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System of Funnels Framework for Robust Global Non-linear Control

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***Abstract*— There exist various methods for planning nominal trajectories to guide desired behaviours of non-linear systems, along with constructive methods for computing finite-time invariant sets, termed funnels, about locally-stabilized nominal trajectories. In order to achieve a desired behaviour defined by a set of nominal trajectories and their corresponding funnels, one has to switch from one local control to another at the right instances. This paper presents a general hybrid-control framework which is designed for correct switching between locally stabilizing controllers and can be used in conjunction with various approaches for funnel computation. Our framework prescribes exact connectivity conditions to be satisfied by the different funnels used such that the desired behaviour is achieved globally and in a robust manner. Due to its generality, the framework can be applied to implement a wide class of dynamic behaviours. An example of a periodic behaviour governed by our framework is provided.**

I. INTRODUCTION

Often dynamic systems are required to demonstrate non-trivial behaviours such as dexterous manoeuvres, under-actuated manipulation or transitions from one type of periodic behaviour to another [1], [2]. The execution of such behaviours requires a robust feedback control that guarantees achievement of the desired task even in the presence of modelling errors or external disturbances. Additionally, a global treatment is usually required to account for various possible initial conditions. Global tracking approaches are normally limited by their restrictive assumptions. Instead, one can use local tracking or stabilization control in combination with a global, higher-level, switching strategy that dictates which local controller should be activated at a given time.This paper presents such a hybrid control framework, with a guarantee to achieve a desired behaviour globally and robustly for a large class of non-linear systems. We generalize the notion of a funnel and suggest a suitable hybrid controller that orchestrates the switching process.

Our work is inspired by the LQR-Trees method for global stabilization of an equilibrium state of a non-linear system [3]. The LQR-Trees method constructs a tree of trajectories, each encompassed with funnels [4]. The funnels, that cover the entire state space, are connected to each other in a manner that allows cascading trajectories with any initial conditions towards the target equilibrium state, i.e. the root of the tree. The LQR-Trees approach was mainly enabled due to the ability to compute conservative regions of finite-time invariance using SOS programs. A drawback of the LQR-Trees approach is that the switching mechanism is not appropriately designed to avoid chattering and instability when the high-level algorithm has to decide which controller should be active and the state of the system lies on the boundary between neighbouring funnels.

Another approach which inspired our work is Throw-and-Catch [5]. In this approach, a directed tree of open-loop trajectories connecting equilibria states of the system is constructed such that the root of the tree is again the goal equilibrium state. Discrete logic states are used in a hybrid control algorithm to achieve robust global stabilization of the goal state. The robustness is attributed to overlaps between the regions where various controllers can safely operate and the careful design of the hybrid switching controller.

In this work, we suggest a framework that prescribes a switching controller and accompanying sufficient conditions for appropriate connections between funnels to guarantee robust execution of the desired behaviour. The framework is suitable for funnels arranged in any directed graph structure and is thus far more general in its applicability. Proofs for the theorems stated can be found in [6].

II. RESULTS

A. Fundamental Definitions

In this section, we generalize the definition of a funnel that appears in [4]. Our definition includes three basic objects that are used in the system-of-funnels framework: a funnel, an entrance and an outlet. The separate treatment of a funnel and its entrance allows a robust design of the hybrid control system and the generalised outlet definition allows more possibilities of switching from one funnel to another.

Consider the non-linear dynamic control system:

$$\dot{z} = f(z, u), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Consider a set-valued mapping $\mathbf{F} : \mathbb{R} \rightrightarrows \mathbb{R}^n$ and define $\mathrm{dom}(\mathbf{F}) := \{\tau \mid \mathbf{F}(\tau) \neq \emptyset\}$ as its domain. Denote by $\mathcal{F}$ the graph of $\mathbf{F}$, i.e., $\mathcal{F} := \mathrm{gph}(\mathbf{F}) := \{(\tau, z) \mid z \in \mathbf{F}(\tau)\}$. $\mathcal{F}$ (or $\mathbf{F}$) is called a *funnel* if both:

(a) $\mathrm{dom}(\mathbf{F}) = [0, \infty)$ or $\mathrm{dom}(\mathbf{F}) = [0, T]$ for some $T > 0$.
(b) There exist a function $u : \mathcal{F} \to \mathbb{R}^m$ such that $\tau \to u(\tau, z)$ is piecewise continuous and $z \to u(\tau, z)$ is continuous, and a set $\mathcal{E} \subset \mathcal{F}$ such that for each $(\tau_0, z_0) \in \mathcal{E}$, each solution of (1) with $u = u(\tau, z)$ and $\dot{\tau} = 1$ starting at (τ_0, z_0) at time $t = 0$ does not exhibit finite escape time and satisfies $z(t) \in \mathrm{int}(\mathbf{F}(\tau(t)))$ for all t where the solution is defined.[1]

[1] $z(t)$ is defined for all t such that $\tau(t) = \tau_0 + t \in \mathrm{dom}(\mathbf{F})$.

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If $\mathbf{F}$ is outer semi-continuous (osc), we have that $\mathcal{F}$ is a closed set and we say that the funnel is closed. We refer to such set $\mathcal{E}$ as the *entrance of the funnel* $\mathcal{F}$, and to such function u as the *control policy associated with* $\mathcal{F}$.

Let $\mathcal{O}$ be a closed subset of $\mathcal{F}$ and define a set-valued mapping $\mathbf{O} : \mathcal{E} \rightrightarrows \mathrm{dom}(\mathbf{F})$ such that for each $\xi_0 := (\tau_0, z_0) \in \mathcal{E}$, $\mathbf{O}(\xi_0)$ is the set of all values of τ in $\mathrm{dom}(\mathbf{F})$ for which a trajectory ξ that starts from ξ_0 and flows in the funnel belongs to $\mathcal{O}$. $\mathcal{O}$ is termed the *outlet* of a funnel $\mathcal{F}$ if it satisfies the following conditions:

(a) If $\mathrm{dom}(\mathbf{F})$ is $[0, \infty)$, then $\mathcal{O} := \emptyset$.
(b) If, otherwise, $\mathrm{dom}(\mathbf{F}) = [0, T]$, then, for each $\xi_0 \in \mathcal{E}$, at least one of the following two conditions is satisfied:
  - $T \in \mathbf{O}(\xi_0)$.
  - $\mathbf{O}(\xi_0)$ contains an open interval I (a connected subset of $\mathrm{dom}(\mathbf{F})$), such that $\inf I \geq \tau_0$.

Roughly speaking, an outlet is significant only for funnels with a finite temporal depth and must meet the condition that any trajectory ξ that starts in the entrance of the funnel will eventually reach its outlet such that it will not be able to keep flowing in the set $\mathcal{F} \setminus \mathcal{O}$ either at all or for some non-zero amount of time.

B. A System of Funnels

We define below a system of funnels with respect to a set of edges, which holds information about possible connections between funnels.

Define a set of funnel indices as $\mathbb{K} := \{1, ..., K\}$, where $K \in \bar{\mathbb{N}} \setminus \{0\}$. Let $\Upsilon \subseteq \mathbb{K} \times \mathbb{K}$ be a set of directed edges. Let $\Sigma := \{\mathcal{F}_k\}_{k \in \mathbb{K}}$ be a set of K funnels, such that each funnel $\mathcal{F}_k \in \Sigma$, $k \in \mathbb{K}$, is generated via the mapping $\mathbf{F}_k : \mathbb{R} \rightrightarrows \mathbb{R}^n$, and let $\mathcal{E}_k$, $\mathcal{O}_k$ and u_k be the entrance, the outlet and the control policy associated with $\mathcal{F}_k$, respectively.

A set of funnels Σ is called a *system of funnels* relative to the set of edges Υ, if for each $k \in \mathbb{K}$, the condition $(\tau, z) \in \mathcal{O}_k$ implies the existence of j and r such that $(k, j) \in \Upsilon$ and $(r, z) \in \mathcal{E}_j$.

C. Properties of a System of Funnels

Let Σ be a system of funnels. For every $k \in \mathbb{K}$, define $J_k := \{j \mid (k, j) \in \Upsilon\}$, that is, the set of indices of funnels connected to $\mathcal{F}_k$ via outgoing edges from Υ.

Σ is said to be *locally bounded* if for each $k \in \mathbb{K}$ and for each compact set $S_k \subset \mathcal{O}_k$, the set $G_k^S := \{(\tau, r, z, j) \mid (\tau, z) \in S_k,\ (r, z) \in \mathcal{E}_j,\ j \in J_k\}$ is bounded.

Σ is said to be *closed* if for each $k \in \mathbb{K}$ the set $G_k^{\mathcal{O}} := \{(\tau, r, z, j) \mid (\tau, z) \in \mathcal{O}_k,\ (r, z) \in \mathcal{E}_j,\ j \in J_k\}$ is closed.

D. The Hybrid Controller Design

Consider a system of funnels, Σ, defined with respect to a set of edges Υ. We define the following hybrid controller with respect to Σ.

1) Hybrid State Variable: We use $\xi = (\tau, z) \in \mathbb{R} \times \mathbb{R}^n$ as a hybrid state variable, where τ is a state of the dynamic, hybrid controller and can be viewed as the current *temporal depth* inside a funnel (whose domain is the set of all admissible temporal depths), and z is the state of the dynamic system (1). The aim of the controller is to ensure that ξ flows inside funnels or switches from one funnel to another once reaching outlets of funnels. In addition, for a given system of funnels Σ, we define the *hybrid state* $x = (\xi, k) \in \mathbb{R}^{n+1} \times \mathbb{R}$, where k is the index of a funnel $\mathcal{F}_k \in \Sigma$.

2) Flow Set: First, we wish to define subsets of $\mathbb{R}^{n+1}$ where $\xi = (\tau, z)$ changes continuously, that is, flows. Flowing is allowed when the hybrid state $x = (\tau, z, k)$ is such that $\xi \in \mathcal{F}_k$ and $k \in \mathbb{K}$. Therefore, for each $k \in \mathbb{K}$, we define $\tilde{C}_k \subseteq \mathbb{R}^{n+1}$ as:

$$\tilde{C}_k := \mathrm{cl}(\mathcal{F}_k \setminus \mathcal{O}_k). \tag{2}$$

Finally, we define the set $C \subset \mathbb{R}^{n+2}$ as:

$$C := \bigcup_{k \in \mathbb{K}} \tilde{C}_k \times \{k\}. \tag{3}$$

The set C contains the subset of $\mathbb{R}^{n+2}$ where $x = (\tau, z, k)$ should flow continuously, hence it is called the *flow set*.

3) Flow Map: When $\xi \in \tilde{C}_k$, that is, when $x \in C$, the hybrid trajectory's dynamics evolve according to the flow map $f_k : \mathcal{F}_k \to \mathbb{R}^{n+2}$:

$$\dot{x} = (\dot{\tau}, \dot{z}, \dot{k}) = f_k(\xi), \tag{4}$$

where in general we have:

$$f_k(\xi) := (1, f(z, u_k(\tau, z)), 0). \tag{5}$$

However, in the special case (used in Theorem 3) when

- the domain of the funnel is infinite (that is $\mathrm{dom}(\mathbf{F}_k) = [0, \infty)$), and
- both the funnel, $\mathbf{F}_k$, and the control policy associated with it, u_k, are independent of τ (that is, time invariant),

we use different dynamics for τ:

$$f_k(\xi) := (-\tau, f(z, u_k(z)), 0). \tag{6}$$

The use of $\dot{\tau} = -\tau$ in Eq. (6) is merely technical. It does not affect the dynamics of any other state variable but τ and is chosen to prevent the linear unbounded growth of τ which happens when $\dot{\tau} = 1$. The bounded dynamics of τ in (6) assist us in proving a stability property later on.

4) Jump Set: We wish to enable a switch of the funnel index k and a reset of τ when the state (τ, z, k) is such that (τ, z) has reached the outlet $\mathcal{O}_k$. For each $k \in \mathbb{K}$ we define a set $\tilde{D}_k \subseteq \mathbb{R}^{n+1}$ as:

$$\tilde{D}_k := \mathcal{O}_k. \tag{7}$$

Also define the set $D \subseteq \mathbb{R}^{n+2}$ as:

$$D := \bigcup_{k \in \mathbb{K}} \tilde{D}_k \times \{k\}. \tag{8}$$

The set D is called the *jump set* as it indicates that the trajectory of the hybrid state (τ, z, k) is such that the values of k and τ may jump.

5) Jump Map: For each $(k, j) \in \Upsilon$, we define a set-valued reset map $\mathbf{R}_{k,j} : \mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}$ which specifies the admissible new values for the temporal depth state after a jump from funnel $\mathcal{F}_k$ to funnel $\mathcal{F}_j$ has occurred:

$$\mathbf{R}_{k,j}(\tau,z) := \begin{cases} \{r \in \mathbb{R}_{\geq 0} \mid (r,z) \in \mathcal{E}_j\}, \\ k \in \mathbb{K},\ (\tau,z) \in \mathcal{O}_k,\ j \in J_k, \\ \emptyset, \\ \text{otherwise.} \end{cases} \quad (9)$$

The jump map $\mathbf{G}_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+2}$ for a given value of k is then the union of all possible reset maps for k, augmented by the respective values of z and j:

$$\mathbf{G}_k(\tau,z) := \bigcup_{j \in J_k} (\mathbf{R}_{k,j}(\tau,z), z, j)\,. \quad (10)$$

Note that $(\mathbf{R}_{k,j}(\tau,z), z, j)$ is empty when $\mathbf{R}_{k,j}(\tau,z)$ is empty. Also note that $\mathbf{G}_k(\xi)$ is not empty for $\xi \in \tilde{D}_k$ due to the definition of a system of funnels, since we have that $\xi \in \tilde{D}_k = \mathcal{O}_k$ implies the existence of j and r such that $j \in J_k$ and $(r,z) \in \mathcal{E}_j$.

If Σ is closed, then we obtain that $\mathrm{gph}(\mathbf{G}_k)$ is closed and therefore $\mathbf{G}_k$ is osc. Moreover, $\mathbf{G}_k$ is locally bounded if Σ is locally bounded.

6) The Hybrid Closed-Loop System: Our design of the hybrid control strategy leads to the following hybrid system:

$$\mathcal{H}_{cl} : \begin{cases} \dot{x} = f_k(\xi), & x \in C, \\ x^+ \in \mathbf{G}_k(\xi), & x \in D. \end{cases} \quad (11)$$

Note that f_k takes one of the forms (5) or (6), depending on whether or not the funnel $\mathcal{F}_k$ has the properties required for equation (6) to hold.

E. Theorem 1

Consider a closed system of funnels Σ. If the hybrid system (11) is initialized at a point $x(0,0) = (\tau_0, z_0, k_0) \in C \cup D$ such that $(\tau_0, z_0) \in \mathcal{E}_{k_0}$, then all of its maximal solutions x are complete (i.e., $\mathrm{dom}(x)$ is unbounded) and satisfy $z(t,i) \in \mathrm{int}\left(\mathbf{F}_{k(t,i)}(\tau(t,i))\right)$ for all $(t,i) \in \mathrm{dom}(x)$.

F. bootstrap controller

Let Σ be a system of funnels as defined in Section II-B. Under the assumption that

$$\xi(0,0) \in \mathcal{E}_{k(0,0)}, \quad (12)$$

Theorem 1 guarantees that all maximal solutions $x = (\xi, k)$ to (11) are complete and satisfy $\xi(t,i) \in \mathcal{F}_{k(t,i)}$ for all $(t,i) \in \mathrm{dom}(x)$. In order to extend the result of Theorem 1, we wish to treat some cases when the assumption (12) does not hold. Consider the following two cases:

1) $\xi(0,0) \notin \mathcal{F}_{k(0,0)}$. In this case, neither flow nor jump are possible with system (11) and therefore solutions cannot be extended.
2) $\xi(0,0) \in \mathcal{F}_{k(0,0)} \setminus \mathcal{E}_{k(0,0)}$ and at some $t' \in \left[0, \mathrm{length}(\mathrm{dom}(\mathbf{F}_{k(0,0)}))\right)$ we have that $z(t',0)$ intersects with the boundary of $\mathbf{F}_{k(0,0)}(t')$ and cannot flow or jump any more under the dynamics governed by (11).

In both of these two cases, solutions cannot be extended according to system (11). Therefore, we wish to allow the execution of a *recovery policy* once $x \in \mathrm{cl}(\{x \mid \xi \notin \mathcal{F}_k,\ k \in \mathbb{K}\})$ because this condition may imply that it is impossible for the solution to be extended under the dynamics of system (11). Note that for $k \in \mathbb{K}$, the set $\mathrm{cl}(\{x \mid \xi \notin \mathcal{F}_k,\ k \in \mathbb{K}\})$ includes both the case $\xi \notin \mathcal{F}_k$ as in 1 and the case $\xi \in \partial\mathcal{F}_k$ (where ∂ stands for boundary) which is implied by the scenario that arises in 2. The aim of the recovery policy is to ensure that:

- maximal solutions are complete, even if the initial condition does not satisfy (12); and that
- if the recovery policy is executed, then solutions will satisfy the condition $\xi(t,i) \in \mathcal{E}_{k(t,i)}$ for some finite $t \in \mathbb{R}_{\geq 0}$ and $i \in \{0,1,2\}$.

For each $k \in \mathbb{K}$, define the mapping $\mathbf{E}_k : \mathbb{R} \rightrightarrows \mathbb{R}^n$ such that $\mathcal{E}_k$ is its graph, i.e., $\mathcal{E}_k = \mathrm{gph}(\mathbf{E}_k)$. Define the domain of each $\mathbf{E}_k$ as $\mathrm{dom}(\mathbf{E}_k) := \{\tau \mid \mathbf{E}_k(\tau) \neq \emptyset\}$. We allow switching back to "in-funnel" control when z is such that there exist $k' \in \mathbb{K}$ and $\tau' \in \mathrm{dom}(\mathbf{E}_{k'})$ such that $z \in \mathbf{E}_{k'}(\tau')$. We assume the existence of a continuous function $u_0 : \mathbb{R}^n \to \mathbb{R}^m$, with the property that every solution z of (1) with $u = u_0(z)$ initiated at $z(0) = z_0 \in \mathbb{R}^n$ satisfies $z(T_0) \in \mathrm{int}(\mathbf{E}_{k'}(\tau'))$ for some finite (trajectory dependent) $T_0 \in \mathbb{R}_{\geq 0}$, $k' \in \mathbb{K}$ and $\tau' \in \mathrm{dom}(\mathbf{E}_{k'})$. We refer to such u_0 as a bootstrap control.[2] We use a logic state variable $\ell \in \{0,1\}$ to indicate whether or not the bootstrap control is being used. Note that $\ell = 0$ when the bootstrap is turned **on**. The recovery policy is comprised of up to three stages (depending on the initial state) which are:

1) Once a condition $x \in \mathrm{cl}(\{x \mid \xi \notin \mathcal{F}_k,\ k \in \mathbb{K}\})$ (such as in cases 1 and 2 above) is detected while $\ell = 1$ (i.e., bootstrap control is turned off), the hybrid controller is allowed to switch ℓ to 0 without changing x (which would correspond to one jump). In the case that the trajectory resulting from the dynamics $\dot{\xi} = (1, f(z, u_k(\xi)))$ or $\dot{\xi} = (-\tau, f(z, u_k(z)))$ with $\xi \in \mathcal{F}_k$ is forced outside of the flow set before reaching $\mathcal{O}_k$, then this jump would become mandatory.
2) While $\ell = 0$, the bootstrap control is applied continuously until the z component of the state reaches some value z' for which there exist τ' and k' such that $z' \in \mathbf{E}_{k'}(\tau')$.
3) Once such τ' and k' exist for some value of z' while $\ell = 0$, the hybrid controller sets ℓ, τ and k to 1, τ', and k', respectively. This corresponds to another jump, after which the assumption of Theorem 1 is satisfied.

We define a set-valued mapping $\mathbf{G}_0 : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^{n+2}$ as:

$$\mathbf{G}_0(\xi) := \{(r,\zeta,j) \mid (r,\zeta) \in \mathcal{E}_j,\ \zeta = z,\ j \in \mathbb{K}\}, \quad (13)$$

where $\xi = (\tau, z)$.[3] We make the following two assumptions:

1) All $\mathcal{E}_k$, $k \in \mathbb{K}$, are closed sets. Therefore, $\mathbf{G}_0$ is osc.
2) For each compact set $Z \in \mathbb{R}^n$, the set $\{(r,z,j) \mid z \in Z,\ (r,z) \in \mathcal{E}_j,\ j \in \mathbb{K}\}$ is bounded. Therefore G_0 is locally bounded. This assumption can be relaxed by,

[2] Although the bootstrap control assumption may look too restrictive, it should be noticed that in many cases (e.g., a mechanical system with a non-negligible friction) a bootstrap controller can be easily constructed.

[3] Note that the mapping $\mathbf{G}_0$ is independent of its τ argument. It appears as a dummy argument for a clearer presentation later on.

for instance, introducing a morphism $\widehat{\mathbf{G}}_0 : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ which assigns for each set $\mathbf{G}_0(\xi)$ a bounded set $\widehat{\mathbf{G}}_0(\mathbf{G}_0(\xi)) \subseteq \mathbf{G}_0(\xi)$ and using $\widehat{\mathbf{G}}_0(\mathbf{G}_0(\xi))$ instead of $\mathbf{G}_0(\xi)$.

We define three jump sets:

1) D_0, as the conditions required to switch u_0 off.
2) D_1, as the conditions required to switch u_0 on.
3) D_2, as the conditions when $x = (\xi, k)$ is such that $\xi \in \mathcal{O}_k$, $k \in \mathbb{K}$, while $\ell = 1$.

$$D_0 \; := \; \{x \mid \mathbf{G}_0(\xi) \neq \emptyset\} \times \{0\}, \tag{14}$$
$$D_1 \; := \; \mathrm{cl}(\{x \mid \xi \notin \mathcal{F}_k,\ k \in \mathbb{K}\}) \times \{1\}, \tag{15}$$
$$D_2 \; := \; \{x \mid \xi \in \mathcal{O}_k, k \in \mathbb{K}\} \times \{1\} = D \times \{1\}. \tag{16}$$

We also define two flow sets:

1) C_0, as the region in which the trajectory flows with bootstrap control and $\ell = 0$.
2) C_1, as the region in which the trajectory flows "in funnels", i.e., such that $\xi \in \mathcal{F}_k$, $k \in \mathbb{K}$ and $\ell = 1$.

$$C_0 \; := \; \mathrm{cl}(\{x \mid \mathbf{G}_0(\xi) = \emptyset\}) \times \{0\}, \tag{17}$$
$$C_1 \; := \; \{x \mid \xi \in \mathcal{F}_k,\ k \in \mathbb{K}\} \times \{1\} = C \times \{1\}. \tag{18}$$

With these definitions above, we can now redefine the closed-loop hybrid control system as:

$$H_{cl} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\ell} \end{pmatrix} = \begin{pmatrix} f_{k\ell}(\xi) \\ 0 \end{pmatrix} & \begin{pmatrix} x \\ \ell \end{pmatrix} \in C_0 \cup C_1, \\ \begin{pmatrix} x^+ \\ \ell^+ \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} & \begin{pmatrix} x \\ \ell \end{pmatrix} \in D_1, \\ \begin{pmatrix} x^+ \\ \ell^+ \end{pmatrix} \in \begin{pmatrix} \mathbf{G}_{k\ell}(\xi) \\ 1 \end{pmatrix} & \begin{pmatrix} x \\ \ell \end{pmatrix} \in D_0 \cup D_2, \end{cases} \tag{19}$$

where $k\ell$ (the index of $f_{k\ell}$ and $\mathbf{G}_{k\ell}$) is the product of $k \in \mathbb{K}$ and $\ell \in \{0, 1\}$ and $f_0 := (1, f(z, u_0(\tau, z)), 0)$.

Note that there is a non-empty intersection between D_1 and D_2 (corresponding to $\ell = 1$ and $\xi \in \partial\mathcal{O}_k$), so that when (x, ℓ) belongs to this intersection, any of the jumps in the second and third lines of (19) can take place.

G. Theorem 2

Every solution (x, ℓ) to the hybrid system (19) starting at $(x_0, \ell_0) := (x(0,0), \ell(0,0)) = (\tau_0, z_0, k_0, \ell_0) \in \mathbb{R}^{n+1} \times \mathbb{K} \times \{0, 1\}$ is complete, and for each such solution there exist $t' \in \mathbb{R}_{\geq 0}$ and $i' \in \{0, 1, 2\}$ such that (x, ℓ) is guaranteed to satisfy $(\tau(t,i), z(t,i)) \in \mathcal{F}_{k(t,i)}$ with $k(t,i) \in \mathbb{K}$ and $\ell = 1(t,i)$ for all $(t,i) \in \{(t,i) \in \mathrm{dom}(x)\}$ satisfying $t \geq t'$ and $i \geq i'$.

H. Theorem 3

Consider the hybrid system (19) defined for a closed system of funnels Σ. Assume that the following conditions apply for Σ:

1) The index set $\mathbb{K}$ is finite, i.e., $K < \infty$.
2) The entrance set for each funnel, $\mathcal{E}_k$ ($k \in \mathbb{K}$), is compact.
3) For each $k \in \mathbb{K}$ such that the funnel $\mathcal{F}_k$ has an infinite temporal depth (i.e., $\mathrm{dom}(\mathbf{F}_k) = [0, \infty)$), we have that:
   a) The domain $\mathrm{dom}(\mathbf{E}_k)$ is of the form $[0, T_\mathcal{E}]$, where $T_\mathcal{E} \in \mathbb{R}_{\geq 0}$.
   b) The controller u_k, the funnel $\mathcal{F}_k$ and its entrance $\mathcal{E}_k$ are time invariant, that is $u_k = u_k(z)$ and the mappings $\mathbf{F}_k$ and $\mathbf{E}_k$ are constant on their domains.[4] For simplicity, we define: $F_k := \mathbf{F}_k(\tau)$ for all $\tau \in \mathrm{dom}(\mathbf{F}_k)$ and $E_k := \mathbf{E}_k(\tau)$ for all $\tau \in \mathrm{dom}(\mathbf{E}_k)$.
   c) For the system:
   $$\dot{z} = f(z, u_k(z)), \quad z \in F_k, \tag{20}$$
   there are no finite escape times and, for each compact set $S_k \subset F_k$ there exists a time $T \geq 0$ so that each solution starting in S_k either terminates within time T or else, reaches E_k within time T.
4) The bootstrap control $u_0(z)$ is such that every solution z to (1) with $u = u_0(z)$ eventually reaches the set $\mathcal{U} := \bigcup_{k \in \mathbb{K}} \bigcup_{\tau \in \mathrm{dom}(\mathbf{E}_k)} \mathrm{int}(\mathbf{E}_k(\tau))$ in finite time.[5]

Then, there exists a compact set $\mathcal{X} \subset \mathbb{R}^{n+1} \times \mathbb{K} \times \{1\}$ such that the ω-limit set of any compact set of initial conditions $S \subset \mathbb{R}^{n+1} \times \mathbb{K} \times \{1, 2\}$ is a subset of $\mathcal{X}$, that is, $\omega(S) \subseteq \mathcal{X}$, and is therefore uniformly globally asymptotically stable (UGAS).

I. Corollary 1 (Reduced Set $\mathcal{X}$)

In theorem 3 we showed that under some assumptions, solutions of the system from any compact set S of initial conditions will eventually reach a compact set $\mathcal{X}$ and remain there forever. Before further specifying, $\mathcal{X}$ was defined as a union of the infinite-time reachability sets of all trajectories (ξ, k, ℓ) starting with $\xi \in \mathcal{E}_k$ and $\ell = 1$, for any $k \in \mathcal{K}$. We showed that $\mathcal{X}$ contains a UGAS set to which all trajectories converge. The smaller the set $\mathcal{X}$ is, the tighter is the bound on the UGAS set. However, it is intuitive to see that for some systems of funnels, it can be inferred that $\mathcal{X}$ can be smaller than its most general definition, if we take into account the graph structure with respect to which the system of funnels is defined. Consider the system of funnels Σ for which $\mathbb{K}$ is the set of funnel indices and Υ is its set of edges. Define $\mathcal{G}(\mathbb{K}, \Upsilon)$ to be a directed graph of which the elements of $\mathbb{K}$ are the vertices and the elements of Υ are the edges.

Define $\mathcal{G}^c(\mathbb{K}^c, \Upsilon^c) \subseteq \mathcal{G}(\mathbb{K}, \Upsilon)$ to be the *union of all cycles* in $\mathcal{G}$. Also, define $\mathbb{K}^\infty$ as the *union of all sinks* in $\mathbb{K}$. Define a union of all vertices from cycles and all sinks in $\mathcal{G}$ as $\bar{\mathbb{K}} := \mathbb{K}^c \cup \mathbb{K}^\infty$. Clearly, $\bar{\mathbb{K}} \subseteq \mathbb{K}$. Let $\hat{\mathbb{K}}$ be the set of all vertices from $\mathbb{K}$ which are elements of any path connecting any two vertices from $\bar{\mathbb{K}}$, that is:

$$\hat{\mathbb{K}} := \{k \in \mathbb{K} \mid \exists v \in \bar{\mathbb{K}},\ \exists w \in \bar{\mathbb{K}},\ \exists P_\mathcal{G} \ \text{s.t.}\ \{v, k, w\} \subset P_\mathcal{G}\} \tag{21}$$

We conclude that $\hat{\mathbb{K}}$ is the reduced set of funnel indices which is essential to define $\mathcal{X}$. Now, the definition of $\mathcal{X}$ can be replaced from $\bigcup_{k \in \mathbb{K}} \mathcal{R}_k$ to $\bigcup_{k \in \hat{\mathbb{K}}} \mathcal{R}_k$, where $\mathcal{R}_k$ is defined

[4] Note that in this case, equation (6) describes the flow map.

[5] This assumption is equivalent to the original bootstrap assumption.

as the infinite-horizon reachable set of the dynamic system

$$\begin{cases} \dot{\tau} &= \begin{cases} 1 & \text{if } \operatorname{dom}(\mathcal{F}_k) = [0, T_k] \\ -\tau & \text{if } \operatorname{dom}(\mathcal{F}_k) = [0, \infty) \end{cases} \\ \dot{z} &= f(z, u_k(z)) \\ \dot{k} &= 0 \\ \dot{\ell} &= 0 \end{cases} \tag{22}$$

III. DEMONSTRATING EXAMPLE

A. Wing Swing Motion - Task Description

We demonstrate the use of our framework on a toy problem of inducing repetitive wing-swing motion for a simple pendulum from any initial condition. The pendulum's dynamics are described by the equation of motion:

$$I\ddot{\theta} + b\dot{\theta} + mgl\sin(\theta) = M, \tag{23}$$

where θ is the angle of the pendulum measured counter-clockwise from its stable rest position, M is the joint torque, b is viscous friction, m is the pendulum's mass, l is the pendulum's length, g is the gravity constant and I is the pendulum's inertia. The values of the parameters are those used in [3]. The task is to achieve an indefinite motion from a low angle of $\theta_L := 30^\circ$ to a high angle of $\theta_H := 150^\circ$ and vice versa.

B. Hybrid Controller Design Using the Systems-of-Funnels Framework

For simplicity, the bootstrap control is $u_0 = 0$. Therefore, we must consider the top and bottom rest configurations, corresponding to $\theta_\pi := 180^\circ$ and $\theta_0 := 0^\circ$, in our system-of-funnels design. The state of the system is defined as $z := (\theta, \dot{\theta}) \in \mathbb{R}^2$, and the input is $u := M \in \mathbb{R}$. Initially, we find 4 nominal open-loop trajectories for z using a Bi-directional RRT algorithm (see [6] for details):

1) A trajectory from $(\theta_0, 0)$ to $(\theta_L, 0)$, denoted φ_1
2) A trajectory from $(\theta_L, 0)$ to $(\theta_H, 0)$, denoted φ_2
3) A trajectory from $(\theta_H, 0)$ to $(\theta_L, 0)$, denoted φ_3
4) A trajectory from $(\theta_\pi, 0)$ to $(\theta_L, 0)$, denoted φ_4

Note that both angles θ_L and θ_H can maintain their values in equilibrium when the input torque is $0.5mgl$. Figure 1 shows the four obtained nominal trajectories in state space.

We now construct a system of four funnels: $\Sigma = \{\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3,\ \mathcal{F}_4\}$. Each of the funnels is constructed about one of the obtained nominal trajectories. The purpose of $\mathcal{F}_1$ is to drive trajectories from the vicinity of $(\theta_0, 0)$ to the vicinity of $(\theta_L, 0)$. The purpose of $\mathcal{F}_2$ is to drive trajectories from the vicinity of $(\theta_L, 0)$ to the vicinity of $(\theta_H, 0)$. The purpose of $\mathcal{F}_2$ is to drive trajectories from the vicinity of $(\theta_H, 0)$ to the vicinity of $(\theta_L, 0)$. The purpose of $\mathcal{F}_2$ is to drive trajectories from the vicinity of $(\theta_\pi, 0)$ to the vicinity of $(\theta_L, 0)$. This can be summarized by the set of edges for this system of funnels:

$$\Upsilon := \{(1,2),\ (2,3),\ (3,2),\ (4,2)\} \tag{24}$$

These tasks can be achieved using the directed graph shown in Fig. 2 with respect to which the system-of-funnels

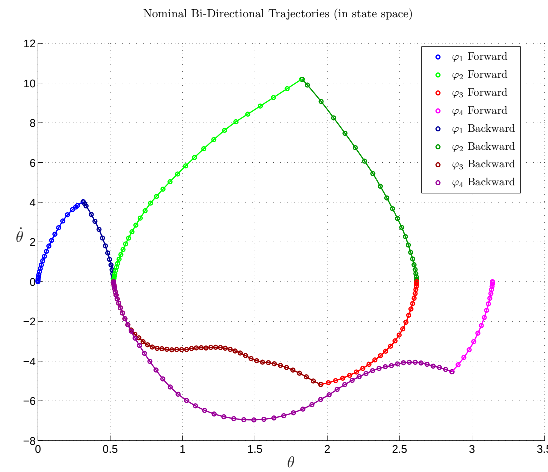

Fig. 1. All four trajectories in state space

is constructed. One can notice that there are no sinks in this graph and there is only one cycle, comprised of funnels $\mathcal{F}_2$ and $\mathcal{F}_3$ and the edges connecting them. This shows that when using the hybrid controller proposed in this paper, trajectories of the hybrid control system starting from any initial condition will eventually cycle through these two funnels.

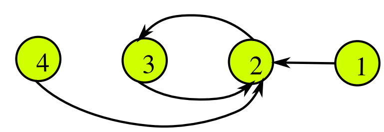

Fig. 2. A graph of 4 funnels for wing swing motion of a pendulum.

We chose to define the domain of the entrances to each of the funnels as $\{0\}$. Therefore, $\mathbf{E}_k(0)$ is the only non-empty set of the mapping $\mathbf{E}_k$ ($k \in \mathbb{K} := \{1, 2, 3, 4\}$) and it is in the interior of the set $\mathbf{F}_k(0)$.

Since funnels $\mathcal{F}_2$ and $\mathcal{F}_3$ are interconnected, and have to satisfy the co-dependent conditions:

- the outlet of funnel 3 is in the entrance of funnel 2, and
- the outlet of funnel 2 is in the entrance of funnel 3,

we begin by designing these two funnels first. Since the end states of each nominal trajectory are equilibria, the funnels' outlets $\mathcal{O}_2$ and $\mathcal{O}_3$ were chosen by obtaining conservative estimates (using an SOS program) of the regions of attraction about the equilibria $z_{e,L} := (30^\circ, 0)$ and $z_{e,H} := (150^\circ, 0)$ (both maintained by the nominal torque $u_{e,L} := u_{e,L} := 0.5mgl$) when using a locally stabilizing time-invariant LQR control. Then, funnels $\mathcal{F}_2$ and $\mathcal{F}_3$ were computed backwards in time using SOS programming as in [3]. The input u_k associated with the funnel k was a time-varying LQR control designed for local tracking of the nominal trajectory about which the funnel is constructed. When using this control law and this funnel construction, trajectories starting inside $\mathbf{E}_k$ do not reach the boundaries of $\mathbf{F}_k(\tau)$ for any $\tau \in \operatorname{dom}(\mathbf{F}_k)$.

This initial design does not guarantee that funnels $\mathcal{F}_2$ and $\mathcal{F}_3$ are properly connected according to the co-dependent conditions stated 1. and 2. above. In order to solve this problem, we extend the durations of the nominal trajectories at their target equilibrium state such that effectively a local stabilization about each of the equilibria is performed for some finite time duration. For instance, the new nominal trajectory φ_2 will be a concatenation of the original trajectory φ_2 and an additional constant trajectory with the angle value of θ_H and zero velocity for some finite amount of time, to allow trajectories to get closer to the equilibrium $(\theta_H, 0)$, thus shrink the outlet $\mathcal{O}_2$ without affecting the entrance $\mathcal{E}_2$ at all. This extension of the trajectories was done by trial and error - just enough to have a sufficient margin between the entrances and the outlets of the corresponding funnels (in practical applications, the size of the margin should be chosen by taking into account the expected disturbances and noise in the system).

With the appropriately extended trajectories, the resulting funnels have a smaller outlet which fits well inside the entrance of the opposite funnel. Figure 3 shows the projections of funnel $\mathcal{F}_2$ into the state space about the augmented nominal trajectory φ_2. The green and the blue colors of the funnel represent the forward and the backwards sections for the bi-directional trajectory. These were constructed such that the last green section is inside the first blue section.

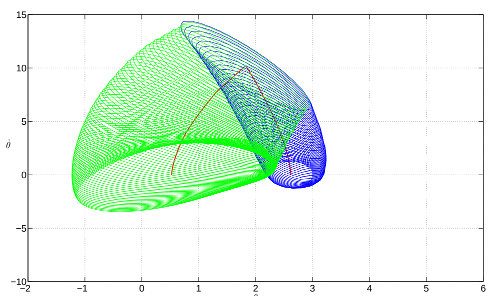

Fig. 3. $\mathcal{F}_2$ - state-space plot

In order to design funnels $\mathcal{F}_1$ and $\mathcal{F}_4$, we notice that both of them have to connect into funnel $\mathcal{F}_2$. Therefore, we chose the outlets $\mathcal{O}_1$ and $\mathcal{O}_4$ to be in the interior of the entrance $\mathcal{E}_2$, and then constructed the funnels $\mathcal{F}_1$ and $\mathcal{F}_4$ backwards in time as usual. Figure 4 shows the boundaries of $\mathbf{F}(0)$ and $\mathbf{F}(T)$ for all the four funnels and demonstrates that all of the conditions that have to be satisfied for $\Sigma = \{\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3,\ \mathcal{F}_4\}$ to be a system-of-funnels are indeed satisfied.

In order to simplify coding, we chose to allow a switch from bootstrap control to 'in-funnel' control only when the state of the system (z) reaches the sets $\mathbf{E}_1(0)$ or $\mathbf{E}_4(0)$, in which case a jump of k will be to either 1 or 4, respectively. Therefore, we have that $\mathbb{K}' := \{1, 4\}$ utilised in $\mathbf{G}_0$ instead of $\mathbb{K}$.

C. Simulated Motion

Figure 5 shows the trajectory of the closed-loop system when started from the initial conditions $(\tau_0, z_0, k_0, \ell_0) = (0, -90°, 5, 1, 1)$ as well as the entrance to funnel 1. Initially, the hybrid controller detects that $(\tau, z) \notin \mathcal{F}_k$ and that $\mathbf{G}_0(\tau, z)$ is empty. As a result a flow with bootstrap control occurs, until the value of z is inside the set $\mathbf{E}_1(0)$, when the bootstrap control is switched off and the trajectory flows in $\mathcal{F}_1$ for T_1 time, then cycles through funnels $\mathcal{F}_2$ and $\mathcal{F}_3$ as desired.

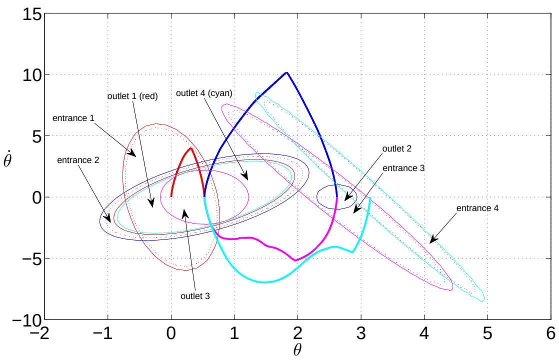

Fig. 4. Final construction of all four Funnels - only start and end sections (and entrances)

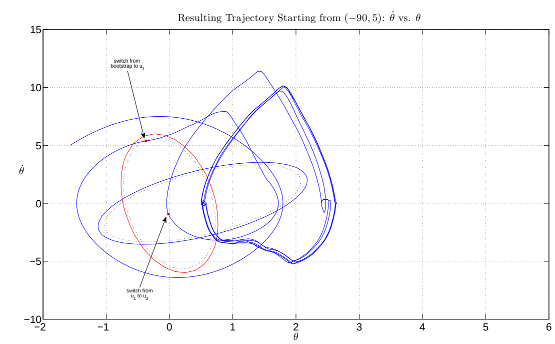

Fig. 5. A trajectory from the initial condition $(-90°, 5)$ - state-space plot

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