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Closeness of solutions and averaging for nonlinear systems on Riemannian manifolds*

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***Abstract*— An averaging result for periodic dynamical systems evolving on Euclidean spaces is extended to those evolving on (differentiable) Riemannian manifolds. Using standard tools from differential geometry, a perturbation result for time-varying dynamical systems is developed that measures closeness of trajectories via a suitable metric on a finite time horizon. This perturbation result is then extended to bound excursions in the trajectories of periodic dynamical systems from those of their respective averages, on an infinite time horizon, yielding the specified averaging result. Some simple examples further illustrating this result are also presented.**

I. Introduction

Averaging is a powerful perturbation based tool [8], [10], [21] that has applications in the study of time-varying linear and nonlinear dynamical systems. Where applicable, averaging can provide closeness of solutions bounds for the trajectories of such systems relative to those of a corresponding time-invariant averaged system. As the trajectories of time-invariant averaged systems can typically be analysed more easily than those of the original time-varying systems, such bounds can lead to averaging results that simplify stability analysis considerably. Averaging results have been developed for numerous classes of dynamical systems and differential inclusions [3], [5], [7], [18], [19], [25], [27], [28], including dynamical systems on Lie groups [15]–[17].

Riemannian manifolds have been considered as configuration spaces for many dynamical systems, see [1], [2], [4], [6], [22]. The analyses of such systems require differential geometric tools, see [4], [6]. In this paper, a class of dynamical systems evolving on Riemannian manifolds is considered. Based on [24], averaging results for finite and infinite time horizons are studied, see also [2], [4], [6], [15]–[17]. These results exploit closeness of solutions arguments for dynamical systems evolving on Riemannian manifolds over an infinite time horizon, and notions of asymptotic and exponential stability (generalized from Euclidean spaces to Riemannian manifolds).

In terms of organization, Section II of this paper presents some required mathematical preliminaries concerning differential geometry, etc. Section III reports the main averaging results for dynamical systems on Riemannian manifolds for the finite time horizon case, as developed in [24]. Section IV presents two illustrative examples to show closeness of solutions for dynamical systems evolving on a sphere and the compact Lie group $SO(3)$ respectively. For convenience, some standard differential geometric notation is summarized in Table I.

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TABLE I
Symbols and Their Descriptions

Symbol	Description
$\mathbb{R}_{>0}$	$(0,\infty)$
$\mathbb{R}_{\geq 0}$	$[0,\infty)$
M	differentiable manifold
$C^{\infty}(M)$	space of smooth functions on M
$\mathfrak{X}(M)$	space of smooth vector fields on M
$\mathfrak{X}(M\times\mathbb{R})$	space of smooth time-varying vector fields on M
T_xM	tangent space at $x\in M$
T_x^*M	cotangent space at $x\in M$
$\frac{\partial}{\partial x_i}$	basis tangent vectors at $x\in M$
dx_i	basis cotangent vectors at $x\in M$
∇	Levi-Civita connection on M
g	Riemannian metric on M
d	Riemannian distance on M
(M,g)	Riemmanian manifold
$f(x,t)$	vector field on M
$\|\|f\|\|$	Riemannian norm of f
Φ_f	flow associated with f

II. Preliminaries

Attention is restricted to time-varying dynamical systems evolving on finite dimensional Riemannian manifolds. In order to formalize this setting, some standard concepts and results are reviewed from the differential geometry literature (see for example [1], [9], [11]–[14], [20], etc).

A. Riemannian manifolds

Definition 1: (see [14], Chapter 3) A Riemannian manifold (M,g) is a differentiable manifold M together with a Riemannian metric g, where g is defined for each $x\in M$ via an inner product $g_x : T_xM\times T_xM\to\mathbb{R}$ on the tangent space T_xM (to M at x) such that the function defined by $x\mapsto g_x(X(x),Y(x))$ is smooth for any vector fields $X,Y\in\mathfrak{X}(M)$. In addition,

(i) (M,g) is n-dimensional if M is n-dimensional;
(ii) (M,g) is connected if for any $x,y\in M$, there exists a piecewise smooth curve connecting them.

■

(Note that in the special case where $M\doteq\mathbb{R}^n$, the Riemannian metric g is defined everywhere by $g_x=\sum_{i=1}^{n}dx_i\otimes dx_i$, where $\otimes$ is the tensor product on $T_x^*M\times T_x^*M$, see [14].)

As formalized in Definition 1, connected Riemannian manifolds possess the property that any pair of points $x,y\in$

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M can be connected via a path $\gamma \in \mathscr{P}(x,y)$, where

$$\mathscr{P}(x,y) \doteq \left\{ \gamma : [a,b] \to M \;\middle|\; \begin{array}{l} \gamma \text{ piecewise smooth,} \\ \gamma(a) = x\,,\ \gamma(b) = y \end{array} \right\} \tag{1}$$

Theorem 1: ([12], Page 94) Suppose (M,g) is an n-dimensional connected Riemannian manifold. Then, for any $x, y \in M$, there exists a piecewise smooth path $\gamma \in \mathscr{P}_{x,y}$ that connects x to y. ■

The existence of connecting paths (via Theorem 1) between pairs of elements of an n-dimensional connected Riemannian manifold (M,g) facilitates the definition of a corresponding Riemannian distance. In particular, the Riemannian distance $d : M \times M \to \mathbb{R}$ is defined as the infimal path length between any two elements of M, with

$$d(x,y) \doteq \inf_{\gamma \in \mathscr{P}(x,y)} \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t),\, \dot\gamma(t))}\, dt\,. \tag{2}$$

(Note that in the special case where $M \doteq \mathbb{R}^n$, the Riemannian distance (2) simplifies to $d(x,y) = \|x - y\|$.)

Using the definition of Riemannian distance d of (2), it may be shown that (M,d) defines a metric space.

Theorem 2: ([12], Page 94) Any n-dimensional connected Riemannian manifold (M,g) defines a metric space (M,d) via the Riemannian distance d of (2). Furthermore, the induced topology of (M,d) is the same as the manifold topology of (M,g). ■

B. Systems evolving on Riemannian manifolds

A time-varying dynamical system on a Riemannian manifold (M,g) is a system of time-dependent first order nonlinear differential equations defined in terms of a differentiable time-dependent vector field f by

$$\dot x(t) = f(x(t), t)\,, \quad x(t_0) = x_0 \in M\,, \tag{3}$$

in which $x(t) \in M$ denotes the state at time $t \in [t_0,\, t_f]$ evolved from an initial state $x_0 \in M$, where the time interval $[t_0, t_f] \subset \mathbb{R}$ is such that $f(x(t),t) \in T_{x(t)}M$ for all $t \in [t_0, t_f]$. The time-dependent (integral) flow $\Phi_f : [t_0, t_f] \times [t_0, t_f] \times M \mapsto M$ of system (3) associated with the differentiable time-dependent vector field f is the map defined by

$$\Phi_f(s, t_0, x_0) \doteq x(s) \;\Big|\; \text{(3) holds with } s \in [t_0, t_f]\,. \tag{4}$$

Given a smooth vector field f, it may be shown [14] that $\Phi_f(s, t_0, \cdot)$ is a local diffeomorphism.

Definition 2: A vector field f is *complete* if the flow $\Phi_f(t, t_0, x_0)$ defined by (3) and (4) exists for all $t \in [t_0, \infty)$. (That is, if $t_f = \infty$.) ■

In examining the closeness of solutions for systems evolving on a Riemannian manifold (M,g), it is useful to note that the trajectories of such systems describe curves on M that may be parametrized smoothly with respect to perturbations. Families of such curves are considered *admissible* provided such smoothness conditions hold.

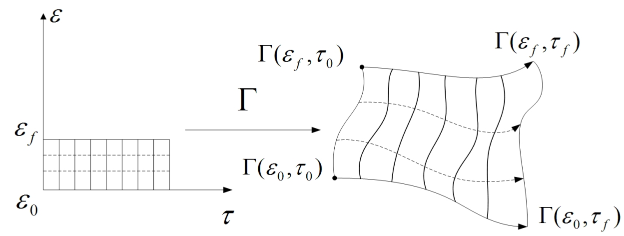

Fig. 1. Admissible family of curves.

Definition 3: ([12], see also Figure 1) An *admissible family* of curves on M is a continuous map $\Gamma : (\epsilon_0, \epsilon_f) \times [\tau_0, \tau_f] \to M$ such that $\Gamma(\epsilon, \tau)$ is smooth with respect to $\epsilon \in (\epsilon_0, \epsilon_f)$ and $\tau \in [\tau_0, \tau_f]$. ■

The tangent vectors to M defined by an admissible family of curves Γ on M (as per Definition 3) may formally be written as

$$\partial_\epsilon \Gamma(\epsilon, \tau) \doteq \frac{\partial}{\partial \epsilon}\Gamma(\epsilon,\tau)\,, \qquad \partial_\tau \Gamma(\epsilon, \tau) \doteq \frac{\partial}{\partial \tau}\Gamma(\epsilon,\tau)\,. \tag{5}$$

However, in general, $\partial_\epsilon \Gamma(\epsilon,\tau)$ and $\partial_\tau \Gamma(\epsilon,\tau)$ do not define vector fields on M, since the image of Γ may not cover M. This problem arises more generally when seeking to characterize the effects of perturbations on vector fields defined on M. In particular, the ability to differentiate one vector field with respect to another is useful. However, the utility of these derivatives is limited by the fact that they need not define vector fields on M, as per the derivatives $\partial_\epsilon \Gamma(\epsilon,\tau)$ and $\partial_\tau \Gamma(\epsilon,\tau)$ above. This problem is addressed via the concept of a *linear connection*, see [12].

Definition 4: ([12]) A *linear connection* on a manifold M is a mapping $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, written as $(X,Y) \mapsto \nabla_X Y$ for any smooth vector fields $X, Y \in \mathfrak{X}(M)$, satisfying the following properties:

(i) $\nabla_X Y$ is linear over $C^\infty(M)$ in X, i.e.,

$$\nabla_{f X_1 + h X_2} Y = f\, \nabla_{X_1} Y + h\, \nabla_{X_2} Y, \tag{6}$$

for all $f, h \in C^\infty(M)$, where $X_1, X_2, Y \in \mathfrak{X}(M)$;

(ii) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y, i.e.,

$$\nabla_X (a\, Y_1 + b\, Y_2) = a\, \nabla_X Y_1 + b\, \nabla_X Y_2, \tag{7}$$

for all $a, b \in \mathbb{R}$, $X, Y_1, Y_2 \in \mathfrak{X}(M)$;

(iii) ∇ satisfies the product rule, i.e.,

$$\nabla_X (f\, Y) = f\, \nabla_X Y + (X\, f)\, Y, \tag{8}$$

for all $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$.

■

(Note that linear connections are also sometimes referred to as *affine* connections.) Linear connections can be further specialized in the case where (M,g) is a Riemannian manifold.

Definition 5: ([12]) A linear connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ on a Riemannian manifold (M,g) is

1) *compatible* with Riemannian metric g if

$$\nabla_X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{9}$$

for all $X, Y, Z \in \mathfrak{X}(M)$;

2) *symmetric* if it is torsion-free, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad (10)$$

for all $X, Y \in \mathfrak{X}(M)$, where

$$[X, Y](f) \doteq X(Y(f)) - Y(X(f)), \quad f \in C^\infty(M).$$

■

On Riemannian manifolds, a unique linear connection which satisfies the properties above may be characterized by the following theorem.

Theorem 3: (Fundamental Lemma of Riemannian Geometry, [12] p.68) Given a Riemannian manifold (M, g), there exists a *unique* linear connection ∇ on M that is both

(i) *compatible* with the Riemannian metric g; and

(ii) *symmetric*. ■

The unique, linear, compatible and symmetric connection specified by Theorem 3 is known as the *Levi-Civita* connection. We employ the Levi-Civita connection to analyze the variation of piecewise smooth curves under different vector fields on (M, g) to relate the closeness of solutions for the state trajectory of a nominal time-varying system and its averaged system.

III. Averaging on Riemannian manifolds

As is the case for systems evolving on Euclidean spaces [18], [19], [25], [28], the analysis of time-varying systems of the form (3) evolving on Riemannian manifolds can be substantially simplified via *averaging*. In particular, averaging allows properties of the time-varying system (3) to be inferred via corresponding properties of a suitably defined time-invariant *averaged* system, combined with closeness of solutions between that time-invariant averaged system and the original time-varying system (3).

Definition 6: A time varying vector field $f \in \mathfrak{X}(M \times \mathbb{R})$ is T *periodic* if

$$f(x, t + T) = f(x, t). \quad (11)$$

■

For a complete T periodic vector field $f \in \mathfrak{X}(M \times [t_0; \infty])$, the averaged system $\hat{f} \in \mathfrak{X}(M)$ is defined as $\dot{x}(t) = \hat{f}(x(t))$, where

$$\hat{f}(x) \doteq \frac{1}{T} \int_0^T f(x, s)\, ds. \quad (12)$$

We derive the propagation equations for a single point under two different vector fields in order to bound the variation of the distance between different state trajectories.

Assumption 1: We assume that the vector fields f_1, $f_2 \in \mathfrak{X}(M \times \mathbb{R})$ are complete (as per Definition 2). ■

Theorem 4 (Perturbation Theorem): Consider the following time varying dynamical systems on M:

$$\begin{aligned} &\dot{x}(t) = f_1(x(t), t), \\ &\dot{y}(t) = f_2(y(t), t), \\ &x(t_0) = y(t_0) = x_0, \quad f_1, f_2 \in \mathfrak{X}(M \times \mathbb{R}). \end{aligned} \quad (13)$$

Then,

$$\begin{aligned} &d(\Phi_{f_1}(t, t_0, x_0), \Phi_{f_2}(t, t_0, x_0)) \leq \\ &K(t_1 - t_0) \exp[C(t - t_0)], \quad t \in [t_0, t_1], \end{aligned} \quad (14)$$

for some $K, C \in \mathbb{R}_{\geq 0}$.

Proof: Here we give a sketch of the proof. For a detailed proof, see [23], [24]. Consider a piecewise smooth curve $\gamma \in \mathscr{P}(x_0, x_0)$ as follows (the existence of $\gamma(.)$ is guaranteed by Theorem 1):

$$\mathscr{P}(x, y) \doteq \left\{ \gamma : [0, 1] \to M \;\middle|\; \begin{array}{c} \gamma \text{ piecewise smooth,} \\ \gamma(0) = x_0, \ \gamma(1) = x_0 \end{array} \right\} \quad (15)$$

Define a (time, parameter) varying vector field $X \in \mathfrak{X}(\mathbb{R} \times \mathbb{R} \times M)$ by

$$\begin{aligned} &X(\tau, t, x) = f_2(x, t) + \tau(f_1(x, t) - f_2(x, t)), \\ &\tau \in [0, 1], t \in \mathbb{R}, x \in M. \end{aligned} \quad (16)$$

It is clear that $X(0, t, x) = f_2(x, t)$, $X(1, t, x) = f_1(x, t)$ and X is smooth with respect to τ, t and x. Hence, $\Phi_{X(0,t,x)}(t, t_0, x_0) = \Phi_{f_2}(t, t_0, x_0)$ and $\Phi_{X(1,t,x)}(t, t_0, x_0) = \Phi_{f_1}(t, t_0, x_0)$. An admissible family of curves, Γ, corresponding to $\Phi_X(t, t_0, \gamma(\tau))$ is given by

$$\Gamma : [0, 1] \times \mathbb{R} \to M, \quad \Gamma(\tau, t) \doteq \Phi_X(t, t_0, \gamma(\tau)). \quad (17)$$

The variation of $l(\Gamma(t))$ with respect to t, where $l : M \to \mathbb{R}_{\geq 0}$ is the length function on M $\left(l(\Gamma(t)) \doteq l(\Gamma(\cdot, t)) = \int_0^1 ||\partial_\tau \Gamma(\tau, t)|| d\tau\right)$, is given by

$$\begin{aligned} \frac{d}{dt} l(\Gamma(t)) &= \frac{d}{dt} \int_0^1 ||\partial_\tau \Gamma(\tau, t)|| d\tau \\ &\leq \int_0^1 ||\nabla_{\frac{\partial}{\partial \tau}} \partial_t \Gamma(\tau, t)|| d\tau \\ &= \int_0^1 ||\nabla_{\frac{\partial}{\partial \tau}} X(\tau, t, \Gamma(\tau, t))|| d\tau, \end{aligned} \quad (18)$$

where we applied the variation of admissible curves, see [12], the Cauchy-Schwarz inequality and the Levi-Civita connection whose existence is guaranteed by Theorem 3, to obtain (18). Finally we obtain (see [23])

$$\begin{aligned} &l(\Gamma(t)) \leq l(\Gamma(t_0)) + \int_{t_0}^t \int_0^1 ||\nabla_{\frac{\partial}{\partial \tau}} f_2(\Gamma(\tau, s), s)|| d\tau ds + \\ &\int_{t_0}^t \int_0^1 ||f_1(\Gamma(\tau, s), s) - f_2(\Gamma(\tau, s), s)|| d\tau ds + \\ &\int_{t_0}^t \int_0^1 \tau ||\nabla_{\frac{\partial}{\partial \tau}} (f_1(\Gamma(\tau, s), s) - f_2(\Gamma(\tau, s), s))|| d\tau ds. \end{aligned} \quad (19)$$

We note that since $l(\Gamma(t_0)) = l(\gamma)$, we can choose $\gamma(\tau) = x_0, \tau \in [0, 1]$, so that $l(\Gamma(t_0)) = 0$. Hence, without loss of generality, we have

$$d(\Phi_{f_1}(t,t_0,x_0),\Phi_{f_2}(t,t_0,x_0)) \leq l(\Gamma(t)) \leq \int_{t_0}^{t}\int_0^1 ||\nabla_{\frac{\partial}{\partial\tau}} f_2(\Gamma(\tau,s),s)||d\tau ds + \int_{t_0}^{t}\int_0^1 ||(f_1(\Gamma(\tau,s),s)-f_2(\Gamma(\tau,s),s))||d\tau ds + \int_{t_0}^{t}\int_0^1 \tau||\nabla_{\frac{\partial}{\partial\tau}}(f_1(\Gamma(\tau,s),s)-f_2(\Gamma(\tau,s),s))||d\tau ds. \tag{20}$$

Define

$$D_\Gamma \doteq \bigcup_{\tau\in[0,1],s\in[t_0,t_1]} \Gamma(\tau,s) \subset M. \tag{21}$$

Since Γ is continuous on $[0,1],\times[t_0,t_1]$, D_Γ is compact in the topology of M. By our hypotheses $f_1, f_2 \in \mathfrak{X}(M\times\mathbb{R})$ and Γ is continuous by the way it is constructed. Therefore $||f_1(x,s)-f_2(x,s)||$ attains its maximum on $D_\Gamma\times[t_0,t_1]$, which is denoted by

$$K_\Gamma \doteq \max_{(x,t)\in D_\Gamma\times[t_0,t_1]} ||f_1(x,t)-f_2(x,t)||. \tag{22}$$

As is well known, the covariant differential of a vector field $X\in\mathfrak{X}(M)$, i.e. $\nabla X(x),\ \ x\in M$, is a linear operator, with

$$\nabla X(x) : (T_xM, ||.||) \to (T_xM, ||.||). \tag{23}$$

We denote the norm of $\nabla X(x)$ by $||\nabla X(x)||$ then, $\exists C_i \in (0,\infty)$, such that

$$C_i \doteq \sup_{x\in D_\Gamma, t\in[t_0,t_1]} ||\nabla f_i(x,t)||, \ \ i=1,2. \tag{24}$$

Hence,

$$\nabla_{\frac{\partial}{\partial\tau}} f_i(\Gamma(\tau,t),t) \leq C_i||\partial_\tau\Gamma(\tau,t)||, \tag{25}$$

and finally we get

$$l(\Gamma(t)) \ \ \leq \ \ K_\Gamma(t_1-t_0)+\int_{t_0}^{t}\big(C_1+2C_2\big)l(\Gamma(s))ds. \tag{26}$$

The inequality (26) is an appropriate form for an application of the Gronwall inequality, so that we have

$$d(\Phi_{f_1}(t,t_0,x_0),\Phi_{f_2}(t,t_0,x_0)) \leq l(\Gamma(t)) \leq K_\Gamma(t_1-t_0)\exp[(C_1+2C_2)(t-t_0)], \tag{27}$$

which completes the proof for $K=K_\Gamma$ and $C=(C_1+2C_2)$. ■

Now consider the perturbed vector fields f_1^ϵ and f_2^ϵ defined by

$$f_1^\epsilon(x,t)=\epsilon f_1(x,t), \ \ f_2^\epsilon(x,t)=\epsilon f_2(x,t), \ \ \epsilon\in\mathbb{R}_{\geq 0}. \tag{28}$$

The following lemma is an extension of the results of Theorem 4 to perturbed dynamical systems on M.

Lemma 1: Consider the dynamical systems

$$\dot{x}(t)=\epsilon f_1(x(t),t), \quad \dot{y}(t)=\epsilon f_2(y(t),t), \quad x(t_0)=y(t_0)=x_0, \ \ f_1,f_2\in\mathfrak{X}(M\times\mathbb{R}). \tag{29}$$

on M. Suppose there exists $\epsilon_1\in\mathbb{R}_{>0}$ such that the integral flows $\Phi_{\epsilon f_i}(\cdot,t_0,x_0),\ \ i=1,2$, exist on $[t_0,t_1]$ for $\epsilon\in(0,\epsilon_1]$. Then for a time interval of order $O(1)$,

$$d(\Phi_{\epsilon f_1}(t,t_0,x_0),\Phi_{\epsilon f_2}(t,t_0,x_0)) = O(\epsilon), \ \ t\in[t_0,t_1]. \tag{30}$$

Proof: We define $\Gamma(\tau,t,\epsilon)$ as an admissible family of curves defined by

$$X(\tau,t,x,\epsilon)=\epsilon f_2(x,t)+\epsilon\tau(f_1(x,t)-f_2(x,t))\in T_xM, \quad \tau\in[0,1], t\in[t_0,t_1], x\in M, \tag{31}$$

such that

$$\Gamma(\tau,t,\epsilon)\doteq\Gamma_\epsilon(\tau,t)=\Phi_X(t,t_0,\gamma(\tau))\in M, \quad \gamma(\tau)=x_0, \tau\in[0,1], \tag{32}$$

where $\Gamma_\epsilon(\tau,t)$ is an admissible family of curves. By construction, Γ is continuous with respect to (τ,t). Employing the results of [1], it can be shown that Γ is continuous with respect to ϵ. Hence, $\hat{D}_\Gamma$ is compact, where

$$\hat{D}_\Gamma \doteq \bigcup_{\tau\in[0,1],t\in[t_0,t_1],\epsilon\in[0,\epsilon_1]} \Gamma(\tau,t,\epsilon). \tag{33}$$

Following the results of [24], we define K_Γ and $C_i,\ \ i=1,2$, as follows:

$$\hat{K}_\Gamma \doteq \sup_{(x,t)\in\hat{D}_\Gamma\times[t_0,t_1]} ||f_1(x,t)-f_2(x,t)||, \quad \hat{C}_i \doteq \sup_{(x,t)\in\hat{D}_\Gamma\times[t_0,t_1]} ||\nabla f_i(x,t)||, \ \ i=1,2. \tag{34}$$

Therefore, by Theorem 4 and the results of [24],

$$d(\Phi_{f_1}(t,t_0,x_0),\Phi_{f_2}(t,t_0,x_0)) \leq \epsilon\hat{K}_\Gamma(t_1-t_0)\exp[\epsilon_1(\hat{C}_1+2\hat{C}_2)(t-t_0)] = O(\epsilon), \tag{35}$$

which completes the proof. ■

Lemma 1 ensures closeness of solutions on compact time intervals. Using this result, the following theorem states the main averaging result on compact Riemannian manifolds for time intervals of order $\frac{1}{\epsilon}$.

Theorem 5 (Averaging Theorem): For a smooth n dimensional compact Riemannian manifold (M,g), let $f\in\mathfrak{X}(M\times\mathbb{R})$ be a T-periodic smooth vector field with the averaged vector field $\hat{f}\in\mathfrak{X}(M)$. Then, for any given $t_1\in[t_0,\infty)$, such that $t_1-t_0=O(\frac{1}{\epsilon}),\ \ \epsilon\in(0,\epsilon_1]$, for some $\epsilon_1\in\mathbb{R}_{>0}$,

$$d(\Phi_{\epsilon f}(t,t_0,x_0),\Phi_{\epsilon\hat{f}}(t,t_0,x_0)) = O(\epsilon). \tag{36}$$

Proof: See [24]. ■

IV. Examples

A. Example 1 – a system evolving on the unit sphere $\mathbf{S}^2$

Consider a parametrization of $\mathbf{S}^2$ which is given by

$$(x_1,x_2)\to(\cos(x_1)\sin(x_2),\sin(x_1)\sin(x_2),\cos(x_2)), \quad \mathbf{S}^2\in\mathbb{R}^3, x_1,x_2\in[0,2\pi), \tag{37}$$

where the induced Riemannian metric is given by

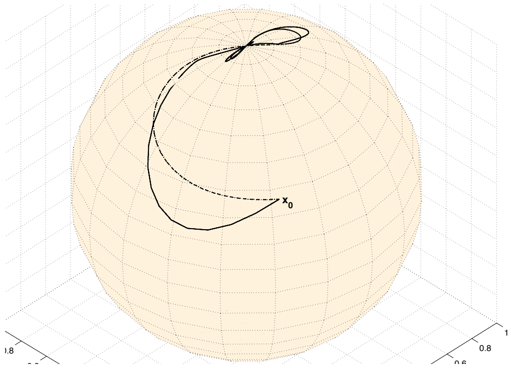

Fig. 2. State trajectories on the sphere (Nominal system: solid line, Averaged system: dashed line), $\epsilon = .3$. (Example 1)

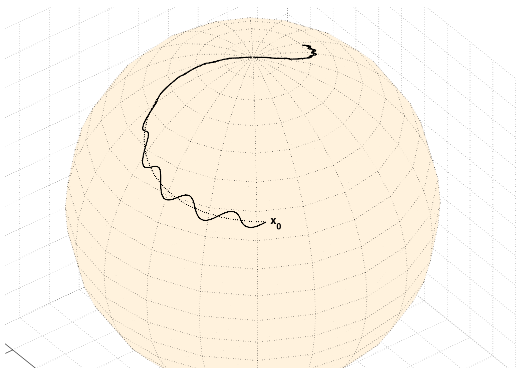

Fig. 3. State trajectories on the sphere (Nominal system: solid line, Averaged system: dashed line), $\epsilon = .05$. (Example 1)

$$g_x = dx_1 \otimes dx_1 + \sin^2(x_1) dx_2 \otimes dx_2, \tag{38}$$

and $\otimes$ is the tensor product, see [14]. Note that $\mathbf{S}^2$ is a smooth connected compact 2-dimensional manifold. The dynamical equations are as follows:

$$f : \begin{cases} \dot{x_1}(t) = \epsilon(x_2(t) - \sin(x_1(t))\cos(t)) \\ \dot{x_2}(t) = \epsilon\Big(-\dfrac{1}{2}x_2(t) - \dfrac{1}{4}\sin(x_1(t)) + \\ \qquad x_2(t)\cos(x_1(t))\cos(t) - \\ \qquad \sin(x_1(t))\cos(x_1(t))\cos^2(t)\Big) \end{cases} \tag{39}$$

By applying (12) to (39), the averaged system is given by

$$\hat{f} : \begin{cases} \dot{x_1}(t) = \epsilon x_2(t) \\ \dot{x_2}(t) = \epsilon(-\dfrac{1}{2}x_2(t) - \dfrac{1}{4}\sin(x_1(t)) - \\ \qquad \dfrac{1}{4}\sin(2x_1(t))) \end{cases} \tag{40}$$

Figures 2 and 3 show the closeness of solutions for the nominal and averaged systems above for $\epsilon = .3$, and $.05$ respectively for $t \in [0, 10]$ as expected by the results of Perturbation Theorem 4 and Lemma 1.

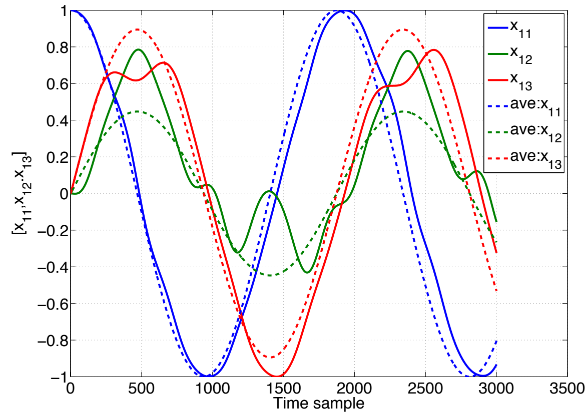

Fig. 4. State trajectories on $SO(3)$ (Nominal system: solid line, Average system: dashed line), $\epsilon = 0.3$ (Example 2)

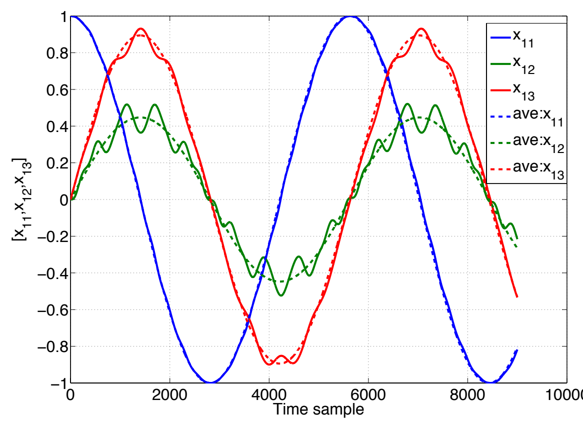

Fig. 5. State trajectories on $SO(3)$ (Nominal system: solid line, Average system: dashed line), $\epsilon = 0.1$ (Example 2)

B. *Example 2 – a system evolving on $SO(3)$*

In this section we present an example on $SO(3)$ (note that $SO(3)$ is a compact Lie group), see [14]. The Lie algebra $\mathcal{L}$ of a Lie group G is the tangent space at the identity element e with the associated Lie bracket defined on the tangent space of G, i.e. $\mathcal{L} = T_e G$. We recall that $SO(3)$ is the rotation group in $\mathbb{R}^3$ given by

$$SO(3) = \{x \in GL(3) \,|\quad x \cdot x^T = I,\ det(x) = 1\}, \tag{41}$$

where $GL(n)$ is the set of nonsingular $n \times n$ matrices. The Lie algebra of $SO(3)$ which is denoted by $so(3)$ is given by (see [26])

$$so(3) = \{X \in \mathbb{R}^{3\times 3} \,|\quad X + X^T = 0\}, \tag{42}$$

where $\mathbb{R}^{n\times n}$ is the space of all $n \times n$ matrices. The Lie group operation $\star$ is given by the matrix multiplication.

A left invariant dynamical system on $SO(3)$ is given by (for the definition of left invariant dynamical systems see [6])

$$\dot{x}(t) = xX, \quad x(t_0) = x_0, \ \ X \in so(3). \tag{43}$$

The Lie algebra bilinear operator is defined as the commuter of matrices, i.e.

$$[X,Y] = XY - YX, \quad X,Y \in so(3). \tag{44}$$

A controlled left invariant system on $SO(3)$ is then defined by

$$\dot{x}(t) = x(t)\begin{pmatrix} 0 & u_1(t) & u_3(t) \\ -u_1(t) & 0 & u_2(t) \\ -u_3(t) & -u_2(t) & 0 \end{pmatrix}, \\ x(t) \in SO(3), (u_1,u_2,u_3) \in \mathbb{R}^3.$$

The Lie algebra $so(3)$ is spanned by

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ and } \\ e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{45}$$

Consider the following perturbed left invariant dynamical system on $SO(3)$:

$$\dot{x}(t) = \epsilon x(t)\begin{pmatrix} 0 & \sin^2(t) & 1 \\ -\sin^2(t) & 0 & \cos(t) \\ -1 & -\cos(t) & 0 \end{pmatrix}. \tag{46}$$

The average dynamical systems is given by

$$\dot{x}(t) = \epsilon x(t)\begin{pmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{47}$$

Figures 4 and 5 show closeness of solutions for the nominal and averaged systems above for $\epsilon = .3$, and .1 respectively for $t \in [0,20]$ and $t \in [0,40]$ as expected by Lemma 1.

References

[1] R. Abraham, J. E. Marsden, and T. S. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Springer, 1988.

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer, 1989.

[3] R. R. Bitmead and C. R. Jr. Johnson. *Discrete averaging principles and robust adaptive identification, control and dynamics: advances in theory and applications*. Academic Press, 1987.

[4] A. M. Bloch. *Nonholonomic Mechanics and Control*. Springer, 2000.

[5] F. Bullo. Averaging and vibrational control of mechanical systems. *SIAM J. Control and Optimization*, 41(2):542–562, 2002.

[6] F. Bullo and A.D. Lewis. *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Mechanical Control Systems*. Springer, 2005.

[7] T. Donchev and G. Grammel. Averaging of functional differential inclusions in banach spaces. *Journal of Mathematical Analysis and Applications*, 311:402–416, 2005.

[8] J. Guckenheimer and P. Holmes. *Nonlinear Oscillatory, Dynamical Systems, and Bifurcations of Vector Fields*. Springer, 1990.

[9] J. Jost. *Riemannian Geometry and Geometrical Analysis*. Springer, 2004.

[10] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.

[11] W.P.A. Klingenberg. *Riemannian Geometry*. de Gruyter Studies in Mathematics, 1995.

[12] J. M. Lee. *Riemannian Manifolds, An Introduction to Curvature*. Springer, 1997.

[13] J. M. Lee. *Introduction to Topological Manifolds*. Springer, 2000.

[14] J. M. Lee. *Introduction to Smooth Manifolds*. Springer, 2002.

[15] N. E. Leonard. *Averaging and Motion Control of Systems on Lie Groups*. PhD Thesis, University of Maryland, 1994.

[16] N. E. Leonard and P. S. Krishnaprasad. Averaging for attitude control and motion planning. In *Proceedings of the 32nd IEEE Conference on Decision and Control*, pages 3098–3104, December, 1993.

[17] N. E. Leonard and P. S. Krishnaprasad. High-order averaging on Lie groups and control of an autonomous underwater vehicle. In *Proceedings of the American Control Conference*, pages 157–162, December, 1994.

[18] D. Nešić and P. M. Dower. Input-to-state stability and averaging of systems with inputs. *IEEE Trans. Automatic Control*, 46(11):1760–1765, 2001.

[19] D. Nešić and A. R. Teel. Input-to-state stability for nonlinear time-varying systems via averaging. *Mathematics of Control, Signals, and Systems*, 14:257–280, 2001.

[20] P. Petersen. *Riemannian Geometry*. Springer, 1998.

[21] J. A. Sanders and F. Verhulst. *Averaging Methods in Nonlinear Dynamical Systems*. Springer, 1985.

[22] S. Sastry. *Nonlinear Systems: Analysis, Stability and Control*. Springer, 1999.

[23] F. Taringoo, D. Nešić, Y. Tan, and P.M. Dower. *Averaging for nonlinear systems evolving on Riemannian manifolds*. Submitted to *Automatica*.

[24] F. Taringoo, D. Nešić, Y. Tan, and P.M. Dower. *Averaging for nonlinear systems on Riemannian manifolds*. To appear, Proc. 52nd IEEE Conference on Decision and Control, 2013, Florence, Italy.

[25] A. R. Teel and D. Nešić. Averaging for a class of hybrid systems. *Dynamics of Continuous, Discrete and Impulsive Systems*, 17(6):829–851, 2010.

[26] V. Varadarajan. *Lie groups, Lie algebras, and their representations*. Springer, 1984.

[27] V. M. Volosov. *Averaging in Systems of Ordinary Differential Equations*. Russian Math Surveys, 1962.

[28] W. Wang and D. Nešić. Input-to-state stability and averaging of linear fast switching systems. *IEEE Trans. Automatic Control*, 55:1274–1279, 2010.