Journal of Statistical Physics (2023) 190:160
https://doi.org/10.1007/s10955-023-03171-x

Expanding the Fourier Transform of the Scaled Circular Jacobi β Ensemble Density

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Received: 5 June 2023 / Accepted: 22 September 2023 / Published online: 18 October 2023

Abstract

The family of circular Jacobi β ensembles has a singularity of a type associated with Fisher and Hartwig in the theory of Toeplitz determinants. Our interest is in the Fourier transform of the corresponding $N \to \infty$ bulk scaled spectral density about this singularity, expanded as a series in the Fourier variable. Various integrability aspects of the circular Jacobi β ensemble are used for this purpose. These include linear differential equations satisfied by the scaled spectral density for $\beta = 2$ and $\beta = 4$, and the loop equation hierarchy. The polynomials in the variable $u = 2/\beta$ which occur in the expansion coefficents are found to have special properties analogous to those known for the structure function of the circular β ensemble, specifically in relation to the zeros lying on the unit circle $|u| = 1$ and interlacing. Comparison is also made with known results for the expanded Fourier transform of the density about a guest charge in the two-dimensional one-component plasma.

Keywords Random matrices · Circular beta ensemble · Structure function · Loop equations

1 Introduction

1.1 Definition of the Eigenvalue PDF and Quantities of Interest

In random matrix theory, the joint eigenvalue probability density function (PDF) proportional to

Communicated by Pierpaolo Vivo.

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$$\prod_{l=1}^{N} w^{(\mathrm{cJ})}(\theta_l) \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^{\beta}, \ w^{(\mathrm{cJ})}(\theta) = e^{q\theta} |1 + e^{i\theta}|^{\beta p} \ (-\pi < \theta \le \pi), \quad (1.1)$$

is referred to as the generalised circular Jacobi β ensemble. This reduces to the PDF specifying the circular β ensemble (see [27, Sect. 2.8]) upon setting $p = q = 0$. With $q = 0$ but $p \neq 0$, an interpretation of the factor $|1 + e^{i\theta}|^{\beta p}$ is as being due to a spectrum singularity of degeneracy p at $\theta = \pm\pi$, and the resulting PDF has been termed the circular Jacobi ensemble; see [27, Sect. 3.9]. With $q \neq 0$ also, the factor $e^{q\theta}$ is discontinuous comparing values at $\theta = \pm\pi$, extending the spectrum singularity to the class introduced by Fisher and Hartwig in the theory of asymptotics of Toeplitz determinants [26]. We remark that generally the study of Fisher-Hartwig singularities in random matrix theory and its applications is active to this present day, with recent references including [1, 6, 9, 11, 14, 16, 21, 22, 42, 67, 71]. The naming "circular Jacobi" comes from the fact that the orthogonal polynomials in $z = e^{i\theta}$ with respect to the inner product $(f, g) = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)} w^{(\mathrm{cJ})}(\theta)|_{\beta=2}\, d\theta$ are given by the hypergeometric polynomials $\{{}_2F_1(-n, b, b + \bar{b}; 1 - z)\}$, where $b = p - iq$, which in term can be written in terms of Jacobi polynomials [63]. The construction of a random Hessenberg matrix with eigenvalue PDF (1.1) has been given in [10].

The shift $\theta_j \mapsto \theta_j + \pi$ in (1.1) gives the modified generalised circular β ensemble, denoted $\widetilde{\mathrm{cJ}}$, with weight $w^{(\widetilde{\mathrm{cJ}})}(\theta) = e^{q(\theta-\pi)}|1 - e^{i\theta}|^{\beta p}$. In the present paper we focus our attention on the density for the (modified) generalised circular Jacobi β ensemble, $\rho_{(1),N}^{(\widetilde{\mathrm{cJ}})}(\theta; \beta, p, q)$ say, and the analogue of the corresponding moments, both of which are to be appropriately scaled with $N \to \infty$. The density is defined by integrating out all but one of the θ_l in the PDF specified by (1.1). There being N choices of the one eigenvalue not integrated out effectively contributes a factor of N—see (2.1) below with $k = 1$. The moments are the coefficients $\{c_k^{(\widetilde{\mathrm{cJ}})}(N, \beta, p, q)\}$ of the Fourier series for the density,

$$\rho_{(1),N}^{(\widetilde{\mathrm{cJ}})}(\theta; \beta, p, q) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k^{(\widetilde{\mathrm{cJ}})}(N, \beta, p, q) e^{-ik\theta}. \quad (1.2)$$

The particular $N \to \infty$ scaled limit of interest is

$$c_\infty^{(\widetilde{\mathrm{cJ}})}(\tau; \beta, p, q) = \lim_{N,k\to\infty} c_k^{(\widetilde{\mathrm{cJ}})}(N, \beta, p, q)\Big|_{\tau=2\pi k/N}, \qquad (\tau \neq 0), \quad (1.3)$$

which is expected to be well defined for general real τ.

Associated with the scaled limit (1.3) is a scaling limit in the neighbourhood of the spectrum singularity of the density itself, first studied for $q = 0$ and $\beta = 1, 2$ and 4 in [38], and extended to all even β and general q in [37, 53]. For general $\beta > 0$ this can also be regarded as a bulk scaling limit, with origin at the spectrum singularity, specified by taking $N \to \infty$ and the angular coordinates θ_j scaled $\theta_j \mapsto 2\pi x_j/L$ so that the mean spacing between eigenvalues L/N is a constant, which we take to be unity by choosing $L = N$. The Fourier series for finite N in (1.2), upon subtracting the constant $c_0^{(\mathrm{cJ})} = 2\pi N$, then tends to a well defined Fourier transform relating to $c_\infty^{(\widetilde{\mathrm{cJ}})}$. Thus the scaling limit of the density in the neighbourhood of the spectrum singularity is specified by

$$\rho_{(1),\infty}^{(\mathrm{cJ})}(x; \beta, p, q) := \lim_{N\to\infty} \frac{2\pi}{N} \rho_{(1),N}^{(\mathrm{cJ})}(-\pi\, \mathrm{sgn}(x) + 2\pi x/N; \beta, p, q), \quad (1.4)$$

or in the setting of the weight (1.1)

$$\rho_{(1),\infty}^{(\widetilde{\mathrm{cJ}})}(x;\beta,p,q) := \lim_{N\to\infty}\frac{2\pi}{N}\rho_{(1),N}^{(\widetilde{\mathrm{cJ}})}(2\pi\chi_{x<0}+2\pi x/N;\beta,p,q). \tag{1.5}$$

Applying this limit to the Fourier series (1.2) we obtain the Fourier transform expression

$$\rho_{(1),\infty}^{(\widetilde{\mathrm{cJ}})}(x;\beta,p,q) - 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_{\infty}^{(\widetilde{\mathrm{cJ}})}(\tau;\beta,p,q)e^{-i\tau x}\,d\tau, \tag{1.6}$$

where $c_{\infty}^{(\widetilde{\mathrm{cJ}})}$ is given by (1.3), and thus by Fourier inversion

$$c_{\infty}^{(\widetilde{\mathrm{cJ}})}(\tau;\beta,p,q) = \int_{-\infty}^{\infty}\Big(\rho_{(1),\infty}^{(\widetilde{\mathrm{cJ}})}(x;\beta,p,q)-1\Big)e^{i\tau x}\,dx. \tag{1.7}$$

1.2 First Motivation—Analytic Structure of the Generalised Spectral Form Factor

In the case $q = 0$, $p = 1$, (1.7) can be identified with the bulk structure function for the limiting statistical state defined by (1.1) with $p = q = 0$ [15, 28, 33]. Thus for finite N the structure function $S_N(k;\beta)$ can be defined by the covariance of the pair of linear statistics $\sum_{j=1}^{N} e^{ik\lambda_j}, \sum_{j=1}^{N} e^{-ik\lambda_j}$,

$$S_N(k;\beta) = \Big\langle\Big|\sum_{j=1}^{N} e^{ik\lambda_j}\Big|^2\Big\rangle - \Big|\Big\langle\sum_{j=1}^{N} e^{ik\lambda_j}\Big\rangle\Big|^2, \tag{1.8}$$

where the averages are with respect to (1.1) in the case $p = q = 0$. In the bulk scaling limit, this can be written in terms of the truncated (connected) two-particle correlation function $\rho_{(2),\infty}^{T}(x_1,x_2)$ according to

$$S_\infty(\tau;\beta) := \lim_{\substack{N,k\to\infty\\ \tau\ \mathrm{fixed}}} S_N(k;\beta)\Big|_{\tau=2\pi k/N} = \int_{-\infty}^{\infty}\Big(\rho_{(2),\infty}^{T}(x,0)+\delta(x)\Big)e^{i\tau x}\,dx. \tag{1.9}$$

But by the definition of the truncated two-particle correlation function in the case $p = q = 0$ we have

$$\rho_{(2),\infty}^{T}(x,0) = \rho_{(1),\infty}^{(\widetilde{\mathrm{cJ}})}(x;\beta,1,0) - 1, \tag{1.10}$$

where the RHS comes about since one of the fixed eigenvalues involved in the definition of the two-particle correlation induces a spectrum singularity of degeneracy 1. Comparing (1.9) with the substitution (1.10) to (1.7) then tells us that

$$S_\infty(\tau;\beta) - 1 = c_{\infty}^{(\widetilde{\mathrm{cJ}})}(\tau;\beta,1,0). \tag{1.11}$$

We know from [33] that the quantity

$$f(\tau;\beta) := \frac{\pi\beta}{|\tau|}S_\infty(\tau;\beta), \qquad 0<\tau<\min(2\pi,\pi\beta), \tag{1.12}$$

extends to an analytic function of τ for $|\tau| < \min(2\pi,\pi\beta)$ and furthermore satisfies the functional equation

$$f(\tau;\beta) = f\Big(-\frac{2\tau}{\beta};\frac{4}{\beta}\Big). \tag{1.13}$$

The power series expansion of $f(\tau;\beta)$ in (1.13) is known to have the form [28, 33, 70]

$$f(\tau;\beta) = 1 + \sum_{j=1}^{\infty} p_j(2/\beta)(\tau/2\pi)^j \tag{1.14}$$

where

$$p_j(u) = \begin{cases} (1-u)^2 \sum_{l=0}^{j-2} b_{j,l} u^l & (b_{j,0}=1,\ b_{j,l}=b_{j,j-2-l})\ j \text{ even} \\ (1-u) \sum_{l=0}^{j-1} b_{j,l} u^l & (b_{j,0}=1,\ b_{j,l}=b_{j,j-1-l})\ j \text{ odd.} \end{cases} \tag{1.15}$$

Thus the coefficient of τ^j is a palindromic or anti-palindromic polynomial in $u = 2/\beta$. Note that this structure is consistent with (1.13). The explicit form of $p_j(u)$ has been determined up to and including $j = 9$ in [33, 70], while the explicit form of $p_{10}(u)$ was calculated recently in [28]. It is found numerically that all the zeros of these polynomials are on the unit circle in the complex u-plane, and furthermore that the zeros interlace as the degree is increased.

The equality (1.11) tells us that $(\pi\beta/|\tau|)(c_\infty^{(\widetilde{\mathrm{cJ}})}(\tau;\beta,1,0)+1)$ shares the same expansion in τ as $f(\tau;\beta)$. Given that the latter is a higher structured quantity, we pose the question as to the nature of the the small τ expansion of $c_\infty^{(\widetilde{\mathrm{cJ}})}(\tau;\beta,p,q)$ for general p and q. This is our first motivation for studying the Fourier transform (1.7).

Remark 1.1 In relation to the large but finite N structure function for general random matrix ensembles, one notes that the two-point quantity $\langle|\sum_{j=1}^N e^{ik\lambda_j}|^2\rangle$ on the RHS of (1.8) has been the subject of some recent attention in literature applying random matrix theory to the study of many body quantum chaos [18, 64] and the scrambling of information in black holes [17, 24]; many more references could be given. Of interest is its graphical shape as a function of k, which assuming a non-constant spectral density exhibits dip-ramp-plateau features as k varies as a function of N, with corresponding dynamical significance. Following on from the applications, several theoretical works have quantified dip-ramp-plateau for various model random matrix ensembles. With such concerns being in common with the present work when viewed broadly, we give a comprehensive list of references [15, 28–30, 34, 55, 59, 65, 66].

1.3 Second Motivation—Comparison with the Analogous Expansion for the Two-Dimensional One-Component Plasma with an Impurity

Further motivation for our interest in (1.2) is that the case $q = 0$ has previously been the subject of some literature for the corresponding two-dimensional generalisation of (1.1), which is up to proportionality specified by the PDF

$$\prod_{l=1}^{N} |z_l|^{\beta Q} e^{-\beta\pi|z_l|^2/2} \prod_{1\le j<k\le N} |z_k - z_j|^\beta, \qquad z_l \in \mathbb{C}. \tag{1.16}$$

The PDF (1.16) has the interpretation as the Boltzmann factor for the two-dimensional one-component plasma, consisting of N mobile two-dimensional unit charges, repelling via the pair potential $-\log|z-z'|$, with a smeared out uniform neutralising background of charge density -1 in the region $|z| < \sqrt{N/\pi}$; see [12, Sect. 4] for a recent review. At the origin is a so-called host charge (or impurity) of strength Q.

Let $\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)$ denote the density at point $\mathbf{r}$ in the limit $N\to\infty$ of (1.16). Charge neutrality of the screening cloud implies

$$\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)-1)\,d\mathbf{r}=-Q. \tag{1.17}$$

Two distinct derivations have been given for a sum rule specifying the second moment of the screening cloud [47, 61]

$$\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)-1)|\mathbf{r}|^2\,d\mathbf{r}=-\frac{2}{\pi\beta}\Big((1-\beta/4)+(\beta/4)Q\Big)Q. \tag{1.18}$$

Most recently, under the assumption of analyticity in Q, the fourth moment of the screening cloud has been shown to obey [62]

$$\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)-1)|\mathbf{r}|^4\,d\mathbf{r}=\frac{1}{(\pi\beta)^2}\Big(b_0(\beta)+b_1(\beta)Q+b_2(\beta)Q^2\Big)Q, \tag{1.19}$$

where

$$b_0(\beta)=-(\beta-6)(\beta-8/3),\qquad b_1(\beta)=\frac{2}{3}\beta(2\beta-7),\qquad b_2(\beta)=-\frac{1}{3}\beta^2. \tag{1.20}$$

Earlier [13, 48] the polynomial $b_0(\beta)/\beta^2$ appeared as the coefficient of $|\mathbf{k}|^6$ in the expansion of the structure function $S^{\rm OCP}_\infty(\mathbf{k};\beta)$. From the spherical symmetry of $\rho^{\rm OCP}_{(1)}(\mathbf{r};Q)-1$ we have for its Fourier transform

$$\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)-1)e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r}=\sum_{j=0}^\infty\frac{(-1)^j}{(j!)^2}\Big(\frac{|\mathbf{k}|^2}{4}\Big)^j\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)-1)|\mathbf{r}|^{2j}\,d\mathbf{r}, \tag{1.21}$$

and so the results (2.21)–(1.19) give the small $|\mathbf{k}|$ expansion up and including order $|\mathbf{k}|^4$, and exhibit a polynomial structure in Q and $4/\beta$,

$$\begin{aligned}\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)-1)e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r}&=-Q+\frac{|\mathbf{k}|^2}{2\pi\beta}\Big((1-\beta/4)+(\beta/4)Q\Big)Q\\&\quad+\frac{|\mathbf{k}|^4}{64(\pi\beta)^2}\Big(b_0(\beta)+b_1(\beta)Q+b_2(\beta)Q^2\Big)Q+O(|\mathbf{k}|^6).\end{aligned} \tag{1.22}$$

At order $|\mathbf{k}|^6$ there is evidence that the polynomial structure in $4/\beta$ breaks down, at least for the coefficient of Q as the latter is equivalent to the coefficient of $|\mathbf{k}|^8$ in the small $|\mathbf{k}|$ expansion of the structure function $S^{\rm OCP}_\infty(\mathbf{k};\beta)$ ([62]; see also Sect. 5 below) which is expected to consist of an infinite series in powers of $4/\beta$ [48]. In contrast, in [62] it is argued that the coefficient of $|\mathbf{k}|^{2j}$ is always a polynomial in Q of degree $j+1$.

1.4 Summary of Results

Analogous to the expansion (1.14), we will show that there are coefficients $\{h_j(\beta,p,q)\}$ and $\{\tilde h_j(\beta,p,q)\}$ such that

$$c^{\rm (cJ)}_\infty(\tau;\beta,p,q)=\sum_{j=0}^\infty h_j(\beta,p,q)\tau^j\,{\rm sgn}\,\tau+\sum_{j=0}^\infty\tilde h_j(\beta,p,q)\tau^j,\qquad|\tau|<|\tau^*|, \tag{1.23}$$

where $|\tau^*|$ is the radius of convergence of the series. We are most interested in properties of the coefficients as a function of β, p and q. One notes that the symmetry of the discontinuous factor of (1.1) in the neighbourhood of $\theta = \pm\pi$, namely $e^{q(-\pi+\theta)}$ $(\theta > 0)$ and $e^{q(\pi+\theta)}$ $(\theta < 0)$, being unchanged by the mapping $\theta \mapsto -\theta$ and $q \mapsto -q$, implies $\rho_{(1),\infty}^{(\mathrm{cJ})}(x; \beta, p, q) = \rho_{(1),\infty}^{(\mathrm{cJ})}(-x; \beta, p, -q)$. Hence from (1.7) we have the functional properties

$$h_j(\beta, p, q) = (-1)^{j+1} h_j(\beta, p, -q), \qquad \tilde{h}_j(\beta, p, q) = (-1)^j \tilde{h}_j(\beta, p, -q). \tag{1.24}$$

In addition, the fact that $\rho_{(1),\infty}^{(\mathrm{cJ})}(x; \beta, p, q)$ is real tells us that

$$\overline{h_j(\beta, p, q)} = (-1)^{j+1} h_j(\beta, p, q), \qquad \overline{\tilde{h}_j(\beta, p, q)} = (-1)^j \tilde{h}_j(\beta, p, q). \tag{1.25}$$

It follows from these equations together that h_j alternates pure imaginary then real, while $\tilde{h}_j$ alternates real and then pure imaginary.

Our study of analytic properties of the Fourier transform (1.6) and its small τ expansion (1.23) begins in Sect. 2 where we specialise to $\beta = 2$. By doing this use can be made of a third order linear differential equation which then characterises $\rho_{(1),\infty}^{(\mathrm{cJ})}$. Our main finding is a second order difference equation for the odd indexed expansion coefficients of the non-analytic terms in (1.23)—see Proposition 2.11. For p a positive integer the implied sequence terminates for indices great than $2p-1$. The even indexed expansion coefficients of the non-analytic terms all vanish as does the coefficients of the analytic terms vanish except for $\tilde{h}_0$. The considerations of Sect. 2 are repeated in Sect. 3, but now for $\beta = 4$. This is possible due to the derivation of an explicit fifth order linear differential equation characterising $\rho_{(1),\infty}^{(\mathrm{cJ})}$. Proposition 3.2 then gives a fourth order difference equation satisfied by the expansion coefficients in (1.23), which in general are all non-zero.

A different approach to studying the expansion coefficients of the Fourier transform is introduced in Sect. 4. This is to derive and apply a hierarchical set of equations referred to as loop equations for the generating function of the Fourier coefficients. In this method $\beta > 0$ is arbitrary. However the computational complexity increases with the order of the expansion coefficient, which restricts the number of terms which can be computed. Five orders are specified in Proposition 4.2. A discussion of the explicit functional forms obtained by the loop equation analysis, in the context of known properties of the expansion coefficients of the structure function $S_\infty(\tau; \beta)$, is given in Sect. 5. This includes identities relating the case $q = 0$ of (1.23) with $p \to 0$, and the case $p = 0$ of (1.23) with $q \to 0$, to the case $(p, q) = (1,$ features of the $\beta \to \infty$ limit, and features of the coefficients in the expansion (1.23) generalising those of the polynomials in (1.14). Section 5 also contains a comparison between our findings and analogous results known for the two-dimensional one-component plasma with an impurity charge (recall Sect. 1.3).

2 The Case $\beta = 2$

2.1 The Confluent Hypergeometric Kernel and a Bessel Kernel

Denote the normalised form of the PDF proportional to (1.1) by $p_{N,\beta}^{(\mathrm{cJ})}(\theta_1, \dots, \theta_N)$. The corresponding k-point correlation function, $\rho_{(N),k}^{(\mathrm{cJ})}(\theta_1, \dots, \theta_k)$ say, is up to normalisation obtained by integrating over all but the first k of the variables in $p^{(\mathrm{cJ})}_{N,\beta}$,

$$\rho^{(\mathrm{cJ})}_{(k),N}(\theta_1,\dots,\theta_k) = N(N-1)\cdots(N-k+1)\int_{(-\pi,\pi]^{N-k}} d\theta_{k+1}\cdots d\theta_N\, p^{(\mathrm{cJ})}_{N,\beta}(\theta_1,\dots,\theta_N). \tag{2.1}$$

In keeping with (1.4) in this we replace $\{\theta_j\}$ in favour of $\{x_j\}$ and further define

$$\begin{aligned}&\rho^{(\mathrm{cJ})}_{(k),\infty}(x_1,\dots,x_k;\beta,p,q)\\ &\quad= \lim_{N\to\infty}\left(\frac{2\pi}{N}\right)^k \rho^{(\mathrm{cJ})}_{(k),N}(-\pi\,\mathrm{sgn}(x_1)+2\pi x_1/N,\dots,-\pi\,\mathrm{sgn}(x_k)+2\pi x_k/N).\end{aligned} \tag{2.2}$$

The case $\beta = 2$ of (2.1) and (2.2) is special. The k-point correlation function then has a determinantal structure, with the elements of the determinant moreover independent of k. Specifically, in relation to (2.2) we have

$$\rho^{(\mathrm{cJ})}_{(k),\infty}(x_1,\dots,x_k;\beta,p,q)\Big|_{\beta=2} = \det\left[K^{(p,q)}_\infty(x_j,x_l)\right]_{j,l=1,\dots,k}, \tag{2.3}$$

where [7, after the change of variables $1/x_l \mapsto \pi x_l$] (see also the introduction of [23] for further references, and the recent work [37] for an independent derivation)

$$\begin{aligned}K^{(p,q)}_\infty(x,y) &= \frac{(2\pi)^{2p}|\Gamma(p+1-iq)|^2}{\pi\Gamma(2p+2)\Gamma(2p+1)} e^{-i\pi(x+y)-q\pi(\mathrm{sgn}\,x+\mathrm{sgn}\,y)/2}\frac{(xy)^{p+1}}{x^2(x-y)}\\ &\quad\times\Big(x\,{}_1F_1(p+1-iq;2p+2;2i\pi x)\,{}_1F_1(p-iq;2p;2i\pi y)-(x\leftrightarrow y)\Big).\end{aligned} \tag{2.4}$$

Here ${}_1F_1(a;c;z)$ denotes the confluent hypergeometric function in standard notation. In particular, taking the limit $y\to x$ using L'Hôpital's rule gives for the bulk scaled density

$$\begin{aligned}&\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)\Big|_{\beta=2}\\ &= \frac{(2\pi)^{2p}|\Gamma(p+1-iq)|^2}{\pi\Gamma(2p+2)\Gamma(2p+1)} e^{-2i\pi x-q\pi\,\mathrm{sgn}\,\mathrm{x}}|x|^{2p}\bigg(\frac{d}{dx}\Big(x\,{}_1F_1(p+1-iq;2p+2;2i\pi x)\Big)\\ &\quad\times{}_1F_1(p-iq;2p;2i\pi x) - x\,{}_1F_1(p+1-iq;2p+2;2i\pi x)\frac{d}{dx}{}_1F_1(p-iq;2p;2i\pi x)\bigg).\end{aligned} \tag{2.5}$$

We remark that use of the Kummer transformation ${}_1F_1(a,b;z) = e^z\,{}_1F_1(b-a,b;z)$ shows that (2.5) is unchanged by the mapping $x\mapsto -x, q\mapsto -q$, which must hold for general $\beta>0$ as noted in the discussion above (1.24).

2.2 The Case $q = 0$—Bessel Function Asymptotics

Use of the expression for a particular ${}_1F_1$ in terms of a Bessel function,

$${}_1F_1(p;2p;2iX) = \Gamma(p+1/2)\left(\frac{X}{2}\right)^{-p+1/2} e^{iX} J_{p-1/2}(X), \tag{2.6}$$

shows [56]

$$K^{(p,q)}_\infty(x,y)\Big|_{q=0} = \pi(xy)^{1/2}\frac{J_{p+1/2}(\pi x)J_{p-1/2}(\pi y)-J_{p-1/2}(\pi x)J_{p+1/2}(\pi y)}{2(x-y)}. \tag{2.7}$$

Taking the limit $y \to x$ and making use of Bessel function identities then gives [27, Eq. (7.49) with $\rho = 1, a = p$]

$$\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)|_{\substack{\beta=2\\q=0}} = \frac{\pi^2|x|}{2}\Big((J_{p-1/2}(\pi x))^2 + (J_{p+1/2}(\pi x))^2 - \frac{2p}{\pi x} J_{p-1/2}(\pi x)J_{p+1/2}(\pi x)\Big). \tag{2.8}$$

As a check, it follows from (2.8) and trigonometric formulas for the Bessel functions at half integer order that

$$\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)|_{\substack{\beta=2\\p=q=0}} = 1. \tag{2.9}$$

We see that this is as required by the rotation invariance of the PDF (1.1) for $p = q = 0$ and the normalisation implied by bulk scaling. Setting now $p = 1$, shows

$$\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)|_{\substack{\beta=2\\p=1,q=0}} = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2. \tag{2.10}$$

This is in keeping with (1.10) upon recalling the standard fact that in relation to (1.1) with $p = q = 0$, $\rho^{(\mathrm{cJ})}_{(2),\infty}(x, 0;\beta,p,q)|_{\beta=2}$ is given by the RHS of (2.10); see e.g. [27, Eq. (7.2) with $\rho = 1$]. Generally, substituting any positive integer for p in (2.8) reduces the Bessel function to a trigonometric form involving $\sin 2\pi x$, $\cos 2\pi x$. This is further illustrated by the next simplest example after (2.10), which is to set $p = 2$ with the result

$$\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)|_{\substack{\beta=2\\p=2,q=0}} = 1 - \frac{2}{\pi^2x^2} - \frac{3}{2\pi^4x^4} + \frac{3-2\pi^2x^2}{2\pi^4x^4}\cos 2\pi x + \frac{3}{\pi^3x^3}\sin 2\pi x. \tag{2.11}$$

In the notation of (1.7) and (1.23)

$$\begin{aligned} c^{(\mathrm{cJ})}_{\infty}(\tau;\beta,p,q)\Big|_{\beta=2} &= \int_{-\infty}^{\infty}\Big(\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)|_{\beta=2} - 1\Big)e^{i\tau x}\,dx \\ &= \sum_{j=0}^{\infty} h_j(\beta,p,q)|_{\beta=2}\mathrm{sgn}(\tau)\tau^j + \sum_{j=0}^{\infty}\tilde{h}_j(\beta,p,q)|_{\beta=2}\tau^j. \end{aligned} \tag{2.12}$$

Starting with (2.5), we don't know how to give an explicit special function evaluation of the integral in (2.12). Nonetheless, we can use (2.5), expanded for large x, to deduce the $h_j(\beta,p,q)$ for j odd in the corresponding small τ expansion. Our ability to do this comes about from Fourier transform theory [52]. Thus if

$$\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)|_{\substack{\beta=2\\q=0}} - 1 \underset{x\to\infty}{\sim} \sum_{n=1}^{\infty}\frac{c_{2n}}{x^{2n}}, \tag{2.13}$$

where only non-oscillatory terms are recorded on the RHS, then

$$\int_{-\infty}^{\infty}\Big(\rho^{(\mathrm{cJ})}_{(1),\infty}(x;\beta,p,q)|_{\substack{\beta=2\\q=0}} - 1\Big)e^{i\tau x}\,dx \underset{\tau\to 0}{\sim} \pi\sum_{n=1}^{\infty}\frac{(-1)^n c_{2n}}{(2n-1)!}|\tau|^{2n-1}. \tag{2.14}$$

Generally this approach can access the terms in (2.12) singular in τ only. The analytic terms are not accessible via this method.

According to (2.5), the expansion (2.13) as applies to the case $q = 0$ can be deduced from knowledge of the large x asymptotic expansion of the Bessel function. The latter is given by [57, Eq. (10.17.3)]

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2}\left(\cos\omega\sum_{k=0}^{\infty}(-1)^k\frac{a_{2k}(\nu)}{z^{2k}} - \sin\omega\sum_{k=0}^{\infty}(-1)^k\frac{a_{2k+1}(\nu)}{z^{2k+1}}\right), \tag{2.15}$$

where with $(u)_s := u(u-1)\cdots(u+s-1)$

$$\omega = z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi, \quad a_k(\nu) = \frac{1}{(-2)^k k!}\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k. \tag{2.16}$$

Proposition 2.1 *Consider the large x asymptotic expansion of $\rho_{(1),\infty}^{(\mathrm{cJ})}(x;\beta,p,q)|_{\beta=2,q=0}$ restricted to non-oscillatory terms. This is given by (2.13) with $c_{2n} = c_{2n}(p)$, where*

$$\begin{aligned} c_{2n}(p) = \frac{(-1)^n}{\pi^{2n}}\Bigg(&\frac{1}{2}\sum_{s=0}^{n}\Big(a_{2s}(p-1/2)a_{2(n-s)}(p-1/2) + a_{2s}(p+1/2)a_{2(n-s)}(p+1/2)\Big) \\ &-\frac{1}{2}\sum_{s=0}^{n-1}\Big(a_{2s+1}(p-1/2)a_{2(n-s)-1}(p-1/2) + a_{2s+1}(p+1/2)a_{2(n-s)-1}(p+1/2)\Big) \\ &+p\sum_{s=0}^{n-1}\Big(a_{2s}(p-1/2)a_{2(n-s)-1}(p+1/2) - a_{2s+1}(p-1/2)a_{2(n-s)-2}(p+1/2)\Big)\Bigg), \end{aligned} \tag{2.17}$$

with $\{a_k(\nu)\}$ as in (2.16).

Proof We substitute (2.15) in (2.8). Applying simple trigonometric identities allows the non-oscillatory terms to be separated from the oscillatory terms. Finally, the coefficient of $1/x^{2n}$ is extracted using the formula for multiplication of power series. □

Remark 2.2 The use of computer algebra to compute (2.17) for small values of n and general p suggests the simplified form

$$c_{2n}(p) = -\frac{\alpha_n}{\pi^{2n}}\prod_{l=0}^{n-1}(p^2 - l^2), \tag{2.18}$$

where α_n is a rational number. Regarding the latter, the computer algebra computation gives

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{8}, \quad \alpha_3 = \frac{1}{16}, \quad \alpha_4 = \frac{5}{128}, \quad \alpha_5 = \frac{7}{256}, \quad \alpha_6 = \frac{21}{1024}. \tag{2.19}$$

We note in particular that (2.18) implies $c_{2n}(p) = 0$ for $n > p$, and that $c_{2n}(p)$ is a polynomial in p^2 of degree $n-1$. In fact this simplified form, together with the explicit value of α_n, can be obtained by taking a different approach to the large x asymptotic expansion of $\rho_{(1),\infty}^{(\mathrm{cJ})}(x;\beta,p,q)|_{q=0,\beta=2}$, namely via a differential equation, which is to be done next.

2.3 The Case $q = 0$—A Third Order Linear Differential Equation

In the work [39, Sect. 3.1.2] the scaled density $(1/\pi)\rho^{\rm cJ}_{(1),\infty}(x/\pi;\beta,p,q)|_{\beta=2,q=0}$ was shown to satisfy the third order linear differential equation

$$x^2 R'''(x) + 4x R''(x) + (2 - 4p^2 + 4x^2)R'(x) - \frac{4p^2}{x} R(x) = 0. \tag{2.20}$$

This can be used to provide an alternative approach to the computation of the coefficients in the expansion (2.13).

Proposition 2.3 *The expansion* (2.13) *holds with*

$$c_2 = -\frac{1}{2\pi^2} p^2, \qquad c_{2n} = -\frac{1}{\pi^{2n}} \frac{(2n-3)!!}{(2n)!!} \prod_{l=0}^{n-1} (p^2 - l^2), \quad (n \geq 2). \tag{2.21}$$

Proof Due to the scaling of x in the density assumed in the derivation of (2.20), the expansion (2.13) transforms to

$$R(x) - \frac{1}{\pi} \underset{x\to\infty}{\sim} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\pi^{2n} c_{2n}}{x^{2n}}. \tag{2.22}$$

Substituting (2.22) in (2.20) and equating like (negative) powers of x implies

$$c_2 = -\frac{1}{2\pi^2} p^2, \qquad c_{2n+2} = \frac{1}{\pi^2} \frac{2n-1}{2(n+1)} (p^2 - n^2) c_{2n}, \quad (n \geq 1). \tag{2.23}$$

Iterating the first order recurrence (2.21) follows. □

Remark 2.4 In the notation of (2.18), (2.21) gives $\alpha_1 = 1/2$ and

$$\alpha_n = \frac{(2n-3)!!}{(2n)!!}$$

for $n \geq 2$. This indeed reproduces the values obtained in (2.19).

With $r(x) := R(x) - 1/\pi = \mathrm{O}(x^{-2})$ as implied by (2.22), there is a well defined meaning to taking the Fourier transform of (2.20). Upon integration by parts and an additional differentiation with respect to τ this reads

$$-\frac{d^3}{d\tau^3}\Big((\tau^3 - 4\tau)\hat{r}(\tau)\Big) + 4\frac{d^2}{d\tau^2}(\tau^2 \hat{r}(\tau)) - 2(1-2p^2)\frac{d}{d\tau}(\tau \hat{r}(\tau)) - 4p^2 \hat{r}(\tau) = 0, \tag{2.24}$$

where

$$\hat{r}(\tau) := \int_{-\infty}^{\infty} r(x) e^{i\tau x}\, dx.$$

Proposition 2.5 *Let*

$$b_0 = \int_{-\infty}^{\infty} \Big(R(x) - \frac{1}{\pi}\Big)\, dx, \qquad b_1 = \frac{p^2}{2\pi} \tag{2.25}$$

and specify $\{c_{2n}\}$ *by Proposition* 2.3. *The small* τ *expansion*

$$\hat{r}(\tau) = \sum_{n=0}^{\infty} b_n (\pi\tau)^n, \qquad \tau > 0, \tag{2.26}$$

substituted in (2.24) *has the unique solution*

$$b_{2n-1} = \pi \frac{(-1)^n c_{2n}}{(2n-1)!} \quad (n \geq 1), \qquad b_{2n} = 0 \quad (n \geq 1). \tag{2.27}$$

Proof Substituting (2.26) in (2.24) gives

$$b_{n+2} = -\frac{n}{\pi^2 (n+3)(n+2)(n+1)} \Big(p^2 - \frac{(n+1)^2}{4} \Big) b_n \quad (n \geq 0).$$

The value of b_0 is not restricted by the differential equation, nor is the value of b_1, while for n even and positive the recurrence gives $b_n = 0$. For odd subscripts, replacing n by $2n-1$ allows the recurrence to be related to that in (2.23), which implies (2.27). □

Remark 2.6 1. The implied coefficient of the third derivative in (2.24) has a zero at $\tau = 0$ and $\tau = \pm 2$, telling us that the radius of convergence of (2.26) is 2.

2. Since $r(x)$ as relates to (2.24) is an even function of x, $\hat{r}(\tau)$ must be an even function of τ. Hence for $\tau < 0$ (2.26) is now a power series in $|\tau|$, as is consistent with (2.12). In particular the coefficient of τ must then relate to the coefficient of $1/x^2$ in the large x expansion of $r(x)$ as implied by the relation between (2.13) and (2.14). This justifies the choice of b_1 in (2.25) which as already mentioned otherwise is not determined by (2.24).
3. The log-gas interpretation of the factor $|1 + e^{i\theta}|^{\beta p}$ in the circular Jacobi weight $w^{(\mathrm{cJ})}(\theta)$ of (1.1), otherwise interpreted as a spectrum singularity of degeneracy p at $\theta = \pi$, is as a fixed charge of strength p. From this log-gas picture, perfect screening of the fixed charge (see e.g. [27, Sect. 14.1]) implies

$$\int_{-\infty}^{\infty} \Big(R(x) - \frac{1}{\pi} \Big) \, dx = -p \tag{2.28}$$

and thus from (2.25) that $b_0 = -p$.

Corollary 2.7 *Specify* $\{c_{2j}\}$ *by Proposition* 2.3 *supplemented by* $c_0 = 0$*. For* $\beta = 2$ *and* $q = 0$*, the expansion* (1.23) *holds true with*

$$h_{2j}(\beta, p, q)\Big|_{\substack{\beta=2\\ q=0}} = c_{2j} \ \ (j \geq 0), \quad h_{2j-1}(\beta, p, q)\Big|_{\substack{\beta=2\\ q=0}} = \tilde{h}_j(\beta, p, q)\Big|_{\substack{\beta=2\\ q=0}} = 0 \ \ (j \geq 1). \tag{2.29}$$

The perfect screening sum rule (2.28) *gives* $\tilde{h}_0(\beta, p, q)|_{\beta=2, p=0} = -p$.

2.4 Third Order Linear Differential Equation for the Case $q \neq 0$

A third order linear differential equation satisfied by the scaled density $(1/\pi)\rho^{\mathrm{cJ}}_{(1),\infty}(x/\pi; \beta, p, q)|_{\beta=2}$ for general p, q can be obtained by following the same procedure as used in [39] to deduce (2.20), suitably generalised to involve the extra parameter q. In doing this, use is made of a general relation between the generalised circular Jacobi ensemble, and the generalised Cauchy ensemble.

In relation to this, one remarks that the PDF (1.1) is an example of a β ensemble with the eigenvalues supported on the unit circle in the complex plane. In contrast, a β ensemble on the real line is specified by a joint eigenvalue PDF of the form

$$\prod_{l=1}^{N} w(x_l) \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta}. \tag{2.30}$$

Generally an Hermitian matrix H can be constructed from a unitary matrix U by way of the Cayley transform

$$H = i(\mathbb{I}_N - U)(\mathbb{I}_N + U)^{-1},$$

valid provided -1 is not in the spectrum of U and thus $\mathbb{I}_N + U$ is invertible. This was introduced into random matrix theory by Hua [46]. The corresponding mapping of the eigenvalues is

$$x = i\frac{1 - e^{i\theta}}{1 + e^{i\theta}} = \tan(\theta/2), \tag{2.31}$$

which is recognised as specifying a stereographic projection between the unit circle and the real line. Applying the inverse of this mapping, up to proportionality the PDF (1.1) transforms to the functional form (2.30) with

$$w(x) = \frac{1}{(1 - ix)^c (1 + ix)^{\bar{c}}}, \qquad c = \beta(N + p - 1)/2 + 1 - iq, \tag{2.32}$$

which is then referred to as the generalised Cauchy β ensemble; see [27, Eq. (3.124)].

Proposition 2.8 *The scaled density* $(1/\pi)\rho^{\rm cJ}_{(1),\infty}(x/\pi; \beta, p, q)|_{\beta=2}$ *obeys the third order linear differential equation*

$$x^3 R'''(x) + 4x^2 R''(x) + 2x(1 - 2p^2 - 4qx + 2x^2)R'(x) - 4(p^2 + qx)R(x) = 0. \tag{2.33}$$

Proof For the Cauchy ensemble with weight (2.32) and $\beta = 2$ let $\rho^{(Cy)}_{(1),N}(t)$ denote the corresponding density. With $g(t) = (1 + t^2)\rho^{(Cy)}_{(1),N}(t)$ we know from [39, Eq. (3.33) with $r(t) = g(t), \alpha_1 = p, \alpha_2 = -q$] that

$$\begin{aligned}&(1 + t^2)^2 g''' + 2t(1 + t^2)g'' + 4\Big((p^2 t - (N + p)q)(g - tg') \\ &\quad + ((N + p)^2 + (N + p)qt - p^2 - q^2)g'\Big) = 0.\end{aligned} \tag{2.34}$$

From (2.31) and associated theory it follows $2\,g(t)|_{t=\tan(\theta/2)} = \rho^{(\rm cJ)}_{(1),N}(\theta)$. We want to scale θ about the spectrum singularity at $\theta = \pi$, and as a normalisation we require that at large distances in the scaled variable the density is $1/\pi$. These requirements are met by choosing $\theta = \pi + 2X/N$. With $t = \tan(\theta/2)$ and N large this implies $t \sim -N/X$. We make this substitution in (2.34). Equating terms at leading order in N in the resulting equation gives (2.33). □

For $q \neq 0$, it follows from the functional form (1.1) that $\rho^{(\rm cJ)}_{(1),\infty}(x; \beta, p, q)$ is not an even function of x, in distinction to the case $q = 0$. In place of (2.13) we now expect

$$\rho^{(\rm cJ)}_{(1),\infty}(x; \beta, p, q)|_{\beta=2} - 1 \underset{x \to \pm\infty}{\sim} \sum_{n=1}^{\infty} \frac{d_n}{x^n}, \tag{2.35}$$

for the asymptotic form of the non-oscillatory terms. The analogue of (2.14) for the non-analytic terms in the small τ form of the Fourier transform is then [52]

$$\int_{-\infty}^{\infty} \Big(\rho^{(\rm cJ)}_{(1),\infty}(x; \beta, p, q)|_{\beta=2} - 1\Big)e^{i\tau x}\,dx \underset{\tau \to 0}{\sim} \pi \sum_{n=1}^{\infty} d_n \frac{i^n}{(n-1)!}\tau^{n-1}\,\mathrm{sgn}\,\tau. \tag{2.36}$$

As for the result of Proposition 2.3, the differential equation (2.33) can be used to determine the coefficients d_n in (2.35).

Proposition 2.9 *The expansion* (2.35) *holds with*

$$d_1 = -\frac{q}{\pi}, \quad d_2 = -\frac{p^2+q^2}{2\pi^2}, \quad d_{n+2} = \frac{1}{\pi}\frac{2n+1}{n+2} q d_{n+1} + \frac{1}{\pi^2}\frac{n-1}{n+2}\Big(p^2 - \frac{n^2}{4}\Big) d_n \ \ (n \geq 1). \tag{2.37}$$

Proof As with (2.22), to relate the expansion (2.35) to an expansion of $R(x)$ in (2.33) requires a scaling $x \mapsto x/\pi$ to give

$$R(x) - \frac{1}{\pi} \underset{x\to\infty}{\sim} \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\pi^n d_n}{x^n}. \tag{2.38}$$

The result now follows by substituting (2.38) in (2.33) and equating like (negative) powers of x. ◻

We note that setting $q = 0$ in (2.37) reclaims the result of (2.21). However unlike the situation with $q = 0$ the recurrence in (2.37) does not admit a simple functional form for its solution. Structural points of interest are that d_{2n} is equal to (p^2+q^2) times a polynomial in p^2 and q^2 of degree $n-1$, while for $n \geq 2$, d_{2n-1} is equal to $q(p^2+q^2)$ times a polynomial in p^2 and q^2 of degree $n-2$. Specifically, as some low order examples

$$d_3 = -\frac{q(p^2+q^2)}{2\pi^3}, \quad d_4 = \frac{(p^2+q^2)(1-p^2-5q^2)}{8\pi^4}, \quad d_5 = \frac{q(p^2+q^2)(5-3p^2-7q^2)}{8\pi^5}. \tag{2.39}$$

And as we know from (2.36), knowledge of $\{d_n\}$ gives the explicit form of the coefficients $\{h_n(\beta, p, q)|_{\beta=2}\}$ in the functional form (1.23) according to

$$h_{n-1}(\beta, p, q)|_{\beta=2} = \pi d_n \frac{i^n}{(n-1)!}, \quad n \geq 1. \tag{2.40}$$

Remark 2.10 1. Starting from (2.5), and using the appropriate generalisation of (2.15) as can be found in [57], we have independently verified that with respect to non-oscillatory terms

$$R(x) - \frac{1}{\pi} \underset{x\to\infty}{\sim} -\frac{q}{\pi x}, \tag{2.41}$$

as is consistent with Proposition 2.9.

2. Assuming the validity of (2.28) for $q \neq 0$, the result (2.37) for d_1 used in (2.36) implies

$$\lim_{\tau\to 0^{\pm}} \int_{-\infty}^{\infty} \Big(R(x) - \frac{1}{\pi}\Big) e^{i\tau x}\, dx = -p - iq \operatorname{sgn}\tau, \tag{2.42}$$

telling us that for $q \neq 0$ the Fourier transform is a discontinuous function of τ, and that in (1.23), $\tilde{h}_0(\beta, p, q)|_{\beta=2} = -p$.

3. Difference equations satisfied by moments of the spectral density for classical ensembles is a rich theme in random matrix theory; references include [19, 20, 35, 36, 43, 45, 50, 51, 54, 69]. Excluding the case of the Cauchy ensemble, a feature of these settings is a global scaling for which the spectral density has compact support. However in the present setting of the non-compactly supported bulk scaled spectral density about a spectrum singularity, there is no literal meaning of the moments due to the slow decay at infinity. Hence the non-analytic terms in the expansion (1.23) of the Fourier series. Note that this is in contrast to the two-dimensional case as exhibited in (1.21), even though the support there also non-compact.

It remains to investigate the coefficients $\{\tilde{h}_j(\beta, p, q)|_{\beta=2}\}$ in the expansion (1.23) for $j \geq 1$. For this purpose we introduce $r(x) = R(x) - 1/\pi$ and proceed to take the Fourier transform of (2.33) to deduce as the $q \neq 0$ generalisation of (2.24)

$$-\frac{d^3}{d\tau^3}\Big((\tau^3 - 4\tau)\hat{r}(\tau)\Big) + \frac{d^2}{d\tau^2}\Big((4\tau^2 + 8iq\tau)\hat{r}(\tau)\Big) - \frac{d}{d\tau}\Big((2(1-2p^2)\tau + 4iq)\hat{r}(\tau)\Big) \\ -4p^2\hat{r}(\tau) = 0. \tag{2.43}$$

The following result now follows by direct substitution.

Proposition 2.11 *The small* τ *expansion*

$$\hat{r}(\tau) = \sum_{n=0}^{\infty} e_n \Big(\frac{\tau}{\pi}\Big)^n, \qquad \tau > 0, \tag{2.44}$$

substituted in (2.43) *implies the recurrence for* $\{e_n\}$

$$(n+3)(n+2)(n+1)e_{n+2} + iq(2n+3)(n+1)e_{n+1} + n\Big(p^2 - \frac{(n+1)^2}{4}\Big)e_n = 0, \quad (n \geq 0). \tag{2.45}$$

The values of $e_2, e_3, \dots$ implied by (2.45) are independent of e_0 but depend on e_1. However in keeping with the circumstances of Proposition 2.5 the value of e_1 is not determined by the differential equation (2.43). We note that writing $e_n = \tilde{d}_{n+1} i^{n+1}/n!$ in (2.45) shows $\{\tilde{d}_n\}$ satisfies the same recurrence relation as $\{d_n\}$ in Proposition 2.9. In the context of the expansion (1.23) this implies that for $\beta = 2$ the sequences $\{h_j(\beta, p, q)\}$ and $\{\tilde{h}_j(\beta, p, q)\}$ satisfy the same recurrence. While all terms in the sequence $\{h_j(\beta, p, q)|_{\beta=2}\}$ are known according to Proposition 2.9 and (2.40), the sequence members $\tilde{h}_j(\beta, p, q)|_{\beta=2}$ for $q \neq 0$ and $j \geq 2$ depend on the value of $\tilde{h}_1(\beta, p, q)|_{\beta=2}$ for $q \neq 0$, which is not known from the above considerations. However results from Sect. 4—specifically by taking the imaginary part of $\alpha_1|_{\beta=2}$ in (4.21)—allow us to deduce that we have $\tilde{h}_1(\beta, p, q)|_{\beta=2} = 0$ for general q and hence $\tilde{h}_j(\beta, p, q)|_{\beta=2} = 0$ for $j \geq 1$.

3 The Case $\beta = 4$

The scaled statistical state in the neighbourhood of the spectrum singularity for $\beta = 4$ is a Pfaffian point process, rather than the simpler determinantal point process for $\beta = 2$ [37, 38]. Perhaps surprisingly then, the functional form of the of the $\beta = 4$ density is functionally related to the $\beta = 2$ density. This is simplest to state with $q = 0$, for which from [38, Eq. (3.34) with $a = p$ and $X = Y$] reads

$$\rho_{(1),\infty}^{(\mathrm{cJ})}(x; \beta, p, q)\Big|_{\substack{\beta=4\\ q=0}} = \rho_{(1),\infty}^{(\mathrm{cJ})}(2x; \beta, p, q)\Big|_{\substack{\beta=2\\ q=0,\, p\mapsto 2p}} \\ -\pi p \frac{J_{2p-1/2}(2x)}{x^{1/2}} \int_0^x s^{-1/2} J_{2p+1/2}(2s)\, ds, \tag{3.1}$$

where for convenience it is assumed $x > 0$ (as previously remarked, for $q = 0$ the density is an even function). An analogous result for $q \neq 0$ is given in [37]. However for our interest in the expansion (1.23) we will not make use of such an explicit expression. Rather the key feature of the $\beta = 4$ density for our purposes is that it, like the $\beta = 2$ density, satisfies a linear differential equation albeit now of degree 5.

Proposition 3.1 *Let $\tilde{p} = p - 2p^2$. The scaled density $(1/\pi)\rho^{\rm cJ}_{(1),\infty}(x/\pi; \beta, p, q)|_{\beta=4}$ satisfies the differential equation*

$$\begin{aligned} x^5 R^{(5)}(x) &+ 10x^4 R^{(4)}(x) + x^3(20x^2 - 20qx + 22 + 10\tilde{p})R'''(x) \\ &+ x^2(64x^2 - 76qx + 4 + 44\tilde{p})R''(x) \\ &+ 4x(16x^4 - 32qx^3 + 4(4q^2 + 4\tilde{p} + 1)x^2 - q(6 + 16\tilde{p})x + 4\tilde{p}^2 + 7\tilde{p} - 1)R'(x) \\ &+ 8(-4qx^3 + 4(q^2 + \tilde{p})x^2 - q(6\tilde{p} - 1)x + 2\tilde{p}^2)R(x) = 0. \end{aligned}$$

Proof A fifth order linear differential equation for the $\beta = 4$ Jacobi ensemble density has been given in [60, Th. 2]. Making use of the relation between the Jacobi and Cauchy averages [39]

$$\langle f(1 - ix_1, \dots, 1 - ix_N)\rangle^{\rm (Cy)} = \langle f(2x_1, \dots, 2x_N)\rangle^{\rm (J)}\Big|_{\substack{\lambda_1 = -\beta(N+p-1)/2-1+iq, \\ \lambda_2 = -\beta(N+p-1)/2-1-iq}} \quad (3.2)$$

valid for f a multivariable polynomial, and where the average on the RHS is to be understood in the sense of analytic continuation, we can deduce from this a fifth order differential equation for the quantity $(1+t^2)\rho^{(Cy)}_{(1),N}(t)$, where $\rho^{(Cy)}_{(1),N}(t)$ denotes the $\beta = 4$ Cauchy ensemble density corresponding to the weight (2.32). As in the proof of Proposition 2.8, the computation is concluded by substituting $t = -N/X$ as corresponds to a hard edge scaling and equating to leading order in N. Due to the large number of terms involved, the required steps were all carried out using computer algebra. □

As for the differential equation (2.33), the differential equation of Proposition 3.1 has a unique solution of the form (2.38).

Proposition 3.2 *Substituting the expansion* (2.38) *in the differential equation of Proposition* 3.1*, but with the coefficients renamed from* $\{d_n\}$ *to* $\{g_n\}$ *for distinction, one obtains that*

$$\begin{aligned} g_1 &= -\frac{q}{2\pi}, \quad g_2 = \frac{1}{8\pi^2}(2\tilde{p} - q^2), \quad g_3 = -\frac{1}{16\pi^3}q\left(-1 - 2\tilde{p} + q^2\right) \\ g_4 &= \frac{1}{128\pi^4}\left(-16\tilde{p} - 4\tilde{p}^2 + 19q^2 + 12\tilde{p}q^2 - 5q^4\right), \end{aligned}$$

with the higher order coefficients $g_5, g_6, \dots$ then determined by the fourth order recurrence relation

$$\begin{aligned} &64(n + 4)\pi^4 g_{n+4} - 32q(11 + 4n)\pi^3 g_{n+3} \\ &\quad + 4\left(32 + 54n + 29n^2 + 5n^3 + 8\tilde{p}(3 + 2n) + 24q^2 + 16nq^2\right)\pi^2 g_{n+2} \\ &\quad - 4q\left(4\tilde{p}(1 + 4n) + n\left(4 + 11n + 5n^2\right)\right)\pi g_{n+1} \\ &\quad + \left((n - 1)\left(16\tilde{p}^2 + 2\tilde{p}n(5n - 2) + n^2\left(-2 + n + n^2\right)\right)\right) g_n = 0. \end{aligned}$$

As with going from (2.35) to (2.36), then to (2.40), knowledge of $\{g_n\}$ tells us that

$$h_{n-1}(\beta, p, q)|_{\beta=4} = \pi g_n \frac{i^n}{(n - 1)!}. \quad (3.3)$$

Remark 3.3 1. A fifth order linear differential equation is also known for the density of the Jacobi ensemble in the case $\beta = 1$ [60]. However this is not independent of the corresponding differential equation in the case $\beta = 4$. Writing $g_n = g_n(\beta, p, q)$, where $\{g_n\}$ is used in the same sense as Proposition 3.2, the dependence implies

$$g_n(\beta, p, q)|_{\beta=1} = (-2)^{n+1} g_n(\beta, -p/2, -2q)|_{\beta=4}. \tag{3.4}$$

This functional equation is generalised in (4.33) below.

2. The fifth order differential equation of Proposition 3.1 can be transformed into a fifth order differential equation for the Fourier transform of $(R(x) - 1/\pi)$. Now substituting the small τ expansion (2.44) gives a fifth order recurrence for the coefficients $\{e_n\}$. Upon the substitution $e_n = \tilde{g}_{n+1} i^{n+1}/n!$, this recurrence becomes that of Proposition 3.2 with $\{g_n\}$ replaced by $\{\tilde{g}_n\}$. As in the discussion below Proposition 2.11 in the case $\beta = 2$, this implies that for $\beta = 4$ the sequences $\{h_j(\beta, p, q)\}$ and $\{\tilde{h}_j(\beta, p, q)\}$ of the expansion (1.23) satisfy the same recurrence. According to (3.3) and Proposition 3.2 the first of these is fully determined. However the latter requires specification of $\tilde{g}_2, \tilde{g}_3, \tilde{g}_4$ as initial conditions, which are not determined by the differential equation. However, we will see that these initial conditions can be accessed using result deduced from results of the next section (specifically (4.31) and (4.32)), which imply

$$\tilde{g}_2 = -\frac{q}{\pi^2}, \quad \tilde{g}_3 = \frac{1}{8\pi^3}(-\tilde{p} + q^2), \quad \tilde{g}_3 = \frac{1}{16\pi^4} q(-1 - 3\tilde{p} + 2q^2). \tag{3.5}$$

3. The identity (3.2) was used in [39] to obtain linear differential equations satisfied by the density in the Cauchy ensemble with $\beta = 1, 2$ and 4, and to study the corresponding moments. This line of study was initiated in [3]. Another application of (3.2) was given in [31], where it was used to study the distribution of the trace in the Cauchy β ensemble; see also [2–5, 68] for the case $\beta = 2$. As the latter references indicate, one of the interests in this is that it relates to the joint moments of the characteristic polynomial of a random unitary random matrix with Haar measure and its derivative, which in turn is relevant to the study of the Riemann zeta function.

4 A Loop Equation Approach

4.1 Connected Correlators

Consider the Stieltjes transform

$$\overline{W}_1(x; N, \beta, p, q) := \int_0^{2\pi} \frac{\widetilde{\rho_{(1),N}^{(\mathrm{cJ})}}(\theta; \beta, p, q)}{x - e^{i\theta}} \, d\theta, \qquad x \notin \mathcal{C}_1, \tag{4.1}$$

where $\mathcal{C}_1$ denotes the unit circle in the complex plane. Expanding for large x shows that this quantity relates to the Fourier components of the density as specified in (1.2),

$$\overline{W}_1(x; N, \beta, p, q) = \frac{1}{x} \sum_{k=0}^{\infty} \frac{c_k^{(\widetilde{\mathrm{cJ}})}(N, \beta, p, q)}{x^k}, \quad |x| > 1. \tag{4.2}$$

One recalls that a primary aim of this paper is to obtain the small τ expansion of the Fourier transform $c_\infty^{(\mathrm{cJ})}(\tau; \beta, p, q)$ as specified in (1.7). To see how $\overline{W}_1(x; N, \beta, p, q)$ relates to this aim, suppose that this quantity admits a $1/N$ expansion

$$\overline{W}_1(x; N, \beta, p, q) = \frac{N}{x} + N \sum_{l=1}^{\infty} \frac{\overline{W}_1^l(x; \beta, p, q)}{N^l}. \tag{4.3}$$

Suppose furthermore that each $W_1^l(x;\beta,p,q)$ when expanded for large x analogous to (4.2) has coefficients which are polynomials in k of degree $l-1$. Denote the leading coefficient in the polynomial by $\alpha_{l-1}(\beta,p,q)$ so that

$$\overline{W}_1^l(x;\beta,p,q)=\frac{1}{x}\sum_{k=1}^{\infty}\frac{1}{x^k}\Big(\alpha_{l-1}(\beta,p,q)k^{l-1}+\mathrm{O}(k^{l-2})\Big). \tag{4.4}$$

It then follows from the definition (1.3) that

$$\widetilde{c_\infty^{(\mathrm{cJ})}}(\tau;\beta,p,q)=\sum_{l=0}^{\infty}\Big(\frac{\tau}{2\pi}\Big)^l\alpha_l(\beta,p,q). \tag{4.5}$$

In fact this general approach has been used to determine $\{\alpha_l(\beta,p,q)|_{p=1,q=0}\}$ up to and including $l=10$ in [70], thereby reclaiming the explicit form of the polynomials $\{p_j(y)\}$ in (1.14) for $j=1,\dots,9$, first obtained in [33].

In the loop equation formalism, the computation of the coefficients in the series of (4.3) up to the j-th requires knowledge of the large N expansion of particular truncations of the multipoint correlators

$$W_n(x_1,\dots,x_n;N,\beta,p,q):=\Big\langle G(x_1)\cdots G(x_n)\Big\rangle,\qquad G(x):=\sum_{j=1}^{N}\frac{1}{x-e^{i\theta_j}}. \tag{4.6}$$

These truncations, denoted $\overline{W}_n$, have for $n=2$ the simple covariance form

$$\overline{W}_2(x_1,x_2)=\Big\langle(G(x_1)-\langle G(x_1)\rangle)(G(x_2)-\langle G(x_2)\rangle)\Big\rangle, \tag{4.7}$$

and similarly for $n=3$. For general $n\geq 2$ the truncations, referred to as connected correlators, are symmetric linear combinations of $\{W_m(x_{j_1},\dots,x_{j_m};N,\beta,p,q)\}_{m=1,\dots,n}$ with $1\leq j_1<\cdots<j_m\leq n$ defined so that they are generating functions for the mixed cumulants; see e.g. [32, Eq. (1.17)]. A key feature of the connected correlators is that their order of decay in a $1/N$ expansion increases as n increases,

$$\overline{W}_n(x_1,\dots,x_n;N,\beta,p,q)=N^{2-n}\sum_{l=0}^{\infty}\frac{\overline{W}_n^l(x_1,\dots,x_n;\beta,p,q)}{N^l}; \tag{4.8}$$

see [8, 70]. To compute $\overline{W}_1^l(x;\beta,p,q)$ in (4.3) up to order $l=j$, we will require knowledge of the expansion (4.8) of $\overline{W}_n$ for $n=2,\dots,j+1$ up to order $l=n-j-1$.

4.2 The Loop Equation Hierarchy

The connected correlators have previously been analysed in the case of the Jacobi β ensemble based on (2.30) with the weight

$$w(x)=x^{\lambda_1}(1-x)^{\lambda_2}\chi_{0<x<1}, \tag{4.9}$$

where $\chi_A=1$ for A true, $\chi_A=0$ otherwise. They satisfy a hierarchy of equations—referred to as loop equations—specified by [40, Eq. (4.6)]

$$0=\left((\kappa-1)\frac{\partial}{\partial x_1}+\left(\frac{\lambda_1}{x_1}-\frac{\lambda_2}{1-x_1}\right)\right)\overline{W}_n^{\mathrm{J}}(x_1,J_n)-\frac{n-1}{x_1(1-x_1)}\overline{W}_{n-1}^{\mathrm{J}}(J_n)$$

$$+ \frac{\chi_{n=1}}{x_1(1-x_1)}\Big((\lambda_1+\lambda_2+1)N+\kappa N(N-1)\Big) - \frac{\chi_{n\neq 1}}{x_1(1-x_1)}\sum_{k=2}^{n} x_k \frac{\partial}{\partial x_k}\overline{W}^{\mathrm{J}}_{n-1}(J_n)$$
$$+ \chi_{n\neq 1}\sum_{k=2}^{n}\frac{\partial}{\partial x_k}\left\{\frac{\overline{W}^{\mathrm{J}}_{n-1}(x_1,\ldots,\hat{x}_k,\ldots,x_n)-\overline{W}^{\mathrm{J}}_{n-1}(J_n)}{x_1-x_k}+\frac{1}{x_1}\overline{W}^{\mathrm{J}}_{n-1}(J_n)\right\}$$
$$+ \kappa\left[\overline{W}^{\mathrm{J}}_{n+1}(x_1,x_1,J_n)+\sum_{J\subseteq J_n}\overline{W}^{\mathrm{J}}_{|J|+1}(x_1,J)\overline{W}^{\mathrm{J}}_{n-|J|}(x_1,J_n\setminus J)\right]. \tag{4.10}$$

Here the subscripts J on the $\overline{W}^{\mathrm{J}}_m$ indicate the connected correlations with respect to the Jacobi β ensemble, $\kappa := \beta/2$ and $J_n := \{x_2,\ldots,x_n\}$ with $J_1 = \emptyset$. Also, the tuple $(x_1,\ldots,\hat{x}_k,\ldots,x_n)$ consists of $x_1,\ldots,x_n$ in order with x_k excluded.

There is a mapping between averages in the generalised circular Jacobi and Jacobi ensembles, which reads [27, corollary of Prop. 3.9.1]

$$\langle f(-e^{i\theta_1},\ldots,-e^{i\theta_N})\rangle^{(\mathrm{cJ})} = \langle f(x_1,\ldots,x_N)\rangle^{(\mathrm{J})}\Big|_{\substack{\lambda_1=-\beta(N+p-1)/2-1-iq\\ \lambda_2=\beta p}}. \tag{4.11}$$

Here f is a multivariable polynomial, and the identity is to be interpreted in the sense of analytic continuation; cf. (3.2). The extension of the validity of (4.11) for f analytic follows by appropriate summation. Note that shifting the integration variables $\theta_j \mapsto \theta_j + \pi$ on the LHS of (4.11) removes the minus signs in the arguments of f to give

$$\langle f(e^{i\theta_1},\ldots,e^{i\theta_N})\rangle^{(\widetilde{\mathrm{cJ}})} = \langle f(x_1,\ldots,x_N)\rangle^{(\mathrm{J})}\Big|_{\substack{\lambda_1=-\beta(N+p-1)/2-1+iq\\ \lambda_2=\beta p}}. \tag{4.12}$$

The mapping (4.12) allows the loop equations (4.10) for the generalised Jacobi β ensemble to be transformed to the loop equations for the circular Jacobi β ensemble with weight (1.1) by a simple substitution of λ_1, λ_2. With this done, the large N expansion (4.8) can be substituted and like powers of N equated to obtain a triangular system of equations for $\{W_n^{(l)}\}$.

Corollary 4.1 *Let $\kappa = \beta/2$, $\tilde{\lambda}_1 = \kappa - 1 - \kappa p - qi$, $\tilde{\lambda}_2 = 2\kappa p$. We have*

$$-\frac{1}{x}\overline{W}^0_1(x) + (\overline{W}^0_1(x))^2 = 0, \tag{4.13}$$
$$-\frac{\kappa}{x_1}\overline{W}^0_2(x_1,x_2) - \frac{1}{x_1(1-x_1)}\overline{W}^0_1(x_2) - \frac{1}{x_1(1-x_1)}x_2\frac{\partial}{\partial x_2}\overline{W}^0_1(x_2)$$
$$+\frac{\partial}{\partial x_2}\left(\frac{\overline{W}^0_1(x_1)-\overline{W}^0_1(x_2)}{x_1-x_2}+\frac{1}{x_1}\overline{W}^0_1(x_2)\right) + 2\kappa\overline{W}^0_1(x_1)\overline{W}^0_2(x_1,x_2) = 0, \tag{4.14}$$
$$(\kappa-1)\frac{\partial\overline{W}^0_1(x)}{\partial x} + \left(\frac{\tilde{\lambda}_1}{x} - \frac{\tilde{\lambda}_2}{1-x}\right)\overline{W}^0_1(x) - \frac{\kappa}{x}\overline{W}^1_1(x)$$
$$+\frac{\kappa p - iq}{x(1-x)} + \kappa\overline{W}^0_1(x,x) + 2\kappa\overline{W}^0_1(x)\overline{W}^1_1(x) = 0, \tag{4.15}$$
$$-\frac{\kappa}{x_1}\overline{W}^{l+1}_1(x_1) + \left[(\kappa-1)\frac{\partial}{\partial x_1} + \frac{\tilde{\lambda}_1}{x_1} - \frac{\tilde{\lambda}_2}{1-x_1}\right]\overline{W}^l_1(x_1)$$
$$+\kappa\overline{W}^{l-1}_2(x_1,x_1) + \kappa\sum_{m=0}^{l+2}\overline{W}^m_1(x_1)\overline{W}^{l+2-m}_1(x_1) = 0, \quad l \geq 1, \tag{4.16}$$

and

$$
\begin{aligned}
&-\frac{\kappa}{x_1}\overline{W}_2^{l+1}(x_1,x_2)+\left[(\kappa-1)\frac{\partial}{\partial x_1}+\frac{\tilde\lambda_1}{x_1}-\frac{\tilde\lambda_2}{1-x_1}\right]\overline{W}_2^{l}(x_1,x_2)\\
&-\frac{1}{x_1(1-x_1)}\left(x_2\frac{\partial}{\partial x_2}+1\right)\overline{W}_1^{l+1}(x_2)+\frac{\partial}{\partial x_2}\left\{\frac{\overline{W}_1^{l+1}(x_1)-\overline{W}_1^{l+1}(x_2)}{x_1-x_2}+\frac{1}{x_1}\overline{W}_1^{l+1}(x_2)\right\}\\
&+\kappa\overline{W}_3^{l-1}(x_1,x_1,x_2)+2\kappa\sum_{m=0}^{l+1}\overline{W}_1^{m}(x_1)\overline{W}_2^{l+1-m}(x_1,x_2)=0,\quad l\geq 0. \qquad (4.17)
\end{aligned}
$$

In addition, an equation analogous to (4.17) holds which expresses $\overline{W}_n^{l+1}$ for $n \geq 3$ in terms of lower order terms in the triangular system.

A hand calculation readily suffices to solve the above triangular system up to and including $\overline{W}_1^2(x)$. Thus we find

$$
\overline{W}_1^0(x)=\frac{1}{x},\quad \overline{W}_1^1(x)=\frac{\kappa p+iq}{\kappa x(1-x)},\quad \overline{W}_1^2(x)=\frac{(\kappa p-iq-\kappa+1)(\kappa p+iq)}{\kappa^2(x-1)^2}, \qquad (4.18)
$$

with the intermediate results

$$
\overline{W}_2^0(x,y)=\overline{W}_2^1(x,y)=0,\qquad \overline{W}_3^0(x,y,z)=0. \qquad (4.19)
$$

The evaluations of $\overline{W}_1^0(x)$, $\overline{W}_1^1(x)$ and $\overline{W}_2^0(x,y)$ can in fact be obtained by specialising parameters in results from [40, Prop. 4.8]. The first of the equations in (4.18) is equivalent to the normalisation

$$
\int_0^{2\pi}\rho_{(1),N}^{(\widetilde{\mathrm{cJ}})}(\theta;\beta,p,q)\,d\theta=N=\frac{N}{2\pi}\delta_{k=0}\int_0^{2\pi}e^{ik\theta}\,d\theta,\quad k\in\mathbb{Z}. \qquad (4.20)
$$

As commented in the sentence above (1.4), this last expression, with the restriction to $k=0$ removed, is to be subtracted from the Fourier coefficient $\int_0^{2\pi}\rho_{(1),N}^{(\widetilde{\mathrm{cJ}})}(\theta;\beta,p,q)e^{ik\theta}\,d\theta$ to arrive at (1.7). With this done, according to the definition of $\{\alpha_l\}$ given in (4.4), we read off from the second and third equations in (4.18) that

$$
\alpha_0=-p-\frac{iq}{\kappa},\qquad \alpha_1=\left(\frac{1}{\kappa}-1\right)p+p^2+\frac{q(-q+(\kappa-1)i)}{\kappa^2}, \qquad (4.21)
$$

where $\kappa:=\beta/2$.

The use of computer algebra allows the evaluations (4.19) to be extended, and thus similarly for (4.21). Specifically, in relation to the latter, this shows

$$
\begin{aligned}
\alpha_2=&\frac{\alpha(1+\bar\alpha-\kappa)(-2+\alpha-\bar\alpha+2\kappa)}{2\kappa^3}, \qquad (4.22)\\
\alpha_3=&\frac{\alpha(1+\bar\alpha-\kappa)\left(6+\alpha^2+\bar\alpha^2-3\alpha(2+\bar\alpha-2\kappa)-5\bar\alpha(\kappa-1)-11\kappa+6\kappa^2\right)}{6\kappa^4}, \qquad (4.23)\\
\alpha_4=&\frac{1}{24\kappa^5}\alpha(1+\bar\alpha-\kappa)\left(\alpha^3-\bar\alpha^3-6\alpha^2(2+\bar\alpha-2\kappa)+9\bar\alpha^2(\kappa-1)\right.\\
&+\bar\alpha\left(-26+47\kappa-26\kappa^2\right)\\
&\left.+\alpha\left(34+6\bar\alpha^2-29\bar\alpha(\kappa-1)-63\kappa+34\kappa^2\right)+12\left(-2+5\kappa-5\kappa^2+2\kappa^3\right)\right), \qquad (4.24)\\
\alpha_5=&\frac{1}{120\kappa^6}\alpha(1+\bar\alpha-\kappa)\left(\alpha^4+\bar\alpha^4-10\alpha^3(2+\bar\alpha-2\kappa)-14\bar\alpha^3(\kappa-1)\right.
\end{aligned}
$$

$$+ 5\alpha^2 \left(22 + 4\bar{\alpha}^2 - 19\bar{\alpha}(\kappa - 1) - 41\kappa + 22\kappa^2\right) + \bar{\alpha}^2 \left(71 - 127\kappa + 71\kappa^2\right)$$
$$+ \bar{\alpha} \left(154 - 377\kappa + 377\kappa^2 - 154\kappa^3\right) + 4 \left(30 - 91\kappa + 124\kappa^2 - 91\kappa^3 + 30\kappa^4\right)$$
$$- 5\alpha \left(42+2\bar{\alpha}^3 - 17\bar{\alpha}^2(\kappa - 1) - 107\kappa + 107\kappa^2 - 42\kappa^3 + \bar{\alpha} \left(47 - 86\kappa + 47\kappa^2\right)\right)\Big), \quad (4.25)$$

where $\alpha := \kappa p + iq$. We are now in a position to specify the coefficients in (1.23) up to and including index label $j = 5$.

Proposition 4.2 *Let $\alpha_0, \ldots, \alpha_5$ be given by (4.21)–(4.25). We have*

$$h_j(\beta, p, q) = \frac{i}{(2\pi)^j} \mathrm{Im}\, \alpha_j, \quad j \text{ even} \qquad h_j(\beta, p, q) = \frac{1}{(2\pi)^j} \mathrm{Re}\, \alpha_j, \quad j \text{ odd} \quad (4.26)$$

and

$$\tilde{h}_j(\beta, p, q) = \frac{1}{(2\pi)^j} \mathrm{Re}\, \alpha_j, \quad j \text{ even} \qquad \tilde{h}_j(\beta, p, q) = \frac{i}{(2\pi)^j} \mathrm{Im}\, \alpha_j, \quad j \text{ odd}. \quad (4.27)$$

Explicity, for indices up to and including $j = 3$,

$$h_0(\beta, p, q) = -\frac{iq}{\kappa}, \quad h_1(\beta, p, q) = \frac{1}{2\pi\kappa^2}(q^2 + \kappa p + \kappa^2(-1 + p)p), \quad (4.28)$$

$$h_2(\beta, p, q) = \frac{iq}{4\pi^2\kappa^3}(-1 + q^2 + \kappa(2 + p) + \kappa^2(-1 - p + p^2)), \quad (4.29)$$

$$h_3(\beta, p, q) = \frac{1}{48\pi^2\kappa^4}\Big(17q^2 - 5q^4 + \kappa(-33q^2 - 6p(-1 + q^2)) + \kappa^2(17q^2 + p^2(5 - 6q^2)$$
$$+ p(-17 + 6q^2)) + \kappa^3 p(17 - 9p - 2p^2) - \kappa^4 p(6 - 5p - 2p^2 + p^3)\Big) \quad (4.30)$$

and

$$\tilde{h}_0(\beta, p, q) = -p, \quad \tilde{h}_1(\beta, p, q) = \frac{iq(1 - \kappa)}{2\pi\kappa^2}, \quad \tilde{h}_2(\beta, p, q)$$
$$= -\frac{(1 - \kappa)}{4\pi^2\kappa^3}(2q^2 + \kappa p + \kappa^2(-1 + p)p), \quad (4.31)$$

$$\tilde{h}_3(\beta, p, q) = -\frac{iq(1 - \kappa)}{48\pi^3\kappa^4}(-6 + 16q^2 + \kappa(11 + 12p) + 6\kappa^2(-1 - 2p + 2p^2)). \quad (4.32)$$

Proof This follows by comparing (1.23) and (4.5), then making use of the property noted in the sentence below (1.25). □

Remark 4.3 1. Based on the loop equations (4.10) it has been shown in [32] that the Jacobi β ensemble connected correlators $\overline{W}_n^{\mathrm{J}}$ satisfy the functional equation

$$\overline{W}_n^{\mathrm{J}}(x_1, \ldots, x_n; N, \kappa, \lambda_1, \lambda_2) = (-\kappa)^{-n}\overline{W}_n^{\mathrm{J}}(x_1, \ldots, x_n; N, 1/\kappa, -\kappa\lambda_1, -\kappa\lambda_2),$$

whereas in Proposition 4.2, $\kappa := \beta/2$. The mapping (4.11) then induces the functional equation for the circular Jacobi β ensemble connected correlators $\overline{W}_n^{\widetilde{\mathrm{cJ}}}$,

$$\overline{W}_n^{\widetilde{\mathrm{cJ}}}(x_1, \ldots, x_n; N, \kappa, p, q) = (-\kappa)^{-n}\overline{W}_n^{\widetilde{\mathrm{cJ}}}(x_1, \ldots, x_n; -\kappa N, 1/\kappa, -\kappa p, q/\kappa).$$

Now setting $n = 1$, expanding in $1/N$ as in (4.3), then proceeding from this to (4.5) gives the functional equation

$$c_\infty^{\widetilde{\mathrm{cJ}}}(\tau; \beta, p, q) = -\frac{2}{\beta}c_\infty^{\widetilde{\mathrm{cJ}}}(-2\tau/\beta; 4/\beta, -\beta p/2, -2q/\beta)$$

or equivalently

$$h_j(\beta, p, q) = \Big(-\frac{2}{\beta}\Big)^{j+1} h_j\Big(\frac{4}{\beta}, -\frac{\beta p}{2}, -\frac{2q}{\beta}\Big),$$
$$\tilde{h}_j(\beta, p, q) = \Big(-\frac{2}{\beta}\Big)^{j+1} \tilde{h}_j\Big(\frac{4}{\beta}, -\frac{\beta p}{2}, -\frac{2q}{\beta}\Big). \tag{4.33}$$

The functional equations (4.33) are readily exhibited on the explicit functional forms (4.28)–(4.32).

2. In the case $q = 0$ the density $\rho^{(\mathrm{cJ})}_{(1),\infty}(x; \beta, p, q)$ is an even function of x and consequently $c^{(\widetilde{\mathrm{cJ}})}_{\infty}(\tau; \beta, p, q)$ is an even function of τ. In the expansion (1.23) we must then have that $h_j(\beta, p, q)|_{q=0} = 0$ for j even and $\tilde{h}_j(\beta, p, q)|_{q=0} = 0$ for j odd, which can also be viewed as corollaries of (1.24). This is indeed a feature of the explicit functional forms (4.28)–(4.32).

5 Discussion

Setting $q = 0$ we read off from (1.7), (1.23) and the results of Proposition 4.2 that

$$\int_{-\infty}^{\infty} \Big(\rho^{(\mathrm{cJ})}_{(1),\infty}(x; \beta, p, q)|_{q=0} - 1\Big) e^{i\tau x}\, dx$$
$$= -p + \frac{p}{2\pi}((1-\kappa) + \kappa p)\frac{|\tau|}{\kappa} - \frac{p(1-\kappa)}{4\pi^2}((1-\kappa) + \kappa p)\frac{\tau^2}{\kappa^2} + \mathrm{O}(\tau^3). \tag{5.1}$$

It is of interest to compare this expansion to the analogue for the two-dimensional one-component plasma (1.21). With Q identified as p, we see that the first term on the RHS, which corresponds to the perfect screening of the external charge, is the same for both. In two-dimensions the generalised Fourier transform of the logarithmic potential $V^{\mathrm{OCP}}(\mathbf{r}) = -\log|\mathbf{r}|$ is $\hat{V}^{\mathrm{OCP}}(\mathbf{k}) = 2\pi/|\mathbf{k}|^2$ and the dimensionless thermodynamic pressure is $\beta P^{\mathrm{OCP}} = 1 - \beta/4$. Hence the second term in the expansion (1.22) can be written

$$\frac{Q}{\hat{V}^{\mathrm{OCP}}(\mathbf{k})\beta}(\beta P^{\mathrm{OCP}} + (\beta/4)Q). \tag{5.2}$$

In relation to the circular β ensemble (CβE), the generalised Fourier transform of the logarithmic potential $V^{\mathrm{C}\beta\mathrm{E}}(r) = -\log|r|$ is $\hat{V}^{\mathrm{C}\beta\mathrm{E}}(\tau) = \pi/|\tau|$ and the dimensionless thermodynamic pressure is $\beta P^{\mathrm{C}\beta\mathrm{E}} = 1 - \beta/2$. Hence the second term in the expansion (5.1) can be written

$$\frac{p}{\hat{V}^{\mathrm{C}\beta\mathrm{E}}(\tau)\beta}(\beta P^{\mathrm{C}\beta\mathrm{E}} + (\beta/2)p). \tag{5.3}$$

The structural similarity between (5.2) and (5.3) is evident. However when it comes to comparing the third terms on the RHSs of (5.1) and (1.22) structural differences show. Specifically the former is Q times a quadratic polynomial in Q, while the latter is p times a linear polynomial in p. This is perhaps not unsurprising as the argument used in [62] to predict the Q^2 term relies in part on the screening cloud having a fast decay with well defined moments, which is not true in one-dimension. The dependence on β is quadratic in $1/\beta$ in both cases, although in (5.1) there is a stand alone factor of $(1/\kappa - 1)$ with no analogue in (1.22).

A linear response argument is used in [62] to deduce that

$$\lim_{Q\to 0}\frac{1}{Q}\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)-1)e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r} = -\beta\tilde{V}^{\rm OCP}(\mathbf{k})S^{\rm OCP}_{\infty}(\mathbf{k};\beta)$$
$$= -\beta\tilde{V}^{\rm OCP}(\mathbf{k})\Big(1+\int_{\mathbb{R}^2}(\rho^{\rm OCP}_{(1),\infty}(\mathbf{r};Q)|_{Q=1}-1)e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r}\Big), \tag{5.4}$$

where the second equality follows from the two-dimensional analogue of (1.11). As noted in [62], this sum rule is readily checked to be a feature of the expansion (1.22). The same linear response argument can be applied to the one-dimensional log-gas to obtain

$$\lim_{p\to 0}\frac{1}{p}\int_{-\infty}^{\infty}(\rho^{\rm OCP}_{(1),\infty}(x;p,q)|_{q=0}-1)e^{i\tau x}\,dx = -\beta\tilde{V}^{\rm C\beta E}(\tau)S^{\rm C\beta E}_{\infty}(\tau;\beta)$$
$$= -\beta\tilde{V}^{\rm C\beta E}(\tau)\Big(1+\int_{-\infty}^{\infty}(\rho^{\rm OCP}_{(1),\infty}(x;p,q)|_{\substack{p=1\\q=0}}-1)e^{i\tau x}\,dx\Big), \tag{5.5}$$

or equivalently, using (1.6) and (1.11), the relation

$$\lim_{p\to 0}\frac{1}{p}c_{\infty}^{(\widetilde{\rm cJ})}(\tau;\beta,p,0) = -\frac{\pi\beta}{|\tau|}\Big(1+c_{\infty}^{(\widetilde{\rm cJ})}(\tau;\beta,1,0)\Big). \tag{5.6}$$

From the results of Proposition 4.2 we can check the validity of (5.6) in a small τ expansion up to and including order τ^4.

Linear response also applies in the limit $q\to 0$. To facilitate this, define $V(x)=\pi\,\mathrm{sgn}(x)$. Noting as an improper integral that

$$i\int_{-\infty}^{\infty}\frac{e^{-ix\tau}}{\tau}\,d\tau = \pi\,\mathrm{sgn}(x)$$

allows us to take the inverse Fourier transform to conclude $\hat{V}(\tau)=2\pi i/\tau$. With this knowledge we can deduce the analogues of (5.5) and (5.6). Specifically, in relation to the latter we obtain

$$\lim_{q\to 0}\frac{1}{q}c_{\infty}^{(\widetilde{\rm cJ})}(\tau;\beta,0,q) = -\frac{2i}{\tau}\Big(1+c_{\infty}^{(\widetilde{\rm cJ})}(\tau;\beta,1,0)\Big). \tag{5.7}$$

As with (5.6), using the results of Proposition 4.2 we can check the validity of (5.7) in a small τ expansion up to and including order τ^4.

In Proposition 4.2 let us replace q by κq. Then h_j and $\tilde{h}_j$ are each polynomials of degree j in $1/\kappa$. As remarked in the Introduction below (1.15), in the case $p=1$, $q=0$ these polynomials have the property of being palindromic or anti-palindromic in $u:=1/\kappa$. Moreover, it has been observed from the explicit form of these polynomials that all the zeros lie on the unit circle $|u|=1$ in the complex u-plane and moreover exhibit an interlacing property [28, 33], which is a feature too of the polynomials appearing in the $1/N$ expansion of the moments of the spectral density for the Gaussian β ensemble [69]. (More on this has been communicated to the senior author by Michael A. La Croix, who among other things highlights the references [25, 44, 49]—one should add too the recent work [58]—although any sort of explanation is still lacking.) Our results of Proposition 4.2 reveal these same properties for the polynomials in $u:=1/\kappa$ which result as coefficients by expanding h_j or $\tilde{h}_j$ in a power series in p for $q=0$, or in q for $p=0$. For example, from (4.25) and (4.25) we compute that in relation to $h_5(\beta,p,q)|_{q=0}$ the coefficients of $p,\dots,p^5$ in order are, up to proportionality, the (anti-)palindromic polynomials

$$(u-1)(30-91u+124u^2-91u^3+30u^4),\quad 32-75u+90u^2-75u^3+32u^4,$$

$$(u-1)(11-20u+11u^2), \quad 5-9u+5u^2, \quad u-1,$$

and which indeed can be checked to have successively interlacing zeros all on the unit circle in the complex u-plane.

The results of Proposition 4.2 take on a simpler form in the case $q=0$ with $\beta \to \infty$ (low temperature limit),

$$\lim_{\beta\to\infty} c_\infty^{(\widetilde{\text{cJ}})}(\tau;\beta,p,0) = -p + \tilde{p}\frac{|\tau|}{2\pi} + \tilde{p}\Big(\frac{\tau}{2\pi}\Big)^2 + \tilde{p}\Big(1-\frac{1}{6}\tilde{p}\Big)\Big(\frac{|\tau|}{2\pi}\Big)^3 + \tilde{p}\Big(1-\frac{1}{3}\tilde{p}\Big)\Big(\frac{\tau}{2\pi}\Big)^4 + \tilde{p}\Big(1-\frac{7}{15}\tilde{p}+\frac{1}{60}\tilde{p}^2\Big)\Big(\frac{\tau}{2\pi}\Big)^5 + \cdots, \tag{5.8}$$

where $\tilde{p} := p(p-1)$. The factor of $\tilde{p}$ in all terms but the first makes sense as the ground state is an equally spaced lattice for both $p=0$ and $p=1$, which is when $\tilde{p}$ vanishes. Also, the leading term in $\tilde{p}$ of each power of τ is suggestive of the summation in τ

$$\lim_{\beta\to\infty} c_\infty^{(\widetilde{\text{cJ}})}(\tau;\beta,p,0) + p \underset{\tilde{p}\to 0}{\sim} \frac{\tilde{p}|\tau|/(2\pi)}{1-|\tau|/(2\pi)}, \tag{5.9}$$

which makes explicit the form of the singularity in τ as $|\tau| \to (2\pi)^-$. In fact for $p \to 0$ (and thus $\tilde{p} \to -p \to 0$) this can be anticipated from the first equality in (5.5). The required input is the large β form of $S_\infty^{\text{C}\beta\text{E}}(\tau;\beta)$, which we know from [33, Sect. 5] to be given by the RHS of (5.9) with $\tilde{p}$ replaced by $2/\beta$. Substituting in the first equality of (5.5) gives a result equivalent to (5.9) for $p \to 0$. We remark that with $q=0$ a result of [41] tells us that with bulk spectrum singularity scaling, in the limit $\beta \to \infty$ the eigenvalues crystallise at the zeros of the Bessel function $J_{p-1/2}(\pi x)$.

Acknowledgements The contribution of Menglin Wang in helping with some of the calculations in Sect. 2 is acknowledged. Also, we are appreciative of the feedback provided by the reviewers.

Funding This research is part of the program of study supported by the Australian Research Council Discovery Project grant DP210102887. In particular the grant partially supported the visit of Bo-Jian Shen to the University of Melbourne to work on this project. The research of Bo-Jian Shen is also supported by the National Natural Science Foundation of China (Grant Nos. 12175155 and 12371251) and the Shanghai Frontier Research Institute for Modern Analysis.

Data Availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest There are no conflicts of interest associated with this work.

References

1. Akemann, G., Aygün, N., Würfel, T.R.: Generalised unitary group integrals of Ingham-Siegel and Fisher-Hartwig type, arXiv:2305.19852
2. Assiotis, T., Keating, J., Warren, J.: On the joint moments of the characteristic polynomials of random unitary matrices. Int. Math. Res. Not. **2022**, 4564–14603 (2022)
3. Assiotis, T., Bedert, B., Gunes, M.A., Soor, A.: Moments of generalised Cauchy random matrices and continuous-Hahn polynomials. Nonlinearity **34**, 4923 (2021)
4. Assiotis, T., Bedert, B., Gunes, M.A., Soor, A.: On a distinguished family of random variables and Painlevé equations. Probab. Math. Phys. **2**, 613–642 (2021)
5. Assiotis, T., Gunes, M.A., Soor, A.: Convergence and an explicit formula for the joint moments of the circular Jacobi β-ensemble characteristic polynomial. Math. Phys. Anal. Geom. **25**, 15 (2022)
6. Berestycki, N., Webb, C., Wong, M.D.: Random Hermitian matrices and Gaussian multiplicative chaos. Probab. Theory Relat. Fields **172**, 103–189 (2018)
7. Borodin, A., Olshanski, G.: Infinite random matrices and ergodic measures. Commun. Math. Phys. **223**, 87–123 (2001)
8. Borot, G., Guionnet, A.: Asymptotic expansion of β matrix models in the one-cut regime. Commun. Math. Phys. **317**, 447–483 (2013)
9. Bourgade, P., Falconet, H.: Liouville quantum gravity from random matrix dynamics, arXiv: 2206.03029
10. Bourgade, P., Nikeghbali, A., Rouault, A.: Circular Jacobi ensembles and deformed Verblunsky coefficients. Int. Math. Res. Not. IMRN **23**, 4357–4394 (2009)
11. Byun, S.-S., Charlier, C.: On the characteristic polynomial of the eigenvalue moduli of random normal matrices, arXiv:2205.04298
12. Byun, S.-S., Forrester, P.J.: Progress on the study of the Ginibre ensembles I: GinUE, arXiv:2211.16223
13. Can, T., Laskin, M., Wiegmann, P.: Geometry of quantum Hall states: gravitational anomaly and kinetic coefficients. Ann. Phys. **362**, 752–794 (2015)
14. Charlier, C., Gharakhloo, R.: Asymptotics of Hankel determinants with a Laguerre-type or Jacobi-type potential and Fisher-Hartwig singularities. Adv. Math. **383**, 107672 (2021)
15. Cipolloni, C., Erdös, L., Schröder, D.: On the spectral form factor for random matrices. Commun. Math. Phys. (2023). https://doi.org/10.1007/s00220-023-04692-y
16. Claeys, T., Glesner, G., Minakov, A., Yang, M.: Asymptotics for averages over classical orthogonal ensembles. Int. Math. Res. Not. **2022**, 7922–7966 (2022)
17. Cotler, J.S., Gur-Ari, G., Hanada, M., Polchinski, J., Saad, P., Shenker, S.H., Stanford, D., Streicher, A., Tezuka, M.: Black holes and random matrices. JHEP **1705**, 118 (2017); Erratum: [JHEP **1809** (2018), 002]
18. Cotler, J.S., Hunter-Jones, N., Liu, J., Yoshida, B.: Chaos, complexity, and random matrices. J. High Energy Phys. **1711**, 048 (2017)
19. Cohen, P., Cunden, F., O'Connell, N.: Moments of discrete orthogonal polynomial ensembles. Electron. J. Probab. **25**, 1–19 (2020)
20. Cunden, F., Mezzadri, F., O'Connell, N., Simm, N.: Moments of random matrices and hypergeometric orthogonal polynomials. Commun. Math. Phys. **369**, 1091–1145 (2019)
21. Deaño, A., Simm, N.: Characteristic polynomials of complex random matrices and Painlevé transcendents. Int. Math. Res. Not. **2022**, 210–264 (2022)
22. Deift, P., Its, A., Krasovsky, I.: Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model: some historic and some recent results. Commun. Pure Appl. Math. **66**, 1360–1438 (2013)
23. Deift, P., Krasovsky, I., Vasilevska, J.: Asymptotics for a determinant with a confluent hypergeometric kernel. Int. Math. Res. Not. **2011**, 2117–2160 (2011)
24. del Campo, A., Molina-Vilaplana, J., Sonner, J.: Scrambling the spectral form factor: unitarity constraints and exact results. Phys. Rev. D **95**, 126008 (2017)
25. Dołga, M., Féray, Valentin: Cumulants of Jack symmetric functions and the b-conjecture. Trans. Am. Math. Soc. **369**, 9015–9039 (2017)
26. Fisher, M.E., Hartwig, R.E.: Toeplitz determinants—some applications, theorems and conjectures. Adv. Chem. Phys. **15**, 333–353 (1968)
27. Forrester, P.J.: Log-Gases and Random Matrices. Princeton University Press, Princeton (2010)
28. Forrester, P.J.: Differential identities for the structure function of some random matrix ensembles. J. Stat. Phys. **183**, 33 (2021)
29. Forrester, P.J.: Quantifying dip-ramp-plateau for the Laguerre unitary ensemble structure function. Commun. Math. Phys. **387**, 215–235 (2021)
30. Forrester, P.J.: Moments of the ground state density for the d-dimensional Fermi gas in an harmonic trap. Random Matrices **10**, 2150018 (2021)

31. Forrester, P.J.: Joint moments of a characteristic polynomial and its derivative for the circular β ensemble. Probab. Math. Phys. **3**, 145–170 (2022)
32. Forrester, P.J.: High-low temperature dualities for the classical β-ensembles. Random Matrices **11**, 2250035 (2022)
33. Forrester, P.J., Jancovici, B., McAnally, D.S.: Analytic properties of the structure function for the one-dimensional one-component log-gas. J. Stat. Phys. **102**, 737–780 (2000)
34. Forrester, P.J., Kieburg, M., Li, S.-H., Zhang, J.: Dip-ramp-plateau for Dyson Brownian motion from the identity on $U(N)$, arXiv:2206.14950
35. Forrester, P.J., Kumar, S.: Differential recurrences for the distribution of the trace of the β-Jacobi ensemble. Physica D **434**, 133220 (2022)
36. Forrester, P.J., Li, S.-H., Shen, B.-J., Ye, G.-F.: q-Pearson pair and moments in q-deformed ensembles. Ramanujan J. **60**, 195–235 (2023)
37. Forrester, P.J., Li, S.-H., Trinh, A.K.: Asymptotic correlations with corrections for the circular Jacobi β-ensemble. J. Approx. Theory **271**, 105633 (2021)
38. Forrester, P.J., Nagao, T.: Correlations for the Cauchy and generalized circular ensemble with orthogonal and symplectic symmetry. J. Phys. A **34**, 7919–7932 (2001)
39. Forrester, P.J., Rahman, A.A.: Relations between moments for the Jacobi and Cauchy random matrix ensembles. J. Math. Phys. **62**, 073302 (2021)
40. Forrester, P.J., Rahman, A.A., Witte, N.S.: Large N expansions for the Laguerre and Jacobi β ensembles from the loop equations. J. Math. Phys. **58**, 113303 (2017)
41. Forrester, P.J., Rogers, J.B.: Electrostatics and the zeros of the classical polynomials. SIAM J. Math. Anal. **17**, 461–468 (1986)
42. Fyodorov, Y.V., Simm, N.J.: On the distribution of the maximum value of the characteristic polynomial of GUE random matrices. Nonlinearity **29**, 2837–2855 (2016)
43. Gisonni, M., Grava, T., Ruzza, G.: Jacobi ensemble, Hurwitz numbers and Wilson polynomials. Lett. Math. Phys. **111**, 67 (2021)
44. Goulden, I.P., Jackson, D.M.: Connection coefficients, matchings, maps and combinatorial conjectures for Jack symmetric functions. Trans. Am. Math. Soc. **348**, 873–892 (1996)
45. Harer, J., Zagier, D.: The Euler characteristic of the moduli space of curves. Inven. Math. **85**, 457–485 (1986)
46. Hua, L.K.: Analysis of Functions of Several Complex Variables in the Classical Domains. American Mathematical Society, Providence (1963)
47. Jancovici, B., Šamaj, L.: Guest charge and potential fluctuations in two-dimensional classical Coulomb systems. J. Stat. Phys. **131**, 613–629 (2008)
48. Kalinay, P., Markoš, P., Šamaj, L., Travěnec, I.: The sixth-moment sum rule for the pair correlations of the two-dimensional one-component plasma: exact result. J. Stat. Phys. **98**, 639–666 (2000)
49. La Croix, M.A.: The combinatorics of the Jack parameter and the genus series for topological maps, Ph.D. thesis, University of Waterloo (2009)
50. Ledoux, M.: Differential operators and spectral distributions of invariant ensembles from the classical orthogonal polynomials. The continuous case. Electron. J. Probab. **9**, 177–208 (2004)
51. Ledoux, M.: A recursion formula for the moments of the Gaussian orthogonal ensemble. Ann. Inst. Henri Poincaré Probab. Stat. **45**, 754–769 (2009)
52. Lighthill, J.: Introduction to Fourier Analysis and Generalized Functions. Cambridge University Press, Cambridge (1958)
53. Liu, D.-Z.: Limits for circular Jacobi beta-ensembles. J. Approx. Theory **215**, 40–67 (2017)
54. Mezzadri, F., Simm, N.J.: τ-function theory of quantum chaotic transport with $\beta = 1, 2, 4$. Commun. Math. Phys. **324**, 465–513 (2013)
55. Mukherjee, A., Hikami, S.: Spectral form factor for time-dependent matrix model. J. High Energy Phys. **2021**, 071 (2021)
56. Nagao, T., Slevin, K.: Laguerre ensembles of random matrices: nonuniversal correlation functions. J. Math. Phys. **34**, 2317–2330 (1993)
57. NIST Digital Library of Mathematical Functions
58. Novaes, M.: Time delay statistics for finite number of channels in all symmetry classes. Europhys. Lett. **139**, 21001 (2022)
59. Okuyama, K.: Spectral form factor and semi-circle law in the time direction. J. High Energy Phys. **2019**, 161 (2019)
60. Rahman, A.A., Forrester, P.J.: Linear differential equations for the resolvents of the classical matrix ensembles. Random Matrices Theory Appl. **10**, 2250003 (2021)
61. Šamaj, L.: A generalization of the Stillinger-Lovett sum rules for the two-dimensional jellium. J. Stat. Phys. **128**, 1415–1428 (2007)

62. Šamaj, L.: Fourth moment of the charge density induced around a guest charge in two-dimensional jellium. J. Stat. Phys. **175**, 1066–1079 (2019)
63. Sri Ranga, A.: Szegö polynomials from hypergeometric functions. Proc. Am. Math. Soc. **138**, 4243–4247 (2010)
64. Torres-Herrera, E.J., García-García, A.M., Santos, L.F.: Generic dynamical features of quenched interacting quantum systems: survival probability, density imbalance, and out-of-time-ordered correlator. Phys. Rev. B **97**, 060303 (2018)
65. Vleeshouwers, W.L., Gritsev, V.: Topological field theory approach to intermediate statistics. SciPost Phys. **10**, 146 (2021)
66. Vleeshouwers, W.L., Gritsev, V.: The spectral form factor in the 't Hooft limit—intermediacy versus universality. SciPost Phys. Core **5**, 051 (2022)
67. Webb, C., Wong, M.D.: On the moments of the characteristic polynomial of a Ginibre random matrix. Proc. Lond. Math. Soc. **118**, 1017–1056 (2019)
68. Winn, B.: Derivative moments for characteristic polynomials from CUE. Commun. Math. Phys. **315**, 532–562 (2012)
69. Witte, N.S., Forrester, P.J.: Moments of the Gaussian β ensembles and the large N expansion of the densities. J. Math. Phys. **55**, 083302 (2014)
70. Witte, N.S., Forrester, P.J.: Loop equation analysis of the circular ensembles. J. High Energy Phys. **2015**, 173 (2015)
71. Xu, S., Zhao, Y.: Gap probability of the circular unitary ensemble with a Fisher-Hartwig singularity and the coupled Painlevé V system. Commun. Math. Phys. **377**, 1545–1596 (2020)