Singular Values of Products of Ginibre Random Matrices

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The squared singular values of the product of M complex Ginibre matrices form a biorthogonal ensemble, and thus their distribution is fully determined by a correlation kernel. The kernel permits a hard edge scaling to a form specified in terms of certain Meijer G-functions, or equivalently hypergeometric functions ${}_0F_M$, also referred to as hyper-Bessel functions. In the case $M = 1$ it is well known that the corresponding gap probability for no squared singular values in $(0, s)$ can be evaluated in terms of a solution of a particular sigma form of the Painlevé III' system. One approach to this result is a formalism due to Tracy and Widom, involving the reduction of a certain integrable system. Strahov has generalised this formalism to general $M \geq 1$, but has not exhibited its reduction. After detailing the necessary working in the case $M = 1$, we consider the problem of reducing the 12 coupled differential equations in the case $M = 2$ to a single differential equation for the resolvent. An explicit 4-th order nonlinear is found for general hard edge parameters. For a particular choice of parameters, evidence is given that this simplifies to a much simpler third order nonlinear equation. The small and large s asymptotics of the 4-th order equation are discussed, as is a possible relationship of the $M = 2$ systems to so-called 4-dimensional Painlevé-type equations.

1 Introduction

1.1 Fredholm determinant

Let $X(1), \ldots, X(M)$, $M \geq 1$ be a sequence of rectangular matrices $X(m) \in \mathbb{C}^{N_m \times N_{m-1}}$ with $1 \leq m \leq M$. We define the parameters $\nu_m = N_m - N_0$, $m = 0, 1, \ldots, M$ and will assume that $\nu_m \geq 0$. Each of the $X(m)$ are drawn from the Ginibre ensemble where their elements are i.i.d standard complex Gaussian random variables $X(m)_{j,k} \in N[0,1] + iN[0,1]$ and each $X(m)$ is independent of the others. We form the matrix product $Y_M = X(M) \ldots X(1) \in \mathbb{C}^{N_M \times N_0}$ and the associated positive definite form $Y_M^\dagger Y_M \in \mathbb{C}^{N_0 \times N_0}$. Our primary interest is in integrable structures, in particular differential equations, characterising the smallest eigenvalue of $Y_M^\dagger Y_M$ in the so-called hard edge limit. In the case $M = 1$ it is well known that the integrable structures relate to the Painlevé III equation (1), (2). Underlying the integrable structures is the explicit form of the joint distribution of all the eigenvalues, given by Akemann, Ipsen and Kieburg in 2013 for arbitrary N_m.

Theorem 1.1. *(3) The squared singular values of Y_M, $Spec(Y_M^\dagger Y_M) = (x_1, \ldots, x_{N_0})$, form a determinantal point process on $\mathbb{R}_{>0}$. This determinantal point process is a bi-orthogonal ensemble with a joint probability density function (jPDF)*

$$P^{(M)}(x_1, \ldots, x_{N_0}) = \frac{1}{Z_{N_0}} \prod_{1 \leq j < k \leq N_0} (x_k - x_j) \det\left(w_{k-1}^{(M)}(x_j) \right)_{1 \leq j,k \leq N_0},$$

where $x_k \in \mathbb{R}_{>0}$, $k = 1, \ldots, N_0$, Z_{N_0} is the normalisation constant, and the functions $w_k^{(M)}$ are

$$w_k^{(M)}(x) = G^{M,0}_{0,M}(x|\nu_M, \nu_{M-1}, \ldots, \nu_2, \nu_1 + k),$$

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in terms of Meijer's G-function.

We denote the correlation kernel of the determinantal point process defined by the jPDF above by $K_{N_0}^{(M)}(x,y)$, meaning that the n-point correlation function

$$\rho_{(n)}(x_1,\ldots,x_n) = N_0(N_0-1)\cdots(N_0-n+1) \times \int_0^\infty P^{(M)}(x_1,\ldots,x_n,x_{n+1},\ldots,x_{N_0})\,dx_{n+1}\cdots dx_{N_0}, \quad (1.1)$$

is given by

$$\rho_{(n)}(x_1,\ldots,x_n) = \det[K_{N_0}^{(M)}(x_j,x_k)]_{j,k=1,\ldots,n}. \quad (1.2)$$

It turns out that for large N_0 the eigenvalues near the origin, referred to as the hard edge since the spectral density is strictly zero for $x<0$, are spaced on distances of order $1/N_0$. Scaling the eigenvalues by this factor and taking $N_0\to\infty$ whilst keeping the ν_m fixed defines the hard edge limit. The explicit form of the correlation kernel in this limit was calculated by Kuijlaars and Zhang in 2014.

Theorem 1.2. *(4) Let $K_{N_0}^{(M)}(x,y)$ be the correlation kernel of the above determinantal point process. Its hard edge scaled limit is given by*

$$\lim_{N_0\to\infty}\frac{1}{N_0}K_{N_0}^{(M)}\left(\frac{x}{N_0},\frac{y}{N_0}\right) = K_M(x,y),$$

where

$$K_M(x,y) = \frac{\mathcal{B}\left(G^{1,0}_{0,M+1}(x|-\nu_0,-\nu_1,\ldots,-\nu_M),G^{M,0}_{0,M+1}(y|\nu_1,\ldots,\nu_M,\nu_0)\right)}{x-y}. \quad (1.3)$$

Here $\mathcal{B}(\cdot,\cdot)$ is a bilinear operator defined by

$$\mathcal{B}(f(x),g(y)) := (-1)^{M+1}\sum_{j=0}^{M}(-1)^j\left(x\frac{d}{dx}\right)^j f(x)\sum_{i=0}^{M-j}\alpha_{i+j}\left(y\frac{d}{dy}\right)^i g(y).$$

The constants α_i are determined from

$$\prod_{m=0}^{M}(x-\nu_m) = x\sum_{i=0}^{M}\alpha_i x^i.$$

The kernel functions f, g are defined in terms of the Meijer G-functions by

$$f(x) = G^{1,0}_{0,M+1}(x| - \nu_0, -\nu_1, \ldots, -\nu_M), g(y) = G^{M,0}_{0,M+1}(y|\nu_1, \ldots, \nu_M, \nu_0).$$

The Meijer G-functions are specified in terms of certain Mellin-Barnes integrals. More important to us is the fact that they satisfy certain linear differential equations of degree $M+1$ (5).

Proposition 1.1. *The functions f and g satisfy the linear differential equations*

$$\prod_{j=0}^{M}\left(x\frac{d}{dx}+\nu_j\right)f(x) = -xf(x), \quad \prod_{j=0}^{M}\left(y\frac{d}{dy}-\nu_j\right)g(y) = (-1)^M yg(y).$$

Also useful in the ensuing theory are the sequence of related functions which we define for $0 \le j \le M$

$$\phi_j(x) = (-1)^{M-j+1}\left(x\frac{d}{dx}\right)^j f(x), \quad \psi_j(y) = \sum_{i=0}^{M-j}\alpha_{i+j}\left(y\frac{d}{dy}\right)^i g(y). \quad (1.4)$$

Thus the above kernel (1.4) can be written as a generalised "integrable" kernel, see e.g. (4),

$$K_M(x,y) = \frac{\sum_{j=0}^{M}\phi_j(x)\psi_j(y)}{x-y}, \qquad \sum_{j=0}^{M}\phi_j(x)\psi_j(x) = 0.$$

We remark that the latter orthogonality relation is far from obvious, given the definitions made.

We now come to the central objects of our study, the hard edge gap probabilities. Let $0 \le a_1 < a_2 < \cdots < a_{2L-1} < a_{2L} < \infty$ be the endpoints of a collection of L intervals of $\mathbb{R}_+$, and denote their union $J = \cup_{l=1}^{L}(a_{2l-1}, a_{2l})$.

The probability that there are no eigenvalues in J is referred to as the gap probability and denoted $E_M(0; J)$. A standard result for a determinant point process tells us that (see e.g. (6, §9.1))

$$E_M(0; J) = \det(\mathbb{1} - \mathbb{K}_M), \tag{1.5}$$

where $\mathbb{K}_M$ is an integral operator acting on $L^2((0, \infty))$ with kernel $K_M(x, y)\chi_J(y)$, where K_M is given by (1.4) and $\chi_J(y)$ is the characteristic function of the interval J.

1.2 Strahov's extension of Tracy-Widom theory

In distinction to the case $M = 1$, the kernel (1.4) for $M \geq 2$ is not symmetric. Thus, in addition to the integral kernel

$$\mathbb{K}_M \doteq K_M(x, y)\chi_J(y), \tag{1.6}$$

(here the symbol $\doteq$ denotes ”with kernel“), one requires the additional integral operators

$$\mathbb{K}'_M \doteq K_M(y, x)\chi_J(y), \qquad \mathbb{K}^T_M \doteq K_M(y, x)\chi_J(x).$$

From these integral operators we define the primary variables $0 \leq m \leq M$ and $1 \leq l \leq L$

$$\begin{aligned} x_m^{(2l)} &:= \sqrt{-1}(1 - \mathbb{K}_M)^{-1}\phi_m(a_{2l}), \quad & y_m^{(2l)} &:= \sqrt{-1}(1 - \mathbb{K}'_M)^{-1}\psi_m(a_{2l}), \\ x_m^{(2l-1)} &:= (1 - \mathbb{K}_M)^{-1}\phi_m(a_{2l-1}), \quad & y_m^{(2l-1)} &:= (1 - \mathbb{K}'_M)^{-1}\psi_m(a_{2l-1}). \end{aligned}$$

This essentially means a doubling of the number of primary variables over that occurring in the Tracy and Widom theory. Furthermore we require the auxiliary variables, which are constructed as inner products of the primary variables

$$
\begin{aligned}
\xi_m &:= (-1)^M \sum_{l=1}^{L} \int_{a_{2l-1}}^{a_{2l}} dx \, \phi_0(x) \, (1-\mathbb{K}'_M)^{-1} \, \psi_m(x) \\
&\qquad + (-1)^{M+1-m} e_{M+1-m}(\nu_0, \ldots, \nu_M), \\
\eta_m &:= (-1)^M \sum_{l=1}^{L} \int_{a_{2l-1}}^{a_{2l}} dx \, \phi_m(x) \, (1-\mathbb{K}'_M)^{-1} \, \psi_M(x).
\end{aligned}
$$

Here $e_k(\{x\})$ denotes the k-th elementary symmetric polynomial in the variables $\{x\}$. It follows that the gap probability for $M \geq 1$, and say for a single interval $J = (0, s)$, i.e. $L = 1$, is determined by a certain product of the primary variables[1]

$$
\begin{aligned}
\det(1-\mathbb{K}_M) &= \exp\left\{ (-1)^{M+1} \int_0^s dt \log(\frac{s}{t}) x_0(t) y_M(t) \right\} \\
&= \exp\left\{ \int_0^s dt \, t^{-1} \eta_0(t) \right\}.
\end{aligned}
\tag{1.7}
$$

Strahov also observed that the foregoing system is a Hamiltonian system with $2L$ Hamiltonians H_j and $(2L+1)(M+1)$ canonical conjugate pairs of co-ordinates $x_m^{(k)}, y_m^{(k)}$ and ξ_m, η_m. The Hamiltonian equations of motion are then

$$
a_j \frac{\partial}{\partial a_j} x_m^{(k)} = \frac{\partial H_j}{\partial y_m^{(k)}}, \quad a_j \frac{\partial}{\partial a_j} y_m^{(k)} = -\frac{\partial H_j}{\partial x_m^{(k)}},
$$

and

$$
\frac{\partial}{\partial a_j} \xi_m = \frac{\partial H_j}{\partial \eta_m}, \quad \frac{\partial}{\partial a_j} \eta_m = -\frac{\partial H_j}{\partial \xi_m},
$$

for $1 \leq j, k \leq 2L$ and $0 \leq m \leq M$. The Hamiltonians are given explicitly by

$$
\begin{aligned}
H_j = -x_0^{(j)} \left(\sum_{m=0}^{M} \eta_m y_m^{(j)} \right) + \left(\sum_{m=0}^{M} \xi_m x_m^{(j)} \right) y_M^{(j)} + (-1)^{M+1} a_j x_0^{(j)} y_M^{(j)} \\
- \sum_{m=0}^{M-1} x_{m+1}^{(j)} y_m^{(j)} + \sum_{k=1, k \neq j}^{2L} \frac{a_k}{a_j - a_k} \sum_{m', m=0}^{M} x_m^{(j)} x_{m'}^{(k)} y_m^{(k)} y_{m'}^{(j)}.
\end{aligned}
\tag{1.8}
$$

[1]This differs from the formulae of Prop. 3.9 and §4.5 of (7) in the sign of the integral. This is due to the omission of $\sqrt{-1}$ factors in the relations following the first paragraph at the beginning of §4.3, when substituted into Eq. (4.42) of that work.

Also noted in (7) is the fact that this Hamiltonian system is an isomonodromic system with a natural representation as $(M+1)\times(M+1)$ matrices. One makes the following definitions,

$$E=(-1)^{M+1}\begin{pmatrix}0&0&\dots&0\\0&0&\dots&0\\\vdots&&&0\\1&0&\dots&0\end{pmatrix},$$

$$C=\begin{pmatrix}-\eta_0&-1&0&\dots&0\\-\eta_1&0&-1&\dots&0\\\vdots&&&&\vdots\\-\eta_{M-1}&0&0&\dots&-1\\-\eta_M+\xi_0&\xi_1&\xi_2&\dots&\xi_M\end{pmatrix},$$

and constructs the residue matrices thus

$$A^{(l)}=\begin{pmatrix}x_0^{(l)}\\x_1^{(l)}\\\vdots\\x_M^{(l)}\end{pmatrix}\otimes\left(y_0^{(l)},\quad y_1^{(l)},\quad\dots,\quad y_M^{(l)}\right).$$

Then the first member of the Lax pair for $\Psi(z;a_1,\dots,a_{2L})$ is

$$\frac{\partial\Psi}{\partial z}=\left\{E+\frac{C-\sum_{j=1}^{2L}A^{(j)}}{z}+\sum_{j=1}^{2L}\frac{A^{(j)}}{z-a_j}\right\}\Psi, \tag{1.9}$$

and the second members are for $1\le j\le 2L$

$$\frac{\partial\Psi}{\partial a_j}=-\frac{A^{(j)}}{z-a_j}\Psi. \tag{1.10}$$

The compatibility relations of (1.9) and (1.10) now leads to Schlesinger equations, which are precisely the same as those derived from the Hamilton equations of motion.

1.3 Plan of the paper

In Section 2 we detail the analysis required to reduce the Hamiltonian system in the case $M = 1$ down to a single nonlinear equation characterising the Hamiltonian and thus the gap probability in the case $L = 1$. This characterisation is a known result (1), but its derivation via the formalism of §1.2 involves some subtleties, the appreciation of which is essential to progress to the new territories of $M \geq 2$. The case $M = 2$ is addressed in Section 3. The corresponding Hamiltonian system consists of 12 coupled equations. Calling on the experience gained from Section 2, and with the essential aid of computer algebra, a reduction is found of the 12 coupled equations down to a single nonlinear equation determining the gap probability. This equation is of fourth order, and is given in Proposition 3.4. Both the small and large s asymptotics of this equation can be determined, and from the latter the corresponding large spacing asymptotic form of the gap probability is deduced; see Corollary 3.1. In the special case $\nu_1 = -1/2$, $\nu_2 = 0$ evidence is found that the fourth order equation of Proposition 3.4 can be reduced to a specific third order equation, (3.81) below. We conclude by discussing a possible relationship of the $M = 2$ systems to the recently introduced theory of so-called 4-dimensional Painlevé equations.

2 $M = 1$ Tracy-Widom Theory at the Hard Edge

The original Tracy and Widom theory must be equivalent to the $M = 1$ and $L = 1$ case of the preceding theory, although this is not immediate. Therefore it is instructive to consider this case first, primarily because it will provide essential guidance for the $M \geq 2$ cases. This will also serve to clarify some misunderstanding present in the existing literature relating to this point.

From Prop. 3.9 of (7) for $J = (0, s)$, $a_1 = 0, a_2 = s$, i.e $x_j = x_j^{(2)}$, $y_j = y_j^{(2)}$ and $M = 1$, we read off the following system of coupled quasi-linear ODEs

($' = d/ds$) with respect to s

$$sx_0' = -\eta_0 x_0 - x_1, \tag{2.1}$$
$$sx_1' = -\eta_1 x_0 + sx_0 + \xi_0 x_0 + \xi_1 x_1, \tag{2.2}$$
$$sy_1' = -\xi_1 y_1 + y_0, \tag{2.3}$$
$$sy_0' = -\xi_0 y_1 - sy_1 + \eta_0 y_0 + \eta_1 y_1, \tag{2.4}$$
$$\xi_0' = x_0 y_0, \tag{2.5}$$
$$\xi_1' = x_0 y_1, \tag{2.6}$$
$$\eta_0' = x_0 y_1, \tag{2.7}$$
$$\eta_1' = x_1 y_1. \tag{2.8}$$

In this case the Hamiltonian (1.8) simplifies to

$$H = -\eta_0 x_0 y_0 + (\xi_0 - \eta_1 + s) x_0 y_1 - x_1 y_0 + \xi_1 x_1 y_1,$$

and the Hamiltonian equations of motion

$$sx_j' = \frac{\partial}{\partial y_j} H, \quad sy_j' = -\frac{\partial}{\partial x_j} H, \quad j = 0, 1$$

$$\eta_j' = \frac{\partial}{\partial \xi_j} H, \quad \xi_j' = -\frac{\partial}{\partial \eta_j} H, \quad j = 0, 1$$

furnish the system (2.1)-(2.8) above. Note that (1.7) with $M = 1$ gives

$$\det(\mathbb{1} - \mathbb{K}_1) = \exp\Big(\int_0^s dt \, \frac{\eta_0(t)}{t} \Big). \tag{2.9}$$

In the matrix formulation of the isomonodromic problem we recall the definitions

$$E := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} -\eta_0 & -1 \\ \xi_0 - \eta_1 & \xi_1 \end{pmatrix},$$

and

$$A := A^{(2)} = \begin{pmatrix} x_0 y_0 & x_0 y_1 \\ x_1 y_0 & x_1 y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 & y_1 \end{pmatrix},$$

where $A^{(2)}$ is a rank 1 matrix so $\det A^{(2)} = 0$. The Schlesinger equations are now

$$sA^{(2)\prime} = \left[C + sE, A^{(2)}\right], \qquad C' = \left[E, A^{(2)}\right]. \tag{2.10}$$

Proposition 2.1. *The isomonodromic system $\Psi(x,s)$ has a singularity pattern $\frac{3}{2}$+1+1 where the Riemann-Papperitz symbol is*

$$\left\{\begin{matrix} 0 & 1 & \multicolumn{2}{c}{\infty(\frac{1}{2})} \\ -\nu_0 & 0 & i\sqrt{s} & -\frac{1}{2} \\ -\nu_1 & 0 & -i\sqrt{s} & \nu_0+\nu_1 \end{matrix}\right\}. \tag{2.11}$$

We have the resonant or ramified case, see (8), (9) and (10).

Proof. The isomonodromic system (2.11) differs from the one in (1.9) and (1.10) through the transformation of the independent variable $z \mapsto sz$ and $\Psi(sz,s) \mapsto \Psi(z,s)$, which become

$$\frac{\partial \Psi}{\partial z} = \left\{ sE + \frac{C - A^{(1)}}{z} + \frac{A^{(1)}}{z-1} \right\} \Psi, \tag{2.12}$$

$$\frac{\partial \Psi}{\partial s} = \left\{ s^{-1}Ez + s^{-1}C \right\} \Psi. \tag{2.13}$$

The effect of this is to place the regular singularities at the canonical positions 0 and 1. The resonant or ramified case arises because E is nilpotent with eigenvalues $0,0$; the eigenvalues of $C - A^{(1)}$ are $-\nu_0, -\nu_1$ whilst those of $A^{(1)}$ are $0,0$. Let us denote the matrix in braces on the right-hand side of (2.12) by A. The Jordan decomposition of sE is

$$sE = \begin{pmatrix} 0 & s^{-1} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & s^{-1} \\ 1 & 0 \end{pmatrix}^{-1}.$$

so we transform the system (2.12) to $B = \begin{pmatrix} 0 & s^{-1} \\ 1 & 0 \end{pmatrix}^{-1} \cdot A \cdot \begin{pmatrix} 0 & s^{-1} \\ 1 & 0 \end{pmatrix}$. We next apply the shearing transformation $S := \mathrm{diag}(1, z^{-g})$ with an arbitrary exponent g and form a new coefficient matrix $C = S^{-1} \cdot B \cdot S - S^{-1} \cdot S'$.

The leading exponent matrix of C is $\begin{pmatrix} 1 & g \\ -g+1 & 1 \end{pmatrix}$ so we achieve off-diagonal balance if we choose $g = 1/2$. Under this choice the leading coefficient of C (of order $z^{-1/2}$) is now diagonalisable if $s \neq 0$

$$\begin{pmatrix} 0 & 1 \\ -s & 0 \end{pmatrix} = \begin{pmatrix} -is^{-1/2} & is^{-1/2} \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} is^{1/2} & 0 \\ 0 & -is^{1/2} \end{pmatrix} \cdot \begin{pmatrix} -is^{-1/2} & is^{-1/2} \\ 1 & 1 \end{pmatrix}^{-1} .$$

So we apply this transformation and define another coefficient matrix $D = \begin{pmatrix} -is^{-1/2} & is^{-1/2} \\ 1 & 1 \end{pmatrix}^{-1} \cdot C \cdot \begin{pmatrix} -is^{-1/2} & is^{-1/2} \\ 1 & 1 \end{pmatrix}$. Associated with the fractional exponent for g we define a new spectral variable $z = \frac{1}{4}w^2$, and perform a large w expansion

$$\frac{1}{2}wD = \begin{pmatrix} is^{1/2} & 0 \\ 0 & -is^{1/2} \end{pmatrix} + w^{-1} \begin{pmatrix} 1/2 - e_1 & 1/2 + e_1 - 2\eta_0 \\ 1/2 + e_1 - 2\eta_0 & 1/2 - e_1 \end{pmatrix} + \mathrm{O}(w^{-2}).$$

The sub-leading term appearing above can also be diagonalised

$$\begin{pmatrix} 1/2 - e_1 & 1/2 + e_1 - 2\eta_0 \\ 1/2 + e_1 - 2\eta_0 & 1/2 - e_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 - 2\eta_0 & 0 \\ 0 & -2e_1 + 2\eta_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} ,$$

and these diagonal elements give us the last column of the Riemann-Papperitz symbol. $\square$

The ramified cases of the isomonodromic systems are quite important because they arise very naturally from random matrix theory applications, and

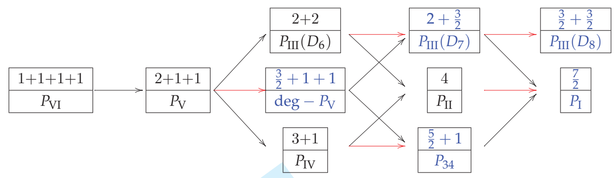

Figure 1: The degeneration scheme of the Painlevé equations interpreted through their isomonodromic deformation problems. The unramified and ramified cases are given in black and blue entries respectively, and the singularity confluence transitions are given by black arrows, while the drop in the Poincaré index transitions (in this case always 1/2) are given by red arrows. The deg $-$ P_{V} system is equivalent to the $P_{\mathrm{III}}(D_6)$ system, whilst P_{34} is equivalent to P_{II}.

the re-interpretation of the degeneration scheme of the Painlevé equations via isomonodromy deformations was completed relatively recently by Kapaev & Hubert 1999 (8), Kapaev 2002 (9) and Ohyama and Okumura 2006 (10). In this expanded scheme, see Fig. 1, there are 5 integer (unramified) types and 5 half-integer types, even though there are only 6 independent transcendents.

In the case $M = 1$ we read off from (1.4), together with knowledge of special cases of the Meijer G-function (see e.g. (11)), that

$$f(x) = G^{1,0}_{0,2}(x|-\nu_0,-\nu_1) = x^{-\frac{1}{2}(\nu_0+\nu_1)} J_{\nu_1-\nu_0}(2\sqrt{x}),$$
$$g(x) = G^{1,0}_{0,2}(x|\nu_1,\nu_0) = x^{\frac{1}{2}(\nu_0+\nu_1)} J_{\nu_1-\nu_0}(2\sqrt{x}).$$

Recalling (1.4), and making use of difference-differential identities for the Bessel functions we thus have

$$\begin{aligned}
\phi_0(x) &= x^{-\frac{1}{2}(\nu_0+\nu_1)} J_{\nu_1-\nu_0}(2\sqrt{x}),\\
\phi_1(x) &= \nu_0 x^{-\frac{1}{2}(\nu_0+\nu_1)} J_{\nu_1-\nu_0}(2\sqrt{x}) + x^{-\frac{1}{2}(\nu_0+\nu_1-1)} J_{\nu_1-\nu_0+1}(2\sqrt{x}),\\
\psi_0(x) &= -\nu_0 x^{\frac{1}{2}(\nu_0+\nu_1)} J_{\nu_1-\nu_0}(2\sqrt{x}) - x^{\frac{1}{2}(\nu_0+\nu_1+1)} J_{\nu_1-\nu_0+1}(2\sqrt{x}),\\
\psi_1(x) &= x^{\frac{1}{2}(\nu_0+\nu_1)} J_{\nu_1-\nu_0}(2\sqrt{x}).
\end{aligned}$$

Remark 2.1. It is obvious from the above that the linear orthogonality relation

$$\psi_0(x)\phi_0(x) + \psi_1(x)\phi_1(x) = 0,$$

holds and in fact one observes the splitting or folding relations in this case $\psi_1(x) = x^{\nu_0+\nu_1}\phi_0(x)$ and $\phi_1(x) = -x^{-\nu_0-\nu_1}\psi_0(x)$.

Using the Bessel function integral identities

$$\begin{aligned}\int_0^x duu^2 J_\nu(u)J_{\nu+1}(u)\\ &= \frac{1}{2}\nu x^2 J_\nu^2(x) - \nu(\nu+1)xJ_\nu(x)J_{\nu+1}(x) + \frac{1}{2}(\nu+1)x^2 J_{\nu+1}^2(x),\end{aligned}$$

$$\int_0^x duuJ_\nu^2(u) = x^2 J_\nu^2(x) - 2\nu xJ_\nu(x)J_{\nu+1}(x) + x^2 J_{\nu+1}^2(x),$$

and the Neumann expansions

$$\begin{aligned}x_j = i\phi_j(s) + i\int_0^s dzK_1(s,z)\phi_j(z)\\ + i\int_0^s dz\int_0^s dz' K_1(s,z)K_1(z,z')\phi_j(z') + \ldots,\end{aligned}$$

$$\begin{aligned}y_j = i\psi_j(s) + i\int_0^s dzK_1(z,s)\psi_j(z)\\ + i\int_0^s dz\int_0^s dz' K_1(z',z)K_1(z,s)\psi_j(z') + \ldots,\end{aligned}$$

we can deduce the behaviour of the variables in the neighbourhood of $s = 0$, which furnishes the initial conditions for the integrals of motion to be deduced below,

$$x_0(s) \underset{s\to 0}{\sim} i\phi_0(s) \sim i\frac{s^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)}, \tag{2.14}$$

$$x_1(s) \underset{s\to 0}{\sim} i\phi_1(s) \sim i\frac{\nu_0 s^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)} + i\frac{(1-\nu_0)s^{-\nu_0+1}}{\Gamma(\nu_1-\nu_0+2)}, \tag{2.15}$$

$$y_0(s) \underset{s\to 0}{\sim} i\psi_0(s) \sim -i\frac{\nu_0 s^{\nu_1}}{\Gamma(\nu_1-\nu_0+1)} + i\frac{(\nu_0-1)s^{\nu_1+1}}{\Gamma(\nu_1-\nu_0+2)}, \tag{2.16}$$

$$y_1(s) \underset{s\to 0}{\sim} i\psi_1(s) \sim i\frac{s^{\nu_1}}{\Gamma(\nu_1-\nu_0+1)}, \tag{2.17}$$

$$\eta_0(s) \underset{s\to 0}{\sim} -\frac{s^{\nu_1-\nu_0+1}}{\Gamma(\nu_1-\nu_0+2)\Gamma(\nu_1-\nu_0+1)}, \tag{2.18}$$

$$\eta_1(s) \underset{s\to 0}{\sim} -\frac{\nu_0 s^{\nu_1-\nu_0+1}}{\Gamma(\nu_1-\nu_0+2)\Gamma(\nu_1-\nu_0+1)} \tag{2.19}$$

$$+\frac{(1-2\nu_0)s^{\nu_1-\nu_0+2}}{\Gamma(\nu_1-\nu_0+3)\Gamma(\nu_1-\nu_0+1)}, \tag{2.20}$$

$$\xi_0(s) \underset{s\to 0}{\sim} \nu_0\nu_1 + \frac{\nu_0(\nu_1-\nu_0+1)s^{\nu_1-\nu_0+1}}{\Gamma^2(\nu_1-\nu_0+2)} \tag{2.21}$$

$$+\frac{(1-2\nu_0)s^{\nu_1-\nu_0+2}}{\Gamma(\nu_1-\nu_0+3)\Gamma(\nu_1-\nu_0+1)}, \tag{2.22}$$

$$\xi_1(s) \underset{s\to 0}{\sim} -\nu_0-\nu_1-\frac{s^{\nu_1-\nu_0+1}}{\Gamma(\nu_1-\nu_0+2)\Gamma(\nu_1-\nu_0+1)}. \tag{2.23}$$

Consequently we note an analogue of the orthogonality relation given in Remark 2.1

$$\frac{y_1}{x_0} \underset{s\to 0}{\sim} s^{\nu_0+\nu_1}, \quad \frac{x_1}{y_0} \underset{s\to 0}{\sim} -s^{-\nu_0-\nu_1}.$$

As was the case in the Tracy and Widom theory we would like to reduce the order of the coupled ODE system and deduce the first integrals of the motion. For convenience we define the elementary symmetric functions $e_j, j=1,2$ of ν_0,ν_1.

Proposition 2.2. *The system possesses the integrals of motion*

$$\xi_1 = \eta_0 - e_1, \quad \mathrm{Tr}C = -e_1; \tag{2.24}$$

the orthogonality relation

$$\mathrm{Tr}A^{(2)} = x_0 y_0 + x_1 y_1 = 0; \tag{2.25}$$

the further integrals of motion

$$\eta_1 + \xi_0 = e_2; \tag{2.26}$$

$$s x_0 y_1 = -\eta_0 \xi_1 + \eta_0 + \xi_0 - \eta_1 - e_2; \tag{2.27}$$

and with η_0 identified as the Hamiltonian, the identity

$$\eta_0 x_0 y_0 + (\eta_1 - \xi_0 - s) x_0 y_1 + x_1 y_0 - \xi_1 x_1 y_1 + \eta_0 = 0. \tag{2.28}$$

Proof. Subtracting (2.7) from (2.6)

$$\xi_1' - \eta_0' = 0,$$

therefore

$$\xi_1 - \eta_0 = -e_1.$$

and (2.24) follows.

Adding $y_0 \times (2.1)$ to $x_0 \times (2.4)$ we find

$$s(x_0 y_0)' = (\eta_1 - \xi_0 - s) x_0 y_1 - x_1 y_0.$$

On the other hand adding $y_1 \times (2.2)$ to $x_1 \times (2.3)$ we deduce

$$s(x_1 y_1)' = (\xi_0 - \eta_1 + s) x_0 y_1 + x_1 y_0.$$

Thus we find their sum vanishes and for $s \neq 0$

$$(x_0 y_0)' + (x_1 y_1)' = 0,$$

and application of the initial values gives (2.25). An immediate consequence of this latter relation and (2.5) and (2.8) is

$$\xi_0' + \eta_1' = 0.$$

Applying the values at $s = 0$ we conclude that (2.26) is satisfied.

Adding $y_1\times$(2.1) and $x_0\times$(2.3) and then employing (2.5) to (2.8) we find

$$s(x_0y_1)' = -\xi_1\eta_0' - \eta_0\xi_1' + \xi_0' - \eta_1'.$$

Utilising (2.7) once again we have the total derivative

$$(sx_0y_1)' - \eta_0' = -\xi_1\eta_0' - \eta_0\xi_1' + \xi_0' - \eta_1'.$$

Employing (2.24) and (2.26) we have (2.27).

Forming $x_0'\times$(2.4) minus $y_0'\times$(2.1) and simplifying we arrive at

$$0 = (\eta_1 - \xi_0 - s)x_0'y_1 + \eta_0(x_0y_0)' + x_1y_0'.$$

Next forming $x_1'\times$(2.3) minus $y_1'\times$(2.2) and simplifying we have

$$0 = (\eta_1 - \xi_0 - s)x_0y_1' - \xi_1(x_1y_1)' + x_1'y_0.$$

Adding these two later relations we compute

$$\begin{aligned}
0 =&(\eta_1 - \xi_0 - s)(x_0y_1)' + \eta_0(x_0y_0)' - \xi_1(x_1y_1)' + (x_1y_0)' \\
=&[(\eta_1 - \xi_0 - s)x_0y_1]' - (\eta_1' - \xi_0' - 1)x_0y_1 + (\eta_0x_0y_0)' - \eta_0'x_0y_0 \\
&- (\xi_1x_1y_1)' + \xi_1'x_1y_1 + (x_1y_0)' \\
=&[(\eta_1 - \xi_0 - s)x_0y_1 + \eta_0x_0y_0 - \xi_1x_1y_1 + x_1y_0]' + x_0y_1 \\
&- (x_1y_1 - x_0y_0)x_0y_1 - x_0y_1x_0y_0 + x_0y_1x_1y_1 \\
=&[(\eta_1 - \xi_0 - s)x_0y_1 + \eta_0x_0y_0 - \xi_1x_1y_1 + x_1y_0]' + x_0y_1 \\
=&[(\eta_1 - \xi_0 - s)x_0y_1 + \eta_0x_0y_0 - \xi_1x_1y_1 + x_1y_0 + \eta_0]' .
\end{aligned}$$

Appealing to the initial conditions we deduce (2.28), and consequently $H = \eta_0$. □

Another feature of the Tracy and Widom theory is the appearance of the σ-forms for the resolvent function (for justification of this terminology, see (6, Section §9.3)) which is also easily deduced in the generalised theory.

Proposition 2.3. *The resolvent function $\eta_0(s)$ (recall (2.9)) satisfies a specialised σ-form equation for Painlevé III'*

$$s^2(\eta_0'')^2 - e_1^2(\eta_0')^2 + 4(\eta_0')^2\left(s\eta_0' - \eta_0 + s + e_2\right) - 4\eta_0\eta_0' = 0, \tag{2.29}$$

subject to the boundary conditions (2.18) at $s = 0$. The resolvent is given in terms of Okamoto's function $h(s; v_1, v_2)$ (see Prop. 4.1 of (2), or Eq. (0.7) of (12))

$$\eta_0(s) = h(s) - \frac{s}{2} - \frac{1}{4}(\nu_1 - \nu_0)^2,$$

for the special case $v_1 = v_2 = \pm(\nu_1 - \nu_0)$.

Proof. We follow the Okamoto prescription and recast the dynamical variables in terms of η_0 and its derivatives. From (2.7) we have $\eta_0'' = x_0'y_1 + x_0y_1'$. On the other hand we deduce from (2.25) and the formulae for y_0 and x_1 using (2.3) and (2.1) respectively that $sx_0y_1' - sx_0'y_1 = e_1\eta_0'$. Combining these two relations we have

$$sx_0y_1' = \frac{1}{2}\left(s\eta_0'' + e_1\eta_0'\right), \quad sx_0'y_1 = \frac{1}{2}\left(s\eta_0'' - e_1\eta_0'\right). \tag{2.30}$$

Now we use the same relations to eliminate y_0 and x_1 in the energy conservation relation (2.28) and we find

$$0 = \eta_0 + (\eta_0 - e_2 - s - sx_0y_1)x_0y_1 - s^2x_0'y_1',$$

where we have utilised (2.27) in the last step. Finally using the identity

$$x_0'y_1' = \frac{(sx_0'y_1)(sx_0y_1')}{s^2x_0y_1},$$

and substituting for the η_0 derivatives we arrive at (2.29). $\square$

Remark 2.2. The Hamiltonian variables can then be computed in terms of the resolvent and are given by

$$x_0^2 = s^{-\nu_0-\nu_1}\eta_0', \quad y_1^2 = s^{\nu_0+\nu_1}\eta_0',$$
$$x_1 = -x_0\left[\frac{1}{2}s\frac{\eta_0''}{\eta_0'} + \eta_0 - \frac{1}{2}(\nu_0+\nu_1)\right],$$
$$y_0 = y_1\left[\frac{1}{2}s\frac{\eta_0''}{\eta_0'} + \eta_0 - \frac{1}{2}(\nu_0+\nu_1)\right],$$
$$\xi_0 = \nu_0\nu_1 - \frac{1}{2}\left[-s\eta_0' + \eta_0(1+\nu_0+\nu_1-\eta_0)\right],$$
$$\xi_1 = \eta_0 - \nu_0 - \nu_1, \quad \eta_1 = \frac{1}{2}\left[-s\eta_0' + \eta_0(1+\nu_0+\nu_1-\eta_0)\right].$$

The relations for ξ_0 and η_1 follow from (2.26) and (2.27).

We now give relations between the two sets (x_0, y_0) and (x_1, y_1) which we call *folding relations* and the proof of these.

Proposition 2.4. *Assume $x_0 \neq 0$. Then x_1, y_1 are related to x_0, y_0 by*

$$x_1 = -s^{e_1}y_0, \quad y_1 = s^{e_1}x_0. \tag{2.31}$$

Proof. Let $y_1 = f(s)x_0$, so that $x_1 = -f^{-1}(s)y_0$ using (2.25). Substituting this into (2.3) we have

$$sx_0f' = f\left(-sx_0' - \xi_1x_0 + f^{-1}y_0\right).$$

Now employing (2.1) into the right-hand side of the above relation we deduce

$$sx_0\frac{f'}{f} = (\eta_0 - \xi_1)x_0 = e_1x_0.$$

Under our assumption $x_0 \neq 0$ we then have $sf' = e_1f$ which has the general solution $f = s^{e_1}$ given the initial condition $y_1/x_0 \to s^{e_1}$ as $s \to 0$. These relations also follow easily from the relations of Remark 2.2. □

Remark 2.3. The relations (2.31) are the non-linear analogues of the relations given in Remark 2.1 for the kernel functions. They also correct in an essential way assertions made in Remark (c) on pg. 9 of (7). Understanding these relations for $M = 1$ is key to that of the more general case $M > 1$.

Having reduced our system to the pair of canonical variables (x_0, y_0) we are at the stage of discussing co-incidence with the original theory of Tracy and Widom (1). For convenience we will set $\nu_0 = 0$ in this discussion.

Lemma 2.1. *Let $\nu_0 = 0$ and $\nu_1 = \nu$. Expressed solely in terms of x_0, y_0 the equations of motion are*

$$sx_0' = -\eta_0 x_0 + s^{-\nu} y_0, \tag{2.32}$$

$$sy_0' = -(2\xi_0 + s)s^{\nu} x_0 + \eta_0 y_0, \tag{2.33}$$

$$\eta_0' = s^{\nu} x_0^2, \tag{2.34}$$

$$\xi_0' = x_0 y_0. \tag{2.35}$$

Equation (2.27) *is now*

$$s^{\nu+1} x_0^2 = 2\xi_0 + \eta_0 - \eta_0(\eta_0 - \nu), \tag{2.36}$$

and (2.28) *is*

$$-s^{-\nu} y_0^2 - (2\xi_0 + s)s^{\nu} x_0^2 + (2\eta_0 - \nu)x_0 y_0 + \eta_0 = 0. \tag{2.37}$$

Proof. Using (2.31) both (2.1) and (2.3) reduce to (2.32), while (2.2) and (2.4) reduce to (2.33). □

Proposition 2.5. *The current system $\nu, s, x_0, y_0, \eta_0, \xi_0$ maps to that of Tracy and Widom (1) $\alpha, t, q(t), p(t), u(t), v(t)$ under the transformations $\nu = \alpha$, $s = \frac{1}{4}t$*

$$x_0(s) = is^{-\nu/2} q(t), \qquad y_0(s) = is^{\nu/2}\left(p(t) - \frac{\alpha}{2} q(t)\right),$$

$$\eta_0(s) = -\tfrac{1}{4} u(t), \qquad \xi_0(s) = \tfrac{1}{4}\left(-v(t) + \frac{\alpha}{2} u(t)\right).$$

Proof. We proceed by way of verification from our own results by direct calculation. Thus we find (2.36) becomes

$$tq^2 = \tfrac{1}{4}u^2 + u + 2v,$$

i.e. Eq. (2.19) of (1); (2.37) becomes

$$u = 4p^2 - (\alpha^2 - t + 2v)q^2 + 2qpu,$$

which is Eq. (2.20) of (1); (2.34) and (2.35) become

$$\frac{d}{dt}u = q^2, \quad \frac{d}{dt}v = qp,$$

which are Eqs. (2.24) and (2.25) of (1) respectively; and finally (2.32) and (2.33) become

$$t\frac{d}{dt}q = p + \tfrac{1}{4}qu, \qquad t\frac{d}{dt}p = (\tfrac{1}{4}\alpha^2 - \tfrac{1}{4}t + \tfrac{1}{2}v)q - \tfrac{1}{4}pu,$$

which match Eqs. (2.22) and (2.23) of (1) respectively. □

3 $M = 2$ Theory at the Hard Edge

3.1 Fredholm Theory

Here we treat the case $M = 2$ with a single interval $J = (0, s)$ and thus $L = 1$. As before we define the elementary symmetric functions $e_j, j = 1, 2, 3$ of ν_0, ν_1, ν_2. In this case application of Prop. 3.9 of (7) for $J = (0, s)$, $a_1 = 0, a_2 = s$, i.e $x_j = x_j^{(2)}$, $y_j = y_j^{(2)}$ yields the following system of coupled

ODEs

$$
\begin{aligned}
sx_0' &= -\eta_0 x_0 - x_1, && (3.1)\\
sx_1' &= -\eta_1 x_0 - x_2, && (3.2)\\
sx_2' &= -\eta_2 x_0 - s x_0 + \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2, && (3.3)\\
sy_2' &= -\xi_2 y_2 + y_1, && (3.4)\\
sy_1' &= -\xi_1 y_2 + y_0, && (3.5)\\
sy_0' &= -\xi_0 y_2 + s y_2 + \eta_0 y_0 + \eta_1 y_1 + \eta_2 y_2, && (3.6)\\
\xi_0' &= -x_0 y_0, && (3.7)\\
\xi_1' &= -x_0 y_1, && (3.8)\\
\xi_2' &= -x_0 y_2, && (3.9)\\
\eta_0' &= -x_0 y_2, && (3.10)\\
\eta_1' &= -x_1 y_2, && (3.11)\\
\eta_2' &= -x_2 y_2. && (3.12)
\end{aligned}
$$

The Hamiltonian (1.8) is now

$$
\begin{aligned}
H = -\eta_0 x_0 y_0 - \eta_1 x_0 y_1 + (\xi_0 - \eta_2 - s) x_0 y_2 \\
- x_1 y_0 - x_2 y_1 + \xi_1 x_1 y_2 + \xi_2 x_2 y_2, \quad (3.13)
\end{aligned}
$$

and as before the Hamiltonian equations of motion

$$
sx_j' = \frac{\partial}{\partial y_j} H, \quad sy_j' = -\frac{\partial}{\partial x_j} H, \quad j = 0, 1, 2,
$$

$$
\eta_j' = \frac{\partial}{\partial \xi_j} H, \quad \xi_j' = -\frac{\partial}{\partial \eta_j} H, \quad j = 0, 1, 2,
$$

give rise to the previous set of equations (3.1)-(3.12).

In the matrix formulation of the isomonodromy deformation problem we recall the definitions

$$
E := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} -\eta_0 & -1 & 0 \\ -\eta_1 & 0 & -1 \\ \xi_0 - \eta_2 & \xi_1 & \xi_2 \end{pmatrix},
$$

and

$$A := A^{(2)} = \begin{pmatrix} x_0y_0 & x_0y_1 & x_0y_2 \\ x_1y_0 & x_1y_1 & x_1y_2 \\ x_2y_0 & x_2y_1 & x_2y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_0 & y_1 & y_2 \end{pmatrix}.$$

Again $A^{(2)}$ is a rank 1 matrix so $\det A^{(2)} = 0$. The Schlesinger equations take the standard form

$$sA^{(2)\prime} = \left[C + sE, A^{(2)}\right], \tag{3.14}$$

$$C' = \left[E, A^{(2)}\right]. \tag{3.15}$$

Proposition 3.1. *For $M = 2$ the isomonodromic system* (1.9) *and* (1.10) *has the singularity pattern $\frac{4}{3}$+1+1 and its Riemann-Papperitz symbol is*

$$\left\{ \begin{array}{cccc} 0 & 1 & & \infty(\frac{1}{3}) \\ -\nu_0 & 0 & s^{1/3} & -\frac{2}{3} \\ -\nu_1 & 0 & \omega s^{1/3} & -1 \\ -\nu_2 & 0 & \omega^2 s^{1/3} & -\frac{1}{3} + \nu_0 + \nu_1 + \nu_2 \end{array} \right\}, \qquad \omega^3 = 1. \tag{3.16}$$

Proof. After mapping $z \mapsto sz$ and $\Psi(sz, s) \mapsto \Psi(z, s)$ the isomonodromic system become

$$\frac{\partial}{\partial z}\Psi = \left\{ sE + \frac{C - A^{(2)}}{z} + \frac{A^{(2)}}{z-1} \right\} \Psi, \tag{3.17}$$

and

$$\frac{\partial}{\partial s}\Psi = \left\{ s^{-1}Ez + s^{-1}C \right\} \Psi.$$

E is nilpotent with eigenvalues $0, 0, 0$ in Jordan blocks of size 2 & 1, i.e. the resonant or ramified case; the eigenvalues of $C - A^{(2)}$ are $-\nu_0, -\nu_1, -\nu_2$ whilst those of $A^{(2)}$ are $0, 0, 0$. Again let us denote the matrix in braces on the right-hand side of (3.17) by A. The Jordan decomposition of sE is

$$sE = \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1},$$

so we transform the system (3.17) to $B = \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \cdot A \cdot \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. We now apply the first shearing transformation $S := \mathrm{diag}(1, z^{-g}, z^{-2g})$ with an arbitrary exponent g and form the new coefficient matrix $C = S^{-1} \cdot B \cdot S - S^{-1} \cdot S'$. The leading exponent matrix of C is $\begin{pmatrix} 1 & g & 2g+1 \\ -g+2 & 1 & g+1 \\ -2g+1 & -g+1 & 1 \end{pmatrix}$. The smallest positive exponent that allows us to have off-diagonal balance occurs when $-2g+1 = g$, i.e. if we choose $g = 1/3$. Under this choice the leading coefficient of C (which appears at order $z^{-1/3}$) is now

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

A Jordan decomposition of this reveals

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}^{-1},$$

and we now have a 3×3 Jordan block. We apply this decomposition transformation and define another coefficient matrix $D = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \cdot C \cdot \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$. In addition we define a new spectral variable $z = aw^3$ because of the fractional exponent for g and a is a constant to be fixed later. Next we perform a large w expansion of D

$$3aw^2 D = 3a^{2/3} w \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - 3a^{1/3} s \eta_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{O}(w^{-1}).$$

We now apply a second shearing transformation $T := \mathrm{diag}(1, w^{-h}, w^{-2h})$ with another arbitrary exponent h and form the new coefficient matrix $G = T^{-1} \cdot D \cdot T - T^{-1} \cdot T'$. Now the leading exponent matrix of G is $\begin{pmatrix} 1 & h-1 & 2h \\ -h+3 & 1 & h-1 \\ -2h+2 & -h+3 & 1 \end{pmatrix}$.

The smallest positive exponent that allows us to have off-diagonal balance arises when $-2h+2 = h-1$, i.e. if we choose $h = 1$. The integer exponent signals the end of the recursive process of shearing transformations and a diagonalisable matrix. With this value of h we find an expansion for G as $w \to \infty$

$$G = \begin{pmatrix} 0 & 3a^{2/3} & 0 \\ 0 & 0 & 3a^{2/3} \\ -3a^{-1/3}s & 0 & 0 \end{pmatrix} + w^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+3\xi_2 & 0 \\ 0 & 0 & 3-3\eta_0 \end{pmatrix} + \mathrm{O}(w^{-2}).$$

The leading order matrix appearing above is now diagonalisable, and by choosing $a = -1/27$, we can simplify the decomposition in order that the final transformed coefficient matrix H has the expansion as $w \to \infty$

$$H = \begin{pmatrix} s^{1/3} & 0 & 0 \\ 0 & \omega s^{1/3} & 0 \\ 0 & 0 & \omega^2 s^{1/3} \end{pmatrix} + w^{-1} \begin{pmatrix} 2-e_1 & \Omega_2 & \Omega_1 \\ \Omega_1 & 2-e_1 & \Omega_2 \\ \Omega_2 & \Omega_1 & 2-e_1 \end{pmatrix} + \mathrm{O}(w^{-2}),$$

where ω is the third root of unity, $\Omega_1 := \omega\xi_2 - \eta_0 + (1-\omega)/3$ and $\Omega_2 := \omega^2\xi_2 - \eta_0 + (1-\omega^2)/3$. The sub-leading matrix appearing above can also be diagonalised as

$$\begin{pmatrix} \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3-3\eta_0 & 0 \\ 0 & 0 & 1-3e_1+3\eta_0 \end{pmatrix} \cdot \begin{pmatrix} \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \\ 1 & 1 & 1 \end{pmatrix}^{-1},$$

and these diagonal elements give us the last column of the Riemann-Papperitz symbol. □

Definition 3.1. We will define *generic conditions* to be $\nu_2 - \nu_1 \neq \mathbb{Z}$ whether or not ν_0 is zero. However from the perspective of one of the random matrix applications[2] this is precisely the case of interest. In this case we observe that the generic constraint can be lifted in principle with the proper treatment of logarithmic contributions to the initial conditions.

For $M = 2$, the kernel functions are given by

$$f(x) = G^{1,0}_{0,3}(x| -\nu_0, -\nu_1, -\nu_2), \quad g(x) = G^{2,0}_{0,3}(x|\nu_2, \nu_1, \nu_0),$$

however we will employ hyper-Bessel function representations, involving the generalised hypergeometric function ${}_0F_2$. The basic properties of hyper-Bessel functions such as integral representations and as solutions of linear ordinary differential equations have been studied in the works (13), (14), (15), (16) and (17), while their asymptotic expansions for large arguments were investigated in (18), (19) and (20). These functions have arisen in a diversity of applications such as the oblique reflection of long wireless waves from the ionosphere where the earth's magnetic field is regarded as vertical (21) or the reflection and diffraction of atomic deBroglie waves by a travelling evanescent laser wave (22). Using standard relations relating the Meijer G-function to the hyper-Bessel function, and differential-difference identities

[2]However the other random matrix application which we treat later in our work, the Muttalib-Borodin ensembles, does satisfy the generic condition.

of the latter, and recalling the definitions (1.4), we have

$$\phi_0(x) = \\ -\frac{x^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)\Gamma(\nu_2-\nu_0+1)}{}_0F_2\left(;\nu_1-\nu_0+1,\nu_2-\nu_0+1;-x\right),$$

$$\phi_1(x) = \\ -\frac{\nu_0 x^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)\Gamma(\nu_2-\nu_0+1)}{}_0F_2\left(;\nu_1-\nu_0+1,\nu_2-\nu_0+1;-x\right) \\ -\frac{x^{1-\nu_0}}{\Gamma(\nu_1-\nu_0+2)\Gamma(\nu_2-\nu_0+2)}{}_0F_2\left(;\nu_1-\nu_0+2,\nu_2-\nu_0+2;-x\right),$$

$$\phi_2(x) = \\ -\frac{\nu_0^2 x^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)\Gamma(\nu_2-\nu_0+1)}{}_0F_2\left(;\nu_1-\nu_0+1,\nu_2-\nu_0+1;-x\right) \\ +\frac{(1-2\nu_0)\,x^{1-\nu_0}}{\Gamma(\nu_1-\nu_0+2)\Gamma(\nu_2-\nu_0+2)}{}_0F_2\left(;\nu_1-\nu_0+2,\nu_2-\nu_0+2;-x\right) \\ -\frac{x^{2-\nu_0}}{\Gamma(\nu_1-\nu_0+3)\Gamma(\nu_2-\nu_0+3)}{}_0F_2\left(;\nu_1-\nu_0+3,\nu_2-\nu_0+3;-x\right),$$

$$\psi_0(x) = \frac{\Gamma(\nu_2-\nu_1)\,x^{\nu_1}}{\Gamma(\nu_1-\nu_0-1)}{}_0F_2\left(;\nu_1-\nu_0-1,\nu_1-\nu_2+1;x\right) \\ +\frac{\Gamma(\nu_1-\nu_2)\,x^{\nu_2}}{\Gamma(\nu_2-\nu_0-1)}{}_0F_2\left(;\nu_2-\nu_0-1,\nu_2-\nu_1+1;x\right) \\ +(\nu_0-\nu_1-\nu_2+1)\left[\frac{\Gamma(\nu_2-\nu_1)\,x^{\nu_1}}{\Gamma(\nu_1-\nu_0)}{}_0F_2\left(;\nu_1-\nu_0,\nu_1-\nu_2+1;x\right)\right. \\ \left.+\frac{\Gamma(\nu_1-\nu_2)\,x^{\nu_2}}{\Gamma(\nu_2-\nu_0)}{}_0F_2\left(;\nu_2-\nu_0,\nu_2-\nu_1+1;x\right)\right] \\ +\nu_1\nu_2\left[\frac{\Gamma(\nu_2-\nu_1)\,x^{\nu_1}}{\Gamma(\nu_1-\nu_0+1)}{}_0F_2\left(;\nu_1-\nu_0+1,\nu_1-\nu_2+1;x\right)\right. \\ \left.+\frac{\Gamma(\nu_1-\nu_2)\,x^{\nu_2}}{\Gamma(\nu_2-\nu_0+1)}{}_0F_2\left(;\nu_2-\nu_0+1,\nu_2-\nu_1+1;x\right)\right],$$

$$\psi_1(x) = \frac{\Gamma(\nu_2-\nu_1)\,x^{\nu_1}}{\Gamma(\nu_1-\nu_0)}\,{}_0F_2\left(;\nu_1-\nu_0,\nu_1-\nu_2+1;x\right) + \frac{\Gamma(\nu_1-\nu_2)\,x^{\nu_2}}{\Gamma(\nu_2-\nu_0)}\,{}_0F_2\left(;\nu_2-\nu_0,\nu_2-\nu_1+1;x\right) - (\nu_1+\nu_2)\left[\frac{\Gamma(\nu_2-\nu_1)\,x^{\nu_1}}{\Gamma(\nu_1-\nu_0+1)}\,{}_0F_2\left(;\nu_1-\nu_0+1,\nu_1-\nu_2+1;x\right) + \frac{\Gamma(\nu_1-\nu_2)\,x^{\nu_2}}{\Gamma(\nu_2-\nu_0+1)}\,{}_0F_2\left(;\nu_2-\nu_0+1,\nu_2-\nu_1+1;x\right)\right],$$

$$\psi_2(x) = \frac{\Gamma(\nu_2-\nu_1)\,x^{\nu_1}}{\Gamma(\nu_1-\nu_0+1)}\,{}_0F_2\left(;\nu_1-\nu_0+1,\nu_1-\nu_2+1;x\right) + \frac{\Gamma(\nu_1-\nu_2)\,x^{\nu_2}}{\Gamma(\nu_2-\nu_0+1)}\,{}_0F_2\left(;\nu_2-\nu_0+1,\nu_2-\nu_1+1;x\right).$$

Here the linear orthogonality relation

$$\psi_0(x)\phi_0(x)+\psi_1(x)\phi_1(x)+\psi_2(x)\phi_2(x)=0,$$

is not so obvious, and implies an bilinear identity involving hyper-Bessel functions with reflected arguments.

The initial value conditions for the Hamiltonian variables can be imposed through an expansion in the neighbourhood of $s=0$ with restricted argument. Thus we have

$$x_0(s) \sim -\frac{i s^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)\,\Gamma(\nu_2-\nu_0+1)}, \tag{3.18}$$

$$x_1(s) \sim -\frac{i\nu_0 s^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)\,\Gamma(\nu_2-\nu_0+1)} - \frac{i(1-\nu_0)\,s^{1-\nu_0}}{\Gamma(\nu_1-\nu_0+2)\,\Gamma(\nu_2-\nu_0+2)}, \tag{3.19}$$

$$x_2(s) \sim -\frac{i\nu_0^2 s^{-\nu_0}}{\Gamma(\nu_1-\nu_0+1)\,\Gamma(\nu_2-\nu_0+1)} + \frac{i(1-\nu_0)^2\,s^{1-\nu_0}}{\Gamma(\nu_1-\nu_0+2)\,\Gamma(\nu_2-\nu_0+2)}, \tag{3.20}$$

$$y_0(s) \sim \frac{i\nu_0\nu_2\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1}}{\Gamma\left(\nu_1-\nu_0+1\right)} + \frac{i\nu_0\nu_1\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2}}{\Gamma\left(\nu_2-\nu_0+1\right)} - \frac{i\left(\nu_0\nu_2-\nu_0+\nu_1-\nu_2+1\right)\Gamma\left(\nu_2-\nu_1-1\right)s^{\nu_1+1}}{\Gamma\left(\nu_1-\nu_0+2\right)} - \frac{i\left(\nu_0\nu_1-\nu_0+\nu_2-\nu_1+1\right)\Gamma\left(\nu_1-\nu_2-1\right)s^{\nu_2+1}}{\Gamma\left(\nu_2-\nu_0+2\right)}, \quad (3.21)$$

$$y_1(s) \sim -\frac{i\left(\nu_0+\nu_2\right)\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1}}{\Gamma\left(\nu_1-\nu_0+1\right)} - \frac{i\left(\nu_0+\nu_1\right)\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2}}{\Gamma\left(\nu_2-\nu_0+1\right)}, \quad (3.22)$$

$$y_2(s) \sim \frac{i\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1}}{\Gamma\left(\nu_1-\nu_0+1\right)} + \frac{i\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2}}{\Gamma\left(\nu_2-\nu_0+1\right)}, \quad (3.23)$$

$$\eta_0(s) \sim -\frac{\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+1\right)} - \frac{\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+1\right)}, \quad (3.24)$$

$$\eta_1(s) \sim -\frac{\nu_0\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+1\right)} - \frac{\nu_0\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+1\right)} + \frac{\left(-\nu_0^2-\nu_0\nu_1+2\nu_0\nu_2+\nu_1-\nu_2+1\right)\Gamma\left(\nu_2-\nu_1-1\right)s^{\nu_1-\nu_0+2}}{\Gamma\left(\nu_1-\nu_0+3\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)} + \frac{\left(-\nu_0^2-\nu_0\nu_2+2\nu_0\nu_1+\nu_2-\nu_1+1\right)\Gamma\left(\nu_1-\nu_2-1\right)s^{\nu_2-\nu_0+2}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+3\right)\Gamma\left(\nu_2-\nu_0+1\right)},$$

$$
\begin{aligned}
\eta_2(s) \sim &-\frac{\nu_0^2\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+1\right)} \\
&-\frac{\nu_0^2\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+1\right)} \\
&-\big(\nu_0^3-2\nu_0-(2\nu_0^2-2\nu_0+1)\nu_2+(1-\nu_0)^2\nu_1+1\big) \\
&\times\frac{\Gamma\left(\nu_2-\nu_1-1\right)s^{\nu_1-\nu_0+2}}{\Gamma\left(\nu_1-\nu_0+3\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)} \\
&-\big(\nu_0^3-2\nu_0-(2\nu_0^2-2\nu_0+1)\nu_1+(1-\nu_0)^2\nu_2+1\big) \\
&\times\frac{\Gamma\left(\nu_1-\nu_2-1\right)s^{\nu_2-\nu_0+2}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+3\right)\Gamma\left(\nu_2-\nu_0+1\right)},
\end{aligned}
$$

$$
\begin{aligned}
\xi_0(s) \sim &-e_3-\frac{\nu_0\nu_2\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+1\right)} \\
&-\frac{\nu_0\nu_1\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+1\right)} \\
&-\big(\nu_2\nu_0^2-\nu_0^2-2\nu_2^2\nu_0+\nu_1\nu_2\nu_0+\nu_1\nu_0+2\nu_0+\nu_2^2-\nu_1\nu_2-\nu_1-1\big) \\
&\times\frac{\Gamma\left(\nu_2-\nu_1-1\right)s^{\nu_1-\nu_0+2}}{\Gamma\left(\nu_1-\nu_0+3\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)} \\
&-\big(\nu_1\nu_0^2-\nu_0^2-2\nu_1^2\nu_0+\nu_1\nu_2\nu_0+\nu_2\nu_0+2\nu_0+\nu_1^2-\nu_1\nu_2-\nu_2-1\big) \\
&\times\frac{\Gamma\left(\nu_1-\nu_2-1\right)s^{\nu_2-\nu_0+2}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+3\right)\Gamma\left(\nu_2-\nu_0+1\right)},
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
\xi_1(s) \sim &\; e_2+\frac{(\nu_0+\nu_2)\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+1\right)} \\
&+\frac{(\nu_0+\nu_1)\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+1\right)},
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
\xi_2(s) \sim &-e_1-\frac{\Gamma\left(\nu_2-\nu_1\right)s^{\nu_1-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+2\right)\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+1\right)} \\
&-\frac{\Gamma\left(\nu_1-\nu_2\right)s^{\nu_2-\nu_0+1}}{\Gamma\left(\nu_1-\nu_0+1\right)\Gamma\left(\nu_2-\nu_0+2\right)\Gamma\left(\nu_2-\nu_0+1\right)}.
\end{aligned}
\tag{3.27}
$$

Some warning ought to be attached to the above results. Only those terms where the s-exponent has the least (in sign) real part should be admitted as the true lowest order term, depending on the relative sizes of the parameters. Whilst the remaining terms still do contribute there will be additional, higher order terms arising from the leading one, and which will appear at the same order. However such higher order terms haven't been worked out in the above expressions.

3.2 First Integrals

Again the system (3.1)-(3.12) can be reduced in order. This requires some preliminary results.

Lemma 3.1. *Eliminating the variables x_1, x_2, y_0, y_1 successively in favour of x_0, y_2 we have the relations*

$$x_1 = -\eta_0 x_0 - s x_0', \tag{3.28}$$

$$x_2 = -\eta_1 x_0 - s x_0^2 y_2 + (1+\eta_0) s x_0' + s^2 x_0'', \tag{3.29}$$

$$y_1 = \xi_2 y_2 + s y_2', \tag{3.30}$$

$$y_0 = \xi_1 y_2 - s x_0 y_2^2 + (1+\xi_2) s y_2' + s^2 y_2''. \tag{3.31}$$

Consequently we have

$$\xi_1' = -s x_0 y_2' + (\eta_0 - e_1)\eta_0', \tag{3.32}$$

$$\eta_1' = s x_0' y_2 - \eta_0 \eta_0', \tag{3.33}$$

$$\xi_0' = -s^2 x_0 y_2'' - (1 - e_1 + \eta_0) s x_0 y_2' + \eta_0' \xi_1 + s(\eta_0')^2, \tag{3.34}$$

$$\eta_2' = -s^2 x_0'' y_2 - (1+\eta_0) s x_0' y_2 - \eta_0' \eta_1 + s(\eta_0')^2. \tag{3.35}$$

In addition we note

$$\begin{aligned} s^3 x_0''' + (e_1+3) s^2 x_0'' + (e_1 + e_2 + 1 - \eta_0 - 5 s x_0 y_2) s x_0' - s^2 x_0^2 y_2' \\ - (e_1+1) s x_0^2 y_2 - (\xi_0 - \eta_2 - \eta_0 \xi_1 - \xi_2 \eta_1 - s) x_0 = 0, \end{aligned} \tag{3.36}$$

$$s^3y_2''' + (-e_1+3)s^2y_2'' + (-e_1+e_2+1-\eta_0-5sx_0y_2)sy_2' - s^2y_2^2x_0' \\ -(-e_1+1)sx_0y_2^2 + (\xi_0-\eta_2-\eta_0\xi_1-\xi_2\eta_1-s)y_2 = 0. \quad (3.37)$$

Now we have made sufficient preparation for the task of deducing the integrals of the motion.

Proposition 3.2. *Let us assume the generic condition $\nu_2-\nu_1 \neq \mathbb{Z}$ holds. For integral of motions we have the Hamiltonian giving the energy conservation*

$$\eta_0x_0y_0 + \eta_1x_0y_1 + (-\xi_0+\eta_2+s)x_0y_2 - \xi_1x_1y_2 - \xi_2x_2y_2 \\ + x_1y_0 + x_2y_1 + \eta_0 = 0; \quad (3.38)$$

the relations

$$\xi_2 = \eta_0 - e_1, \quad \mathrm{Tr}C = -e_1, \quad (3.39)$$

$$sx_0y_2 = \eta_0\xi_2 + \eta_1 - \xi_1 + e_2 - \eta_0, \quad (3.40)$$

and the orthogonality relation

$$\mathrm{Tr}A^{(2)} = x_0y_0 + x_1y_1 + x_2y_2 = 0. \quad (3.41)$$

In addition the latter relation can be integrated once again to give

$$-3e_3 + e_2(e_1+\eta_0-1) - \eta_0(e_1-\eta_0+1)(e_1+\eta_0-2) \\ - sx_0y_1 + sx_0y_2\,(-2\eta_0+\xi_2+2) + sx_1y_2 \\ + (2e_1-1)\eta_1 + (1-e_1)\xi_1 - 3\,(\eta_2+\xi_0) = 0, \quad (3.42)$$

and furthermore can be split into the two independent integrals

$$3e_3 + e_2(-2e_1+\eta_0-4) + \eta_0(e_1-\eta_0+1)(2e_1-\eta_0+2) \\ -(e_1+1)\eta_1 + (2e_1-3\eta_0+4)\xi_1 \\ + 2sx_0y_1 + sx_0y_2(2e_1-\eta_0+2) + sx_1y_2 + 3\xi_0 = 0, \quad (3.43)$$

and

$$e_2(e_1+\eta_0-1) - \eta_0(e_1-\eta_0+1)(e_1+\eta_0-2) \\ + (-e_1+3\eta_0-4)\eta_1 + (1-e_1)\,\xi_1 \\ - sx_0y_1 - sx_0y_2\,(e_1+\eta_0-2) - 2sx_1y_2 + 3\eta_2 = 0. \quad (3.44)$$

The sum of these later three is zero modulo (3.40) *and* (3.39). *The last integral of the motion is*

$$
\begin{aligned}
e_3 + \xi_0 - \eta_2 - \eta_0\xi_1 - \xi_2\eta_1 & \\
- \xi_1 x_0 y_0 + (\xi_0 - \eta_2 - \xi_2\eta_1)x_0 y_1 + \xi_1\eta_1 x_0 y_2 & \\
- \xi_2 x_1 y_0 + \eta_0\xi_2 x_1 y_1 + (\xi_0 - \eta_2 - \eta_0\xi_1)x_1 y_2 & \\
- x_2 y_0 + \eta_0 x_2 y_1 + \eta_1 x_2 y_2 &= 0. \qquad (3.45)
\end{aligned}
$$

Note that we have revealed the appearance of all of the three elementary symmetric functions of independent parameters ν_0, ν_1, ν_2.

Proof. Comparing (3.9) and (3.10) and noting the initial values for ξ_2 and η_0 as given by (3.24) and (3.27) along with the assumption $\min(\mathrm{Re}(\nu_1 - \nu_0), \mathrm{Re}(\nu_2 - \nu_0)) > -1$ we have (3.39). Considering (3.40) next we compute the derivative of sx_0y_2 using (3.1) and (3.4) and find

$$
\begin{aligned}
(sx_0y_2)' &= s(x_0y_2)' + x_0y_2 \\
&= -\eta_0 x_0 y_2 - \xi_2 x_0 y_2 - x_1 y_2 + x_0 y_1 + x_0 y_2 \\
&= \eta_0\xi_2' + \eta_0'\xi_2 + \eta_1' - \xi_1' - \eta_0' \\
&= (\eta_0\xi_2 + \eta_1 - \xi_1 - \eta_0)'.
\end{aligned}
$$

Assuming again $\min(\mathrm{Re}(\nu_1 - \nu_0), \mathrm{Re}(\nu_2 - \nu_0)) > -1$ we can fix the integration constant and deduce (3.40).

Computing s times the derivative of $x_0y_0 + x_1y_1 + x_2y_2$ using (3.1)-(3.6) we find this vanishes and if $s \neq 0$ then this quantity is a constant. Assuming $\nu_0 \neq 0$, $\min(\mathrm{Re}(\nu_1 - \nu_0), \mathrm{Re}(\nu_2 - \nu_0)) > 0$, or if $\nu_0 = 0$ then this lower bound can be dropped to -1, then the inner product vanishes as $s \to 0$ and thus the constant is in fact zero. Alternatively one can deduce $\mathrm{Tr}A^{(2)} = 0$ from (3.14).

Next we derive (3.42). We first rewrite (3.41) in the following way

$$
\begin{aligned}
0 &= x_0y_0 + x_2y_2 + \frac{(x_1y_2)(x_0y_1)}{x_0y_2} \\
&= -\xi_0' - \eta_2' - \frac{\eta_1'\xi_1'}{\eta_0'}. \qquad (3.46)
\end{aligned}
$$

Now we seek alternative forms for $\eta_1'\xi_1'$ and to this end we re-examine (3.41) from a different point of view. Using the formulae for x_1, x_2, y_0, y_1 given in (3.28),(3.29),(3.31),(3.30) we rewrite the orthogonality relation as

$$\begin{aligned} 0 = x_0 y_2 \left(e_2 - \eta_0 - 3 s x_0 y_2\right) + \left(e_1 + 1\right) s y_2 x_0' + \left(1 - e_1\right) s x_0 y_2' \\ - s^2 x_0' y_2' + s^2 y_2 x_0'' + s^2 x_0 y_2'', \end{aligned} \quad (3.47)$$

and using $\eta_0^{(3)} + 2x_0'y_2' + y_2 x_0'' + x_0 y_2'' = 0$ we can eliminate the last two terms of the above in favour of $x_0'y_2'$ which gives

$$\begin{aligned} 0 = x_0 y_2 \left(e_2 - \eta_0 - 3 s x_0 y_2\right) - s^2 \eta_0^{(3)} \\ + \left(e_1 + 1\right) s y_2 x_0' + \left(1 - e_1\right) s x_0 y_2' - 3 s^2 x_0' y_2'. \end{aligned} \quad (3.48)$$

Now using the identity

$$s^2 x_0' y_2' = \frac{(s y_2 x_0')(s x_0 y_2')}{x_0 y_2}, \quad (3.49)$$

in (3.48) and solutions of (3.32) and (3.33) for $s x_0 y_2'$ and $s y_2 x_0'$ respectively we arrive at an alternative form for the orthogonality relation

$$\begin{aligned} 0 = \left(e_1 - 3\eta_0 - 1\right) \eta_0' \xi_1' + \left(-2e_1 + 3\eta_0 + 1\right) \eta_0' \eta_1' - 3 \eta_1' \xi_1' \\ - \eta_0' \big(- e_1^2 \eta_0' + e_2 \eta_0' + e_1 (3\eta_0 + 1) \eta_0' + s^2 \eta_0^{(3)} + 3 s \eta_0'^2 - 3 \eta_0^2 \eta_0' - 3 \eta_0 \eta_0' \big). \end{aligned}$$

We solve this for $\eta_1'\xi_1'$ and substitute this into (3.46) yielding

$$\begin{aligned} 0 = e_1 \left(\eta_0' + 2\eta_1' - \xi_1'\right) - e_1^2 \eta_0' + e_2 \eta_0' + s^2 \eta_0^{(3)} \\ + 3 s \eta_0 \eta_0'' + 3 s \eta_0'^2 + 3 \eta_0^2 \eta_0' - 3 \eta_0 \eta_0' - \eta_1' + \xi_1' - 3 \eta_2' - 3 \xi_0'. \end{aligned}$$

This is a perfect derivative and when integrated after noting the $s \to 0$ limits of (3.25) and (3.26) (with the proviso $\min(\mathrm{Re}(\nu_1 - \nu_0), \mathrm{Re}(\nu_2 - \nu_0)) > -1$), we obtain

$$\begin{aligned} 0 = -3e_3 - (1 - e_1) e_2 + (-e_1^2 + e_1 + e_2 + 2)\eta_0 - 3\eta_0^2 + \eta_0^3 \\ + (3\eta_0 - 2) s \eta_0' + s^2 \eta_0'' + (2e_1 - 1)\eta_1 + (1 - e_1)\xi_1 - 3\eta_2 - 3\xi_0. \end{aligned} \quad (3.50)$$

Now (3.42) immediately follows by substituting for η_0' and η_0'' using (3.10), (3.1) and (3.4) to clear all the derivatives. However, as alluded to in the proposition, we can go further and split this relation.

We intend to integrate (3.34) in order to prove (3.43). The first thing we do is use the identity for $s^2x_0y_2''$

$$s^2x_0y_2'' = s\left(sx_0y_2'\right)' - sx_0y_2' - s^2x_0'y_2',$$

to replace $s^2x_0y_2''$ in (3.34). Next we replace the term $s^2x_0'y_2'$ using (3.48). This leaves us with terms linear in $sx_0'y_2$ and sx_0y_2' and we replace these last two factors by solving (3.33) and (3.32) respectively. The end result is

$$\begin{aligned}0 = 2e_1^2\eta_0' + e_1\eta_0' + e_2\eta_0' - 6e_1\eta_0\eta_0' + 3\eta_0^2\eta_0' - 3\eta_0\eta_0' + s^2\eta_0^{(3)} \\ - e_1\eta_1' - \eta_1' + 2e_1\xi_1' - 3\xi_1\eta_0' - 3\eta_0\xi_1' + \xi_1' + 3s(sx_0y_2')' + 3\xi_0'.\end{aligned}$$

This is a perfect derivative whose integral is determined as

$$\begin{aligned}0 = -2\left(e_1+2\right)e_2 + 3e_3 + \left(2e_1^2 + 4e_1 + e_2 + 2\right)\eta_0 - 3\left(e_1+1\right)\eta_0^2 + \eta_0^3 \\ - 2s\eta_0' + s^2\eta_0'' + \left(-e_1-1\right)\eta_1 + \left(2e_1 - 3\eta_0 + 4\right)\xi_1 + 3s^2x_0y_2' + 3\xi_0. \quad (3.51)\end{aligned}$$

Here the initial conditions (3.25) and (3.26) have been employed under the assumption $\min(\mathrm{Re}(\nu_1-\nu_0), \mathrm{Re}(\nu_2-\nu_0)) > -1$ and $\nu_0 \neq 0$. Clearing the derivatives of η_0 and subsequent derivatives from this expression gives (3.43). The method for proving (3.44) is similar and will entail integrating (3.35). Here we use the identity for $s^2x_0''y_2$

$$s^2x_0''y_2 = s\left(sx_0'y_2\right)' - sx_0'y_2 - s^2x_0'y_2',$$

to replace $s^2x_0''y_2$ in (3.35). Again we replace the term $s^2x_0'y_2'$ using (3.48). This also leaves us with terms linear in $sx_0'y_2$ and sx_0y_2' and we replace these last two factors by solving (3.33) and (3.32) respectively. Our result this time is

$$\begin{aligned}0 = -e_1^2\eta_0' + e_1\eta_0' + e_2\eta_0' + 3\eta_0^2\eta_0' - 3\eta_0\eta_0' + s^2\eta_0^{(3)} \\ + 3\eta_0\eta_1' + 3\eta_1\eta_0' - e_1\eta_1' - \eta_1' - e_1\xi_1' + \xi_1' + 3s(sx_0'y_2)' + 3\eta_2'.\end{aligned}$$

This is a perfect derivative whose integral is determined as

$$0 = -(1-e_1)\,e_2 + \left(e_1 + e_2 + 2 - e_1^2\right)\eta_0 - 3\eta_0^2 + \eta_0^3 - 2s\eta_0' + s^2\eta_0'' \\ + \eta_1\left(-e_1 + 3\eta_0 - 4\right) + (1-e_1)\,\xi_1 + 3s^2 x_0' y_2 + 3\eta_2. \quad (3.52)$$

Here the initial condition (3.26) has been employed under the previous assumptions. Clearing the derivatives of η_0 and subsequent derivatives from this expression gives (3.44). The last integral of the motion, (3.45), is $\det(C - A^{(2)}) + e_3$. One can verify directly it is a constant using the equations of motion (3.1)-(3.12). □

Proposition 3.3. *Alternatives to the identity* (3.40) *are the relations*

$$e_3 - sx_1y_2 + 2\eta_2 + \xi_0 - \eta_1 + \eta_1\xi_2 \\ = (\eta_0 - 2)\left[-e_2 + sx_0y_2 + \eta_0 - \eta_1 + \xi_1 - \eta_0\xi_2\right] = 0, \quad (3.53)$$

and

$$e_2 - 2e_3 - sx_0y_1 - \eta_2 - 2\xi_0 - \xi_1 + \eta_0\xi_1 \\ = (2 + e_1 - \eta_0)\left[-e_2 + sx_0y_2 + \eta_0 - \eta_1 + \xi_1 - \eta_0\xi_2\right] = 0. \quad (3.54)$$

Proof. The proof employed for sx_0y_2 can be easily adapted to sx_0y_1 and sx_1y_2. We observe

$$\begin{aligned}(sx_0y_1)' &= s(x_0y_1)' + x_0y_1 \\ &= -\eta_0x_0y_1 - x_1y_1 - \xi_1x_0y_2 + x_0y_0 + x_0y_1 \\ &= \eta_0\xi_1' - \xi_0' - \eta_2' + \xi_1\eta_0' - \xi_1' - \xi_0' \\ &= (\eta_0\xi_1 - \xi_1 - 2\xi_0 - \eta_2)',\end{aligned}$$

and

$$\begin{aligned}(sx_1y_2)' &= s(x_1y_2)' + x_1y_2 \\ &= -\eta_1x_0y_2 - x_2y_2 - \xi_2x_1y_2 + x_1y_1 + x_1y_2 \\ &= \eta_1\xi_2' + \eta_2' + \xi_2\eta_1' + \xi_0' + \eta_2' - \eta_1' \\ &= (\eta_1\xi_2 + 2\eta_2 + \xi_0 - \eta_1)'.\end{aligned}$$

These two relations are not independent of (3.40) as can be seen by the following argument. For sx_0y_2 we have

$$(e_1+1)\,\eta_0 - e_2 - s\eta_0' - \eta_0^2 - \eta_1 + \xi_1 = 0,$$

whilst for sx_1y_2

$$\begin{aligned}-\eta_0\,(2e_1+s\eta_0'+2)+(e_1+3)\,\eta_0^2-e_2\,(\eta_0-2)+2s\eta_0'&\\ -\eta_0^3+(\eta_0-2)\,\xi_1+(2-\eta_0)\,\eta_1&=0,\end{aligned}$$

and for sx_0y_1

$$\begin{aligned}&\eta_0\left(e_1^2+3e_1+s\eta_0'+2\right)-(e_1+2)\,s\eta_0'-(2e_1+3)\,\eta_0^2\\ &\quad +e_2\,(-e_1+\eta_0-2)+\eta_0^3+\xi_1\,(e_1-\eta_0+2)+\eta_1\,(-e_1+\eta_0-2)=0.\end{aligned}$$

The factorisation of these two relations gives (3.53) and (3.54). $\square$

Proposition 3.4. *Define the radical F by*

$$F^2 := 4e_1^2\eta_0'^2 - 12e_2\eta_0'^2 + 12\eta_0\eta_0'^2 - 36s\eta_0'^3 + 9s^2\eta_0''^2 - 12s\eta_0'(\eta_0''+s\eta_0^{(3)}). \quad (3.55)$$

The resolvent function $\eta_0(s)$ satisfies a scalar ordinary differential equation *with degrees* $2,3,4,8$ *in* $\eta_0^{(4)},\eta_0^{(3)},\eta_0^{(2)},\eta_0^{(1)}$

$$
\begin{aligned}
&27s^6\eta_0'^2\left(\eta_0^{(4)}\right)^2\\
&\quad+27s^4\Big[-F\eta_0''+3s^2\eta_0''^3+6s\eta_0'^3\eta_0''+2\eta_0'^2\left(\eta_0''+3s\eta_0^{(3)}\right)\\
&\quad-5s\eta_0'\eta_0''\left(\eta_0''+s\eta_0^{(3)}\right)+4\eta_0'^4\Big]\eta_0^{(4)}+81s^6\eta_0'\left(\eta_0^{(3)}\right)^3\\
&\quad+\left[-27e_1^2s^4\eta_0'^2+81e_2s^4\eta_0'^2+18Fs^4-54s^6\eta_0''^2-162s^5\eta_0'\eta_0''\right.\\
&\quad\left.+567s^5\eta_0'^3-81s^4\eta_0\eta_0'^2+243s^4\eta_0'^2\right]\left(\eta_0^{(3)}\right)^2\\
&\quad-3s^2\left[F\left(15s\eta_0''-2\eta_0'\left(e_1^2-3\left(e_2+3s\eta_0'-7\eta_0\right)\right)\right)\right.\\
&\quad+9s^2\eta_0'\eta_0''^2\left(-2e_1^2+6e_2+54s\eta_0'-6\eta_0+11\right)\\
&\quad+4\eta_0'\left(9s\left(e_1^2-3e_2+3\eta_0-3\right)\eta_0'^3\right.\\
&\quad\left.-\left(27\left(e_3+s\right)+2e_1^3-9e_2e_1\right)\eta_0'-108s^2\eta_0'^4+27\eta_0\right)\\
&\quad\left.-18s\eta_0'^2\eta_0''\left(-e_1^2+3e_2+25s\eta_0'-3\eta_0+3\right)-45s^3\eta_0''^3\right]\eta_0^{(3)}\\
&\quad+27s^4\left(-e_1^2+3e_2+27s\eta_0'-3\eta_0+1\right)\eta_0''^4\\
&\quad-54s^3\eta_0'\left(-e_1^2+3e_2+24s\eta_0'-3\eta_0+1\right)\eta_0''^3\\
&\quad-9s^2\left[F\left(-e_1^2+3e_2+18s\eta_0'+6\eta_0+1\right)\right.\\
&\quad-3s\left(4e_1^2-12e_2+12\eta_0+17\right)\eta_0'^3+3\left(e_1^2-3e_2+3\eta_0-1\right)\eta_0'^2\\
&\quad\left.+\left(27\left(e_3+s\right)+2e_1^3-9e_2e_1\right)\eta_0'+108s^2\eta_0'^4-27\eta_0\right]\eta_0''^2\\
&\quad+6s\eta_0'\left[F\left(e_1^2-3\left(e_2+6s\eta_0'-7\eta_0\right)\right)-18s\left(e_1^2-3e_2+3\eta_0-1\right)\eta_0'^3\right.\\
&\quad\left.+2\left(27\left(e_3+s\right)+2e_1^3-9e_2e_1\right)\eta_0'+270s^2\eta_0'^4-54\eta_0\right]\eta_0''\\
&\quad-4\eta_0'^2\left[F\left(e_1^2-3e_2-9s\eta_0'+3\eta_0\right)\left(e_1^2-3\left(e_2+3s\eta_0'-4\eta_0\right)\right)\right.\\
&\quad+27s^2\left(e_1^2-3e_2+3\eta_0-1\right)\eta_0'^4-9s\left(27\left(e_3+s\right)+2e_1^3-9e_2e_1\right)\eta_0'^2\\
&\quad+\left(3\left(27\left(e_3+4s\right)+2e_1^3-9e_2e_1\right)\eta_0\right.\\
&\quad\left.+\left(e_1^2-3e_2\right)\left(27(e_3+s)+2e_1^3-9e_2e_1\right)\right)\eta_0'\\
&\quad\left.-27\eta_0(e_1^2-3e_2+3\eta_0)-243s^3\eta_0'^5\right]=0. \qquad (3.56)
\end{aligned}
$$

Here F *is defined as the positive root of* (3.55).

Proof. For the sake of notational simplicity we define the abbreviations

$$U := sx_0y_2', \quad V := sx_0'y_2, \quad W := s^2x_0y_2'', \quad Z := s^2x_0''y_2.$$

Our derivation entails two steps. The first step is to express the auxiliary variables $\xi_1, \eta_1, \xi_0, \eta_2$ in terms of U, V, W, Z and for this we need four relations to solve. We take these four to be the relations (3.40), (3.43), (3.44) and (3.45). In each of these we replace the bi-linear products x_jy_k using, for example

$$x_0y_1 = -(\eta_0 - e_1)\eta_0' + U, \tag{3.57}$$

$$x_1y_2 = \eta_0\eta_0' - V, \tag{3.58}$$

$$x_0y_0 = -s{\eta_0'}^2 - \eta_0'\xi_1 + (1 + \eta_0 - e_1)U + W, \tag{3.59}$$

$$x_2y_2 = \eta_0'\eta_1 - s{\eta_0'}^2 + (1 + \eta_0)V + Z, \tag{3.60}$$

which follow by writing x_1, x_2, y_1, y_0 using (3.28),(3.29), (3.30) and (3.31) and then rewriting the derivatives of x_0, y_2 using the abbreviations. In this way we have four independent inhomogeneous relations which are linear in $\xi_1, \eta_1, \xi_0, \eta_2$, and have a unique solution assuming $\eta_0'(1 + e_1\eta_0' + U - V) \neq 0$.

In the second step our strategy is to seek an elimination scheme for U, V, W, Z and for this we require four equations involving these variables. Firstly we differentiate (3.10) and this gives us

$$U + V + s\eta_0'' = 0. \tag{3.61}$$

Next we employ (3.49) and (3.10) in (3.48) and deduce

$$\frac{3UV}{\eta_0'} + (1 - e_1)\,U + (e_1 + 1)\,V - \eta_0'\,(e_2 - \eta_0 + 3s\eta_0') - s^2\eta_0^{(3)} = 0. \tag{3.62}$$

These two relations allow us to solve for U, V leading to a quadratic equation and the appearance of the radical F. If we differentiate (3.10) once more and utilise (3.49) again we find

$$W + Z - \frac{2UV}{\eta_0'} + s^2\eta_0^{(3)} = 0. \tag{3.63}$$

We construct our last relation by adding x_0 times (3.37) to y_2 times (3.36) and utilise the third derivative of (3.10) to eliminate $x_0'''y_2 + x_0y_2'''$ from this result. We then use the identity

$$x_0''y_2' = \frac{(s^2x_0''y_2)(sx_0y_2')}{s^3x_0y_2},$$

and a similar one for $x_0'y_2''$ to conclude

$$\frac{3(UZ+VW)}{\eta_0'} + 3(W+Z) - e_1(W-Z) \\ + (1+e_2-\eta_0+6s\eta_0')(U+V) - e_1(U-V) - 2s\eta_0'^2 - s^3\eta_0^{(4)} = 0. \quad (3.64)$$

These four relations allow us to eliminate all reference to x_0, y_2 and their derivatives in favour of η_0 and its first four derivatives via the quantities U, V, W, Z. Substituting the solution for $\xi_1, \eta_1, \xi_0, \eta_2$ found in the first step, and then the solution for U, V, W, Z in the second step, into the energy conservation relation (3.38) now expressed as

$$UZ - VW + e_1UV - \eta_0'(s)\left[\eta_0 + s(U-V)\eta_0'(s)\right] \\ + \eta_0'^{\,2}\left[-e_1\eta_1 + \eta_0(\eta_1+\xi_1) + \eta_2 - \xi_0 + s\right] = 0, \quad (3.65)$$

we find that the final result is (3.56). □

Lemma 3.2. *The quantity F^2 is a perfect square and the radical F is*

$$F = -3x_0y_1 - 3x_1y_2 - e_1x_0y_2. \quad (3.66)$$

The sign is chosen here so that $F > 0$ for the appropriate solution to the boundary conditions (3.18)-(3.23).

Proof. We will prove this by way of verification. Let us use the abbreviations U, V as in the previous proof. Now

$$F = -3x_0y_1 - 3x_1y_2 - e_1x_0y_2 \\ = -3x_0(\xi_2y_2 + sy_2') - 3(-\eta_0x_0 - sx_0')y_2 - e_1x_0y_2.$$

This can be further rewritten

$$F = 3x_0y_2(-\xi_2+\eta_0)-e_1x_0y_2-3sx_0y_2'+3sx_0'y_2 = -2e_1\eta_0'+3(V-U). \quad (3.67)$$

Employing the identity $(V-U)^2 = (V+U)^2 - 4UV$ and the above relation for the difference, (3.61) for the sum and (3.62) for the product we readily compute that F satisfies the definition (3.55). □

In the Okamoto theory of the Painlevé equations expressing the Hamiltonian co-ordinates and momenta in terms of the Hamiltonian and its derivatives is an important task. For PIII', or the $M = 1$ systems, this was given in Remark 2.2, and the analogous result for the $M = 2$ system is given in the Appendix.

3.3 Behaviour of η_0 at $s \to 0$ and $s \to \infty$

Having derived the scalar ordinary differential equation (3.56), equivalent to the coupled first order system (3.1) to (3.12), we can employ this form to good advantage in the analysis of the solutions in the neighbourhood of the singular points $s = 0$ and $s = \infty$. We consider the singular point at $s = 0$ first, which in our theoretical construction occupies the special place by defining the precise solutions we seek as one can observe from (3.24). However we will undertake the task of this analysis in the generic situation and therefore encounter other classes which are not directly relevant to the original problem at hand.

Proposition 3.5. *Let us assume the generic condition $\nu_2 - \nu_1 \neq \mathbb{Z}$ holds. About the singular point $s = 0$ (3.56) possesses various solution types of the form*

$$\eta_0 = C_1 s^{\lambda_1} + C_2 s^{\lambda_1+\delta_1} + \mathrm{O}(s^{\lambda_1+\delta_1+\epsilon}), \quad (3.68)$$

where $\mathrm{Re}\,\lambda_1 > 0$, $\mathrm{Re}\,\delta_1 > 0$ and $\mathrm{Re}\,\epsilon > 0$. The first class have exponents *fixed by the parameters in the leading order with*

$$\lambda_1 = \begin{cases} \nu_0 - \nu_1 + 1 \\ -\nu_0 + \nu_1 + 1 \\ \nu_0 - \nu_2 + 1 \\ -\nu_0 + \nu_2 + 1 \\ \nu_1 - \nu_2 + 1 \\ -\nu_1 + \nu_2 + 1 \end{cases}, \tag{3.69}$$

where $C_1 \neq 0$ is arbitrary and include the case at hand of (3.24)*. In addition there is the case $\lambda_1 = 0$ and two further cases with exponents determined by the parameters*

$$\lambda_1 = 1 \pm \frac{2\sqrt{\nu_0^2 + \nu_1^2 + \nu_2^2 - (\nu_1 + \nu_2)\,\nu_0 - \nu_1\nu_2}}{\sqrt{3}}, \tag{3.70}$$

and

$$\lambda_1 = \frac{1}{6}\left(3 \pm \sqrt{3}\sqrt{4\nu_0^2 + 4\nu_1^2 + 4\nu_2^2 - 4\,(\nu_1 + \nu_2)\,\nu_0 - 4\nu_1\nu_2 - 1}\right), \tag{3.71}$$

where again $C_1 \neq 0$ is arbitrary. The last class have rational, i.e. fractional exponents, at the leading order

$$\eta_0 \sim \sqrt[3]{\pm\sqrt{x^2 + y} - x}\; s^{1/3}, \tag{3.72}$$

where C_1 is fixed by the parameters. Here

$$x = (\nu_0 + \nu_1 - 2\nu_2)\,(2\nu_0 - \nu_1 - \nu_2)\,(\nu_0 - 2\nu_1 + \nu_2)\,, \tag{3.73}$$

and

$$y = \frac{1}{27}\left(9\,(\nu_0 - \nu_1)^2 - 4\right)\left(9\,(\nu_0 - \nu_2)^2 - 4\right)\left(9\,(\nu_1 - \nu_2)^2 - 4\right). \tag{3.74}$$

Proof. First let us render the non-linear ODE (3.56) in a form which is a polynomial in all derivatives of η_0. This entails solving (3.56) for the radical

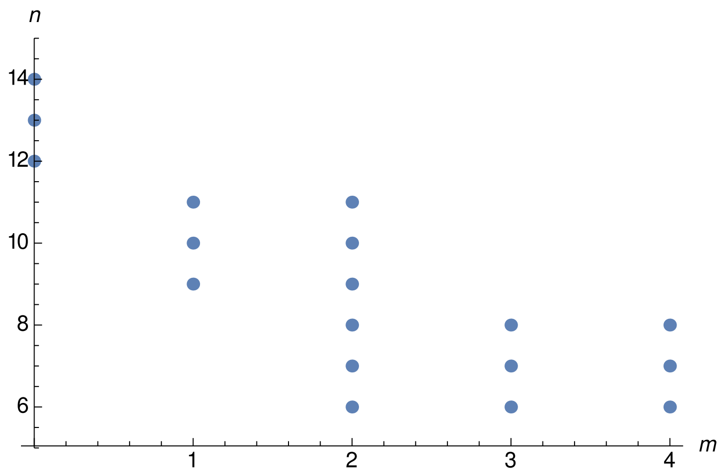

Figure 2: Newton polygon of the exponents (m, n) for the leading term $s^{m+n\lambda_1}$ of an algebraic expansion given by (3.68).

F, squaring the result and equating this to the right-hand side of (3.55). We do not display this because of its size and refer to it as $P\star$. We employ the algebraic expansion (3.68) and examine a region of the convex hull of the points in Fig. 2 on the lower left-hand boundary.

If one takes the lower corner point $6\lambda_1 + 2$ alone then there are 126 terms contributing, which sum to

$$\begin{aligned}27C_1^6\lambda_1^6(\lambda_1+\nu_0-\nu_1-1)(\lambda_1-\nu_0+\nu_1-1)(\lambda_1+\nu_0-\nu_2-1)\\ \times(\lambda_1+\nu_1-\nu_2-1)(\lambda_1-\nu_0+\nu_2-1)(\lambda_1-\nu_1+\nu_2-1)\\ \times(3\lambda_1^2-6\lambda_1-4\nu_0^2-4\nu_1^2-4\nu_2^2+4\nu_0\nu_1+4\nu_0\nu_2+4\nu_1\nu_2+3)^2 s^{6\lambda_1+2}.\end{aligned}$$

These require $C_1 \neq 0$ but otherwise arbitrary and are given in (3.69) and (3.70). Another solution derives from the single point condition at $12\lambda_1$ and the 14 terms give

$$11664C_1^{12}\lambda_1^{12}(3\lambda_1^2-3\lambda_1-\nu_0^2-\nu_1^2-\nu_2^2+\nu_0\nu_1+\nu_0\nu_2+\nu_1\nu_2+1)^2 s^{12\lambda_1}.$$

These solutions are given in (3.71). In addition if the condition at $9\lambda_1 + 1$

applies then we have 37 terms contributing to yield

$$\begin{aligned}216C_1^9\lambda_1^9\,(\nu_0+\nu_1-2\nu_2)\,(2\nu_0-\nu_1-\nu_2)\,(\nu_0-2\nu_1+\nu_2)\\
\times\left(3\lambda_1^2-6\lambda_1-4\nu_0^2-4\nu_1^2-4\nu_2^2+4\nu_0\nu_1+4\nu_0\nu_2+4\nu_1\nu_2+3\right)\\
\times\left(3\lambda_1^2-3\lambda_1-\nu_0^2-\nu_1^2-\nu_2^2+\nu_0\nu_1+\nu_0\nu_2+\nu_1\nu_2+1\right)s^{9\lambda_1+1},\end{aligned}$$

and the solutions (3.70) and (3.71) appear again.

However in addition to these there is another class of non-analytic solutions. If we demand the equality of the three points $12\lambda_1=9\lambda_1+1=6\lambda_1+2$ then we deduce $\lambda_1=\frac{1}{3}$. There are 177 terms contributing at these three points and their sum is

$$\begin{aligned}\frac{16}{177147}\left(3\nu_0^2-3\nu_1\nu_0-3\nu_2\nu_0+3\nu_1^2+3\nu_2^2-3\nu_1\nu_2-1\right)^2 s^4C_1^6\\
\times\left[27C_1^6+54C_1^3\,(2\nu_0-\nu_1-\nu_2)\,(\nu_0-2\nu_1+\nu_2)\,(\nu_0+\nu_1-2\nu_2)\right.\\
\left.-\left(9\,(\nu_0-\nu_1)^2-4\right)\left(9\,(\nu_0-\nu_2)^2-4\right)\left(9\,(\nu_1-\nu_2)^2-4\right)\right],\end{aligned}$$

and the non-trivial solution for C_1 is given by the equation $27C_1^6+54C_1^3x-27y$. These are the *fractional exponent solutions* in (3.72). □

Next we consider $s=\infty$ and examine the generic asymptotic solution developed about this point.

Proposition 3.6. *Let us assume the generic condition $\nu_2-\nu_1\neq\mathbb{Z}$ holds. As $s\to\infty$ and* $\arg(s)<\frac{3}{4}\pi$ *the solution of* (3.56) *for a general resolvent function η_0 permits the asymptotic expansion*

$$\eta_0(s)=-\frac{3}{2^{4/3}}s^{2/3}+\mathrm{O}(s^{1/3},1/\log(s)).\tag{3.75}$$

Proof. Let us determine the necessary conditions for a large-s algebraic solution of the form (3.68) to equation $P\star$. Employing just the first term in $P\star$ we find that s^8 times this expression possesses 1923 terms with an s-dependence of the form $s^{m+n\lambda_1}$ with $m\in\mathbb{Z}_{\geq 0}$ and $n\in\mathbb{N}$. A consolidated plot of these (m,n) values is given in Fig. 2.

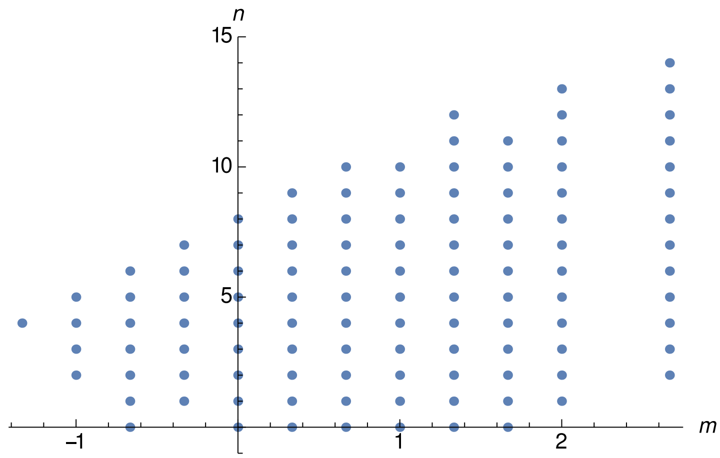

Figure 3: Newton polygon of the exponents $s^{m+n\delta_1}$ for the sub-leading term of asymptotic expansion of (3.68).

From this plot it is clear the line defined by the points $14\lambda_1, 2+11\lambda_1, 4+8\lambda_1$ defines an upper right-hand segment of the convex hull of all these points. Of necessity it must have negative slope. The single mutual solution for λ_1 by equating each of these with the others is $\lambda_1 = 2/3$. There are 14 terms associated with these three points and their sum gives, after making the substitution for the λ_1 solution,

$$\frac{256}{81} s^{28/3} C_1^8 \left(16C_1^3 + 27\right)^2 .$$

The only acceptable, real and non-zero solution for the coefficient is $C_1 = -3 \times 2^{-4/3}$. Proceeding on we introduce an algebraic sub-leading term, as in (3.68), and specialise the values for the exponent and coefficient of leading term found earlier. When we examine $s^{4/3}$ times this resulting expression we find 28042 terms of the form $s^{m+n\delta_1}$ with $m \in \frac{1}{3}\mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. The consolidated plot of these (m, n) values is given in Fig. 3.

Considering this figure we observe that there are two possibilities for lines defining an upper boundary to the convex hull of these points, both with positive slope. The first of the two is defined by the points $\frac{8}{3} + 14\delta_1, 2 + 13\delta_1, \frac{4}{3} + 12\delta_1$ and yields the solution $\delta_1 = -2/3$. However the total of the

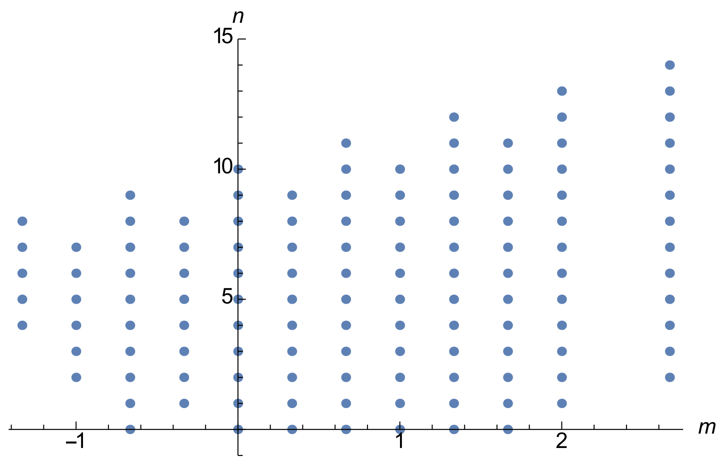

Figure 4: Newton polygon of the exponents $s^{m+n\delta_1}$ for the sub-leading term of asymptotic expansion of (3.76).

378 terms which contribute to this vanish identically with this solution for the exponent and so the coefficient is undetermined. The second of the two lines is defined by the set of 8 points $\frac{4}{3}+12\delta_1, \frac{2}{3}+10\delta_1, \frac{1}{3}+9\delta_1, 8\delta_1, -\frac{1}{3}+7\delta_1, -\frac{2}{3}+6\delta_1, -1+5\delta_1, -\frac{4}{3}+4\delta_1$ and their mutual equality gives the solution $\delta_1=-1/3$. There are 3915 terms which have these exponents and their sum, under evaluation of δ_1, is non-zero.

If we admit an algebraic-logarithmic sub-leading term of the form

$$\eta_0 = C_1 s^{\lambda_1} + C_2 s^{\lambda_1+\delta_1} (\log s)^{\mu_1}\,, \tag{3.76}$$

and employ the solution for the leading term then we have 384003 terms of the form $s^{m+n\delta_1}t^{k+l\mu_1}$ where $t := \log s$ and $\mu_1 \neq 0$.

The set of admissible (m,n) components of the s-exponent is given in Fig. 4 and the upper part of the convex hull of these points is defined by the seven points $14\delta_1+\frac{8}{3}, 13\delta_1+2, 12\delta_1+\frac{4}{3}, 11\delta_1+\frac{2}{3}, 10\delta_1, 9\delta_1-\frac{2}{3}, 8\delta_1-\frac{4}{3}$ and their mutual solution yields $\delta_1=-2/3$.

Given the above solution for δ_1 we consider next the admissible (k,l) components of the t-exponent which are given in Fig. 5. There are two lines defining the upper part of the convex hull of these points, however only

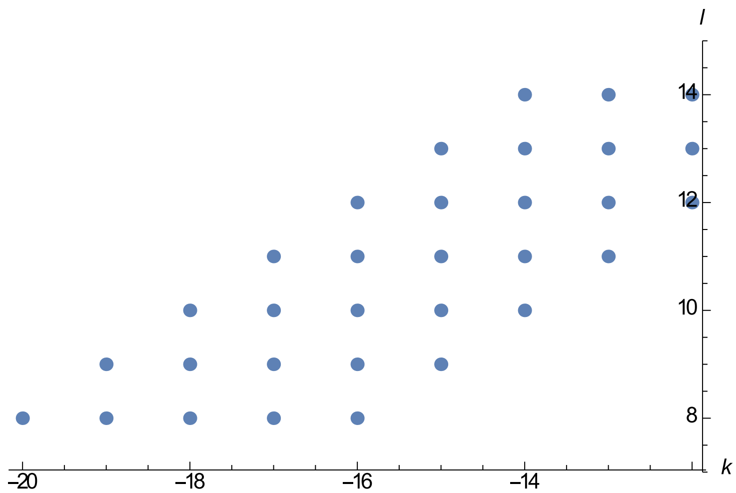

Figure 5: Newton polygon of the exponents $t^{k+l\mu_1}$ for the sub-leading term of asymptotic expansion of (3.76).

the one defined by the seven points $14\mu_1 - 14, 13\mu_1 - 15, 12\mu_1 - 16, 11\mu_1 - 17, 10\mu_1 - 18, 9\mu_1 - 19, 8\mu_1 - 20$ ensures a finite solution, namely $\mu_1 = -1$. In this case we have non-zero solutions for C_2. □

Corollary 3.1. *Let us assume the generic condition $\nu_2 - \nu_1 \neq \mathbb{Z}$ holds. For large s the $M = 2$ gap probability $E_2 := E_{\nu_1,\nu_2}$ has the asymptotic form*

$$E_{\nu_1,\nu_2}(0;(0,s)) = e^{-\frac{9}{2^{7/3}}s^{2/3}+O(s^{1/3})}. \tag{3.77}$$

Remark 3.1. It has been shown in (23), Eq. A.2, that the jpdf of Theorem 1.1 in the large separation limit takes on the simpler functional form proportional to

$$\prod_{l=1}^{N} x_l^{-1/2+1/2M} e^{-Mx_l^{1/M}} \prod_{1\le j<k\le N} (x_k - x_j)(x_k^{1/M} - x_j^{1/M}). \tag{3.78}$$

After the change of variables $x_l \mapsto Mx_l^M$ this specifies the Laguerre Muttalib-Borodin model: see Section 3.4 below. A key feature for present purposes is that exponentiating the product of differences gives the logarithmic pair potential $V_2(x,y) = -\log(|x-y||x^{1/2} - y^{1/2}|)$, which is scale invariant under multiplication of the coordinates. According to Eq. (14.117) of (6) this, together with the fact from Eq. (5.15) of (24) that the hard edge spectral density is proportional to $x^{-M/(M+1)}$, tells us that the leading $s \to \infty$ form of the gap probability at the hard edge is given by $e^{-C_M x^{2/(M+1)}}$ for some C_M. Our analytic result for $M = 2$ (3.77) agrees with this predicted form.

3.4 $\theta = 2$ Muttalib-Borodin Ensembles

The Laguerre Muttalib-Borodin model refers to the eigenvalue PDF proportional to

$$\prod_{l=1}^{N} x_l^c e^{-x_l} \prod_{1 \le j < k \le N} (x_j - x_k)(x_j^\theta - x_k^\theta), \qquad x_l \in \mathbb{R}_{>0}.$$

This is a determinantal point process, and so is fully specified by a correlation kernel, $K^L(x,y)$ say. Define the hard edge scaling limit by

$$K^{(c,\theta)}(x,y) := \lim_{N\to\infty} N^{-1/\theta} K^L(N^{-1/\theta}x, N^{-1/\theta}y).$$

Borodin (25) has obtained the evaluation

$$K^{(c,\theta)}(x,y) = \theta x^c \int_0^1 J_{\frac{c+1}{\theta},\frac{1}{\theta}}(xu) J_{c+1,\theta}((yu)^\theta) u^c \, du,$$

where the function $J_{a,b}(x)$ defines the Wright Bessel function

$$J_{a,b}(x) := \sum_{j=0}^{\infty} \frac{(-x)^j}{j!\Gamma(a+jb)}.$$

In a shift of notation we write $K_M(x,y)$ defined by (1.4) as $K_{\nu_1,\dots,\nu_M}(x,y)$ to emphasize the dependency on the parameter set, and similarly write $E_{\nu_1,\dots,\nu_M}(0;(0,s))$ for the scaled gap probability. These are well defined for all $\nu_i > -1$. We know from Kuijlaars and Zhang (4) and from Forrester and Wang (26) (see Eqs. (1.1), (1.5) and (5.8)) that for $\theta \in \mathbb{Z}_+$

$$x^{1/\theta - 1} K^{(c,\theta)}(\theta x^{1/\theta}, \theta y^{1/\theta}) = K_{\nu_1,\dots,\nu_\theta}(x,y),$$

where

$$\nu_j = \frac{c+j}{\theta} - 1, \qquad 1 \le j \le \theta.$$

Another relevant work on the connection between the two kernels is (27). Thus we can deduce that the gap probabilities (1.7) for $c = 0, 1$ and $\theta = 2$ satisfies the identities

$$\begin{aligned} E_{-1/2,0}(0;(0,s)) &= E^{(0,2)}(0;(0,2\sqrt{s})), \\ E_{0,1/2}(0;(0,s)) &= E^{(1,2)}(0;(0,2\sqrt{s})), \end{aligned} \tag{3.79}$$

where $E^{(c,\theta)}$ denotes the gap probability for the hard edge scaled Laguerre Muttalib-Borodin model.

The significance of this is that the kernels $K^{(c,2)}$ are analytic, so we can apply Bornemann's numerical scheme (28), (29) to evaluate the gap probabilities in the large s regime and test numerically the asymptotic behaviour given in Prop. 3.6. In this situation Bornemann's method converges exponentially fast and we can obtain accurate values for the gap probabilities in this regime. We have implemented the Bornemann method employing 9 nodes in the Clenshaw-Curtis quadrature rule with a precision of 20 decimal digits and truncating the Wright Bessel function series at 100 terms. A table of $\log E^{(c,2)}$ versus $r = 2\sqrt{s}$ is given in the first columns of Table 1 for both cases. We then compute a fit of $\log E^{(c,2)}$ given on a range of r values to the assumed form $a_1 r^{4/3} + b_1 r^{2/3} + c_1$. The values of a_1 are tabulated for the range of $r = 4, \ldots, 14$ and the extrapolated value in the second columns of Table 1 for both $c = 0, 1$. This range was chosen, that is to say limited to these values, because at $r = 15$ the value of $E^{(0,2)}$ is already 8.917166×10^{-15} and larger values are unreliable due to underflow. The extrapolated values should be compared to the predicted value (see (3.75)) of $9 \cdot 2^{-11/3} \sim 0.708705590566$.

We can also independently check the small s expansions generated by the non-linear analogue of the σ-form (3.56) by arbitrary high-accuracy small s expansions using the Neumann expansions of the right-hand sides of (3.79).

Table 1: Computed values of $\log E^{(c,2)}(0;(0,r))$ and the coefficient a_1 versus r for $c = 0, 1$ and extrapolated values of a_1.

	$c = 0$		$c = 1$	
r	$\log E^{(0,2)}(0;(0,r))$	a_1	$\log E^{(1,2)}(0;(0,r))$	a_1
4	-5.96549338586	-0.70729888196	-3.2910182568186667	-0.7050253349947
5	-7.7702165574578	-0.707506362179	-4.6175115857278	-0.7059621770238
6	-9.666703768133	-0.707671636400	-6.06617567204249	-0.7065523127608
7	-11.6460744648319	-0.707802917979	-7.6216467824166	-0.706953478338
8	-13.701343595761	-0.707908414200	-9.2725398209570	-0.707241379027
9	-15.826846765594	-0.70799443184	-11.010033902389	-0.70745656369
10	-18.017880484821	-0.70806558203	-12.8270650595890	-0.7076225663
11	-20.27046470121	-0.7081252605	-14.7178300927	-0.7077538862
12	-22.58117923782	-0.7081762248	-16.6774638565	-0.7078597927
13	-24.9470471656	-0.7082218856	-18.70181973197	-0.7079460684
14	-27.3654492473	-0.70827084	-20.7873147490	-0.7080153465
∞		-0.7088		-0.7083172

For $\nu_1 = -1/2, \nu_2 = 0$ we compute that the initial terms are

$$\begin{aligned}\eta_0(s) = -2\frac{\sqrt{s}}{\sqrt{\pi}} - 2(4-\pi)\frac{s}{\pi} - \frac{32}{3}(3-\pi)\frac{s^{3/2}}{\pi^{3/2}} \\ - \frac{16}{9}(72 - 32\pi + 3\pi^2)\frac{s^2}{\pi^2} - \frac{64}{45}(360 - 200\pi + 27\pi^2)\frac{s^{5/2}}{\pi^{5/2}} \\ - \frac{512}{675}(2700 - 1800\pi + 347\pi^2 - 15\pi^3)\frac{s^3}{\pi^3} + \mathrm{O}(s^{7/2}).\end{aligned}$$

Using computer algebra, we have extended this series to high order, and have computed as well the series expansion of F as implied by (3.55) to high order. We find the remarkable but not understood relation

$$6 - 2F = \eta_0'. \tag{3.80}$$

Substituting this in (3.55) we find the even more remarkable, and similarly not understood result that the resolvent function satisfies the much simpler third-order non-linear ODE

$$\begin{aligned}-12s^2\eta_0'\eta_0^{(3)} + 9s^2\eta_0''^2 - 12s\eta_0'\eta_0'' \\ + \frac{3}{4}\eta_0'\left[\eta_0'(-48s\eta_0' + 16\eta_0 + 1) + 4\right] - 9 = 0.\end{aligned} \tag{3.81}$$

Remark 3.2. We can check that (3.81) is consistent with (3.75). It is furthermore the case that a large s analysis of (3.81) allows (3.77), with $\nu_1 = -1/2$, $\nu_2 = 0$ to be strengthened to read

$$E_{-1/2,0}(0;(0,s)) = e^{-\frac{9}{2^{7/3}}s^{2/3} - \frac{3}{2^{5/3}}s^{1/3} + O(s^{1/6})}. \tag{3.82}$$

3.5 Higher order analogues of Painlevé III

A program to enumerate all of the higher order analogues of the Painlevé equations, at least to the next level of four-dimensional or four accessory parameters, has been initiated by H. Kawakami, A. Nakamura and H. Sakai in the period 2012-15. The first phase of the task was achieved with the construction of isomonodromy deformation problems for Fuchsian differential

Table 2: The unramified cases of the Fuji-Suzuki family.

art.eps

equations, extending the four singularity case corresponding to Painlevé VI, in (30) by the techniques of addition and middle convolution. This yielded the four master cases: the Garnier systems, the Fuji-Suzuki systems, the Sasano systems and the matrix Painlevé systems, which we tabulate below -

$$\boxed{1+1+1+1+1}$$
$$\boxed{\begin{array}{c} 11, 11, 11, 11, 11 \\ H^{1+1+1+1+1}_{\text{Garnier}} \end{array}}$$

$$\boxed{1+1+1+1}$$
$$\boxed{\begin{array}{c} 21, 21, 111, 111 \\ H^{A_5}_{\text{Fuji}-\text{Suzuki}} \end{array}}$$

$$\boxed{1+1+1+1}$$
$$\boxed{\begin{array}{c} 31, 22, 22, 1111 \\ H^{D_6}_{\text{Sasano}} \end{array}}$$

$$\boxed{1+1+1+1}$$
$$\boxed{\begin{array}{c} 22, 22, 22, 211 \\ H^{\text{Matrix}}_{\text{VI}} \end{array}}$$

These four master cases were extended by constructing from them the degeneration schemes of singularity confluence in (31) and (32), and yields four families. Of the four families found the only family relevant to our case, certain higher order analogues of Painlevé III, is the Fuji-Suzuki family which have 3×3 Lax pairs. There are nine cases in this degeneration scheme, see Table 2. **Q1**

However such a classification treats only the unramified cases and only very recently have ramified cases been studied, and a partial list of results has been given in (33). In addition to the nine shown above another seven ramified cases are given. However of those only one is a possibility, namely the one with the singularity pattern $\frac{4}{3}+1+1$ and spectral type $(1)_3, 21, 111$ and has a Riemann-Papperitz symbol

$$\left\{ \begin{array}{cccc} 0 & 1 & & \infty(\frac{1}{3}) \\ 0 & 0 & t^{1/3} & \theta_1^\infty/3-\frac{2}{3} \\ \theta_1^0 & 0 & \omega t^{1/3} & \theta_1^\infty/3-\frac{2}{3} \\ \theta_2^0 & \theta^1 & \omega^2 t^{1/3} & \theta_1^\infty/3-\frac{2}{3} \end{array} \right\}, \tag{3.83}$$

with $\theta_1^0+\theta_2^0+\theta^1+\theta_1^\infty=0$ and $\omega^3=1$. The comparison that must be made here is with our system (3.16), and there are several differences to note. One is that while the indicial exponents at the $z=1$ singularity of (3.16) are all zero (only two are independent) this is just an artifact of the Fredholm theory, which always leads to these exponents vanishing whereas the general integrable system possesses a full set of exponents. Thus we suspect that the generalisation of our system actually has one or possibly two additional, free non-zero parameters here and thus either two or all three are different. However in (3.83) two of the parameters are locked together (here they are conventionally set to zero). Another difference arises also, where the sub-leading spectral data at $z=\infty$ are all equal, whereas in our application these are not equal even in special cases. In summary we believe that there are additional ramified cases to be found in the Fuji-Suzuki family, and that our system is a special case of one such system. Such a system might arise from the unramified system with singularity pattern $2+1+1$ and spectral type $(2)(1), 111, 111$ by a transition involving a fractional drop in the Poincaré index.

Acknowledgements

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Appendix A

Here the Hamiltonian variables in the case $M = 2$ are expressed in terms of η_0 and its derivatives. **Q2**

Proposition .7. *All the dynamical variables can be recovered from the resolvent function η_0 and its derivatives in the following list of formulae. Here F should be interpreted as the positive square root of* (3.55)*.*

$$
\begin{aligned}
\xi_0 = &-(3+e_1-3\eta_0)\frac{((3e_2-e_1^2)F+3(9-F)\eta_0)}{162\eta_0'} \\
&+\frac{1}{162}\Big[9e_1\left(3e_2\left(\eta_0-1\right)+3e_3+(3-F)s-3\left(\eta_0-1\right)\eta_0\right) \\
&+27\left(\eta_0\left(4e_2-(3-F)s+\left(\eta_0-2\right)\eta_0+1\right)-3e_3\left(\eta_0+1\right)+3s\right) \\
&-6e_1^3\left(\eta_0-1\right)-9e_1^2\left(e_2+2\eta_0\right)+2e_1^4\Big] \\
&-\frac{1}{6}s\left(e_1-3\eta_0+1\right)\eta_0' \\
&+\left[(3+e_1-3\eta_0)\frac{s}{108\eta_0'^2}\left(\frac{36s\eta_0'^3}{F}+F\right)+\frac{s^2}{6}\right]\eta_0'' \\
&+(3+e_1-3\eta_0)\left[-\frac{s^2\left(e_1^2-3e_2+3\eta_0-3\right)}{18F\eta_0'}+\frac{s^3}{F}-\frac{Fs^2}{72\eta_0'^3}\right]\eta_0''^2 \\
&-(3+e_1-3\eta_0)\frac{s^3\eta_0''^3}{4F\eta_0'^2}+(3+e_1-3\eta_0)\frac{s^4\eta_0''^4}{8F\eta_0'^3} \\
&+(3+e_1-3\eta_0)\left[-\frac{s^4\eta_0''^2}{4F\eta_0'^2}+\frac{s^3\eta_0''}{2F\eta_0'}+\frac{Fs^2}{108\eta_0'^2}\right]\eta_0{}^{(3)} \\
&+(3+e_1-3\eta_0)\frac{s^4\eta_0''\eta_0{}^{(4)}}{6F\eta_0'},
\end{aligned}
\tag{.84}
$$

$$
\begin{aligned}
\xi_1 = \frac{(3e_2 - e_1^2)F + 3(9-F)\eta_0}{54\eta_0'} \\
+ \frac{1}{54}\Big[-9\left(-6e_2 + 3e_3 + (3-F)s - 3\left(\eta_0 - 1\right)\eta_0\right) \\
+ 9e_1\left(e_2 - 4\eta_0\right) - 2e_1^3\Big] \\
+ \frac{1}{2}s\eta_0' - \left[\frac{s^2\eta_0'}{F} + \frac{Fs}{36{\eta_0'}^2}\right]\eta_0'' \\
+ \left[\frac{s^2\left(e_1^2 - 3e_2 + 3\eta_0 - 3\right)}{6F\eta_0'} - \frac{3s^3}{F} + \frac{Fs^2}{24{\eta_0'}^3}\right]\eta_0''^2 \\
+ \frac{3s^3\eta_0''^3}{4F{\eta_0'}^2} - \frac{3s^4\eta_0''^4}{8F{\eta_0'}^3} - \left[-\frac{3s^4\eta_0''^2}{4F{\eta_0'}^2} + \frac{3s^3\eta_0''}{2F\eta_0'} + \frac{Fs^2}{36{\eta_0'}^2}\right]{\eta_0}^{(3)} \\
- \frac{s^4\eta_0''}{2F\eta_0'}{\eta_0}^{(4)}, \quad (.85)
\end{aligned}
$$

$$
\begin{aligned}
\eta_1 = \frac{(9-F)\eta_0}{18\eta_0'} \\
+ \frac{1}{54}\left[-9\left(3e_3 + (3-F)s + 3\left(\eta_0 - 1\right)\eta_0\right) + 9e_1\left(e_2 + 2\eta_0\right) - 2e_1^3\right] \\
- \frac{1}{2}s\eta_0' - (e_1^2 - 3e_2)(e_1^2 - 3e_2 + 3\eta_0)\frac{2}{27F}\eta_0' + (e_1^2 - 3e_2)\frac{2s}{3F}{\eta_0'}^2 \\
+ \left(e_1^2 - 3\left(e_2 + \eta_0\right)\right)\frac{s\eta_0''}{9F} \\
+ \left[\frac{s^2\left(e_1^2 - 3e_2 + 6\eta_0 - 1\right)}{6F\eta_0'} - \frac{9s^3}{2F}\right]\eta_0''^2 \\
+ \left[\frac{s^2\left(e_1^2 - 3\left(e_2 + \eta_0\right)\right)}{9F} - \frac{5s^3\eta_0''}{6F\eta_0'} + \frac{s^3\eta_0'}{F}\right]{\eta_0}^{(3)} \\
+ \frac{s^4{\eta_0}^{(3)2}}{3F\eta_0'} - \frac{s^4{\eta_0}^{(4)}\eta_0''}{2F\eta_0'}, \quad (.86)
\end{aligned}
$$

$$
\begin{aligned}
\eta_2 = {} & \frac{\eta_0 \left[9\left(2e_1 - 3\eta_0 + 3\right) - F\left(2e_1 - 3\eta_0\right)\right]}{54\eta_0'} \\
& + \frac{1}{162}\Big[- 4e_1^4 + 6\left(\eta_0 - 1\right)e_1^3 + 18\left(e_2 + 2\eta_0\right)e_1^2 \\
& - 9\left(2(3-F)s + 6e_3 + 3e_2\left(\eta_0 - 1\right) + 6\left(\eta_0 - 1\right)\eta_0\right)e_1 \\
& + 27\left(-3s + 3e_3\left(\eta_0 - 1\right) + \eta_0\left(3s - 2e_2 + \left(\eta_0 - 2\right)\eta_0 + 1\right)\right)\Big] \\
& - \frac{1}{162F}\Big[8e_1^5 - 12\left(\eta_0 - 1\right)e_1^4 + 24\left(\eta_0 - 2e_2\right)e_1^3 \\
& + 36\left(2e_2\left(\eta_0 - 1\right) - \left(\eta_0 - 2\right)\eta_0\right)e_1^2 - 18\left(4e_2\left(\eta_0 - e_2\right) - 3Fs\right)e_1 \\
& + 27\left(-4\left(e_2 - \eta_0\right)\left(e_2\left(\eta_0 - 1\right) + \eta_0\right) - Fs\left(3\eta_0 - 1\right)\right)\Big]\eta_0' \\
& - \frac{2\left(9{\eta_0}^2 + 3\left(2e_1^2 - 6e_2 - 3\right)\eta_0 - \left(2e_1 + 3\right)\left(e_1^2 - 3e_2\right)\right)}{9F}s{\eta_0'}^2 \\
& + \frac{6s^2\eta_0{\eta_0'}^3}{F} + \Bigg[\frac{2s^2\eta_0\eta_0'}{F} \\
& - \frac{s\left(-9Fs - 12\left(2e_1 + 3\right) + \left(2e_1 - 3\eta_0 + 3\right)\left(6\left(e_2 + \eta_0 + 2\right) - 2e_1^2\right)\right)}{54F}\Bigg]\eta_0'' \\
& - \Bigg[\frac{3\left(2e_1 - 2\eta_0 + 3\right)s^3}{2F} \\
& + \frac{\left(6e_1 + \left(2e_1 - 3\eta_0 + 3\right)\left(-e_1^2 + 3e_2 - 6\eta_0 - 2\right) + 9\right)s^2}{18F\eta_0'}\Bigg]\eta_0''^2 \\
& + \Bigg[\frac{\left(2e_1 + 3\eta_0 + 3\right)\eta_0's^3}{3F} - \frac{5\left(2e_1 - 3\eta_0 + 3\right)\eta_0''s^3}{18F\eta_0'} \\
& - \frac{\left(\left(3\left(e_2 + \eta_0 + 2\right) - e_1^2\right)\left(2e_1 - 3\eta_0 + 3\right) - 6\left(2e_1 + 3\right)\right)s^2}{27F}\Bigg]{\eta_0}^{(3)} \\
& + \left(2e_1 - 3\eta_0 + 3\right)s^4\frac{{\eta_0}^{(3)2}}{9F\eta_0'} - \left(2e_1 - 3\eta_0 + 3\right)s^4\frac{\eta_0''{\eta_0}^{(4)}}{6F\eta_0'},
\end{aligned}
\tag{.87}
$$

$$
x_0y_1 = \frac{1}{6}\left[-F + 4e_1\eta_0' - 6\eta_0\eta_0' - 3s\eta_0''\right], \tag{.88}
$$

$$
x_1y_2 = \frac{1}{6}\left[-F - 2e_1\eta_0' + 6\eta_0\eta_0' + 3s\eta_0''\right], \tag{.89}
$$

$$x_0 y_2 = -\eta_0', \tag{.90}$$

$$\begin{aligned} x_0 y_0 = {} & \frac{1}{54}\left[\left(e_1\left(e_1+3\right)-3e_2\right)F-3(2F+9)\eta_0\right] \\ & + \frac{1}{54}\Big[9\left(2e_1+1\right)\eta_0+9\left(3\left(e_3+s\right)-4e_2\right) \\ & + e_1\left(2e_1\left(e_1+3\right)-9e_2\right)-9Fs-27{\eta_0}^2\Big]\eta_0' \\ & - \frac{1}{2}s{\eta_0'}^2+\frac{2s{\eta_0'}^3}{F} \\ & + \left[\frac{1}{6}s\left(e_1-3\eta_0-1\right)-\frac{s\left(e_1^2-3e_2+3\eta_0-3\right)\eta_0'}{3F}+\frac{7s^2{\eta_0'}^2}{F}-\frac{Fs}{18\eta_0'}\right]\eta_0'' \\ & + \left[-\frac{s^2\left(e_1^2-3e_2+3\eta_0+6\right)}{6F}+\frac{3s^3\eta_0'}{F}-\frac{Fs^2}{24{\eta_0'}^2}\right]{\eta_0''}^2+\frac{3s^4{\eta_0''}^4}{8F{\eta_0'}^2} \\ & + \left[-\frac{3s^4{\eta_0''}^2}{4F\eta_0'}+\frac{3s^2\eta_0'}{F}+\frac{Fs^2}{36\eta_0'}-\frac{s^2}{6}\right]{\eta_0}^{(3)}+\left[\frac{s^4\eta_0''}{2F}+\frac{s^3\eta_0'}{F}\right]{\eta_0}^{(4)}, \end{aligned} \tag{.91}$$

$$\begin{aligned} x_1 y_1 = {} & \frac{1}{18}e_1F+\frac{1}{9}\left[-3\left(3e_1+1\right)\eta_0+e_1^2+3e_2+9{\eta_0}^2\right]\eta_0' \\ & + s{\eta_0'}^2+\frac{1}{6}s(2-3e_1+6\eta_0)\eta_0''+\frac{1}{3}s^2{\eta_0}^{(3)}, \end{aligned} \tag{.92}$$

$$\begin{aligned} x_2 y_2 = {} & \frac{1}{54}\left[-e_1\left(e_1+6\right)F+3e_2F+3(2F+9)\eta_0\right] \\ & + \frac{1}{54}\Big[9(2e_2-3e_3+(F-3)s-3{\eta_0}^2+\eta_0) \\ & + 9e_1\left(e_2+4\eta_0\right)-2e_1^3-12e_1^2\Big]\eta_0' \\ & - \frac{1}{2}s{\eta_0'}^2-\frac{2s{\eta_0'}^3}{F} \\ & + \left[\frac{s\left(e_1^2-3e_2+3\eta_0-3\right)\eta_0'}{3F}+\frac{1}{6}s\left(2e_1-3\eta_0-1\right)-\frac{7s^2{\eta_0'}^2}{F}+\frac{Fs}{18\eta_0'}\right]\eta_0'' \\ & + \left[\frac{s^2\left(e_1^2-3e_2+3\eta_0+6\right)}{6F}-\frac{3s^3\eta_0'}{F}+\frac{Fs^2}{24{\eta_0'}^2}\right]{\eta_0''}^2-\frac{3s^4{\eta_0''}^4}{8F{\eta_0'}^2} \\ & + \left[\frac{3s^4{\eta_0''}^2}{4F\eta_0'}-\frac{3s^2\eta_0'}{F}-\frac{Fs^2}{36\eta_0'}-\frac{s^2}{6}\right]{\eta_0}^{(3)}+\left[-\frac{s^4\eta_0''}{2F}-\frac{s^3\eta_0'}{F}\right]{\eta_0}^{(4)}. \end{aligned} \tag{.93}$$

Proof. We employ the abbreviations for U, V in (3.51) and (3.52), and together with

$$\xi_1 - \eta_1 = e_2 + \eta_0(\eta_0 - e_1) - \eta_0 + s\eta_0', \tag{.94}$$

and (3.65), we have a system of four linear, independent equations for $\xi_0, \xi_1, \eta_1, \eta_2$ in terms of U, V, W, Z, and η_0 and its derivatives. For the bilinear products we will use the formulae (3.57), (3.58), (3.59) and (3.60). The next step is to solve for U, V, W, Z and in contrast to the proof of (3.56) we employ (3.61), (3.67), (3.63) and (3.64). After some simplifying we arrive at (.84)-(.93). □

One final result should be stated here and this concerns the splitting of $x_0 y_2$ and involves the introduction of a decoupling factor G such that $x_0 := y_2 G$. For $M = 1$ this was a simple algebraic factor but for $M \geq 2$ this is no longer the case.

Proposition .8. *The decoupling factor G satisfies the first-order ordinary differential equation*

$$\left(3s\frac{G'}{G} + 2e_1\right)^2 - 4e_1^2 = 12\left(\eta_0 - e_2 - 3s\eta_0' - s\frac{\eta_0''}{\eta_0'}\right) - 12s^2\left(\frac{\eta_0{}^{(3)}}{\eta_0'} - \frac{3}{4}\left(\frac{\eta_0''}{\eta_0'}\right)^2\right), \tag{.95}$$

and the boundary condition as $s \to 0$

$$G^{-1} \sim -\Gamma(\nu_2 - \nu_1)\Gamma(\nu_2 - \nu_0 + 1)s^{\nu_1+\nu_0} - \Gamma(\nu_1 - \nu_2)\Gamma(\nu_1 - \nu_0 + 1)s^{\nu_2+\nu_0}. \tag{.96}$$

Proof. Clearly $U = sy_2y_2'G$ and $V = sy_2^2G' + sy_2y_2'G$, and together with (3.67) and (3.55), we deduce (.95). The boundary condition is a consequence of (3.18) and (3.23). □

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QUERIES

Q1. ART: proccess this table as a figure.

Q2. CE: Please check appendix.