARY 2). It follows immediately from (47) that

$$\sum_{l=0}^k \gamma(|\phi_x(l, \xi_0, d_0^{l-1}, \mathbf{0})|) \leq \gamma_1(|\xi_0|) \quad (48)$$

for any $k \in \mathbb{Z}_{\geq 0}$, all $\xi_0 \in \bar{\mathcal{X}}_0$ and all sub-sequences d_0^{k-1} of $d_0^\infty \in \mathcal{S}_N^\mathcal{D}$, where the sequence $\mathbf{0}$ is now of length k ; see *Notation and preliminaries*. Since the sum in (48) is bounded it follows

$$\lim_{k \rightarrow \infty} \gamma(|\phi_x(k, \xi_0, d_0^{k-1}, \mathbf{0})|) = 0 \Rightarrow \lim_{k \rightarrow \infty} |\phi_x(k, \xi_0, d_0^{k-1}, \mathbf{0})| = 0$$

which recovers the conclusion from Quevedo et al. (2008b). Note that another way to conclude the latter implication is to use (A.12) as in Quevedo et al. (2008b). ■

Remark 2. In our conference paper Quevedo et al. (2008b) we needed a stronger assumption to derive at the same conclusion. Namely, the property (28) was required to hold for all nodes unlike, in the present case. ■

We finish this subsection with the following observation related to an alternative stability characterisation of Corollary 2. Note that Lemma 2 gives decrease of the Lyapunov function at successful transmission instants which are spaced by at most N steps. Now, these Lyapunov functions can be used to obtain a $\mathcal{KL}$ estimate that bounds the solutions of the system at successful transmission instants. Then, because of boundedness of solutions between successful transmission instants, we can combine the above $\mathcal{KL}$ estimate with inter-sample bounds to obtain a new $\mathcal{KL}$ estimate. This construction would be similar to one documented in Nešić et al. (1999), but, since the formal proof would be more involved due to space limitations it is omitted.

4.2.3. One node case

This particular NCS configuration has been investigated to some extent in Quevedo and Nešić (2011), where ISS was shown. However, partial nonlinear gain ℓ_2 stability was not considered. In the sequel we show that, with assumptions akin to those in Quevedo and Nešić (2011), plus an extra assumption to explicitly consider the time interval up until first successful transmission we can show partial nonlinear gain ℓ_2 stability.

First, we present the model used for generating plant trajectories, given as

$$\bar{x}^+ \triangleq f(\bar{x}, u, \bar{0}_m), \quad \bar{x} = x.$$

Note that buffer contents is not needed any more to make predictions. Further, the cost function is given as

$$J(x, \mathbf{u}) \triangleq g(\phi_{\bar{x}}(N, x, \mathbf{u})) + \sum_{k=0}^{N-1} l(\phi_{\bar{x}}(k, x, \mathbf{u}), u_k)$$

where $\mathbf{u} \triangleq \{u_0, \dots, u_{N-1}\}$ is a sequence of predicted control values. To minimize it, similar constraints as (30), (31) and (32) have to be satisfied and we write them succinctly as

$$\begin{cases} u_k \in \mathcal{U}, \quad \forall k \in \{0, \dots, N-1\}; \\ \phi_{\bar{x}}(k, x, \mathbf{u}) \in \mathcal{X}, \quad \forall k \in \{0, \dots, N-1\}; \\ \phi_{\bar{x}}(N, x, \mathbf{u}) \in \mathcal{X}_f. \end{cases} \quad (49)$$

The optimal value function is now obtained as

$$V(x) \triangleq \min_{\mathbf{u}} J(x, \mathbf{u}) \text{ s.t. (49)}$$

while the sequence of optimal control values over a finite horizon N , defined by Assumption 2, is obtained via

$$\mathbf{u}^* \triangleq \operatorname{argmin}_{\mathbf{u}} J(x, \mathbf{u}) \text{ s.t. (49)}.$$

Note that even though we have only one node (buffer) the structure of the NCS configuration stays the same, see Fig. 1. More precisely, the overall NCS closed-loop system is given by equation (40) by setting $R := 1$. The following robustness result for the NCS at hand complements Theorem 1 in Quevedo and Nešić (2011).

Corollary 3. *Let the following be satisfied: 1) Assumption 1; 2) Assumption 2; 3) Assumption 5 holds for (40) with $R := 1$; 4) Assumption 6 holds for solution of (40) with $R := 1$; 5) The weighting functions satisfy (27) and there exist a terminal control law $\kappa_f : \mathcal{X}_f \rightarrow \mathcal{U}$ such that*

$$\begin{aligned} g(f(x, \kappa_f(x), \bar{0}_m)) - g(x) + l(x, \kappa_f(x)) &\leq 0, \\ f(x, \kappa_f(x), \bar{0}_m) &\in \mathcal{X}_f, \end{aligned}$$

holds for all $x \in \mathcal{X}_f$; 6) There exist a class $\mathcal{K}_\infty$ function χ_{V_1} such that

$$|V(x) - V(y)| \leq \chi_{V_1}(|x - y|), \quad (50)$$

holds for all $x, y \in \mathcal{X}_N$. Then, for a finite horizon N from Assumption 2 the system (40) with $R := 1$ is partially nonlinear gain ℓ_2 stable with respect to sets $\mathcal{S}_N^D$ and $\mathcal{S}^W$. ■

A sketch of the proof is given in the appendix.

4.3. ISS of the overall NCS state

In this section we briefly outline how to ensure ISS of the overall NCS state.

Again, we have R models defined as

$$\bar{\xi}^+ \triangleq (\bar{x}^+, \bar{\beta}^+), \quad (\bar{x}, \bar{\beta}) = (x, \beta) \quad (51)$$

where $\bar{x}^+$ is given by (22) and $\bar{\beta}^+$ by (24). Note that control predictions in $\bar{x}^+$ are obtained using (23). Next, we denote the solution of the overall NCS model (51) k steps into the future starting at the initial state $\xi = (x, \beta)$ under the influence of $\mathbf{u}^{\beta, r}$ by $\phi_{\bar{\xi}}(k, \xi, \mathbf{u}^{\beta, r})$. The cost function is defined as

$$J(\xi, r, \mathbf{u}^r) \triangleq g(\phi_{\bar{\xi}}(N, \xi, \mathbf{u}^{\beta, r})) + \sum_{k=0}^{N-1} l(\phi_{\bar{\xi}}(k, \xi, \mathbf{u}^{\beta, r}), u_k^{\beta, r}) \quad (52)$$

where $r \in \mathcal{N}$. To minimize it, similar constraints (on control predictions and the resulting solutions of the model (51)) as for the minimization of cost function (26) have to be satisfied. Namely, the control predictions (23) are constrained as per

$$u_k^{\beta, r} \in \mathcal{U} \quad (53)$$

for all $k \in \{0, \dots, N-1\}$. Further, the solutions of states of the model (51) are constrained as

$$\phi_{\bar{\xi}}(k, \xi, \mathbf{u}^{\beta, r}) \in \bar{\mathcal{X}} \quad (54)$$

for all $k \in \{0, \dots, N-1\}$ while the final NCS state has to satisfy

$$\phi_{\bar{\xi}}(N, \xi, \mathbf{u}^{\beta, r}) \in \bar{\mathcal{X}}_f. \quad (55)$$

Again, we first minimize each cost function (recall that there are R of them) over the corresponding control sequence $\mathbf{u}^r$, yielding

$$V(\xi, r) \triangleq \min_{\mathbf{u}^r} J(\xi, r, \mathbf{u}^r) \text{ s.t. (51)\&(53)-(55)}.$$

Then, we take the minimum of the set $\{V(\xi, r)\}_{r \in \mathcal{N}}$ resulting in

$$V(\xi) \triangleq \min_r V(\xi, r)$$

which represents the optimal value function. The optimal node and the corresponding sequence of optimally predicted control values are then obtained via

$$r^* \triangleq \underset{r}{\operatorname{argmin}} V(\xi, r),$$

$$(\mathbf{u}^r)^* \triangleq \underset{\mathbf{u}^{r^*}}{\operatorname{argmin}} J(\xi, r^*, \mathbf{u}^{r^*}) \text{ s.t. (51)\&(53)-(55).}$$

Finally, for the cost function of the form (52), using the approach from Quevedo and Nešić (2011), the following result can be obtained.

Theorem 2. *Let the following be satisfied: 1) Assumption 2; 2) Assumption 6; 3) At the first successful transmission we have that $\xi_{k_0} \in \bar{\mathcal{X}}_N$; 4) The set of disturbances $\mathcal{W}$ is compact, hence, there exist*

$$|\mathbb{W}| \triangleq \max_{w \in \mathcal{W}} (|w|) < \infty; \quad (56)$$

5) There exist class $\mathcal{K}$ functions χ_ξ and χ_{w_2} such that

$$|F(\xi_1, d, w) - F(\xi_2, d, \bar{0}_m)| \leq \chi_\xi(|\xi_1 - \xi_2|) + \chi_{w_2}(|w|), \quad (57)$$

holds for all $\xi_1, \xi_2 \in \bar{\mathcal{X}}_N$, all $d \in \mathcal{S}_N^D$ and all $w \in \mathcal{W}$;

6) There exist class $\mathcal{K}_\infty$ function such that

$$|V(\xi_1) - V(\xi_2)| \leq \chi_{V_\xi}(|\xi_1 - \xi_2|) \quad (58)$$

holds for all $\xi_1, \xi_2 \in \bar{\mathcal{X}}_N$; 7) The weighting functions from the cost function (52) satisfy

$$g(\xi) \geq 0, \quad \forall \xi \in \bar{\mathcal{X}}_f, \quad g(\bar{0}_{n+pN}) = 0, \quad (59a)$$

$$l(\xi, u) \geq \alpha(|\xi|), \quad \forall \xi \in \bar{\mathcal{X}}_N, \forall u \in \mathcal{U}, \quad l(\bar{0}_{n+pN}, \bar{0}_p) = 0, \quad (59b)$$

where a function $\alpha \in \mathcal{K}_\infty$ and the set $\bar{\mathcal{X}}_N$ is the set of all feasible initial states, i.e., plant states such that (52) is bounded. Moreover, for some $r \in \mathcal{N}$ there exists a terminal control law $\kappa^r : \mathcal{X}_f \rightarrow \mathcal{U}_r$ such that the sequence of images of mapping κ^r over a horizon N denoted as κ_ξ^r satisfies

$$\begin{aligned} & g\left(\left[\begin{array}{c} f(x, W(\Psi(r, d)S\beta + \Upsilon(\kappa_\xi^r, r, d)), \bar{0}_m) \\ \Psi(r, d)S\beta + \Upsilon(\kappa_\xi^r, r, d) \end{array}\right]\right) \\ & - g(\xi) + l\left(\xi, W(\Psi(r, d)S\beta + \Upsilon(\kappa_\xi^r, r, d))\right) \leq 0, \end{aligned} \quad (60a)$$

$$\begin{aligned} & \left[\begin{array}{c} f(x, W(\Psi(r, d)S\beta + \Upsilon(\kappa_\xi^r, r, d)), \bar{0}_m) \\ \Psi(r, d)S\beta + \Upsilon(\kappa_\xi^r, r, d) \end{array}\right] \in \bar{\mathcal{X}}_f, \\ & \kappa^r(x) \in \mathcal{U}_r, \forall \xi = (x, \beta) \in \bar{\mathcal{X}}_f. \end{aligned} \quad (60b)$$

Then there exist $\beta \in \mathcal{KL}$ and $\eta \in \mathcal{K}$ such that states of the system (51) are bounded via

$$|\xi_k| \leq \beta(|\xi_{k_0}|, k - k_0) + \eta(|\mathbb{W}|), \quad \forall k \geq k_0.$$

■

The proof follows the same lines as the proof for one node case in (Quevedo and Nešić, 2011).

Next, we discuss some assumptions used above and finish by providing some insight into why ISS of the overall NCS state and not Input-Output Stability (IOS), where the output is the plant state, is obtained.

We start with assumption (56) which comes from the definition of ISS; see Definition 4.7 in (Khalil, 2002). Further, assumption (57) reflects the dependency on buffer contents which is caused by dynamic scheduling. The assumption (58) mimics the frequently used assumption when one considers robustness properties of the corresponding system governed by MPC based controllers. For instance, for non-networked systems see (Rawlings and Mayne, 2009; Magni et al., 2006; Grimm et al., 2004), while for the networked cases, see (Quevedo and Nešić, 2011). Next, assumption (59) is a reasonable extension of an assumption used in non-networked cases, e.g., see (Rawlings and Mayne, 2009; Mayne et al., 2000) and networked cases, e.g., see (Quevedo and Nešić, 2011). On the other hand, assumption (60) stems from the fact that the cost function (52) considers buffer contents explicitly, and we can access only one input node at each time instant. It should be noted that, as before, the control law used in (60) is just a construct which is not necessarily used on the plant.

As documented in Ingalls et al. (2001), to show IOS one would need to show that the corresponding system satisfies the *output-uniform asymptotic gain property* (see Definition 1.6 and Theorem 1 in Ingalls et al. (2001)). Due to the specific structure of the NCS configuration we were unable to show that this property is satisfied. Recall that the network allows access to only one input node at each time instant. Hence, buffer contents is used to derive the sequence of optimal control predictions, see Section 3. This fact makes some needed assumptions hard to satisfy. For instance, one would need the following continuity assumption on the optimal value function: “There exist a $\mathcal{K}_\infty$ class function σ such that $|V(\xi_1) - V(\xi_2)| \leq \sigma|x_1 - x_2|$ holds for all $\xi_i = (x_i, \beta_i) \in \bar{\mathcal{X}}_N$, $i \in \{1, 2\}$.”. Now, note that if the optimal value function is quadratic and $\xi_1 = (\bar{0}_n, \beta_1)$ and $\xi_2 = (\bar{0}_n, \beta_2)$ where $\beta_1 \neq \beta_2$, the latter assumption is not satisfied. Thus, we focused on ISS of the overall NCS state and showed that by modifying the cost function and assuming stronger assumptions it is achievable.

5. Simulation Results

In this section we illustrate performance aspects of the proposed method in a stochastic setup, where Assumption 2 can be violated. More precisely, we compare dynamic with static scheduling, and, setting buffer state to zero with setting buffer state to the “last value”, if no data is received over N consecutive time instances; see equation (11) and footnote 1, respectively. We consider the system

$$\begin{aligned} x_1^+ &= x_1 + u_1 \\ x_2^+ &= x_2 + u_2 \\ x_3^+ &= x_3 + x_1 u_2 - x_2 u_1 + w \end{aligned}$$

where $(x_1, x_2, x_3) = x \in \mathcal{C} \subset \mathbb{R}^3$ and $(u_1, u_2) \in \mathcal{U} \subset \mathbb{R}^2$ where $\mathcal{C}$ is a compact set; $x_0 = (-4, 4, 4)$. Disturbance $w \in \mathbb{R}$ is not known to the controller and it will be defined in the sequel. The nominal model (see (22)) is the *discrete nonholonomic integrator* and its dynamics is considered in an MPC framework in Grimm et al. (2005) (see *Example 2*). Using results presented there it follows that by choosing $\bar{\Gamma} > 16/5$ and $N = 4$, one can select stage and terminal cost as

$$l(x, u) = x_1^2 + x_2^2 + 10|x_3| \text{ and } g(x) = 4(x_1^2 + x_2^2 + 10|x_3|).$$

It follows that the corresponding MPC feedback law will globally asymptotically stabilize the origin; for details see Grimm et al. (2005). Disturbance is constructed as follows, for time intervals $\{0, 1, \dots, 20\}$ and $\{40, 41, \dots, 100\}$ it is a random Gaussian process with zero mean and variance of 0.01, while during time interval $\{20, 21, \dots, 40\}$ it is a step of amplitude one plus random Gaussian process with zero mean and variance of 0.3.

First, we consider a network which introduces *i.i.d.* dropouts with 0.2 probability where probability distribution is discrete Bernoulli distribution. Figure 3 illustrates comparison between dynamic (solid line) and static (dashed line) scheduling when buffer is set to zero while Figure 4 captures the scenario when buffer is set to the “last value”. In all figures below, in first subplot we illustrate $\mathcal{L}_2$ norms of the state of the considered system while in subplots 2 (dynamic case) and 3 (static case) we depict corresponding scheduling (solid line) and packet dropouts (diamonds) for channels 1 and 2. Finally, all subplots in each figure share the same time denoted at the bottom of the last subplot. First subplots of Fig. 3 and Fig. 4 clearly illustrate better performance when dynamic scheduling is applied regardless of how buffer state is set when no data is received over N consecutive time instances.

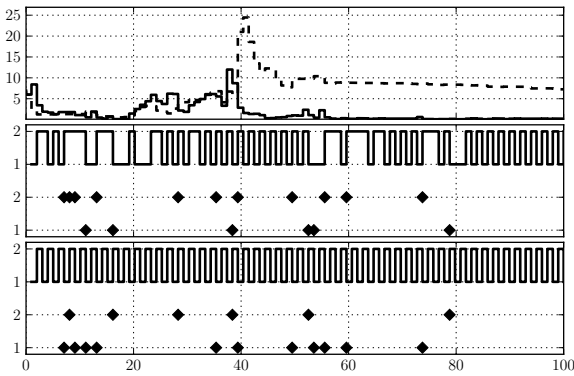


Figure 3: Dynamic and static scheduling comparison; buffer set to zero value; dropout probability 0.2.

We finish this section by increasing the dropout probability to 0.6. Note that in this scenario the buffers will more frequently run out of data due to frequent 4 or more consecutive packet dropouts as documented in second and

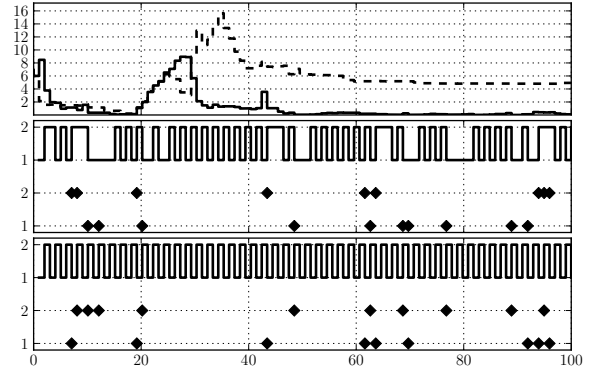


Figure 4: Dynamic and static scheduling comparison; buffer set to the “last value”; dropout probability 0.2.

third subplot of Fig. 5 and Fig. 6. Interestingly, even in this severe dropout scenario, and despite the assumptions of the stability theorem not being satisfied, our method gives good behavior as illustrated in first subplots of Fig. 5 and Fig. 6. Again, dynamic scheduling gives better performance when compared to static scheduling.

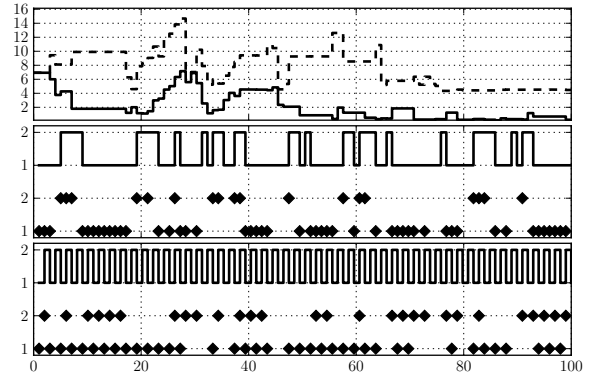


Figure 5: Dynamic and static scheduling comparison; buffer set to zero value; dropout probability 0.6.

We conclude by noting that our method showed good performance, outperformed static scheduling and illustrated great potential.

6. Conclusions and future work

Robustness properties were investigated for a NCS configuration where a data-like network is used in controller-actuator links. Two undesirable network issues, packet dropouts and access constraints, were addressed within an MPC framework by incorporating their effects into the model used for generating predictions. Several stability results were presented with the central requirement that the number of consecutive packet dropouts is uniformly

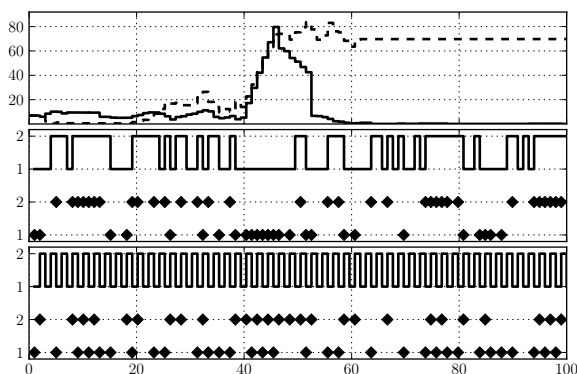


Figure 6: Dynamic and static scheduling comparison; buffer set to the “last value”; dropout probability 0.6.

bounded. To describe robustness properties of plant trajectories only, the concept of nonlinear gains was used. Due to dynamic scheduling, ISS was obtained for the aggregated state of plant and buffer state. Simulation results have illustrated good performance of the proposed method and noticeable improvement when compared to schemes using static scheduling. Future work will include the study of more general NCS's where also plant outputs are transmitted over an unreliable network and delays and quantization issues are taken into account.

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Appendix A.

We start by introducing additional notation and noting some simple observations which should aid in understanding the proofs. Consider any $d_0^\infty \in \mathcal{S}_N^D$, then, for any finite sub-sequence d_0^k we make the following partition

$$d_0^k \triangleq \{d_0^{k_0-1}, d_{k_0}^{k_1-1}, \dots, d_{k_i}^{k_{i+1}-1}, \dots, d_{k_m}^k\}. \quad (\text{A.1})$$

The reason for making this partition we demonstrate via an example. For instance, let $N = 5$, $k = 11$ and let $d_0^{11} = \{0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0\}$ be a finite sub-sequence of some infinite sequence $d_0^\infty \in \mathcal{S}_N^D$. Now, according to (A.1) we write $d_0^{11} = \{d_0^2, d_3^7, d_8^9, d_9^{11}\}$, where $k_0 = 3$, $k_1 = 8$ and $k_2 = 9$. Moreover, the notation (A.1) enables us to rewrite (41) as

$$\begin{aligned} \phi_\xi(k, \xi_0, d_0^{k-1}, w_0^{k-1}) &:= \phi_\xi(k - k_i, \xi_{k_i}, d_{k_i}^{k-1}, w_{k_i}^{k-1}), \\ \xi_{k_i} &:= \phi_\xi(k_i, \xi_0, d_0^{k_i-1}, w_0^{k_i-1}) \end{aligned} \quad (\text{A.2})$$

where $k \in \{k_i, \dots, k_{i+1}\}$ and $k_i, k_{i+1} \in K_N$. Additionally, towards the end, we will use the property (A.2) for the mapping ϕ_x as well. Further, we introduce the (open-loop) iterated mapping of (40) over a horizon N via (A.3). In a similar fashion, the (open-loop) iterated mapping of (2) with implicit input (39) over a horizon N is introduced via (A.4). Also, note that

$$\xi_j = \phi_\xi(j - k_i, \xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) = F^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) \quad (\text{A.6})$$

for all $j \in \{k_i, \dots, k_i + N\}$ and all $k_i \in K_N$. Finally, from (1) it follows that

$$-\alpha(a + b) \leq -\frac{1}{2}\alpha(a) + \frac{1}{2}\alpha(b), \quad (\text{A.7})$$

and,

$$\alpha\left(\sum_{l=i}^j a_l\right) \leq \sum_{l=i}^j \alpha(2^{j-l} a_l), \quad (\text{A.8})$$

for all $i, j \in \mathbb{Z}_{\geq 0}$ where $i \leq j^3$; while

$$\sum_{j=j_1}^{j_2} \alpha\left(\sum_{l=i}^j a_l\right) \leq (j_2 - j_1 + 1) \sum_{l=i}^{j_2} \alpha(2^{j_2-l} a_l) \quad (\text{A.9})$$

for all $i, j, j_1, j_2 \in \mathbb{Z}_{\geq 0}$ where $i \leq j \leq j_1 \leq j_2$.

³In the sequel, we use the convention $\sum_{l=i}^j a_l = 0$ whenever $i > j$.

PROOF (LEMMA 1). The conclusion of Assumption 2 implies that $k_0 \leq N$, hence, from (42) it follows that $\xi_{k_0} \in \bar{\mathcal{X}}_N$. Finally, by using (43) with the notation (A.2), one can easily show, by induction, that $\xi_{k_i} \in \bar{\mathcal{X}}_N$ for all $k_i \in K_N$.

PROOF (LEMMA 2). Let us start by splitting the left hand side of (44) into two parts, the one which involves the disturbances and the one which does not, namely

$$\begin{aligned} V(\xi_{k_{i+1}}) - V(\xi_{k_i}) &= V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, w_{k_i}^{k_{i+1}-1})) - V(\xi_{k_i}) \\ &= V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})) - V(\xi_{k_i}) + V(F^{k_{i+1}}(\xi_{k_i}, \\ &\quad d_{k_i}^{k_{i+1}-1}, w_{k_i}^{k_{i+1}-1})) - V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})). \end{aligned}$$

Note that $d_{k_i}^{k_{i+1}-1} \subset d_0^\infty \in \mathcal{S}_N^D$ and $w_{k_i}^{k_{i+1}-1} \subset w_0^\infty \in \mathcal{S}^W$. Next, we focus on the term $V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})) - V(\xi_{k_i})$. Before we proceed, note that from Assumption 2 and equation (7) it follows that $m_i \leq N - 1$. For the sake of clarity we will first consider case $m_i < N - 1$, followed then by the case $m_i = N - 1$.

Case $m_i < N - 1$: The idea is to find a suitable *feasible* control sequence which will provides us with the appropriate bound on the corresponding term. Towards obtaining such a sequence, consider a node $\tilde{r}_{k_{i+1}} \in \mathcal{N}$ from Assumption 3. Note that this node is not necessarily node $r_{k_i}^*$. Also, worth emphasizing is that in Quevedo et al. (2008b) second part of Assumption 3 had to hold for *all* nodes while in current case it has to hold only for *some* node. Consequently, consider control sequence

$$\tilde{\mathbf{u}}_{k_{i+1}} = \{u_{k_{i+1}}^{\beta_{k_i}, r_{k_i}^*}, \dots, u_{k_i+N-1}^{\beta_{k_i}, r_{k_i}^*}, \tilde{u}_{k_i+N}, \dots, \tilde{u}_{k_{i+1}+N-1}\} \quad (\text{A.10})$$

where first $N - m_i - 1$ elements are the control values from the buffers (recall that these are the result of previous optimizations; see Section 3). Now, since node $\tilde{r}_{k_{i+1}} \in \mathcal{N}$ (which satisfies Assumption 3) is considered, the remaining $m_i + 1$ elements are given according to

$$\tilde{u}_{k_{i+1}+j} = \kappa^{\tilde{r}_{k_{i+1}}}(\tilde{x}_{k_{i+1}+j}), \forall j \in \{N - m_i - 1, \dots, N - 1\}$$

where $\tilde{x}^+ = f(\tilde{x}, \kappa^{\tilde{r}_{k_{i+1}}}(\tilde{x}), \bar{\mathbf{0}}_m)$ with initial state $\tilde{x} := f^{k_i+N}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})$. Finally, we conclude that control sequence (A.10) is a feasible control sequence. Now, direct calculation give that

$$\begin{aligned} J(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0}), \tilde{r}_{k_{i+1}}, \tilde{\mathbf{u}}_{k_{i+1}}) \\ = V(\xi_{k_i}) - \sum_{j=k_i}^{k_i+m_i} l(f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0}), u_j^{\beta_{k_i}, r_{k_i}^*}) \\ + \sum_{j=k_i+N}^{k_{i+1}+N-1} (g(f(\tilde{x}_j, \tilde{u}_j, \mathbf{0})) - g(\tilde{x}_j) + l(\tilde{x}_j, \tilde{u}_j)ig) \\ \leq V(\xi_{k_i}) - \sum_{j=k_i}^{k_i+m_i} l(f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0}), u_j^{\beta_{k_i}, r_{k_i}^*}) \end{aligned} \quad (\text{A.11})$$

where $\tilde{\mathbf{u}}_{k_{i+1}} = \{W_{p_{\tilde{r}_{k_{i+1}}}} S \beta_{k_i+m_i+1}, \dots, W_{p_{\tilde{r}_{k_{i+1}}}} S \beta_{k_i+N-1}, \tilde{u}_{k_{i+1}+N}, \dots, \tilde{u}_{k_{i+1}+N-1}\}$. Due to optimality

$$\begin{aligned} V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})) \\ \leq J(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0}), \tilde{r}_{k_{i+1}}, \tilde{\mathbf{u}}_{k_{i+1}}) \end{aligned}$$

$$F^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) \triangleq \begin{cases} \xi_{k_i}, & \text{for } j = k_i, \\ F(F^{j-1}(\xi_{k_i}, d_{k_i}^{j-2}, w_{k_i}^{j-2}), d_{j-1}, w_{j-1}), & \text{for } k_i + 1 \leq j \leq k_i + N. \end{cases} \quad (\text{A.3})$$

$$f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) \triangleq \begin{cases} x_{k_i}, & \text{for } j = k_i, \\ f(f^{j-1}(\xi_{k_i}, d_{k_i}^{j-2}, w_{k_i}^{j-2}), u_{j-1}^{\beta_{k_i}, r_{k_i}^*}, w_{j-1}), & \text{for } k_i + 1 \leq j \leq k_i + N, \end{cases} \quad (\text{A.4})$$

$$\text{where } u_{j-1}^{\beta_{k_i}, r_{k_i}^*} = W(\Psi(d_{j-1} r_{k_i}^*) S \beta_{j-1} + d_{j-1} \Upsilon((\mathbf{u}^r)_{k_i}^*, r_{k_i}^*)), \forall j \in \{k_i + 1, \dots, k_i + N\}, \quad (\text{A.5})$$

$$\beta^+ = \Psi(dr^*) S \beta + d \Upsilon((\mathbf{u}^r)^*, r^*), \quad \beta = \beta_{k_i}, \quad ((\mathbf{u}^r)^*, r^*) = ((\mathbf{u}^r)_{k_i}^*, r_{k_i}^*).$$

hence,

$$\begin{aligned} & V(F^{k_i+1}(\xi_{k_i}, d_{k_i}^{k_i+1-1}, \mathbf{0})) - V(\xi_{k_i}) \\ & \leq - \sum_{j=k_i}^{k_i+m_i} l(f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0}), u_j^{\beta_{k_i}, r_{k_i}^*}) \\ & \leq - \sum_{j=k_i}^{k_i+m_i} \alpha(|f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})|) \end{aligned} \quad (\text{A.12})$$

as desired.

Case $m_i = N - 1$: In this case we consider a *feasible* control sequence $\{\tilde{u}_{k_i+1}, \dots, \tilde{u}_{k_i+1+N-1}\}$. Using the steps as above, one can easily show that $\sum_{j=k_i+N}^{k_i+N-1} l(f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0}), u_j^{\beta_{k_i}, r_{k_i}^*}) = 0$ (see footnote 3). Finally, concluding (A.11) follows similarly as for $m_i < N - 1$.

Now, we switch focus on the second term, i.e., $V(F^{k_i+1}(\xi_{k_i}, d_{k_i}^{k_i+1-1}, w_{k_i}^{k_i+1-1})) - V(F^{k_i+1}(\xi_{k_i}, d_{k_i}^{k_i+1-1}, \mathbf{0}))$. It follows, from application of (35a) from Assumption 4 that

$$\begin{aligned} & V(F^{k_i+1}(\xi_{k_i}, d_{k_i}^{k_i+1-1}, w_{k_i}^{k_i+1-1})) - V(F^{k_i+1}(\xi_{k_i}, d_{k_i}^{k_i+1-1}, \mathbf{0})) \\ & \leq \chi_V(|f^{k_i+1}(\xi_{k_i}, d_{k_i}^{k_i+1-1}, w_{k_i}^{k_i+1-1}) - f^{k_i+1}(\xi_{k_i}, d_{k_i}^{k_i+1-1}, \mathbf{0})|). \end{aligned} \quad (\text{A.13})$$

Further, even though it is obvious, it might be useful notice that both, $F^{k_i+1}(\cdot, \cdot, w_{k_i}^{k_i+1-1})$ and $F^{k_i+1}(\cdot, \cdot, \mathbf{0})$, have the same buffer contents; also, recall that we consider the sequence of dropout outcomes $d_{k_i}^{k_i+1-1} = \{1, 0, \dots, 0\}$. Now, according to the Assumption 1 and with the aid of notation (A.4) we can state the following observation about the argument of the right hand side function of (A.13) as

$$\begin{aligned} & |f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) - f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})| = |f(f^{j-1}(\xi_{k_i}, d_{k_i}^{j-2}, w_{k_i}^{j-2}, u_{j-1}^{\beta_{k_i}, r_{k_i}^*}, w_{j-1}) - f(f^{j-1}(\xi_{k_i}, d_{k_i}^{j-2}, \mathbf{0}), u_{j-1}^{\beta_{k_i}, r_{k_i}^*}, 0)| \\ & \leq \chi(|f^{j-1}(\xi_{k_i}, d_{k_i}^{j-2}, w_{k_i}^{j-2}) - f^{j-1}(\xi_{k_i}, d_{k_i}^{j-2}, \mathbf{0})|) + \delta(|w_{j-1}|) \end{aligned}$$

for all $j \in \{k_i, \dots, k_i+1\}$. Note that for $j = k_i$, it follows $|x_{k_i} - x_{k_i}| = 0$; see (A.4). Now, by using the latter inequality one can easily show that

$$|f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) - f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})| \leq \sum_{l=1}^{j-k_i} v_l(|w_{j-l}|)$$

where $v_{l+1} = \chi \circ 2 \cdot \text{Id} \circ v_l \in \mathcal{K}$ and $v_1 = \delta \in \mathcal{K}$ for all $j \in \{k_i, \dots, k_i+1\}$; recall that $\mathcal{K}$ functions χ and δ originate from Assumption 1. Further, due to Assumption

2, $k_{i+1} - k_i \leq N$, moreover, since χ and δ are class $\mathcal{K}$ functions, we can always, if needed, bound them with new class $\mathcal{K}$ functions so that $v_{(\cdot)}$ functions satisfy $v_{l+1} > v_l$. It follows then $v_N = \max_{l \in \{1, \dots, N\}} v_l$. Now, with the latter observation the latter inequality becomes

$$|f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) - f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})| \leq \sum_{l=k_i}^{j-1} v_N(|w_l|), \quad (\text{A.14})$$

for all $j \in \{k_i, \dots, k_{i+1}\}$. Setting $j = k_{i+1}$ results in the following upper bound

$$\begin{aligned} & |f^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, w_{k_i}^{k_{i+1}-1}) - f^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})| \\ & \leq \sum_{l=k_i}^{k_{i+1}-1} v_N(|w_l|). \end{aligned}$$

Recall that due to Assumptions 3 and 6 $V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, w_{k_i}^{k_{i+1}-1}))$ (see also (A.6)) is bounded. Finally, by applying (A.8) to the inequality (A.13) yields

$$\begin{aligned} & V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, w_{k_i}^{k_{i+1}-1})) - V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})) \\ & \leq \chi_V \left(\sum_{l=k_i}^{k_{i+1}-1} v_N(|w_l|) \right) \leq \sum_{l=k_i}^{k_{i+1}-1} \chi_V(2^{k_{i+1}-1-k_i} v_N(|w_l|)) \\ & \leq \sum_{l=k_i}^{k_{i+1}-1} \chi_V(2^{N-1} v_N(|w_l|)). \end{aligned} \quad (\text{A.15})$$

Towards combining the latter bound with the bound (A.12) to get the desired bound we do the following. We first add and subtract the term $|f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1})|$ to the argument of the α function on the right hand side of the inequality (A.12). Then we use observation (A.7) and the fact that $|x| - |y| \leq |x - y|$, $\forall x, y \in \mathbb{R}^n$ to rewrite the bound (A.12) as

$$\begin{aligned} & V(F^{k_{i+1}}(\xi_{k_i}, d_{k_i}^{k_{i+1}-1}, \mathbf{0})) - V(\xi_{k_i}) \\ & \leq - \sum_{j=k_i}^{k_i+m_i} \alpha(|f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})|) \\ & = - \sum_{j=k_i}^{k_i+m_i} \alpha(|f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})| + |f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1})| - |f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1})|) \\ & \leq - \sum_{j=k_i}^{k_i+m_i} \frac{1}{2} \alpha(|f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1})|) + \sum_{j=k_i}^{k_i+m_i} \frac{1}{2} \alpha(|f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) - f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})|). \end{aligned} \quad (\text{A.16})$$

Moreover, by using (A.14) we can bound second term on the right hand side of the last inequality in (A.16) as

$$\begin{aligned} \sum_{j=k_i}^{k_i+m_i} \frac{1}{2} \alpha(|f^j(\xi_{k_i}, d_{k_i}^{j-1}, w_{k_i}^{j-1}) - f^j(\xi_{k_i}, d_{k_i}^{j-1}, \mathbf{0})|) \\ \leq \sum_{j=k_i}^{k_i+m_i} \frac{1}{2} \alpha \left(\sum_{l=k_i}^{j-1} v_N(|w_l|) \right). \quad (\text{A.17}) \end{aligned}$$

Now, noticing that for $j = k_i$ the inner sum in (A.17) on the right hand side, according to our convention, is equal to zero (see footnote 3) and applying observation (A.9) results in

$$\begin{aligned} \sum_{j=k_i}^{k_i+m_i} \frac{1}{2} \alpha \left(\sum_{l=k_i}^{j-1} v_N(|w_l|) \right) &\leq m_i \sum_{l=k_i}^{k_i+m_i-1} \frac{1}{2} \alpha(2^{m_i-1} v_N(|w_l|)) \\ &\leq \sum_{l=k_i}^{k_i+1-1} \frac{N}{2} \alpha(2^{N-1} v_N(|w_l|)). \quad (\text{A.18}) \end{aligned}$$

Finally, first we replace j with l in the first sum on the right hand side of the last inequality of (A.16); then we use (A.18) and (A.17) in (A.16) and combine the new bound with (A.15); this is followed by defining bound functions $\alpha_1 \triangleq \frac{1}{2} \cdot \text{Id} \circ \alpha \in \mathcal{K}_\infty$, $\alpha_2 \triangleq \max\{\frac{N}{2} \cdot \text{Id} \circ \alpha \circ 2^{N-1} \cdot \text{Id} \circ v_N, \chi_V \circ 2^{N-1} \cdot \text{Id} \circ v_N\} \in \mathcal{K}$; at the end we use (A.6) (recall that $\phi_\xi = (\phi_x, \phi_\beta)$) to obtain desired inequality (44). ■

PROOF (LEMMA 3). First we consider time interval $\{k_0, \dots, k_i\}$. We do that by applying Lemma 2 i times resulting in i formulas which cover time interval of interest. Then, we add these i formulas which yields

$$\begin{aligned} V(\xi_{k_i}) - V(\xi_{k_0}) &\leq - \sum_{j=0}^{i-1} \sum_{l=k_j}^{k_{j+1}-1} \alpha_1(|\phi_x(l - k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) \\ &\quad + \sum_{j=0}^{i-1} \sum_{l=k_j}^{k_{j+1}-1} \alpha_2(|w_l|) \\ &= - \sum_{l=k_0}^{k_i-1} \alpha_1(|\phi_x(l - k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) + \sum_{l=k_0}^{k_i+1-1} \alpha_2(|w_l|), \end{aligned}$$

Moreover, by using the second inequality from Assumption 4 we get

$$\begin{aligned} V(\xi_{k_i}) + \sum_{l=k_0}^{k_i-1} \alpha_1(|\phi_x(l - k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) \\ \leq V(\xi_{k_0}) + \sum_{l=k_0}^{k_i-1} \alpha_2(|w_l|) \leq \bar{\chi}_V(|\xi_{k_0}|) + \sum_{l=k_0}^{k_i-1} \alpha_2(|w_l|). \quad (\text{A.19}) \end{aligned}$$

Next, we concentrate on the time interval between k_i and

$k \leq k_{i+1}$. First, note that

$$\begin{aligned} V(\xi_{k_i}) &= J(\xi_{k_i}, r_{k_i}^*, (\mathbf{u}^r)_{k_i}^*) \\ &= \sum_{l=k_i}^{k_i+N-1} l(f^l(\xi_{k_i}, d_{k_i}^{l-1}, \mathbf{0}), u_l^{\beta_{k_i}, r_{k_i}^*}) + g(f^{k_i+N}(\xi_{k_i}, d_{k_i}^{k_i+N-1}, \mathbf{0})) \\ &\geq \sum_{l=k_i}^k l(f^l(\xi_{k_i}, d_{k_i}^{l-1}, \mathbf{0}), u_l^{\beta_{k_i}, r_{k_i}^*}) \\ &\geq \sum_{l=k_i}^k \alpha(|f^l(\xi_{k_i}, d_{k_i}^{l-1}, \mathbf{0})|), \end{aligned}$$

where $k_i \leq k \leq k_{i+1} - 1$. Now, adding and subtracting term $|f^l(\xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|$ to the argument of the $\alpha(\cdot)$ function in latter inequality and using the lower bound from (1) results in

$$\begin{aligned} V(\xi_{k_i}) &\geq \sum_{l=k_i}^k \left(\frac{1}{2} \alpha(|f^l(\xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|) \right. \\ &\quad \left. + \frac{1}{2} \alpha(|f^l(\xi_{k_i}, d_{k_i}^{l-1}, \mathbf{0})| - |f^l(\xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|) \right) \\ &\geq \sum_{l=k_i}^k \alpha_1(|f^l(\xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|) \\ &\quad - \sum_{l=k_i}^k \frac{1}{2} \alpha(|f^l(\xi_{k_i}, d_{k_i}^{l-1}, \mathbf{0})| - |f^l(\xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|), \end{aligned}$$

which when used in (A.19) results in

$$\begin{aligned} \sum_{l=k_0}^{k_i-1} \alpha_1(|\phi_x(l - k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) \\ + \sum_{l=k_i}^k \alpha_1(|f^l(\xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|) \\ \leq \bar{\chi}_V(|\xi_{k_0}|) + \sum_{l=k_0}^{k_i-1} \alpha_2(|w_l|) \\ + \sum_{l=k_i}^k \frac{1}{2} \alpha(|f^l(\xi_{k_i}, d_{k_i}^{l-1}, \mathbf{0})| - |f^l(\xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|). \end{aligned}$$

Finally, using (A.6) (recall that that $\phi_\xi = (\phi_x, \phi_\beta)$) and the bound (A.17)-(A.18) provides

$$\begin{aligned} \sum_{l=k_0}^{k-1} \alpha_1(|\phi_x(l - k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) \\ \leq \bar{\chi}_V(|\xi_{k_0}|) + \sum_{l=k_0}^{k_i-1} \alpha_2(|w_l|) + \sum_{l=k_i}^k \frac{N}{2} \alpha(2^{N-1} v_N(|w_l|)), \end{aligned}$$

which gives desired inequality (45) by defining $\alpha_3 \triangleq \max\{\alpha_2, \frac{N}{2} \cdot \text{Id} \circ \alpha \circ 2^{N-1} \cdot \text{Id} \circ v_N\} \in \mathcal{K}$. ■

PROOF (LEMMA 4). We start by observing that according to Assumption 5 it holds that

$$|\phi_\xi(1 - 0, \xi_0, d_0^0, w_0^0)| \leq \bar{\chi}(|\xi_0|) + \bar{\delta}(|w_0|).$$

Continuing in this manner until time k_0 one can easily show that the following holds

$$|\phi_\xi(j, \xi_0, d_0^{j-1}, w_0^{j-1})| \leq \bar{\chi}_j(|\xi_0|) + \sum_{l=1}^j \bar{v}_l(|w_{j-l}|)$$

where $\bar{\chi}_{l+1} = \bar{\chi} \circ 2 \cdot \text{Id} \circ \bar{\chi}_l \in \mathcal{K}$ with $\bar{\chi}_0 = \text{Id}$, $\bar{\chi}_1 = \bar{\chi}$ and $\bar{v}_{l+1} = \bar{\chi} \circ 2 \cdot \text{Id} \circ \bar{v}_l \in \mathcal{K}$ with $\bar{v}_1 = \bar{\delta}$ for all $j \in \{0, \dots, k_0\}$; note that functions $\bar{\chi}$ and $\bar{\delta}$ come from Assumption 5. Further, due to Assumption 2, $k_0 \leq N$; again, with the similar argument about $v_{(\cdot)}$ presented above, we can always ensure that $\bar{\chi}_{l+1} > \bar{\chi}_l$ and $\bar{v}_{l+1} > \bar{v}_l$ which results in $\bar{v}_N = \max_{l \in \{1, \dots, N\}} \bar{v}_l \in \mathcal{K}$ and $\bar{\chi}_N = \max_{l \in \{1, \dots, N\}} \bar{\chi}_l \in \mathcal{K}$, which results in

$$|\phi_\xi(j, \xi_0, d_0^{j-1}, w_0^{j-1})| \leq \bar{\chi}_N(|\xi_0|) + \sum_{l=0}^{j-1} \bar{v}_N(|w_l|).$$

Now, applying a function $\bar{\chi}_V$ from Assumption 4 to latter inequality and summing the corresponding inequality from $j = 0$ to $j = k_0$ with application of (A.9) results in

$$\begin{aligned} & \sum_{j=0}^{k_0} \bar{\chi}_V(|\phi_\xi(j, \xi_0, d_0^{j-1}, w_0^{j-1})|) \\ & \leq (k_0 + 1) \bar{\chi}_V(\bar{\chi}_N(|\xi_0|)) + \sum_{j=0}^{k_0} \bar{\chi}_V\left(\sum_{l=0}^{j-1} \bar{v}_N(|w_l|)\right) \\ & \leq (N + 1) \bar{\chi}_V(\bar{\chi}_N(|\xi_0|)) + k_0 \sum_{l=0}^{k_0-1} \bar{\chi}_V(2^{k_0-1} \bar{v}_N(|w_l|)) \\ & \leq \gamma_1(|\xi_0|) + \sum_{l=0}^{k_0-1} N \bar{\chi}_V(2^{N-1} \bar{v}_N(|w_l|)) \end{aligned}$$

where $\gamma_1 \triangleq (N + 1) \cdot \text{Id} \circ \bar{\chi}_V \circ \bar{\chi}_N \in \mathcal{K}$. Further, note that

$$\begin{aligned} & |\phi_\xi(j, \xi_0, d_0^{j-1}, w_0^{j-1})| \\ & = |(\phi_x(j, \xi_0, d_0^{j-1}, w_0^{j-1}), \phi_\beta(j, \beta_0, d_0^{j-1}))| \\ & \geq |\phi_x(j, \xi_0, d_0^{j-1}, w_0^{j-1})| \end{aligned}$$

due to the definition of the overall NCS (25) and the definition of the Euclidean norm. Hence

$$\begin{aligned} & \sum_{j=0}^{k_0-1} \bar{\chi}_V(|\phi_x(j, \xi_0, d_0^{j-1}, w_0^{j-1})|) + \bar{\chi}_V(|\phi_\xi(k_0, \xi_0, d_0^{k_0-1}, w_0^{k_0-1})|) \\ & \leq \gamma_1(|\xi_0|) + \sum_{l=0}^{k_0-1} \alpha_4(|w_l|). \end{aligned}$$

where $\alpha_4 \triangleq N \cdot \text{Id} \circ \bar{\chi}_V \circ 2^{N-1} \cdot \text{Id} \circ \bar{v}_N \in \mathcal{K}$. Finally, by replacing j with l we obtain (46), as desired. ■

PROOF (COROLLARY 1). The proof follows the same lines as the proof of Theorem 1. Hence, we merely present the resulting functions. They are as follows: $\alpha_1(s) = 2^{-1}s^2$; $\alpha_2(s) = \max\{N2^{-1}, L_{\bar{\chi}_V}\} \cdot 2^{4(N-1)} L_{\bar{\chi}}^{2(N-1)} L_{\bar{\delta}}^2 s^2$; $\alpha_3(s) = \alpha_2(s)$; $\alpha_4(s) = N L_{\bar{\chi}_V} 2^{4(N-1)} L_{\bar{\chi}}^{2(N-1)} L_{\bar{\delta}}^2 s^2$; $\gamma(s) = \min\{L_{\bar{\chi}_V}, 2^{-1}\} \cdot s^2$; $\gamma_1(s) = (N + 1) L_{\bar{\chi}_V} (2 L_{\bar{\chi}})^{2(N-1)} L_{\bar{\chi}}^2 s^2$; $\gamma_2(s) = s$; $\gamma_3(s) = \max\{L_{\bar{\chi}}^{2(N-1)} L_{\bar{\delta}}^2 c, N L_{\bar{\chi}_V} L_{\bar{\chi}}^{2(N-1)} L_{\bar{\delta}}^2\} \cdot 2^{4(N-1)} s^2$. ■

PROOF (COROLLARY 3, SKETCH). We start by observing that inequality (50) implies $V(x) \leq \chi_{V_1}|x|$, $\forall x \in \mathcal{X}_N$. Further, since $\xi = (x, \beta)$, it follows from definition of Euclidean that $V(x) \leq \chi_{V_1}|x| \leq \chi_{V_1}|\xi|$, $\forall \xi \in \bar{\mathcal{X}}_N$. Now, since in comparison to the proof of Theorem 1 the main difference is that now $R = 1$, we just present the changes to the corresponding lemmas. Lemma 1 remains the same. The conclusion of Lemma 2 becomes

$$\begin{aligned} & V(x_{k_{i+1}}) - V(x_{k_i}) \\ & \leq - \sum_{l=k_i}^{k_{i+1}-1} \alpha_1(|\phi_x(l - k_i, \xi_{k_i}, d_{k_i}^{l-1}, w_{k_i}^{l-1})|) + \sum_{l=k_i}^{k_{i+1}-1} \alpha_2(|w_l|) \end{aligned}$$

for all $k_i, k_{i+1} \in K_N$ and all $x_{k_i}, x_{k_{i+1}} \in \mathcal{X}_N$. Note that due to (50) in the definition of a function α_2 a function χ_{V_1} will replace function χ_V . Further, usage of $V(x) \leq \chi_{V_1}|x| \leq \chi_{V_1}|\xi|$ results in the following change in the conclusion of Lemma 3, namely

$$\sum_{l=k_0}^{k-1} \alpha_1(|\phi_x(l - k_0, \xi_{k_0}, d_{k_0}^{l-1}, w_{k_0}^{l-1})|) \leq \bar{\chi}_{V_1}(|\xi_{k_0}|) + \sum_{l=k_0}^{k-1} \alpha_3(|w_l|)$$

for any $\xi_{k_0} \in \bar{\mathcal{X}}_N$. Finally, the conclusion of Lemma 4 takes the form

$$\begin{aligned} & \bar{\chi}_{V_1}(|\xi_{k_0}|) \\ & \leq \gamma_1(|\xi_0|) + \sum_{l=0}^{k_0-1} \alpha_4(|w_l|) - \sum_{l=0}^{k_0-1} \bar{\chi}_{V_1}(|\phi_x(l, \xi_0, d_0^{l-1}, w_0^{l-1})|) \end{aligned}$$

for any $\xi_0 \in \bar{\mathcal{X}}_0$. The proof now follows along the lines as in the proof of Theorem 1 presented above. ■