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EQUILIBRIUM PERIODIC DIVIDEND STRATEGIES WITH NON-EXPONENTIAL DISCOUNTING FOR SPECTRALLY POSITIVE LÉVY PROCESSES

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ABSTRACT. In the dual risk model, we study the periodic dividend problem with a non-exponential discount function which results in a time-inconsistent control problem. Viewing it within the game theoretic framework, we extend the Hamilton-Jacobi-Bellman (HJB) system of equations from the fixed terminal to the time of ruin and derive the verification theorem, and we generalize the theory of classical optimal periodic dividend. Under two special non-exponential discount functions, we obtain the closed-form expressions of equilibrium strategy and the corresponding equilibrium value function in a compound Poisson dual model. Finally, some numerical examples are presented to illustrate the impact of some parameters.

1. Introduction. The spectrally positive Lévy model, known as the dual model, is suitable for modeling companies such as pharmaceuticals and petroleum, and these companies occasionally benefit. In particular, the dual model is more suitable for describing an annuity or pension fund. In such companies, the surplus process is usually assumed to be a Lévy process with positive jumps which are viewed as daily gains. Many researchers have studied the control problem in the dual model. See, for example, Pérez and Yamazaki [24], Li et al. [21], Avanzi and Gerber [5], Foucart et al. [17], Avanzi et al. [4, 6], Zhao et al. [31, 32] and Yin et al. [28] and so on.

In risk theory, the issue of dividend has received widespread attention. The dividend problem was first proposed by De Finetti [15]. The idea of problem is that the company wants to pay a portion of surplus as dividend and determines that dividend strategy maximizes the expected present value of dividends until the time of ruin. Most existing continuous time models assume that dividend can be paid at any time, which lead to very irregular dividend payments. In reality, dividend decisions generally occur at some intervals on a periodic basis. Avanzi et al. [6] studies the optimal periodic dividend problem in a dual model in which the aggregate incomes process follows a compound Poisson process with hyper-exponential jumps.

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Pérez and Yamazaki [24] represents the value function by scale function, and solves the optimal periodic dividend barrier in cases of bounded variation and unbounded variation. Zhao et al. [30] considers the optimal periodic dividend strategy and value function by the fluctuation theory of Lévy processes. Dong et al. [14] assumes that the inter-dividend-decision times follow generalized Erlang distribution, and uses fluctuation identities and scale functions to solve problems.

The above problems are time consistent since the discount rate is constant. In reality, people are more patient with long-term investment and note that non-exponential discount function can well describe this behavior of humans ([22, 26]). Non-exponential discount function leads to a time inconsistent problem. Under the time inconsistent problem, the Bellman optimal principle no longer holds, that is, one strategy is optimal for the initial time, but may not be optimal for the later time. Within the game theoretic framework, Björk and Murgoci [10] and Björk et al. [9] develop the general theory of time inconsistent stochastic control problems. Chunxiang et al. [13], Tian [27] and Grenadier and Wang [18] consider the optimal investment problems with a quasi-hyperbolic discount function. Alia [1] considers equilibrium solution within the class of open-loop controls. Under the above discount function, Chen et al. [11, 12] discuss the optimal dividend problem in the dual model and a general diffusion model, respectively. In a compound Poisson dual model, Li et al. [21] studies the absolutely continuous dividend problem with the mixture of two exponential discount functions. In a diffusion model, Zhao et al. [29] solves the same dividend problem with a mixture of exponential discount functions and a pseudo-exponential discount function.

Although the issue of non-exponential discount functions has recently attracted a lot of attention, the problem of periodic dividend with non-exponential discount functions has not been studied. This paper will study this problem for spectrally positive Lévy processes. For periodic dividend, we assume that the inter-dividend-decision times are exponential [24, 30]. By the theory of Björk et al. [9], we show the extended HJB system of equations and the verification theorem. When the discount rate is constant, the results of this paper are reduced to those in Avanzi et al. [6]. In a compound Poisson dual model, we obtain the closed-form solutions of control problem under a mixture of exponential discount functions and a pseudo-exponential discount function. Then we give effects of the discount rates, the parameters of compound Poisson jump, and the rate of expenses on the equilibrium periodic dividend barrier and the equilibrium value function.

The arrangement of this paper is as follows. In Section 2, we give the dividend problem and definition of equilibrium strategy. The extended HJB system of equations and the verification theorem are given in Section 3. Section 4 discusses the equilibrium strategy and the corresponding equilibrium value function under two special non-exponential discount functions. Section 5 provides our conclusions and proposes some extensions of our work.

2. Model and control problem. First, we will give the definition of spectrally positive Lévy processes and then give the formulation of control problem.

2.1. Spectrally positive Lévy processes. Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, \mathbb{P})$ be a probability space hosting a spectrally positive Lévy process $X = \{X(t)\}_{t \geq 0}$.

For $x \in \mathbb{R}$, we denote by \mathbb{P}_x the law of X when it starts at x and write for convenience \mathbb{P} in place of \mathbb{P}_0 . By the Lévy-Khintchine formula, the process X its

Laplace exponent is given by, for $\theta \geq 0$,

$$\begin{aligned} \psi(\theta) &:= \log \left(E[e^{-\theta X(1)}] \right) \\ &= \frac{\sigma^2}{2}\theta^2 + \tilde{c}\theta + \int_0^\infty (e^{-\theta z} - 1 + \theta z \mathbf{1}_{\{z < 1\}})\nu(dz), \end{aligned}$$

where $\tilde{c} \in \mathbb{R}$, $\sigma \geq 0$, and ν is a measure on $(0, \infty)$ called the Lévy measure of X that satisfies

$$\int_0^\infty (1 \wedge z^2)\nu(dz) < \infty. \tag{1}$$

It is well-known that X has paths of bounded variation if and only if $\sigma = 0$ and $\int_0^1 z\nu(dz) < \infty$, in this case, X can be written as

$$X(t) = -ct + \widetilde{S}(t), \quad t \geq 0,$$

where

$$c := \tilde{c} + \int_0^1 z\nu(dz)$$

and $\{\widetilde{S}(t)\}_{t \geq 0}$ is a driftless subordinator whose Laplace exponent is given by

$$\psi(\theta) = c\theta + \int_0^\infty (e^{-\theta z} - 1)\nu(dz), \quad \theta \geq 0.$$

By the Lévy-Itô decomposition (Theorem 2.1 in Kyprianou [20]), the process X can be represented as

$$X(t) = \tilde{c}t + \sigma B(t) + \int_0^t \int_0^1 y\tilde{\mathcal{N}}(dz, dy) + \int_0^t \int_1^\infty y\mathcal{N}(dz, dy), \quad t \geq 0,$$

where $\{B(t)\}_{t \geq 0}$ is a standard Brownian motion, $\mathcal{N}(dz, dy)$ is an independent Poisson measure on $[0, \infty) \times [0, \infty)$ with intensity measure $\nu(dy)dz$, and $\tilde{\mathcal{N}}(dz, dy) = \mathcal{N}(dz, dy) - \nu(dy)dz$ denotes the compensated random measure. For more details on Lévy process the reader can see Applebaum [2], Barndorff-Nielsen et al. [7] and Schoutens [25].

2.2. Control problem. In a periodic dividend strategy, we assume that dividend payments can only occur at some (typically random) time points. In this paper, dividend decision times are assumed to be governed by a Poisson process $\{N_\gamma(t)\}_{t \geq 0}$ with intensity γ , which is independent of the Lévy process X , i.e.,

$$D(t) = \int_0^t v_z dN_\gamma(z) = \sum_{k=1}^\infty v_{T_k} \mathbf{1}_{\{T_k < t\}},$$

where v_z is the dividend payment at time z . The set of dividend decision times is denoted by $\mathcal{T} = \{T_1, T_2, T_3, \dots\}$, and the quantities $T_{k+1} - T_k, k > 0$, are the inter-dividend-decision times. We restrict ourselves to feedback control strategies, i.e., if the surplus at time t is x , the control v_t is given by

$$v_t = \boldsymbol{\pi}(t, x),$$

where the control law $\boldsymbol{\pi}: [0, \infty) \times [0, \infty) \rightarrow [0, x]$ is a continuous Borel measurable function with respect to t and x .

Definition 2.1. A control law π is said to be admissible iff

- (i) $D^\pi(t)$ is a non-decreasing, left-continuous with right limits and $\{\mathcal{F}(t)\}_{t \geq 0}$ adapted process.
(ii) $0 \leq D^\pi(T_i+) - D^\pi(T_i) = \pi(T_i, X^\pi(T_i)) \leq X^\pi(T_i)$, for $i \geq 1$.

We denote the set of all admissible control laws by Π .

Applying the control law π , we denote the controlled risk process by $\{X^\pi(t)\}_{t \geq 0}$, which is defined by

$$X^\pi(t) := X(t) - D^\pi(t) = X(t) - \sum_{i=1}^{\infty} \pi(T_i, X^\pi(T_i)) \mathbf{1}_{\{T_i < t\}}, \quad t \geq 0.$$

Let

$$T^\pi := \inf\{t \geq 0 : X^\pi(t) \leq 0\}$$

be the time of ruin under the control law π .

The target of the company is to choose a strategy $\pi \in \Pi$ to maximize the expected present value of dividends until ruin

$$J(t, x; \pi) = E_{t,x} \left[\int_t^{T^\pi} \varphi(z-t) \pi(z, X^\pi(z)) dN_\gamma(z) \right], \quad (2)$$

where $E_{t,x}[\cdot] = E[\cdot | X^\pi(t) = x]$ and $\varphi(\cdot)$ is the discount function.

Let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be the discount function, which is continuously differentiable and

$$\varphi(0) = 1, \quad \varphi(t) \geq 0, \quad \varphi'(t) < 0, \quad \lim_{t \rightarrow \infty} \varphi(t) = 0.$$

The optimization problem is time-inconsistent when $\varphi(z-t)$ is non-exponential discount function. By Björk and Murgoci [10], we view the entire problem as a non-cooperative game and look for Nash equilibria for the game. In particular, we consider a game that has one player for each time t , player t can be seen as the future incarnation of the decision maker at time t . Given state (t, x) , player t will choose a control $\pi(t, x)$, and she/he wants to maximize the function $J(t, x; \pi)$. Now, we present a formal definition of equilibrium strategy in the following.

Definition 2.2. Choose a control law $\hat{\pi} \in \Pi$ and a fixed real number $h > 0$. For any fixed initial point $(t, x) \in [0, \infty) \times [0, \infty)$ and any $\pi(s, y) \in \Pi$, we define the control law $\pi_h \in \Pi$ by

$$\pi_h(s, y) = \begin{cases} \pi(s, y), & \text{for } t \leq s < t+h, \quad y \in [0, \infty), \\ \hat{\pi}(s, y), & \text{for } s \geq t+h, \quad y \in [0, \infty). \end{cases}$$

We say that $\hat{\pi}$ is an equilibrium strategy if

$$\liminf_{h \downarrow 0} \frac{J(t, x; \hat{\pi}) - J(t, x; \pi_h)}{h} \geq 0.$$

The equilibrium value function V is defined by $V(t, x) := J(t, x; \hat{\pi})$.

Remark 1. In the above definition, we only care about the value of $\pi(T_i, X^\pi(T_i))$, $i = 1, 2, \dots$, and $\pi(t, X^\pi(t))$, $t \neq T_i$, don't affect dividend. Therefore, when $\pi(t, x)$ is continuous, $\pi(T_i, X^\pi(T_i)) = \hat{\pi}(T_i, X^{\hat{\pi}}(T_i))$, $i = 1, 2, \dots$, are the equilibrium strategy.

3. Extended HJB system of equations and verification theorem. To derive the extended HJB system of equations, we introduce the infinitesimal generator. Any real valued function $f(t, x)$ is called sufficiently smooth meaning that $f(t, x) \in C^{1,1}([0, \infty) \times [0, \infty))$ if X is of bounded variation, otherwise $f(t, x) \in C^{1,2}([0, \infty) \times [0, \infty))$. We define the infinitesimal generator \mathcal{A} by

$$\begin{aligned} \mathcal{A}f(t, x) &= \frac{\partial f}{\partial t}(t, x) - \tilde{c} \frac{\partial f}{\partial x}(t, x) + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2}(t, x) \\ &\quad + \int_0^\infty [f(t, x+z) - f(t, x) - \frac{\partial f}{\partial x}(t, x)z \mathbf{1}_{\{0 < z \leq 1\}}] \nu(dz). \end{aligned}$$

For a constant $\pi \in [0, x]$ and a control law $\boldsymbol{\pi} \in \Pi$, we have

$$\begin{aligned} \mathcal{A}^\pi f(t, x) &= \mathcal{A}f(t, x) + \gamma (f(t, x - \pi) - f(t, x)), \\ \mathcal{A}^{\boldsymbol{\pi}} f(t, x) &= \mathcal{A}f(t, x) + \gamma (f(t, x - \boldsymbol{\pi}) - f(t, x)). \end{aligned}$$

By Avanzi et al. [6] and Björk and Murgoci [10], we give the following extended HJB system of equations.

Definition 3.1. The extended HJB system of equations for V and u is defined by

$$\sup_{0 \leq \pi \leq x} \{ \mathcal{A}^\pi V(t, x) + \gamma \pi + \mathcal{A}^\pi u^t(t, x) - \mathcal{A}^\pi u(t, x) \} = 0, \tag{3}$$

$$\mathcal{A}^{\hat{\boldsymbol{\pi}}} u^s(t, x) + \gamma \varphi(t - s) \hat{\boldsymbol{\pi}}(t, x) = 0,$$

$$V(t, 0) = 0, \tag{4}$$

where

$$u^s(t, x) := u(s, t, x), \quad 0 \leq s \leq t$$

is viewed as a function for two variables t and x . Here $\hat{\boldsymbol{\pi}}$ is the control law which realizes the supremum in the first equation.

A heuristic argument of equation (3) is given in Appendix A. Similar to Björk et al. [9], we give the following definition.

Definition 3.2. For $0 \leq s \leq t, x \geq 0$, the extended HJB system of equations for $u(s, t, x)$ is defined by

$$\sup_{0 \leq \pi \leq x} \{ \mathcal{A}^\pi u^t(t, x) + \gamma \pi \} = 0, \tag{5}$$

$$\mathcal{A}^{\hat{\boldsymbol{\pi}}} u^s(t, x) + \gamma \varphi(t - s) \hat{\boldsymbol{\pi}}(t, x) = 0, \tag{6}$$

$$u(s, t, 0) = 0, \tag{7}$$

where $\hat{\boldsymbol{\pi}}$ is the control law which realizes the supremum in (5).

Remark 2. (i) In a certain sense, the extended HJB system of equations in Definition 3.2 is consistent with Definition 3.1 in Li et al. [21] and Definition 3.1 in Zhao et al. [29]. (ii) If terminal time is fixed, the Definition 3.1 is equivalent to the Definition 3.2, however, in the case of considering bankruptcy, it is not a simple equivalent. Next, we will give the verification theorem according to the Definition 3.2.

Theorem 3.3. (*Verification Theorem*) Assume that $X(t)$ is a square integrable process, if there exists a function $u^s(t, x), s \leq t$, solving the extended HJB system of equations in Definition 3.2 and satisfying

$$\lim_{n \rightarrow \infty} E_{t,x} [u^s(T^n, X^{\hat{\boldsymbol{\pi}}}(T^n))] = 0, \tag{8}$$

$$0 \leq \frac{\partial u^s}{\partial x}(t, x) \leq e^{-\zeta t} \kappa(s), \quad (9)$$

where $T^n = n \wedge T^{\hat{\pi}}$, the constant $\zeta > 0$ and $\kappa(s) \geq 0$ is a function with s . We assume $\varphi(z) \leq e^{-\alpha z}$, $\alpha > 0$. Then $\hat{\pi}$ is an equilibrium strategy and $u^t(t, x) = V(t, x)$ is the corresponding equilibrium value function.

Proof. We give the proof in two steps: 1. We show that $u^t(t, x)$ is the value function corresponding to $\hat{\pi}$, i.e., $u^t(t, x) = J(t, x; \hat{\pi})$; 2. We prove that $\hat{\pi}$ is indeed the equilibrium dividend strategy.

Step1. Applying Itô's formula to the process $u^s(T^n, X^{\hat{\pi}}(T^n))$, we have

$$\begin{aligned} u^s(T^n, X^{\hat{\pi}}(T^n)) &= u^s(t, x) + \int_t^{T^n} \mathcal{A}^{\hat{\pi}} u^s(z, X^{\hat{\pi}}(z)) + \gamma \varphi(z - s) \hat{\pi}(z, X^{\hat{\pi}}(z)) dz \\ &\quad + M_{T^n}^1 + M_{T^n}^2 + M_{T^n}^3 - \int_t^{T^n} \varphi(z - s) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z), \end{aligned} \quad (10)$$

where

$$\begin{aligned} M_n^1 &= \sigma \int_t^n \frac{\partial u^s}{\partial x}(z, X^{\hat{\pi}}(z)) dB(z), \\ M_n^2 &= \int_t^n \int_0^\infty (u^s(z, X^{\hat{\pi}}(z-) + y) - u^s(z, X^{\hat{\pi}}(z-))) (\mathcal{N}(dz, dy) - \nu(dy) dz), \\ M_n^3 &= \int_t^n [u^s(z, X^{\hat{\pi}}(z) - \hat{\pi}(z, X^{\hat{\pi}}(z))) - u^s(z, X^{\hat{\pi}}(z)) \\ &\quad + \varphi(z - s) \hat{\pi}(z, X^{\hat{\pi}}(z))] d(N_\gamma(z) - \gamma z). \end{aligned}$$

Since

$$\begin{aligned} \sup_{n \geq t} E_{t,x}[\langle M^1, M^1 \rangle(n)] &= \sigma^2 \sup_{n \geq t} E_{t,x} \left[\int_t^n \left[\frac{\partial u^s}{\partial x}(z, X^{\hat{\pi}}(z)) \right]^2 d\langle B, B \rangle(z) \right] \\ &= \sigma^2 \sup_{n \geq t} \int_t^n \left[\frac{\partial u^s}{\partial x}(z, X^{\hat{\pi}}(z)) \right]^2 dz \\ &\leq \sigma^2 \sup_{n \geq t} \int_t^n (e^{-\zeta z} \kappa(s))^2 dz < \infty, \end{aligned}$$

by Theorems 8.27, 8.32 and Corollary 7.8 in Klebaner [19], we have that $\{M_n^1\}$ is a square integrable martingale. Therefore, $\{M_{n \wedge T^{\hat{\pi}}}^1\}$ is a zero-mean martingale.

Using the Mean Value Theorem, we have

$$\begin{aligned} &\sup_{n \geq t} E_{t,x}[\langle M^2, M^2 \rangle(n)] \\ &= \sup_{n \geq t} E_{t,x} \left[\int_t^n \int_0^\infty \left[\frac{\partial u^s}{\partial x}(z, X^{\hat{\pi}}(z-) + \varsigma) y \right]^2 \langle \tilde{\mathcal{N}}(dz, dy), \tilde{\mathcal{N}}(dz, dy) \rangle \right] \\ &\leq \sup_{n \geq t} \int_t^n \int_0^\infty (e^{-\zeta z} \kappa(s) y)^2 \nu(dy) dz \\ &= \sup_{n \geq t} \int_t^n (e^{-\zeta z} \kappa(s))^2 dz \int_0^\infty y^2 \nu(dy), \quad 0 < \varsigma < y. \end{aligned}$$

Noting $X(t)$ is a square integrable process and (1), we have

$$\sup_{n \geq t} E_{t,x}[\langle M^2, M^2 \rangle(n)] < \infty.$$

Then we obtain that $\{M^2_{n \wedge T^{\hat{\pi}}}\}$ is a zero-mean martingale. Next, we consider $\{M^3_{n \wedge T^{\hat{\pi}}}\}$,

$$\begin{aligned} & \sup_{n \geq t} E_{t,x}[\langle M^3, M^3 \rangle(n)] \\ &= \sup_{n \geq t} E_{t,x} \left[\int_t^n [u^s(z, X^{\hat{\pi}}(z) - \hat{\pi}(z, X^{\hat{\pi}}(z))) - u^s(z, X^{\hat{\pi}}(z)) \right. \\ & \quad \left. + \varphi(z - s)\hat{\pi}(z, X^{\hat{\pi}}(z))]^2 d\langle \tilde{N}_\gamma, \tilde{N}_\gamma \rangle(z) \right] \\ &\leq \sup_{n \geq t} E_{t,x} \left[\int_t^n [u^s(z, X^{\hat{\pi}}(z) - \hat{\pi}(z, X^{\hat{\pi}}(z))) - u^s(z, X^{\hat{\pi}}(z))]^2 d\langle \tilde{N}_\gamma, \tilde{N}_\gamma \rangle(z) \right] \\ & \quad + \sup_{n \geq t} E_{t,x} \left[\int_t^n [\varphi(z - s)\hat{\pi}(z, X^{\hat{\pi}}(z))]^2 d\langle \tilde{N}_\gamma, \tilde{N}_\gamma \rangle(z) \right], \end{aligned}$$

where $\tilde{N}_\gamma(z) = N_\gamma(z) - \gamma z$. In fact,

$$\begin{aligned} & \sup_{n \geq t} E_{t,x} \left[\int_t^n [\varphi(z - s)\hat{\pi}(z, X^{\hat{\pi}}(z))]^2 d\langle \tilde{N}_\gamma, \tilde{N}_\gamma \rangle(z) \right] \\ &\leq \sup_{n \geq t} E_{t,x} \left[\int_t^n e^{-2\alpha(z-s)} X^2(z) \gamma dz \right] \\ &= \sup_{n \geq t} \int_t^n e^{-2\alpha(z-s)} E_{t,x}(X^2(z)) \gamma dz, \end{aligned}$$

the last equality follows from Fubini's theorem (see Biagini et al. [8]). According to Laplace exponent, we have $E[X^2(z)] = z\psi''(0) + z^2(\psi'(0))^2$, then

$$\begin{aligned} & \sup_{n \geq t} \int_t^n e^{-2\alpha(z-s)} E_{t,x}(X^2(z)) \gamma dz \\ &= \sup_{n \geq t} \int_t^n e^{-2\alpha(z-s)} [z\psi''(0) + z^2(\psi'(0))^2] \gamma dz \\ &< \infty. \end{aligned} \tag{11}$$

Similarly, by the Mean Value Theorem, we obtain

$$\begin{aligned} & \sup_{n \geq t} E_{t,x} \left[\int_t^n [u^s(z, X^{\hat{\pi}}(z) - \hat{\pi}(z, X^{\hat{\pi}}(z))) - u^s(z, X^{\hat{\pi}}(z))]^2 d\langle \tilde{N}_\gamma, \tilde{N}_\gamma \rangle(z) \right] \\ &\leq \sup_{n \geq t} E_{t,x} \left[\int_t^n (e^{-\zeta z} \kappa(s)\hat{\pi}(z, X^{\hat{\pi}}(z)))^2 d\langle \tilde{N}_\gamma, \tilde{N}_\gamma \rangle(z) \right] \\ &\leq \gamma \kappa^2(s) \sup_{n \geq t} \left[\int_t^n e^{-2\zeta z} E_{t,x}(X^2(z)) dz \right] \\ &< \infty. \end{aligned} \tag{12}$$

From (11) and (12), we have $\{M^3_{n \wedge T^{\hat{\pi}}}\}$ is a zero-mean martingale. Taking expectations in both sides of (10) and recalling (6), we get

$$E_{t,x}[u^s(T^n, X^{\hat{\pi}}(T^n))] = u^s(t, x) - E_{t,x} \left[\int_t^{T^n} \varphi(z - s)\hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z) \right].$$

Letting n go to infinity, monotone convergence theorem and (8) give

$$u^s(t, x) = E_{t,x} \left[\int_t^{T^{\hat{\pi}}} \varphi(z-s) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z) \right]. \quad (13)$$

Step2. We only prove that $\hat{\pi}$ is the equilibrium control law which is defined by Definition 2.2. For any $h > 0$, by (59) and (60), we have

$$\begin{aligned} & \liminf_{h \downarrow 0} \frac{J(t, x; \hat{\pi}) - J(t, x; \pi_h)}{h} \\ &= \liminf_{h \downarrow 0} \left\{ \frac{u(t, t, x) - E_{t,x} [u(t, t+h, X^\pi(t+h))]}{h} \right. \\ & \quad \left. - \frac{E_{t,x} \left[\int_t^{t+h} \varphi(z-t) \pi(z, X^\pi(z)) dN_\gamma(z) \right]}{h} \right\} \\ &= - \{ \mathcal{A}^\pi u^t(t, x) + \gamma \pi \} \geq 0. \end{aligned}$$

□

Remark 3. If $u^s(t, x)$ is given by (13), we know

$$\mathcal{A}^\pi u^t(t, x) - \mathcal{A}^\pi u(t, t, x) = E_{t,x} \left[\int_t^{T^{\hat{\pi}}} \varphi'(z-t) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z) \right].$$

Furthermore, if $\varphi(z-t) = e^{-\delta(z-t)}$, we have

$$\mathcal{A}^\pi u^t(t, x) - \mathcal{A}^\pi u(t, t, x) = -\delta u(t, t, x).$$

Due to $V(t, x) = u(t, t, x)$, (3) and (5) become

$$\sup_{0 \leq \pi \leq x} \{ \mathcal{A}^\pi V(t, x) - \delta V(t, x) + \gamma \pi \} = 0.$$

By $u^s(t, x) = e^{-\delta(t-s)} u^t(t, x)$, equation (6) can be written as

$$\mathcal{A}^{\hat{\pi}} V(t, x) - \delta V(t, x) + \gamma \hat{\pi} = 0,$$

since $V(t, x) = V(0, x) := V(x)$, we obtain that (5)-(7) coincide with (3.5) in Avanzi et al. [6].

Remark 4. (i) Assume $u^s(t, x)$ is a concave function for x , due to $X^{\hat{\pi}}(T^{\hat{\pi}}) = 0$ for $T^{\hat{\pi}} < \infty$ and $u^s(t, 0) = 0$, we have

$$\begin{aligned} & E_{t,x} [u^s(T^n, X^{\hat{\pi}}(T^n))] \\ &= E_{t,x} [u^s(T^n, X^{\hat{\pi}}(T^n)) \mathbf{1}_{\{T^{\hat{\pi}} > n\}}] + E_{t,x} [u^s(T^n, X^{\hat{\pi}}(T^n)) \mathbf{1}_{\{T^{\hat{\pi}} \leq n\}}] \\ &\leq E_{t,x} [u^s(n, X^{\hat{\pi}}(n))] + E_{t,x} [u^s(T^{\hat{\pi}}, X^{\hat{\pi}}(T^{\hat{\pi}})) \mathbf{1}_{\{T^{\hat{\pi}} < \infty\}}] \\ &= E_{t,x} [u^s(n, X^{\hat{\pi}}(n))] \\ &\leq E_{t,x} [u^s(n, X(n))] \\ &\leq u^s(n, E_{t,x} [X(n)]) \\ &= u^s(n, x + (n-t)E[X(1)]), \end{aligned}$$

where the last inequality holds by concavity and Jensen's inequality. Letting n go to infinity, we obtain

$$\lim_{n \rightarrow \infty} E_{t,x} [u^s(T^n, X^{\hat{\pi}}(T^n))] \leq \lim_{n \rightarrow \infty} u^s(n, x + (n-t)E[X(1)]) = 0.$$

(ii) If $\varphi(z - s) = e^{-\zeta(z-s)}$, we have

$$\begin{aligned} u^s(t, x) &= E_{t,x} \left[\int_t^{T^{\hat{\pi}}} \varphi(z - s) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z) \right] \\ &= E_{t,x} \left[\int_t^{T^{\hat{\pi}}} e^{-\zeta(z-s)} \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z) \right] \\ &= e^{-\zeta(t-s)} E_x \left[\int_0^{T^{\hat{\pi}}} e^{-\zeta z} \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z) \right]. \end{aligned}$$

From Pérez and Yamazaki [24] and Zhao et al. [30], we have

$$\int_0^{T^{\hat{\pi}}} e^{-\zeta z} \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_\gamma(z)$$

is linear bounded, then $\frac{\partial u^s}{\partial x}(t, x) \leq e^{-\zeta t} \kappa(s)$.

We rewrite the equation (5) as

$$\sup_{0 \leq \pi \leq x} \{ \gamma[\pi + u^t(t, x - \pi) - u^t(t, x)] + \mathcal{A}u^t(t, x) \} = 0. \tag{14}$$

If exists a constant $b \geq 0$ such that $\frac{\partial u^t}{\partial x}(t, x) \geq 1$ when $0 \leq x < b$, while $\frac{\partial u^t}{\partial x}(t, x) < 1$ when $x \geq b$, we obtain

$$\hat{\pi}(t, x) = \begin{cases} 0, & 0 \leq x < b, \\ x - b, & x \geq b. \end{cases} \tag{15}$$

Then the equation (6) becomes

$$\begin{cases} \mathcal{A}u^s(t, x) = 0, & 0 \leq x < b, \\ \mathcal{A}u^s(t, x) + \gamma[(x - b)\varphi(t - s) + u^s(t, b) - u^s(t, x)] = 0, & x \geq b. \end{cases} \tag{16}$$

Remark 5. From Definition 2.1, we know $\hat{\pi}(t, x)$ in (15) belongs to Π . Combined with Pérez and Yamazaki [24] and Avanzi et al. [6], the dividend strategy $\hat{\pi}(t, x)$ in (15) is a periodic barrier dividend strategy.

It is a complicated problem to seek the equilibrium strategy and the corresponding equilibrium value function. We will deal with two special non-exponential discount functions in the following section.

4. Two cases of non-exponential discount functions. Similar to Zhao et al. [29], we try our best to find the solution of the extended HJB system of equations for a mixture of exponential discount functions and a pseudo-exponential discount function.

We assume

$$\varphi_1(t) = \sum_{i=1}^n \omega_i e^{-\rho_i t}, \quad t \geq 0, \tag{17}$$

where $\rho_i > 0$, $\rho_i \neq \rho_j$, for $i \neq j$, and $\omega_i > 0$ satisfies $\sum_{i=1}^n \omega_i = 1$. ω_i , $i = 1, 2, \dots, n$, are the proportion at which the dividends are paid to the shareholders, ρ_i , $i = 1, 2, \dots, n$, are the constant discount rates of the shareholders, respectively.

Let

$$\varphi_2(t) = (1 + \eta t)e^{-\delta t}, \quad t \geq 0, \tag{18}$$

where $\eta > 0$ and $\delta > 0$. To ensure the decrease of $\varphi_2(t)$, we assume that $\eta < \delta$. Note that, in contrast to the more studied case of hyperbolic discount, decision-makers discount the distant future less heavily than the immediate future. More details on a pseudo-exponential discount function, readers can refer to Luttmer and Mariotti [23] and Ekeland and Pirvu [16].

Remark 6. For a mixture of exponential discount functions in (17), we have

$$\varphi_1(t) = \sum_{i=1}^n \omega_i e^{-\rho_i t} \leq e^{-\rho t}, \quad t \geq 0,$$

where $\rho = \min_{1 \leq i \leq n} \{\rho_i\}$. For a pseudo-exponential discount function in (18), we have

$$\varphi_2(t) = (1 + \eta t)e^{-\delta t} \leq e^{-(\delta - \eta)t}, \quad t \geq 0.$$

Then, for two types of discount functions in (17) and (18), there exists a constant $\alpha > 0$ such that

$$\varphi_i(t) \leq e^{-\alpha t}, \quad t \geq 0, \quad i = 1, 2. \quad (19)$$

From Pérez and Yamazaki [24] and Zhao et al. [30], we know $J(t, x; \pi)$ in (2) is linear bounded, furthermore, equilibrium value function is linear bounded.

In this section, we consider a special spectrally positive Lévy process, its Laplace exponent is

$$\psi(\theta) = c\theta + \int_0^\infty (e^{-\theta z} - 1)\nu(dz), \quad \theta \geq 0,$$

where $\nu(dz) = \lambda\beta e^{-\beta z} dz$. In other words, $X(t)$ is a dual process in which the aggregate incomes process follows a compound Poisson process with intensity λ and exponential jumps. We assume that $c > 0$ and the drift of X is positive, i.e.,

$$\mu = E[X(t+1) - X(t)] = \frac{\lambda}{\beta} - c > 0.$$

4.1. A mixture of exponential functions. This subsection assumes a mixture of exponential discount functions $\varphi_1(t)$ in (17). We consider the following ansatz:

$$u^s(t, x) = \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} V_i(x), \quad 0 \leq s \leq t, \quad x \geq 0. \quad (20)$$

Substituting (20) into (16), we have the following integro-differential equations

$$\begin{cases} -cV_i'(x) - \lambda V_i(x) + \lambda \int_0^{b-x} V_i(x+y)\beta e^{-\beta y} dy \\ \quad + \lambda \int_{b-x}^\infty V_i(x+y)\beta e^{-\beta y} dy - \rho_i V_i(x) = 0, & 0 \leq x < b, \\ -cV_i'(x) + \lambda \int_0^\infty V_i(x+y)\beta e^{-\beta y} dy - \lambda V_i(x) \\ \quad + \gamma[V_i(b) - V_i(x)] + \gamma(x-b) - \rho_i V_i(x) = 0, & x \geq b. \end{cases} \quad (21)$$

We denote the roots of the equations $-c\xi^2 + (-\mu\beta - \gamma - \rho_i)\xi + \beta(\gamma + \rho_i) = 0$ and $-c\xi^2 + (-\mu\beta - \rho_i)\xi + \beta\rho_i = 0$ by ξ_{1i} , ξ_{2i} , ξ_{3i} and ξ_{4i} , respectively, i.e.,

$$\xi_{1i} = \frac{(\mu\beta + \gamma + \rho_i) + \sqrt{(\mu\beta + \gamma + \rho_i)^2 + 4c\beta(\gamma + \rho_i)}}{-2c} < 0,$$

$$\begin{aligned} \xi_{2i} &= \frac{(\mu\beta + \gamma + \rho_i) - \sqrt{(\mu\beta + \gamma + \rho_i)^2 + 4c\beta(\gamma + \rho_i)}}{-2c} > 0, \\ \xi_{3i} &= \frac{(\mu\beta + \rho_i) + \sqrt{(\mu\beta + \rho_i)^2 + 4c\beta\rho_i}}{-2c} < 0, \\ \xi_{4i} &= \frac{(\mu\beta + \rho_i) - \sqrt{(\mu\beta + \rho_i)^2 + 4c\beta\rho_i}}{-2c} > 0. \end{aligned}$$

Applying the operator $\frac{d}{dx} - \beta$ to (21) and after some calculations, we obtain

$$V_i(x) = \begin{cases} C_{3i}e^{\xi_{3i}x} + C_{4i}e^{\xi_{4i}x}, & 0 \leq x < b, \\ C_{1i}e^{\xi_{1i}x} + C_{2i}e^{\xi_{2i}x} + \frac{\gamma}{\gamma + \rho_i} [x - b + V_i(b)] + \frac{\gamma\mu}{(\gamma + \rho_i)^2}, & x \geq b, \end{cases}$$

where C_{1i}, C_{2i}, C_{3i} and C_{4i} are constants. By (7) we know $V_i(0) = 0$, furthermore $C_{3i} = -C_{4i} := C_i, i = 1, 2, \dots, n$. By (9) and Remark 6, we have $C_{2i} = 0, i = 1, 2, \dots, n$.

According to the smooth-fit condition at b

$$\begin{cases} u^t(t, b-) = u^t(t, b+), \\ \frac{\partial u^t}{\partial x}(t, b-) = \frac{\partial u^t}{\partial x}(t, b+), \end{cases} \tag{22}$$

we have

$$\begin{cases} \frac{\rho_i}{\gamma + \rho_i} C_i (e^{\xi_{3i}b} - e^{\xi_{4i}b}) = C_{1i}e^{\xi_{1i}b} + \frac{\mu\gamma}{(\gamma + \rho_i)^2}, \\ C_i (\xi_{3i}e^{\xi_{3i}b} - \xi_{4i}e^{\xi_{4i}b}) = C_{1i}\xi_{1i}e^{\xi_{1i}b} + \frac{\gamma}{\gamma + \rho_i}. \end{cases}$$

Then the functions $V_i(x), i = 1, 2, \dots, n$, are given by

$$V_i(x) = \begin{cases} C_i (e^{\xi_{3i}x} - e^{\xi_{4i}x}), & 0 \leq x < b, \\ C_{1i}e^{\xi_{1i}x} + \frac{\gamma}{\rho_i} C_{1i}e^{\xi_{1i}b} + \frac{\gamma}{\gamma + \rho_i} (x - b) + \frac{\gamma\mu}{\rho_i(\gamma + \rho_i)}, & x \geq b, \end{cases} \tag{23}$$

where

$$C_i = -\frac{\gamma}{Z_i} + \frac{\mu\gamma\xi_{1i}}{Z_i(\gamma + \rho_i)}, \tag{24}$$

$$C_{1i} = \frac{\gamma}{\gamma + \rho_i} \frac{(\mu\xi_{3i} - \rho_i)e^{\xi_{3i}b} + (\rho_i - \mu\xi_{4i})e^{\xi_{4i}b}}{Z_i e^{\xi_{1i}b}}, \tag{25}$$

$$Z_i = [\rho_i\xi_{1i} - (\gamma + \rho_i)\xi_{3i}]e^{\xi_{3i}b} + [(\gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}]e^{\xi_{4i}b}. \tag{26}$$

From the last arguments in Section 3, in order to find an equilibrium strategy, we assume $\frac{\partial u^t}{\partial x}(t, b) = 1$, we have

$$\sum_{i=1}^n \omega_i C_i (\xi_{3i}e^{\xi_{3i}b} - \xi_{4i}e^{\xi_{4i}b}) = 1, \tag{27}$$

i.e.,

$$\sum_{i=1}^n \omega_i (\xi_{3i}e^{\xi_{3i}b} - \xi_{4i}e^{\xi_{4i}b}) \left(-\frac{\gamma}{Z_i} + \frac{\mu\gamma\xi_{1i}}{Z_i(\gamma + \rho_i)} \right) = 1. \tag{28}$$

In order to discuss the root of equation (28), we define

$$P_i = -\gamma + \frac{\mu\gamma\xi_{1i}}{\gamma + \rho_i}, \tag{29}$$

and give the following Lemma, which is proved in Appendix B.

Lemma 4.1. Equation (28) has a unique root if and only if

$$\sum_{i=1}^n \omega_i \frac{P_i}{\gamma + \rho_i} + 1 < 0,$$

where P_i is given in (29).

Lemma 4.2. For the discount function $\varphi_1(t)$ in (17), there exists a solution $u^s(t, x)$ of the extended HJB system of equations in Definition 3.2 as following:

(i) If $\sum_{i=1}^n \omega_i \frac{P_i}{\gamma + \rho_i} + 1 \geq 0$, then

$$u^s(t, x) = \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} \left(\frac{\gamma\mu}{(\gamma + \rho_i)^2} (1 - e^{\xi_{1i}x}) + \frac{\gamma}{\gamma + \rho_i} x \right), \quad x \geq 0. \tag{30}$$

(ii) If $\sum_{i=1}^n \omega_i \frac{P_i}{\gamma + \rho_i} + 1 < 0$ and $e^{(\xi_{4i} - \xi_{3i})b} < \frac{\rho_i - \mu\xi_{3i}}{\rho_i - \mu\xi_{4i}}$, then

$$u^s(t, x) = \begin{cases} \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} C_i (e^{\xi_{3i}x} - e^{\xi_{4i}x}), & 0 \leq x < b, \\ \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} \left(C_{1i} e^{\xi_{1i}x} + \frac{\gamma}{\rho_i} C_{1i} e^{\xi_{1i}b} + \frac{\gamma}{\gamma + \rho_i} (x - b) \right. \\ \qquad \qquad \qquad \left. + \frac{\gamma\mu}{\rho_i(\gamma + \rho_i)} \right), & x \geq b, \end{cases} \tag{31}$$

where C_i and C_{1i} , $i = 1, 2, \dots, n$, are given in (24) and (25), respectively.

Proof. See Appendix C. □

Theorem 4.3. For the discount function $\varphi_1(t)$ in (17),

(i) If $\sum_{i=1}^n \omega_i \frac{P_i}{\gamma + \rho_i} + 1 \geq 0$, then $\hat{\pi}(t, x) = x$ is an equilibrium dividend strategy, and

$$V(t, x) = \sum_{i=1}^n \omega_i \left(\frac{\gamma\mu}{(\gamma + \rho_i)^2} (1 - e^{\xi_{1i}x}) + \frac{\gamma}{\gamma + \rho_i} x \right), \quad x \geq 0 \tag{32}$$

is the corresponding equilibrium value function.

(ii) If $\sum_{i=1}^n \omega_i \frac{P_i}{\gamma + \rho_i} + 1 < 0$ and $e^{(\xi_{4i} - \xi_{3i})b} < \frac{\rho_i - \mu\xi_{3i}}{\rho_i - \mu\xi_{4i}}$, then

$$\hat{\pi}(t, x) = \begin{cases} 0, & 0 \leq x < b, \\ x - b, & x \geq b \end{cases}$$

is an equilibrium strategy, and

$$V(t, x) = \begin{cases} \sum_{i=1}^n \omega_i C_i (e^{\xi_{3i}x} - e^{\xi_{4i}x}), & 0 \leq x < b, \\ \sum_{i=1}^n \omega_i \left(C_{1i} e^{\xi_{1i}x} + \frac{\gamma}{\rho_i} C_{1i} e^{\xi_{1i}b} + \frac{\gamma}{\gamma + \rho_i} (x - b) + \frac{\gamma\mu}{\rho_i(\gamma + \rho_i)} \right), & x \geq b \end{cases}$$

is the corresponding equilibrium value function. b is determined by (28), C_i and C_{1i} are given in (24) and (25) for $i = 1, 2, \dots, n$, respectively.

Proof. From Lemma 4.2 we know that the functions $u^s(t, x)$ in (30) and (31) solve the equations (5), (6) and (7), and that there exist $\zeta > 0$ and $\kappa(s) > 0$ such that $0 \leq \frac{\partial u^s}{\partial x}(t, x) \leq e^{-\zeta t} \kappa(s)$. By Remark 4, we have (8) established. \square

4.2. A pseudo-exponential discount function. We now discuss a pseudo-exponential discount function, for the given $\varphi_2(t)$ in (18), we consider the following ansatz:

$$u^s(t, x) = e^{-\delta(t-s)} \{ \eta(t-s) V_3(x) + V_4(x) \}. \tag{33}$$

Substituting (33) into (16) yields

$$\begin{cases} -cV_3'(x) - (\lambda + \delta)V_3(x) + \lambda \int_0^{b-x} V_3(x+y)\beta e^{-\beta y} dy \\ + \lambda \int_{b-x}^\infty V_3(x+y)\beta e^{-\beta y} dy = 0, \quad 0 \leq x < b, \\ -cV_3'(x) + \lambda \int_0^\infty V_3(x+y)\beta e^{-\beta y} dy - (\lambda + \delta)V_3(x) \\ + \gamma[V_3(b) - V_3(x)] + \gamma(x - b) = 0, \quad x \geq b, \\ V_3(0) = 0 \end{cases} \tag{34}$$

and

$$\begin{cases} -cV_4'(x) + \lambda \int_0^\infty V_4(x+y)\beta e^{-\beta y} dy \\ - (\lambda + \delta)V_4(x) + \eta V_3(x) = 0, \quad 0 \leq x < b, \\ \gamma V_4(b) + \gamma(x - b) - cV_4'(x) + \lambda \int_0^\infty V_4(x+y)\beta e^{-\beta y} dy \\ - (\lambda + \gamma + \delta)V_4(x) + \eta V_3(x) = 0, \quad x \geq b, \\ V_4(0) = 0. \end{cases} \tag{35}$$

We denote the roots of the equations $c\theta^2 + (\mu\beta + \delta)\theta - \beta\delta = 0$ and $c\theta^2 + (\mu\beta + \gamma + \delta)\theta - \beta(\gamma + \delta) = 0$ by $\theta_1, \theta_2, \theta_3$ and θ_4 , respectively, i.e.,

$$\begin{aligned} \theta_1 &= \frac{-(\mu\beta + \delta) - \sqrt{(\mu\beta + \delta)^2 + 4c\beta\delta}}{2c} < 0, \\ \theta_2 &= \frac{-(\mu\beta + \delta) + \sqrt{(\mu\beta + \delta)^2 + 4c\beta\delta}}{2c} > 0, \\ \theta_3 &= \frac{-(\mu\beta + \gamma + \delta) - \sqrt{(\mu\beta + \gamma + \delta)^2 + 4c\beta(\gamma + \delta)}}{2c} < 0, \\ \theta_4 &= \frac{-(\mu\beta + \gamma + \delta) + \sqrt{(\mu\beta + \gamma + \delta)^2 + 4c\beta(\gamma + \delta)}}{2c} > 0. \end{aligned}$$

Similar arguments in Subsection 4.1, we have

$$V_3(x) = \begin{cases} \bar{C}(e^{\theta_1 x} - e^{\theta_2 x}), \quad 0 \leq x < b, \\ \bar{B}e^{\theta_3 x} + \frac{\gamma}{\delta}\bar{B}e^{\theta_3 b} + \frac{\gamma}{\gamma + \delta}(x - b) + \frac{\gamma\mu}{\delta(\gamma + \delta)}, \quad x \geq b, \end{cases} \tag{36}$$

where

$$\bar{C} = -\frac{\gamma}{Z} + \frac{\mu\gamma\theta_3}{Z(\gamma + \delta)}, \tag{37}$$

$$\bar{B} = \frac{\theta_1 e^{\theta_1 b} - \theta_2 e^{\theta_2 b}}{\bar{Z} \theta_3 e^{\theta_3 b}} \left(-\gamma + \frac{\mu \gamma \theta_3}{\gamma + \delta} \right) - \frac{\gamma}{(\gamma + \delta) \theta_3 e^{\theta_3 b}}, \tag{38}$$

$$\bar{Z} = [\delta \theta_3 - (\gamma + \delta) \theta_1] e^{\theta_1 b} + [(\gamma + \delta) \theta_2 - \delta \theta_3] e^{\theta_2 b}. \tag{39}$$

Substituting (36) into (35) and solving it, we obtain

$$V_4(x) = \begin{cases} (D_1 + B_1 x) e^{\theta_1 x} + (D_2 + B_2 x) e^{\theta_2 x}, & 0 \leq x < b, \\ (D_3 + B_3 x) e^{\theta_3 x} + D_4 e^{\theta_4 x} + \frac{\gamma \epsilon}{(\gamma + \delta)^2} x + \frac{E}{\beta(\gamma + \delta)} + \frac{\gamma \epsilon (\beta \mu + \gamma + \delta)}{\beta(\gamma + \delta)^3}, & x \geq b, \end{cases} \tag{40}$$

where D_1, D_2, D_3 and D_4 are constants,

$$\begin{aligned} B_1 &= \frac{\eta \bar{C} (\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} < 0, \quad B_2 = \frac{\eta \bar{C} (\beta - \theta_2)}{2c\theta_2 + \mu\beta + \delta} = \frac{\beta - \theta_2}{\beta - \theta_1} B_1 < 0, \\ B_3 &= \frac{\eta \bar{B} (\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} < 0, \quad \epsilon = \gamma + \delta + \eta, \\ E &= \beta \gamma V_4(b) + \frac{\eta \beta \gamma}{\delta} \bar{B} e^{\theta_3 b} - \gamma \beta b \left(1 + \frac{\eta}{\gamma + \delta} \right) + \frac{\gamma \eta}{\gamma + \delta} \left(\frac{\mu \beta}{\delta} - 1 \right) - \gamma. \end{aligned} \tag{41}$$

From (7), we have $V_4(0) = 0$, then $D_1 = -D_2 := \hat{C}$. By (9) and Remark 6, we have $D_4 = 0$.

Applying the principle of smooth fit at b

$$\begin{cases} V_4(b-) = V_4(b+), \\ V_4'(b-) = V_4'(b+), \end{cases} \tag{42}$$

we get

$$\begin{cases} (\hat{C} + B_1 b) e^{\theta_1 b} + (-\hat{C} + k B_1 b) e^{\theta_2 b} = \frac{\gamma + \delta}{\delta} \left(D_3 + B_3 b + \frac{\gamma \eta \bar{B}}{\delta(\gamma + \delta)} \right) e^{\theta_3 b} \\ + \frac{\gamma + \delta}{\delta} \bar{E}, \\ (B_1 + \hat{C} \theta_1 + B_1 \theta_1 b) e^{\theta_1 b} + (-\hat{C} \theta_2 + k B_1 \theta_2 b + k B_1) e^{\theta_2 b} \\ = (D_3 \theta_3 + B_3 \theta_3 b + B_3) e^{\theta_3 b} + \frac{\gamma \epsilon}{(\gamma + \delta)^2}, \end{cases}$$

then the function $V_4(x)$ is given by

$$V_4(x) = \begin{cases} (\hat{C} + B_1 x) e^{\theta_1 x} + (-\hat{C} + k B_1 x) e^{\theta_2 x}, & 0 \leq x < b, \\ (D_3 + B_3 x) e^{\theta_3 x} + \frac{\gamma \epsilon}{(\gamma + \delta)^2} x + \frac{E}{\beta(\gamma + \delta)} + \frac{\gamma \epsilon (\beta \mu + \gamma + \delta)}{\beta(\gamma + \delta)^3}, & x \geq b, \end{cases} \tag{43}$$

where

$$\begin{aligned} D_3 &= \left[\frac{\delta}{\gamma + \delta} e^{(\theta_1 - \theta_3)b} - \frac{\delta}{\gamma + \delta} e^{(\theta_2 - \theta_3)b} \right] \hat{C} + \frac{\delta}{\gamma + \delta} B_1 b e^{-\theta_3 b} (e^{\theta_1 b} + k e^{\theta_2 b}) \\ &\quad - \bar{E} e^{-\theta_3 b} - B_3 b - \frac{\gamma \eta \bar{B}}{\delta(\gamma + \delta)}, \end{aligned} \tag{44}$$

$$\hat{C} = \frac{E_1 e^{\theta_1 b} + E_2 e^{\theta_2 b} + E_3 e^{\theta_3 b} - \bar{E} \theta_3 + \frac{\gamma \epsilon}{(\gamma + \delta)^2}}{(\theta_1 - \frac{\delta}{\gamma + \delta} \theta_3) e^{\theta_1 b} + (\frac{\delta}{\gamma + \delta} \theta_3 - \theta_2) e^{\theta_2 b}}, \tag{45}$$

$$E_1 = B_1 \left(\frac{\delta\theta_3}{\gamma + \delta} b - \theta_1 b - 1 \right), \quad E_2 = kB_1 \left(\frac{\delta\theta_3}{\gamma + \delta} b - \theta_2 b - 1 \right),$$

$$E_3 = B_3 - \frac{\gamma\eta\bar{E}}{\delta(\gamma + \delta)}\theta_3, \quad k = \frac{\beta - \theta_2}{\beta - \theta_1},$$

$$\bar{E} = \frac{\gamma\epsilon\mu}{(\gamma + \delta)^3} + \frac{\eta\gamma\mu}{\delta(\gamma + \delta)^2}.$$

In the following parts, we will consider equilibrium strategy. Assuming that $\frac{\partial u^t}{\partial x}(t, b-) = \frac{\partial u^t}{\partial x}(t, b+) = 1$, i.e.,

$$\left[(\hat{C} + B_1 b)\theta_1 + B_1 \right] e^{\theta_1 b} + \left[(-\hat{C} + kB_1 b)\theta_2 + kB_1 \right] e^{\theta_2 b} - 1 = 0, \tag{46}$$

$$(D_3\theta_3 + B_3\theta_3 b + B_3)e^{\theta_3 b} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} - 1 = 0. \tag{47}$$

Substituting \hat{C} in (45) into (46), we have

$$G(b) = \left[E_1\theta_1 + B_1(b\theta_1 + 1) \left(\theta_1 - \frac{\delta}{\delta + \gamma}\theta_3 \right) \right] e^{2\theta_1 b}$$

$$+ \left[-E_2\theta_2 + kB_1(b\theta_2 + 1) \times \left(\frac{\delta}{\delta + \gamma}\theta_3 - \theta_2 \right) \right] e^{2\theta_2 b}$$

$$+ \left[E_2\theta_1 - E_1\theta_2 + B_1(b\theta_1 + 1) \left(\frac{\delta}{\gamma + \delta}\theta_3 - \theta_2 \right) \right.$$

$$\left. + kB_1(b\theta_2 + 1) \left(\theta_1 - \frac{\delta}{\gamma + \delta}\theta_3 \right) \right] e^{(\theta_1 + \theta_2)b}$$

$$+ \left(-\bar{E}\theta_3\theta_1 + \frac{\gamma\epsilon}{(\gamma + \delta)^2}\theta_1 - \theta_1 + \frac{\delta}{\gamma + \delta}\theta_3 \right) e^{\theta_1 b}$$

$$+ \left(\bar{E}\theta_3\theta_2 - \frac{\gamma\epsilon}{(\gamma + \delta)^2}\theta_2 + \theta_2 - \frac{\delta}{\gamma + \delta}\theta_3 \right) e^{\theta_2 b}$$

$$+ E_3\theta_1 e^{(\theta_1 + \theta_3)b} + (-E_3\theta_2) e^{(\theta_2 + \theta_3)b} = 0. \tag{48}$$

Lemma 4.4. *If*

$$\frac{\frac{\mu\gamma\theta_3}{\gamma + \delta} + \delta}{\frac{\mu\gamma}{\gamma + \delta} \left[\frac{\beta - 1}{2c\theta_3 + \mu\beta + \gamma + \delta} - \frac{2\theta_3}{\gamma + \delta} \right] + \gamma} < \eta \tag{49}$$

and

$$\eta < - \left[\theta_2\theta_3 \left((\gamma + \delta)\theta_2 - \delta\theta_3 \right) \left(\frac{\mu\gamma}{(\gamma + \delta)^2} + \frac{\delta}{\gamma + \delta} \left(\frac{1}{\theta_3} - \frac{1}{\theta_2} \right) \right) \right] N^{-1}, \tag{50}$$

where

$$N = \frac{\gamma}{\gamma + \delta} \left\{ \frac{(\beta - \theta_2)\theta_3}{\mu\beta + \delta + 2c\theta_2} \left(\frac{\mu\delta\theta_3}{\gamma + \delta} - \delta \right) + \frac{(\theta_3 - \beta)\theta_2}{2c\theta_3 + \mu\beta + \gamma + \delta} (\mu\theta_2 - \delta) \right.$$

$$\left. + \frac{\theta_2\theta_3}{\gamma + \delta} \left(2\mu\theta_2 - \frac{(2\delta + \gamma)\mu}{\gamma + \delta}\theta_3 + \gamma + \delta(\gamma + \delta) \right) - \theta_2^2 \right\}.$$

Then (48) has a positive solution.

The proof is shown in Appendix D. In order to study the concavity of $V(t, x)$, we introduce the following lemma.

Lemma 4.5. *If*

$$\eta [k(2\theta_2 - 3\theta_1) - \theta_2] \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) > \frac{\theta_1 - \theta_2}{\beta - \theta_1} \times \delta\beta(\gamma + \delta) \quad (51)$$

and

$$\eta [k(2\theta_2^2 - 3\theta_1\theta_2) - \theta_1\theta_2] \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) > \frac{\theta_1 - \theta_2}{\beta - \theta_1} \times \delta\beta[(\gamma + \delta)\theta_2 - \delta\theta_3], \quad (52)$$

then

$$-\hat{C}\theta_2 + 3kB_1 + kB_1\theta_2b > 0. \quad (53)$$

The proof is given in Appendix E. Next, we will give the main results of this subsection.

Lemma 4.6. *For the discount function $\varphi_2(t)$ in (18), there exists a solution $u^s(t, x)$ of the extended HJB system of equations in Definition 3.2 as following:*

(i) *If $\frac{\theta_3}{2} \leq \eta \left[-\frac{\theta_3}{\gamma + \delta} - \frac{\theta_3 - \beta}{2c\theta_3 + \mu\beta + \gamma + \delta} \right] \leq \frac{(\gamma + \delta)^2 - \gamma\epsilon + \frac{\mu\gamma\epsilon\theta_3}{\gamma + \delta}}{\mu\gamma}$, we have*

$$u^s(t, x) = e^{-\delta(t-s)} \{ \eta(t-s)V_3(x) + V_4(x) \}, \quad (54)$$

where

$$V_3(x) = \frac{\gamma\mu}{(\gamma + \delta)^2} (1 - e^{\theta_3 x}) + \frac{\gamma}{\gamma + \delta} x, \quad x \geq 0, \quad (55)$$

$$V_4(x) = \left[-\frac{\mu\gamma(\eta + \epsilon)}{(\gamma + \delta)^3} - \left(\frac{\mu\gamma}{(\gamma + \delta)^2} \times \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \right) x \right] e^{\theta_3 x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} x + \frac{\gamma}{\gamma + \delta} \left(\frac{\mu\eta}{(\gamma + \delta)^2} - \frac{\eta}{\beta(\gamma + \delta)} - \frac{1}{\beta} \right) + \frac{\gamma\epsilon(\beta\mu + \gamma + \delta)}{\beta(\gamma + \delta)^3}, \quad x \geq 0. \quad (56)$$

(ii) *If the conditions of (49)-(52) are satisfied, we have*

$$u^s(t, x) = e^{-\delta(t-s)} \{ \eta(t-s)V_3(x) + V_4(x) \}, \quad (57)$$

where

$$V_3(x) = \begin{cases} \bar{C}(e^{\theta_1 x} - e^{\theta_2 x}), & 0 \leq x < b, \\ \bar{B}e^{\theta_3 x} + \frac{\gamma}{\delta}\bar{B}e^{\theta_3 b} + \frac{\gamma}{\gamma + \delta}(x - b) + \frac{\gamma\mu}{\delta(\gamma + \delta)}, & x \geq b, \end{cases}$$

$$V_4(x) = \begin{cases} (\hat{C} + B_1x)e^{\theta_1 x} + (-\hat{C} + kB_1x)e^{\theta_2 x}, & 0 \leq x < b, \\ (D_3 + B_3x)e^{\theta_3 x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2}x + \frac{E}{\beta(\gamma + \delta)} + \frac{\gamma\epsilon(\beta\mu + \gamma + \delta)}{\beta(\gamma + \delta)^3}, & x \geq b, \end{cases}$$

parameters \bar{C} , \bar{B} , \hat{C} , B_1 , B_3 , D_3 and E are given in (37), (38), (41), (44) and (45), respectively.

Proof. See Appendix F. □

Remark 7. For $x \geq b$, we have

$$\begin{aligned} V_4'(x) &= (D_3\theta_3 + B_3\theta_3x + B_3)e^{\theta_3 x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} \\ &\geq (D_3\theta_3 + B_3\theta_3b + B_3)e^{\theta_3 x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} \\ &= [\theta_3(D_3 + B_3b) + B_3]e^{\theta_3 x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} \end{aligned}$$

$$= \frac{\delta^2 + \gamma(\delta - \eta)}{(\gamma + \delta)^2} e^{(x-b)\theta_3} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} > 0.$$

From Lemma 4.6, we know $V_4'(x) > 0$ for $0 \leq x < b$, then $V_4(x)$ is an increasing function. From Theorem 4.3, we have $V_3'(x) > 0$ for $x \geq 0$. Then there exists $\zeta > 0$ and $\kappa(s) > 0$ such that

$$0 \leq \frac{\partial u^s}{\partial x}(t, x) = e^{-\delta(t-s)} \{ \eta(t-s)V_3'(x) + V_4'(x) \} \leq e^{-\zeta t} \kappa(s).$$

Theorem 4.7. For the discount function $\varphi_2(t)$ in (18),

(i) If $\frac{\theta_3}{2} \leq \eta \left[-\frac{\theta_3}{\gamma + \delta} - \frac{\theta_3 - \beta}{2c\theta_3 + \mu\beta + \gamma + \delta} \right] \leq \frac{(\gamma + \delta)^2 - \gamma\epsilon + \frac{\mu\gamma\epsilon\theta_3}{\gamma + \delta}}{\mu\gamma}$, $\hat{\pi}(t, x) = x$ is an equilibrium dividend strategy, and

$$V(t, x) = \left[-\frac{\mu\gamma(\eta + \epsilon)}{(\gamma + \delta)^3} - \left(\frac{\mu\gamma}{(\gamma + \delta)^2} \times \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \right) x \right] e^{\theta_3 x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} x + \frac{\gamma}{\gamma + \delta} \left(\frac{\mu\eta}{(\gamma + \delta)^2} - \frac{\eta}{\beta(\gamma + \delta)} - \frac{1}{\beta} \right) + \frac{\gamma\epsilon(\beta\mu + \gamma + \delta)}{\beta(\gamma + \delta)^3}, \quad x \geq 0$$

is the corresponding equilibrium value function.

(ii) If the conditions of (49)-(52) are satisfied, then

$$\hat{\pi}(t, x) = \begin{cases} 0, & 0 \leq x < b, \\ x - b, & x \geq b \end{cases} \tag{58}$$

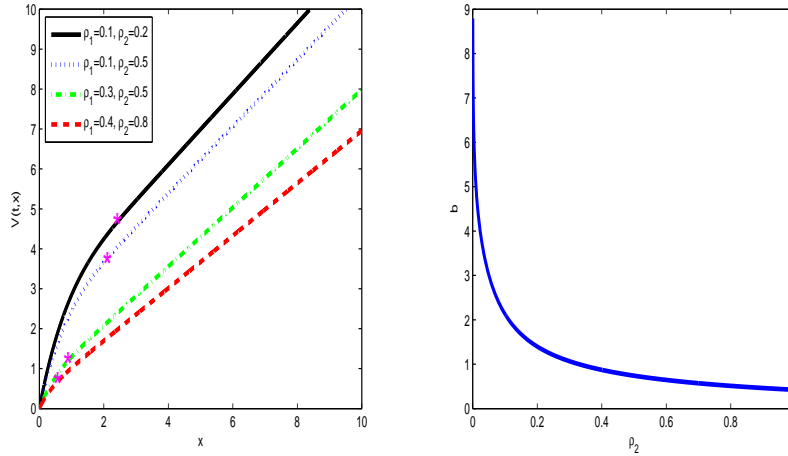
is an equilibrium strategy, and

$$V(t, x) = \begin{cases} (\hat{C} + B_1 x)e^{\theta_1 x} + (-\hat{C} + kB_2 x)e^{\theta_2 x}, & 0 \leq x < b, \\ (D_3 + B_3 x)e^{\theta_3 x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} x + \frac{E}{\beta(\gamma + \delta)} + \frac{\gamma\epsilon(\beta\mu + \gamma + \delta)}{\beta(\gamma + \delta)^3}, & x \geq b \end{cases}$$

is the corresponding equilibrium value function. b is determined by (48), where the parameters \hat{C} , B_1 , B_2 , B_3 , D_3 and E are given in (41), (44) and (45), respectively.

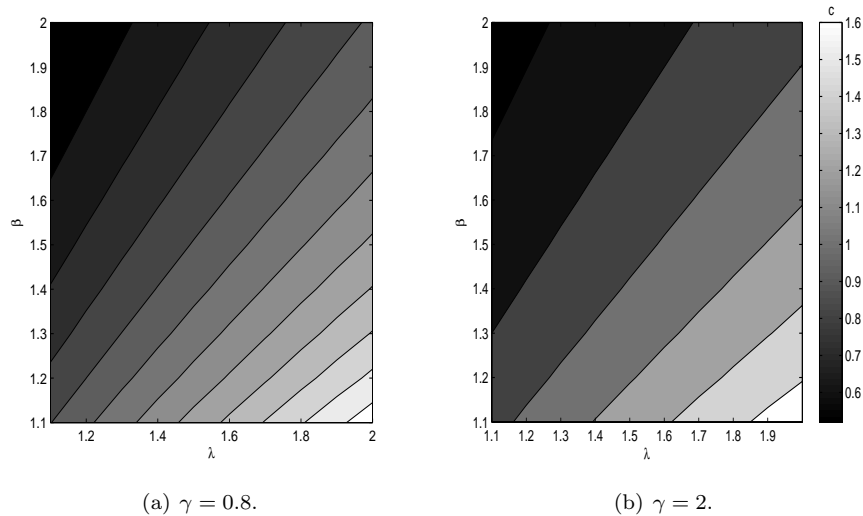
Proof. From Lemma 4.6, we know that the functions $u^s(t, x)$ in (54) and (57) solve the equations (5), (6) and (7), and $u^s(t, x)$ satisfies equation (9) by Remark 7. Note that $\frac{\partial^2 u^s}{\partial x^2}(t, x) = e^{-\delta(t-s)} \{ \eta(t-s)V_3''(x) + V_4''(x) \}$, by Lemma 4.2, we have $V_3''(x) < 0$, then $\frac{\partial^2 u^s}{\partial x^2}(t, x) < 0$. Similar to the arguments of Theorem 4.3, we get (8) holds. \square

4.3. Numerical examples. Let $\beta = 1.2$, $\lambda = 2$, $c = 1$, $\gamma = 1$, $\omega_1 = 0.7$ and $\omega_2 = 1 - \omega_1$, Figure 1 (a) depicts the equilibrium value functions for different ρ_1 and ρ_2 . This picture shows that the larger discount rate results in the smaller function $V(t, x)$. We also derive that the equilibrium periodic dividend barrier b are 2.0813, 1.4574, 0.6576, 0.4224, respectively, for corresponding $(\rho_1, \rho_2) = (0.1, 0.2)$, $(\rho_1, \rho_2) = (0.1, 0.5)$, $(\rho_1, \rho_2) = (0.3, 0.5)$ and $(\rho_1, \rho_2) = (0.4, 0.8)$. These imply that the larger discount rates ρ_1 and ρ_2 result in the less equilibrium periodic dividend barrier b . More details on this conclusion, readers can refer to Figure 1 (b). From the Figure 1 (b), we find that equilibrium periodic dividend barrier b is getting smaller as ρ_2 increases. When $\rho_2 = 0$, that is, the second shareholder does not need to be discounted, the equilibrium periodic dividend barrier b takes the maximum value.



(a) Equilibrium value functions $V(t, x)$. (b) Equilibrium dividend barrier b .

FIGURE 1. Influence of parameters ρ_1 and ρ_2 to equilibrium value function and equilibrium dividend barrier.



(a) $\gamma = 0.8$. (b) $\gamma = 2$.

FIGURE 2. Contour plot of c as a function of β and λ .

In Figure 2, we assume that $\omega_1 = 0.3$, $\rho_1 = 0.07$, $\rho_2 = 0.2$ and $\omega_2 = 1 - \omega_1$. It contains two contour plots that show the combined effects of λ and β in compound Poisson process on the expense rate c such that the equilibrium periodic dividend barrier is strictly positive. We first examine each graph individually and observe that when fixing λ , increasing β results in a lower value of c . The company will require a lower expense rate c to compensate for lower the increment of the company's earnings per time. Similarly, we find that fixing β and increasing λ has a

TABLE 1. Influences of β and λ on b .

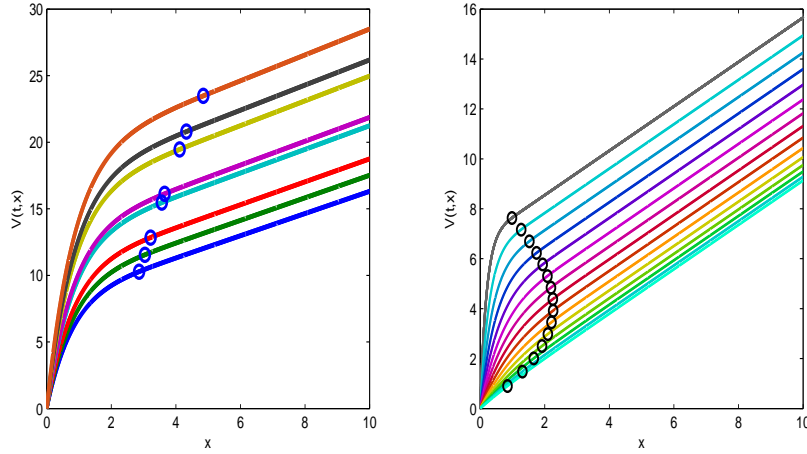
	$c = 0.7$		$\gamma = 1$	$\omega_1 = 0.7$	$\rho_1 = 0.1$	$\rho_2 = 0.3$			
	$\beta = 1.5$				$\lambda = 1.5$				
$\lambda \uparrow$	1.3	1.8	2.1920	3	$\beta \uparrow$	0.8	1	1.3	1.7
$b \curvearrowright$	0.5606	1.5520	1.6794	1.5256	\downarrow	2.2197	1.9983	1.5353	0.6472

increasing impact on c . Next, we illustrate the effect of the expected inter-dividend-decision times γ . Note that the effect of the expected inter-dividend-decision times γ increases from 0.8 in Figure 2 (a) to 2 in Figure 2 (b). As a result, the expense rate c is globally higher in Figure 2 (b), which is evident from the left-up shifting of all the contour lines. This makes sense as more frequent dividends compensate for higher expense rate.

We now consider the influences of λ and β on the equilibrium periodic dividend barriers. Let $c = 0.7$, $\gamma = 1$, $\omega_1 = 0.7$, $\rho_1 = 0.1$, $\rho_2 = 0.3$ and $\omega_2 = 1 - \omega_1$. From Table 1 (\uparrow and \downarrow mean increase and decrease, respectively. \curvearrowright means increase first and then decrease), we find that the equilibrium periodic dividend barrier b will rise first and then fall as the λ increases. In this example, when $\lambda = 2.1920$, b takes the maximum value 1.6794. Next, we consider the effect of β on barrier b . We note that with the increase of β , the increment of the company’s earnings per time decreases. In order to get more dividends, the equilibrium periodic dividend barrier b decreases. Let $\beta = 0.7$, $\gamma = 1$, $\delta = 0.2$, $\lambda = 3$ and $c = 1.7$, the Figure 3 (a) shows the the effect of η changing from 0 to 0.2 on equilibrium value function and equilibrium strategy. The equilibrium value functions for pseudo-exponential discount functions with $\eta = 0$, $\eta = 0.02$, $\eta = 0.04$, $\eta = 0.08$, $\eta = 0.09$, $\eta = 0.14$, $\eta = 0.16$ and $\eta = 0.2$ (from top to bottom), the equilibrium dividend barriers are 2.8609, 3.0419, 3.2180, 3.5644, 3.6518, 4.1157, 4.3262 and 4.8486, respectively. We find that the larger η results in the larger equilibrium dividend barrier b . The case of $\eta = 0$ is time consistent and the equilibrium strategy is optimal in the classical situation. A larger η means a larger discount function, so the value function goes up.

The Figure 3 (b) depicts the effects of expense rate c on the equilibrium value function $V(t, x)$ and the equilibrium dividend barrier b when $\beta = 0.7$, $\gamma = 1$, $\delta = 0.5$, $\lambda = 2$, $\eta = 0.4$ and c gradually increases from the top 0.4 to the bottom 2.5. From this figure, we find that the equilibrium dividend barrier b increases at the beginning and then decreases with the increase of parameter c , which is identified with the classical optimal dividend problem. In addition, the equilibrium value function $V(t, x)$ will decrease as c increases. More precisely, as the expense rate increases, the dividends received by shareholders will decrease.

5. Conclusion. In this paper, the equilibrium period dividend problem is studied for the first time. Under a mixture of exponential discount functions and a pseudo-exponential discount function, we derived the closed-form solutions of time inconsistent control problem, and the general theory of equilibrium period dividend problem is given in a dual model. The equilibrium strategy and the equilibrium value function degenerate into optimal problems under the exponential discount function. Due to difficulties in solving, we only studied the closed-form solutions in



(a) Equilibrium value function $V(t,x)$ for different η . (b) Equilibrium value function $V(t,x)$ for different c .

FIGURE 3. Influence of parameters η and c to equilibrium value function.

a dual process, in which the aggregate incomes process follows a compound Poisson process with exponential jumps. In future research, it would be interesting to consider the equilibrium period dividend problem by the theory of Lévy processes and consider periodic dividend problem with a ruin penalty. Moreover, we can also consider solving by some numerical methods and mathematical tools.

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Appendix

Appendix A. Heuristic argument of equation (3).

Proof. First of all, we define $u(s,t,x) = E_{t,x} \left[\int_t^{T^{\hat{\pi}}} \varphi(z-s) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_{\gamma}(z) \right]$. From the definition of J we have

$$\begin{aligned}
 J(t,x;\pi_h) = & E_{t,x} \left[\int_t^{t+h} \varphi(z-t) \pi(z, X^{\pi}(z)) dN_{\gamma}(z) \right] \\
 & + E_{t,x} \left[\int_{t+h}^{T^{\pi_h}} \varphi(z-t) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_{\gamma}(z) \right].
 \end{aligned}$$

Due to

$$\begin{aligned} J(t+h, X^{\pi_h}(t+h); \pi_h) &= E_{t+h} \left[\int_{t+h}^{T^{\hat{\pi}}} \varphi(z-t-h) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_{\gamma}(z) \right] \\ &= u(t+h, t+h, X^{\pi}(t+h)) \\ &= J(t+h, X^{\pi}(t+h); \hat{\pi}) \\ &= V(t+h, X^{\pi}(t+h)), \end{aligned}$$

where $E_{t+h}[\cdot] = E[\cdot | X^{\pi}(t+h)]$. Taking expectations on the both sides of the above equation yields to

$$\begin{aligned} E_{t,x}[J(t+h, X^{\pi}(t+h); \hat{\pi})] &= E_{t,x}[u(t+h, t+h, X^{\pi}(t+h))] \\ &= E_{t,x}[V(t+h, X^{\pi}(t+h))]. \end{aligned}$$

Since

$$\begin{aligned} &E_{t,x} \left[\int_{t+h}^{T^{\pi_h}} \varphi(z-t) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_{\gamma}(z) \right] \\ &= E_{t,x} E_{t+h} \left[\int_{t+h}^{T^{\hat{\pi}}} \varphi(z-t) \hat{\pi}(z, X^{\hat{\pi}}(z)) dN_{\gamma}(z) \right] = E_{t,x}[u(t, t+h, X^{\pi}(t+h))]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &J(t, x; \pi_h) \\ &= E_{t,x} \left[\int_t^{t+h} \varphi(z-t) \pi(z, X^{\pi}(z)) dN_{\gamma}(z) \right] + E_{t,x}[u(t, t+h, X^{\pi}(t+h))] \quad (59) \\ &= E_{t,x} \left[\int_t^{t+h} \varphi(z-t) \pi(z, X^{\pi}(z)) dN_{\gamma}(z) \right] + E_{t,x}[u(t, t+h, X^{\pi}(t+h))] \\ &\quad + E_{t,x}[V(t+h, X^{\pi}(t+h))] - E_{t,x}[u(t+h, t+h, X^{\pi}(t+h))]. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{E_{t,x} \left[\int_t^{t+h} \varphi(z-t) \pi(z, X^{\pi}(z)) dN_{\gamma}(z) \right]}{h} \\ &= \lim_{h \rightarrow 0} \frac{E_{t,x} \left[\int_t^{t+h} \varphi(z-t) \pi(z, X^{\pi}(z)) \gamma dz \right]}{h} \\ &= \gamma \pi. \end{aligned} \quad (60)$$

Then we obtain

$$\lim_{h \rightarrow 0} \frac{J(t, x; \pi_h) - J(t, x; \hat{\pi})}{h} = \mathcal{A}^{\pi} V(t, x) + \gamma \pi + \mathcal{A}^{\pi} u^t(t, x) - \mathcal{A}^{\pi} u(t, t, x) \leq 0.$$

□

Appendix B. Proof of Lemma 4.1.

Proof. Define

$$F(b) := \sum_{i=1}^n \omega_i \frac{P_i (\xi_{3i} e^{\xi_{3i} b} - \xi_{4i} e^{\xi_{4i} b})}{Z_i} - 1, \quad (61)$$

where $P_i = -\gamma + \frac{\mu\gamma\xi_{1i}}{\gamma+\rho_i}$. Inequality $(P_i + \gamma + \rho_i)\xi_{4i} > \rho_i\xi_{1i}$ is equivalent to $\frac{\mu\gamma}{\rho_i(\gamma+\rho_i)} < \frac{1}{\xi_{4i}} - \frac{1}{\xi_{1i}}$. Firstly, note that

$$\xi_{4i} = \frac{(-\mu\beta - \rho_i) + \sqrt{(\mu\beta + \rho_i)^2 + 4c\beta\rho_i}}{2c}.$$

In addition, according to Cauchy inequality, we have

$$\sqrt{a^2 + c} - a < \frac{c}{2a}, \quad a > 0 \text{ and } c > 0. \tag{62}$$

Then

$$\xi_{4i} < \frac{\beta\rho_i}{\mu\beta + \rho_i},$$

i.e.,

$$\frac{1}{\xi_{4i}} > \frac{\mu\beta + \rho_i}{\beta\rho_i} = \frac{(\mu\beta + \rho_i)(\gamma + \rho_i)}{\beta\rho_i(\gamma + \rho_i)} > \frac{\mu\beta\gamma}{\beta\rho_i(\gamma + \rho_i)} = \frac{\mu\gamma}{\rho_i(\gamma + \rho_i)}.$$

So, we have

$$\frac{1}{\xi_{4i}} - \frac{1}{\xi_{1i}} > \frac{\mu\gamma}{\rho_i(\gamma + \rho_i)}. \tag{63}$$

From (61), we get

$$F(0) = \sum_{i=1}^n \omega_i \frac{P_i(\xi_{3i} - \xi_{4i})}{\rho_i\xi_{1i} - (\gamma + \rho_i)\xi_{3i} + (\gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}} - 1 = - \left[\sum_{i=1}^n \omega_i \frac{P_i}{\gamma + \rho_i} + 1 \right].$$

According to (63), we obtain

$$\begin{aligned} & \lim_{b \rightarrow \infty} F(b) \\ &= \lim_{b \rightarrow \infty} \sum_{i=1}^n \omega_i P_i \frac{\xi_{3i}e^{\xi_{3i}b} - \xi_{4i}e^{\xi_{4i}b}}{[\rho_i\xi_{1i} - (\gamma + \rho_i)\xi_{3i}]e^{\xi_{3i}b} + [(\gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}]e^{\xi_{4i}b}} - 1 \\ &= \sum_{i=1}^n \omega_i P_i \frac{-\xi_{4i}}{(\gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}} - 1 \\ &= - \sum_{i=1}^n \omega_i \left(\frac{P_i\xi_{4i}}{(\gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}} + 1 \right) \\ &= - \sum_{i=1}^n \omega_i \frac{(P_i + \gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}}{(\gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}} \\ &< 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} & F'(b) \\ &= \sum_{i=1}^n \omega_i P_i \frac{(\xi_{3i}^2 e^{\xi_{3i}b} - \xi_{4i}^2 e^{\xi_{4i}b}) Z_i - (\xi_{3i} e^{\xi_{3i}b} - \xi_{4i} e^{\xi_{4i}b}) Z_i'}{Z_i^2} \\ &= \sum_{i=1}^n \omega_i P_i \frac{(\xi_{3i} - \xi_{4i}) e^{(\xi_{3i} + \xi_{4i})b} \{ \xi_{3i} [(\gamma + \rho_i)\xi_{4i} - \rho_i\xi_{1i}] + \xi_{4i} [\rho_i\xi_{1i} - (\gamma + \rho_i)\xi_{3i}] \}}{Z_i^2} \\ &= - \sum_{i=1}^n \omega_i P_i \frac{\rho_i\xi_{1i}(\xi_{3i} - \xi_{4i})^2 e^{(\xi_{3i} + \xi_{4i})b}}{Z_i^2} \\ &< 0, \end{aligned}$$

where $Z'_i = [\rho_i \xi_{1i} - (\gamma + \rho_i) \xi_{3i}] \xi_{3i} e^{\xi_{3i} b} + [(\gamma + \rho_i) \xi_{4i} - \rho_i \xi_{1i}] \xi_{4i} e^{\xi_{4i} b}$. Therefore, Equation (28) has a unique root if and only if $\sum_{i=1}^n \omega_i \frac{P_i}{\gamma + \rho_i} + 1 < 0$. \square

Appendix C. Proof of Lemma 4.2.

Proof. We only need to verify that the equation (5) holds, i.e., $\frac{\partial u^t}{\partial x}(t, x) \geq 1$ when $0 \leq x < b$, while $\frac{\partial u^t}{\partial x}(t, x) < 1$ when $x \geq b$.

(i) Letting $b = 0$ in (23), we obtain (30). Since

$$\begin{aligned} \frac{\partial u^s}{\partial x}(t, 0) &= \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} \left(-\frac{\gamma \mu}{(\gamma + \rho_i)^2} \xi_{1i} + \frac{\gamma}{\gamma + \rho_i} \right) \leq 1, \quad x \geq 0, \\ \frac{\partial^2 u^s}{\partial x^2}(t, x) &= \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} \left(-\frac{\gamma \mu}{(\gamma + \rho_i)^2} \xi_{1i}^2 e^{\xi_{1i} x} \right) < 0, \quad x \geq 0. \end{aligned}$$

We have $\frac{\partial u^s}{\partial x}(t, x) \leq 1$ for $x \geq 0$ and $0 \leq s \leq t$, then the equation (5) holds and the result is proved.

(ii) Firstly, we give the symbols of C_i and C_{1i} , for $i = 1, 2, \dots, n$. Recalling (26), If $\rho_i \xi_{1i} - (\gamma + \rho_i) \xi_{3i} \geq 0$, then $Z_i > 0$; if $\rho_i \xi_{1i} - (\gamma + \rho_i) \xi_{3i} < 0$, then

$$Z_i \geq \rho_i \xi_{1i} - (\gamma + \rho_i) \xi_{3i} + (\gamma + \rho_i) \xi_{4i} - \rho_i \xi_{1i} = (\gamma + \rho_i) \xi_{4i} - (\gamma + \rho_i) \xi_{3i} > 0.$$

If $e^{(\xi_{4i} - \xi_{3i})b} < \frac{\rho_i - \mu \xi_{3i}}{\rho_i - \mu \xi_{4i}}$, we have $C_i < 0$ and $C_{1i} < 0$, $i = 1, 2, \dots, n$.

Next, we discuss the property of $\frac{\partial u^t}{\partial x}(t, x)$. Since

$$\frac{\partial u^s}{\partial x}(t, x) = \begin{cases} \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} C_i (\xi_{3i} e^{\xi_{3i} x} - \xi_{4i} e^{\xi_{4i} x}), & 0 \leq x < b, \\ \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} \left(C_{1i} \xi_{1i} e^{\xi_{1i} x} + \frac{\gamma}{\gamma + \rho_i} \right), & x \geq b \end{cases}$$

and

$$\frac{\partial^2 u^s}{\partial x^2}(t, x) = \begin{cases} \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} C_i (\xi_{3i}^2 e^{\xi_{3i} x} - \xi_{4i}^2 e^{\xi_{4i} x}), & 0 \leq x < b, \\ \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} C_{1i} \xi_{1i}^2 e^{\xi_{1i} x}, & x \geq b, \end{cases}$$

we have $\frac{\partial u^s}{\partial x}(t, x) > 0$ for $x \geq 0$ and $\frac{\partial^2 u^s}{\partial x^2}(t, x) < 0$ for $x \geq b$. Due to

$$\frac{\partial^3 u^s}{\partial x^3}(t, x) = \sum_{i=1}^n \omega_i e^{-\rho_i(t-s)} C_i (\xi_{3i}^3 e^{\xi_{3i} x} - \xi_{4i}^3 e^{\xi_{4i} x}) > 0, \quad 0 \leq x < b,$$

we obtain $\frac{\partial^2 u^s}{\partial x^2}(t, x) < \frac{\partial^2 u^s}{\partial x^2}(t, b+) < 0$ for $0 \leq x < b$. The proof is completed. \square

Appendix D. Proof of Lemma 4.4.

Proof. Note that

$$\begin{aligned} G(0) = & (\theta_1 - \theta_2) \left\{ \eta \left[\frac{\mu \gamma}{(\gamma + \delta)^2} \left(\frac{\beta}{2c\theta_3 + \mu\beta + \gamma + \delta} - \frac{\theta_3}{\gamma + \delta} \right) \right. \right. \\ & \left. \left. - \left(\frac{1}{2c\theta_3 + \mu\beta + \gamma + \delta} + \frac{\theta_3}{\gamma + \delta} \right) \right] + \frac{\gamma}{\gamma + \delta} - \frac{\mu \gamma \theta_3}{(\gamma + \delta)^2} - 1 \right\}. \end{aligned}$$

By $\frac{\theta_2}{\delta\beta} - \frac{1}{\mu\beta+\delta+2c\theta_2} > 0$, $\frac{1}{\mu\beta+\delta+\gamma+2c\theta_3} - \frac{\theta_3}{\beta(\gamma+\delta)} > 0$ (see Lemma A.1. in Zhao et al. [29]) and (49), we have $G(0) < 0$. Now by the definition of B_1 , B_2 and B_3 in (41), we can rewrite $G(b)$ as

$$\begin{aligned}
 G(b) &= \left\{ \frac{\delta\theta_3}{\gamma+\delta} \frac{\eta(\theta_1-\beta)}{2c\theta_1+\mu\beta+\delta} \times \frac{-\gamma+\frac{\mu\gamma\theta_3}{\gamma+\delta}}{[\delta\theta_3-(\gamma+\delta)\theta_1]e^{\theta_1b} + [(\gamma+\delta)\theta_2-\delta\theta_3]e^{\theta_2b}} \right\} \\
 &\quad \times \left(-e^{2\theta_1b} + ke^{2\theta_2b} + (k(\theta_1b-\theta_2b-1) + (1-b\theta_2+b\theta_1))e^{(\theta_1+\theta_2)b} \right) \\
 &\quad + \frac{1}{\theta_3e^{\theta_3b}} \left[\frac{(\theta_1e^{\theta_1b}-\theta_2e^{\theta_2b})(-\gamma+\frac{\mu\gamma\theta_3}{\gamma+\delta})}{[\delta\theta_3-(\gamma+\delta)\theta_1]e^{\theta_1b} + [(\gamma+\delta)\theta_2-\delta\theta_3]e^{\theta_2b}} - \frac{\gamma}{\gamma+\delta} \right] \\
 &\quad \times \left\{ \frac{\eta(\theta_3-\beta)e^{\theta_3b}}{2c\theta_3+\mu\beta+\gamma+\delta} (\theta_1e^{\theta_1b}-\theta_2e^{\theta_2b}) + \frac{\gamma\eta\theta_3e^{\theta_3b}}{\delta(\gamma+\delta)} (\theta_2e^{\theta_2b}-\theta_1e^{\theta_1b}) \right\} \\
 &\quad + \left(\bar{E}\theta_2\theta_3 - \frac{\gamma\epsilon}{(\gamma+\delta)^2}\theta_2 - \frac{\delta}{\gamma+\delta}\theta_3 + \theta_2 \right) e^{\theta_2b} \\
 &\quad + \left(-\bar{E}\theta_1\theta_3 + \frac{\gamma\epsilon}{(\gamma+\delta)^2}\theta_1 - \theta_1 + \frac{\delta}{\gamma+\delta}\theta_3 \right) e^{\theta_1b} \\
 &:= \frac{1}{[\delta\theta_3-(\gamma+\delta)\theta_1]e^{\theta_1b} + [(\gamma+\delta)\theta_2-\delta\theta_3]e^{\theta_2b}} g(b), \tag{64}
 \end{aligned}$$

where

$$\begin{aligned}
 g(b) &:= e^{2\theta_1b} \left\{ -\frac{\gamma\delta\theta_3}{\gamma+\delta} \frac{\eta(\theta_1-\beta)}{2c\theta_1+\mu\beta+\delta} \times \left(\frac{\mu\theta_3}{\gamma+\delta} - 1 \right) + \left(\frac{\theta_3-\beta}{2c\theta_3+\mu\beta+\gamma+\delta} - \frac{\gamma\theta_3}{\gamma+\delta} \right) \right. \\
 &\quad \times \frac{\gamma\eta\theta_1(\mu\theta_1-\delta)}{\gamma+\delta} + [\delta\theta_3-(\gamma+\delta)\theta_1] \times \left(-\bar{E}\theta_1\theta_3 + \frac{\gamma\epsilon\theta_1}{(\gamma+\delta)^2} - \theta_1 + \frac{\delta\theta_3}{\gamma+\delta} \right) \left. \right\} \\
 &\quad + e^{2\theta_2b} \left\{ \frac{k\gamma\delta\theta_3}{\gamma+\delta} \frac{\eta(\theta_1-\beta)}{2c\theta_1+\mu\beta+\delta} \times \left(\frac{\mu\theta_3}{\gamma+\delta} - 1 \right) + \frac{\gamma\eta\theta_2(\mu\theta_2-\delta)}{\gamma+\delta} \right. \\
 &\quad \times \left(-\frac{\gamma\theta_3}{\delta(\gamma+\delta)} + \frac{\theta_3-\beta}{2c\theta_3+\mu\beta+\gamma+\delta} \right) \\
 &\quad \left. + [(\gamma+\delta)\theta_2-\delta\theta_3] \times \left(\bar{E}\theta_2\theta_3 - \frac{\gamma\epsilon\theta_2}{(\gamma+\delta)^2} - \frac{\delta\theta_3}{\gamma+\delta} + \theta_2 \right) \right\} \\
 &\quad + e^{(\theta_1+\theta_2)b} \left\{ \frac{\gamma\delta\theta_3}{\gamma+\delta} \frac{\eta(\theta_1-\beta)}{2c\theta_1+\mu\beta+\delta} \times \left(\frac{\mu\theta_3}{\gamma+\delta} - 1 \right) \right. \\
 &\quad \times [k(\theta_1b-\theta_2b-1) + (1-\theta_2b+\theta_1b)] \\
 &\quad + \left(\frac{\gamma\theta_3}{\delta(\gamma+\delta)} - \frac{\theta_3-\beta}{2c\theta_3+\mu\beta+\gamma+\delta} \right) \times \left(2\frac{\mu\gamma\eta\theta_1\theta_2}{\gamma+\delta} - \frac{\delta\gamma\eta}{\gamma+\delta}(\theta_1+\theta_2) \right) \\
 &\quad + \left[\left(\bar{E}\theta_2\theta_3 - \frac{\gamma\epsilon}{(\gamma+\delta)^2}\theta_2 - \frac{\delta}{\gamma+\delta}\theta_3 + \theta_2 \right) (\delta\theta_3-(\gamma+\delta)\theta_1) \right] \\
 &\quad \left. + \left[\left(-\bar{E}\theta_1\theta_3 + \frac{\gamma\epsilon}{(\gamma+\delta)^2}\theta_1 - \theta_1 + \frac{\delta}{\delta+\gamma}\theta_3 \right) ((\gamma+\delta)\theta_2-\delta\theta_3) \right] \right\}.
 \end{aligned}$$

Then the coefficient of $e^{2\theta_2b}$ is

$$\begin{aligned}
 &\frac{\mu\gamma\eta}{\gamma+\delta} \left\{ \frac{\theta_3^2}{\gamma+\delta} \left(\frac{\beta\delta}{2c\theta_2+\mu\beta+\delta} - \theta_2 - \frac{\theta_2}{2c\theta_2+\mu\beta+\delta} - \frac{\delta\theta_2}{\gamma+\delta} \right) \right. \\
 &\quad \left. + \theta_2^2 \times \left(\frac{\theta_3}{\gamma+\delta} + \frac{-\beta}{2c\theta_3+\mu\beta+\gamma+\delta} + \frac{\theta_3}{\gamma+\delta} + \frac{\theta_3}{2c\theta_3+\mu\beta+\gamma+\delta} \right) \right\} \\
 &\quad + \frac{\eta\gamma}{\gamma+\delta} \left\{ \theta_3 \times \left(\frac{-\delta\beta}{2c\theta_2+\mu\beta+\delta} + \theta_2 + \frac{\theta_2}{2c\theta_2+\mu\beta+\delta} + \frac{\delta(\gamma+\delta-1)\theta_2}{\gamma+\delta} \right) \right.
 \end{aligned}$$

$$\begin{aligned}
 & -\theta_2^2 - \frac{\delta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta}\theta_2 \Big\} \\
 & + \left\{ \theta_2\theta_3 [(\gamma + \delta)\theta_2 - \delta\theta_3] \times \left[\frac{\mu\gamma}{(\gamma + \delta)^2} + \frac{\delta}{\gamma + \delta} \frac{1}{\theta_3} - \frac{\delta}{\gamma + \delta} \frac{1}{\theta_2} \right] \right\}.
 \end{aligned}$$

From lemma 2.1 in Asmussen and Taksar [3], we have

$$\frac{\mu\gamma}{(\gamma + \delta)^2} + \frac{\delta}{\gamma + \delta} \frac{1}{\theta_3} - \frac{\delta}{\gamma + \delta} \frac{1}{\theta_2} < 0.$$

By $\frac{\theta_2}{\delta\beta} - \frac{1}{\mu\beta + \delta + 2c\theta_2} > 0$, $\frac{1}{\mu\beta + \delta + \gamma + 2c\theta_3} - \frac{\theta_3}{\beta(\gamma + \delta)} > 0$ and (50), we get the coefficient of $e^{2\theta_2 b}$ is positive, then we have $\lim_{b \rightarrow \infty} G(b) > 0$. Therefore, the equation $G(b) = 0$ admits a positive solution. \square

Appendix E. Proof of Lemma 4.5.

Proof. Assume that b is a positive solution to $G(x) = 0$. Note that

$$\begin{aligned}
 & -\hat{C}\theta_2 + 3kB_1 + kB_1\theta_2b \\
 & = \frac{-\theta_2 + (1 + b\theta_1)B_1\theta_2e^{\theta_1 b} + (1 + b\theta_2)kB_1\theta_2e^{\theta_2 b} + kB_1(3 + b\theta_2)(\theta_1e^{\theta_1 b} - \theta_2e^{\theta_2 b})}{\theta_1e^{\theta_1 b} - \theta_2e^{\theta_2 b}} \\
 & = \frac{q(b)}{e^{-\theta_1 b}(\theta_1e^{\theta_1 b} - \theta_2e^{\theta_2 b}) [(\delta\theta_3 - (\gamma + \delta)\theta_1)e^{\theta_1 b} + ((\gamma + \delta)\theta_2 - \delta\theta_3)e^{\theta_2 b}]}, \tag{65}
 \end{aligned}$$

where

$$\begin{aligned}
 q(b) := & \left[-\delta\theta_2\theta_3 + (\gamma + \delta)\theta_1\theta_2 + \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \right. \\
 & \times \left. \left[(1 + b\theta_1)\theta_2 + 3k\theta_1 + k\theta_1\theta_2b \right] \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) \right] \\
 & + \left[-(\gamma + \delta)\theta_2^2 + \delta\theta_2\theta_3 + \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) \times (-2k\theta_2) \right] e^{(\theta_2 - \theta_1)b}.
 \end{aligned}$$

Then

$$\begin{aligned}
 q'(b) = & \theta_2(\theta_2 - \theta_1) \left[-(\gamma + \delta)\theta_2 + \delta\theta_3 - \frac{2k\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) \right] e^{(\theta_2 - \theta_1)b} \\
 & + (1 + k)\theta_1\theta_2 \times \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right)
 \end{aligned}$$

and

$$q(0) = (\theta_1 - \theta_2) \frac{\gamma + \delta}{\delta\beta} \left[\frac{\theta_2}{\delta\beta} - \frac{2k\theta_2 - 3k\theta_1 - \theta_2}{\delta\beta(\gamma + \delta)(\theta_1 - \theta_2)} \times \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) \right].$$

If $\eta [k(2\theta_2 - 3\theta_1) - \theta_2] \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) > \frac{\theta_1 - \theta_2}{\beta - \theta_1} \times \delta\beta(\gamma + \delta)$, then it follows from $\frac{\theta_2}{\delta\beta} - \frac{1}{\mu\beta + \delta + 2c\theta_2} > 0$ that

$$q(0) \leq (\theta_1 - \theta_2) \frac{\gamma + \delta}{\delta\beta} \left[\frac{\theta_2}{\delta\beta} - \frac{1}{2c\theta_2 + \mu\beta + \delta} \right] < 0,$$

and we have

$$\begin{aligned}
 q'(0) = & \delta\beta(\theta_1 - \theta_2) [(\gamma + \delta)\theta_2 - \delta\theta_3] \left\{ \frac{\theta_2}{\delta\beta} - \frac{2k\theta_2^2 - 3k\theta_1\theta_2 - \theta_1\theta_2}{\delta\beta(\theta_1 - \theta_2)[(\gamma + \delta)\theta_2 - \delta\theta_3]} \right. \\
 & \times \left. \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta} \right) \right\}.
 \end{aligned}$$

If $\eta [k(2\theta_2^2 - 3\theta_1\theta_2) - \theta_1\theta_2] \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma+\delta}\right) > \frac{\theta_1-\theta_2}{\beta-\theta_1} \times \delta\beta[(\gamma+\delta)\theta_2 - \delta\theta_3]$, by $\frac{\theta_2}{\delta\beta} - \frac{1}{\mu\beta+\delta+2c\theta_2} > 0$, then

$$q'(0) \leq \delta\beta(\theta_1 - \theta_2)[(\gamma + \delta)\theta_2 - \delta\theta_3] \left\{ \frac{\theta_2}{\delta\beta} - \frac{1}{2c\theta_2 + \mu\beta + \delta} \right\} < 0.$$

Thus, it follows from $q'(0) \leq 0$ that

$$\begin{aligned} & \theta_2(\theta_2 - \theta_1) \left\{ -(\gamma + \delta)\theta_2 + \delta\theta_3 + (-2k) \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta}\right) \right\} \\ & \leq -(1 + k)\theta_1\theta_2 \times \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta}\right) < 0. \end{aligned}$$

Therefore,

$$\begin{aligned} q''(b) = & \theta_2(\theta_2 - \theta_1)^2 \left\{ -(\gamma + \delta)\theta_2 + \delta\theta_3 - 2k \frac{\eta(\theta_1 - \beta)}{2c\theta_1 + \mu\beta + \delta} \times \left(-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta}\right) \right\} \\ & \times e^{(\theta_2 - \theta_1)b} < 0. \end{aligned}$$

Then we have $q'(b) < q'(0) < 0$ and $q(b) < 0$. Finally, by (65), we obtain

$$-\hat{C}\theta_2 + 3kB_1 + kB_1\theta_2b > 0.$$

□

Appendix F. Proof of Lemma 4.6.

Proof. Similar to the discussion of Lemma 4.2, we only need to verify that the equation (5) holds, i.e., $V_4'(x) \geq 1$ when $0 \leq x < b$, while $V_4'(x) < 1$ when $x \geq b$.

(i) Letting $b = 0$ in (36) and (43), we obtain (55) and (56). By $\frac{1}{\mu\beta+\delta+\gamma+2c\theta_3} - \frac{\theta_3}{\beta(\gamma+\delta)} > 0$, we know

$$\begin{aligned} V_4'(x) &= \left[-\frac{\mu\gamma(\eta + \epsilon)}{(\gamma + \delta)^3} \theta_3 - \frac{\mu\gamma}{(\gamma + \delta)^2} \times \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} (1 + \theta_3x) \right] e^{\theta_3x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} \\ &> \left[-\frac{\mu\gamma(\eta + \epsilon)}{(\gamma + \delta)^3} \theta_3 - \frac{\mu\gamma}{(\gamma + \delta)^2} \times \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \right] e^{\theta_3x} + \frac{\gamma\epsilon}{(\gamma + \delta)^2} \\ &= \frac{\mu\gamma}{(\gamma + \delta)^2} \left[-\frac{\eta\theta_3}{\gamma + \delta} - \frac{\epsilon\theta_3}{\gamma + \delta} - \frac{\eta\theta_3}{2c\theta_3 + \mu\beta + \gamma + \delta} + \frac{\eta\beta}{2c\theta_3 + \mu\beta + \gamma + \delta} \right] e^{\theta_3x} \\ &\quad + \frac{\gamma\epsilon}{(\gamma + \delta)^2}. \end{aligned}$$

Furthermore, we get

$$\begin{aligned} & -\frac{\epsilon\theta_3}{\gamma + \delta} - \frac{\eta\theta_3}{2c\theta_3 + \mu\beta + \gamma + \delta} \\ &= -\theta_3 - \frac{\eta\theta_3}{\gamma + \delta} - \frac{\eta\theta_3}{2c\theta_3 + \mu\beta + \gamma + \delta} \\ &= -\theta_3 - \eta\theta_3 \left[\frac{1}{\gamma + \delta} + \frac{1}{-\sqrt{(\mu\beta + \gamma + \delta)^2 + 4c\beta(\gamma + \delta)}} \right] > 0. \end{aligned}$$

Then we have $V_4'(x) > 0$.

$$V_4''(x) = -\frac{\mu\gamma(\eta + \epsilon)}{(\gamma + \delta)^3} \theta_3^2 e^{\theta_3x} - (2\theta_3 e^{\theta_3x} + \theta_3^2 x e^{\theta_3x}) \left[\frac{\mu\gamma}{(\gamma + \delta)^2} \times \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \right]$$

$$\begin{aligned}
 &= \left(-\frac{\mu\gamma(\eta + \epsilon)}{(\gamma + \delta)^3} \theta_3 - 2\frac{\mu\gamma}{(\gamma + \delta)^2} \times \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \right) \theta_3 e^{\theta_3 x} \\
 &\quad - \frac{\mu\gamma}{(\gamma + \delta)^2} \times \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \theta_3^2 x e^{\theta_3 x} \\
 &< 0.
 \end{aligned}$$

Therefore $\frac{\partial u^t}{\partial x}(t, x) = V_4'(x) \leq 1$ for $x \geq 0$.

(ii) We only need to verify that $V_4''(x) < 0$, for $x \geq b$, recalling (47),

$$\begin{aligned}
 V_4''(x) &= \theta_3(D_3\theta_3 + B_3\theta_3 x + 2B_3)e^{\theta_3 x} \\
 &\leq \theta_3[\theta_3(D_3 + B_3b) + 2B_3]e^{\theta_3 x} \\
 &= \theta_3 \left[\frac{\delta^2 + \gamma\delta - \gamma\eta}{(\gamma + \delta)^2} e^{-\theta_3 b} + B_3 \right] e^{\theta_3 x} \\
 &< \theta_3 \left[\frac{\delta^2 + \gamma\delta - \gamma\eta}{(\gamma + \delta)^2} e^{-\theta_3 b} + \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \right. \\
 &\quad \left. \times \frac{-\gamma + \frac{\mu\gamma\theta_3}{\gamma + \delta}}{\theta_3 e^{\theta_3 b}} \times \frac{\theta_1 e^{\theta_1 b} - \theta_2 e^{\theta_2 b}}{\bar{Z}} \right] e^{\theta_3 x} \\
 &< \theta_3 \left[\frac{\delta^2 + \gamma\delta - \gamma\eta}{(\gamma + \delta)^2} + \frac{\eta(\theta_3 - \beta)}{2c\theta_3 + \mu\beta + \gamma + \delta} \times \frac{\gamma - \frac{\mu\gamma\theta_3}{\gamma + \delta}}{\theta_3} \right] e^{\theta_3(x-b)} \\
 &< \theta_3 \left[\frac{\delta^2 + \gamma\delta - \gamma\eta}{(\gamma + \delta)^2} + \frac{\eta(\theta_3 - \beta)}{\beta(\gamma + \delta)} \theta_3 \times \frac{\gamma - \frac{\mu\gamma\theta_3}{\gamma + \delta}}{\theta_3} \right] e^{\theta_3(x-b)} < 0.
 \end{aligned}$$

The second last inequality follows from $\frac{1}{\mu\beta + \delta + \gamma + 2c\theta_3} - \frac{\theta_3}{\beta(\gamma + \delta)} > 0$. Now we deal with the case of $0 \leq x < b$,

$$\begin{aligned}
 V_4'''(x) &= \theta_1^2 \left(\hat{C}\theta_1 + B_1\theta_1 x + 3B_1 \right) e^{\theta_1 x} + \theta_2^2 \left(-\hat{C}\theta_2 + kB_1\theta_2 x + 3kB_1 \right) e^{\theta_2 x} \\
 &> \theta_1^2 \left(\hat{C}\theta_1 + 3B_1 \right) e^{\theta_1 x} + \theta_2^2 \left(-\hat{C}\theta_2 + 3kB_1 + kB_1\theta_2 b \right) e^{\theta_2 x}.
 \end{aligned}$$

From Lemma 4.5, then $V_4'''(x) > 0$ for $0 \leq x < b$. Then $V_4''(x)$ is an increasing function and we have $V_4''(x) < V_4''(b-) = V_4''(b+) < 0$. Therefore, $V_4(x)$ is a concave function on $[0, \infty)$. \square

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