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Idealization and External Symbolic Storage: The Epistemic and Technical Dimensions of Theoretic Cognition

Abstract:

This paper explores some of the constructive dimensions and specifics of human theoretic cognition, combining perspectives from (Husserlian) genetic phenomenology and distributed cognition approaches. I further consult recent psychological research concerning spatial and numerical cognition. The focus is on the nexus between the theoretic development of abstract, idealized geometrical and mathematical notions of space and the development and effective use of environmental cognitive support systems. In my discussion, I show that the evolution of the theoretic cognition of space apparently follows two opposing, but in truth, intrinsically aligned trajectories. On the epistemic plane, which is the main focus of Husserl's genetic-phenomenological investigations, theoretic conceptions of space are progressively constituted by way of an idealizing emancipation of spatial cognition from the concrete, embodied intentionality underlying the human organism's perception of space. As a result of this emancipation, it ultimately becomes possible for the human mind to theoretically conceive of and posit space as an ideal entity that is universally geometrical and mathematical. At the same time, by synthesizing a range of literature on spatial and mathematical cognition, I illustrate that for the theoretic mind to undertake precisely this emancipating process successfully, and further, for an ideal and objective notion of geometrical and mathematical space to first of all become fully scientifically operative, the cognitive support provided by a range of specific symbolic technologies is central. These include lettered diagrams, notation systems, and more generally, the technique of formalization, and require for their functioning various cognitively efficacious types of embodiment. Ultimately, this paper endeavors to understand the specific symbolic-technological dimensions that have been instrumental to major shifts in the development of idealized, scientific conceptions of space. The epistemic characteristics of these shifts have been previously discussed in genetic phenomenology, but without devoting sufficient attention to the constructive role of symbolic technologies. At the same time, this paper identifies some of the irreducible phenomenological and epistemic dimensions that characterize the functioning of the historically situated, embodied and distributed theoretic mind.

***Keywords*: Theoretic Cognition – Spatial Cognition – Distributed Cognition – Genetic Phenomenology – Enculturation - Formalization– Idealization – Symbolic Technology – Embodiment**

1. Introduction: The Riddle of Theoretic Cognition

In the ambitious, multidisciplinary project outlined in his book *The Origin of the Modern Mind*, Donald (1991) proposes that human cognition has evolved through four more or less distinct stages, where the later, more abstract and complex cognitive stages build upon the structures of the earlier ones. All these cognitive stages, Donald claims, are characterized by the development of a specific form of memory. First, there is the *episodic stage*, which is the stage of the evolutionary development of episodic memory. An organism's episodic memory is its capacity to recall specific past events and apply them in a straightforward manner to present circumstances. This capacity can be observed both among human and non-human mammals. The episodic stage is followed by the *mimetic stage*. The mimetic stage was brought about by the evolutionary advent of mimetic behavior, which allows, Donald notes, the embodied organism, on a basic level, to 'represent' events. Mimetic behavior is characterized by the voluntary use of one's own body as a communicative and representational device. This is manifested through human abilities such as gestural communication, embodied skill, mime and the imitation of bodily movements of others (Donald 2001, p. 263). It has been suggested that the evolutionary development of the capacity for bodily mimesis has facilitated the establishment of relatively developed forms of intersubjective behavior that involve empathy, shared attention, complex imitation, and ultimately, intentional communication (Tomasello 2008; Zlatev 2008). Mimesis, in the way Donald understands it, appears to be restricted to the human animal, and in its more basic forms, to some other primates. Third, following the mimetic stage, there is the *mythic stage.* The mythic stage is synonymous with the advent of speech, the capacity for complex modes of symbolic reference, and more generally, sophisticated modes of linguistically mediated cognition and communication. Apparently, the capacity to fully engage in such developed, linguistically mediated types of cognitive and communicative activity is limited to human organisms. This would suggest that the advent of the mythic stage constitutes a distinctively human event in the overall process of the evolution of cognition.[1]

Within Donald's account of the evolutionary trajectory that led to the formation of the modern human mind, the mythic stage is finally followed by what Donald labels the *theoretic stage.*[2]

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[1] More recent research suggests that some animals, great apes for instance, are also capable of higher-order forms of symbolic cognition and communication (Savage-Rumbaugh & Lewin 1994). However, in the case of (non-human) primate symbolic cognition and communication, certain limitations still apply, for instance when it comes to acquiring and using a comprehensive vocabulary.

[2] The sequence of the evolutionary stages that Donald identifies here, and that of their distinctive symbolic means, is roughly mirroring that of the decisive stages that can be observed on the level of ontogenetic development (see my discussion in the section *Symbolic Technologies as Cognitive Tools: From Gesture to Notation* (3.1).

This stage, which on an evolutionary scale only seems to appear after the advent of relatively complex, i.e., socially, administratively and functionally differentiated human societies, is characterized by the development of a theoretic culture that facilitates highly abstract, systematized and formalized types of cognitive activity. Donald argues that the innovative development and effective cognitive use of external symbolic storage and representational devices is central to the functioning of theoretic culture. These are devices that take a material, enduring form in the environment, outside of the brain and body of the individual human organism. Donald also refers to these material devices as symbolic technologies; the domain of symbolic technologies, following Donald, comprising those material cultural objects "that are specifically designed to represent, communicate and store knowledge" (Donald 2010, p. 71). Examples for symbolic technologies are pictures, diagrams, and writing systems. Importantly, all of these symbolic technologies are essential for reliably performing a wide range of highly abstract, higher-order cognitive tasks. Moreover, it appears as if the same technologies are, at least in some instances, intrinsic to the actual production and advancement of forms of theoretic knowledge.

The modes of cognitive activity that can be associated with what Donald refers to as theoretic culture possess some distinctive epistemic characteristics. I will briefly list here the four which I consider to be the most important ones. First, there is the *systematization of enquiry.* This systematization expresses itself, e.g., in the explanation of rules, generalization, the use of deductive reasoning, and the logical consistency of formulated arguments. Second, there is the *development of a theoretical attitude*, which presupposes the cognitive detachment of the (theoretic) point of view from the particularity of the experienced 'here and now' of the individual embodied human subject. Third, there is the *invention of the method of idealization.* The method of idealization allows the theoretic mind to posit some cognitive constructs and entities as existing eternally and universally, independently of the domain of concrete and particular empirical phenomena. Fourth, there is the *development of techniques of formalization*, i.e., the development of artificial material-symbolic representations that are cognitively used, e.g., in mathematical thinking, to systematically describe processes in a decontextualized, schematic and highly abstract manner.

This paper explores some of the specific epistemic and technical dimensions of theoretic cognition, and of the theoretic cognition of space in particular. In my discussion I combine perspectives from Husserl's genetic phenomenological account of the genesis of scientific thought and of the form of intentionality constitutive for this thought, and historically informed distributed cognition approaches as well as recent psychological research that explore the cognitive role of both embodied and environmental structures. The specific goal of my discussion is to elucidate the nature of the constructive nexus between (the formation of) abstract theoretical constructs involving epistemic processes of idealization and mathematization, which is the almost exclusive focus of Husserl’s genetic-phenomenological investigations, and the cognitive and technical functioning of material-cultural and embodied structures in highly abstract domains of human thinking. The specific example I use in this paper is the theoretic development of a scientific notion of geometrical and mathematical space. I argue that this notion constitutes an intricate cognitive ‘construct’ whose distinctive formation and structure can only be properly understood if equally considering abstract theoretic as well as concrete embodied and extrasomatic processes and structures.

On the one hand, using Husserl’s discussion referred to above as a reference point, I illustrate that, on an epistemic plane, a truly scientific geometrical and mathematical notion of space is established, as it were ‘internally’, by way of a certain, abstracting cognitive procedure that involves the constitution of a specific, theoretic-scientific type of intentionality. This procedure, by way of systematized processes of idealization, liberates the scope of the human organism’s conceptions of space from the finite structure of spatial bodies concretely encountered in first-person, embodied experience, and ultimately also makes it possible for the human mind to theoretically conceive of and posit space as an abstract and ideal entity which is universally mathematical. On the other hand, drawing on a range of specific historical examples, I demonstrate that for the theoretic mind to fully accomplish this idealizing liberation of its conceptions of space from the embodied experience of concrete bodies, and to arrive at a form of intentionality that lends itself to a scientific understanding of spatial nature, the concrete, 'external' cognitive support provided by a range of innovatively combinatory, visual-spatial symbolic technologies remains central.

Overall, this paper aims to elucidate some of the specific symbolic-technological dimensions that are crucial to those major idealizing progressions in the theoretic cognition of space whose epistemic structure and cognitive implications have been comprehensively discussed in genetic phenomenology. A particular focus in this regard is on the concrete material-symbolic means that have allowed for the effective scientific use of abstract, idealized constructs pertaining to the theoretic cognition of space. In doing so, this paper seeks to rectify the relative neglect of these material-symbolic dimensions in phenomenological literature on the cognition of space, as well as in phenomenological discussions of the formation of a theoretic-scientific form of intentionality. In addition, it seeks to identify and specify some of the irreducible phenomenological and epistemic dimensions that characterize the functioning of the historically situated, embodied and distributed theoretic mind.

2. The Internal Dimensions of Theoretic Cognition: Idealization and Mathematization

It can generally be said that all elementary forms of spatial cognition have their primary basis in the embodied perception and experience of space. This applies both to non-human and human animals alike. A striking example is the cognitive functioning of systems of spatial orientation. For instance, both human organisms as well as other primates directly or indirectly refer to their own body and its place to orient themselves spatially. In some behavioral instances, the individual body may directly constitute the main operative reference point of spatial orientation, in relation to which the surrounding spatial environment is structured. In other instances, where external reference points (e.g., specific, perceptually salient material objects of the environment) are used as means for orientation, the body likewise remains cognitively indispensable as it facilitates the effective cognitive coupling of the organism to such external structures.[3]

In contrast to modes of spatial cognition that are directly informed by the cognitive agent's own embodiment, human organisms are also capable of a type of cognitive behavior that paleoanthropologist Leroi-Gourhan refers to as a "symbolic 'domestication'" of spatial cognition (1993, p. 313). This type of cognitive domestication can be aligned with a general process of the domestication of the human mind that followed the development of symbolic means of

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[3] In many behavioral contexts, the animal organism cognitively structures its spatial environment by way of direct reference to the place of its own, perceiving and moving body. Take the case of the human organism's embodied cognition of space, which is generally characterized by an ego-centric frame of reference. In such a frame of reference, the perceiving individual's own living body, the cognitively indispensable "*medium of all perception*" (Husserl 1989, p. 61), functions, as both phenomenologists and psychologists have traditionally stressed, as a "relational center of all spatial orientations" (Husserl 1997, p. 109; see also Piaget & Inhelder 1956; Ströker 1987). This is due to the fact that this body, in the first-person perception of space, constitutes an "ever-abiding point of reference" that gives rise to the appearance of spatial directions such as left and right, front and back, and above and below (Husserl 1997, p. 66). At the same time, more recent psychological and linguistic research has shown that in various developmental, biological and cultural contexts, (some of) the structures governing the organism's cognition of space, and the organism's spatial orientation in particular, have their basis in allocentric frames of reference (see, e.g., Gentner 2007; Haun, Rapold, Call, Janzen, & Levinson 2006; Levinson 2003; Sinha & de Lopez 2000; see also Acredolo 1978). In this case, a salient and stable feature of the environment, existing independent of the organism's own living body, may serve as the dominant source of reference for orientation in space. Allocentric frames of spatial reference, which can be observed already amongst human infants and also among non-human primates (Acredolo 1978; Gentner 2007; Haun, Rapold, Call, Janzen, & Levinson 2006), nevertheless continue to rely indirectly on the organism's own, living body. This is because in an organism's active spatial comportment, an allocentric frame of reference becomes practically useful for orientational purposes only if the absolute referent it also effectively related to place of the individual organism's own living and moving body.

communication, including that of the technique of writing emerging roughly 5,000-6,000 years ago (see Goody 1979; for further enduringly instructive discussions of the cognitive implications of the development of writing and literacy see, e.g., Goody & Watt 1963; Havelock 1963; Havelock 1982). The phenomenon of the symbolic domestication of space becomes generally manifest through the (enculturated) human organism's ability to effectively use a system of symbols to subject its own, embodied access to space to a process of reflection, thus allowing for transindividual generalizations and comparisons (see Leroi-Gourhan 1993, p. 283). This is why the process of symbolic domestication necessarily involves, in the case of the cognition of space, the partial superimposition of a more or less determinate, systematized and transindividual symbolic framework on the indeterminate and dynamic structures originally governing the individual human organism's own, experiential cognition of its spatial environment.

As indicated, such superimposition is made possible by the evolution of human symbolic thinking and communication. This evolution, it is often suggested, begins with the development of gestural communicative means (Armstrong & Wilcox 2007; Corballis 2002; Donald 1991; Tomasello 2008), and, via the development of the symbolic system of speech, ultimately extends to the development and effective use of artificial material-symbolic storage and representation systems. As I show in more detail later, these material-symbolic systems ultimately facilitate complex and abstract mathematical considerations about ideal yet exact determinations of spatial nature (for comprehensive discussions of the evolution of hominid symbolic thinking and communication see, e.g., Deacon 1997; Donald 1991; Donald 2001; Tomasello 2008). The symbolic 'domestication' of the spatial cognition of human organisms, despite its name, entails a significant cognitive expansion, even liberation. Arguably most important here is that the use of ever more complex symbolic means facilitates a cognitive as well as communicative detachment and liberation from the confines of the spatial situation as it is directly experienced through the senses of the human organism's own body (see Hockett 1960). A striking example of such detachment is what is referred to as displaced reference, i.e., the human ability to use symbolic means to refer to entities and events that are spatially and temporally removed from the immediate situation of the embodied, cognitive and communicative agent. Ultimately, the symbolic domestication of spatial cognition thus allows the human mind to virtually extend its cognitive grasp on the world to spaces that lie beyond the limited confines of the human organism's own immediate embodied, access to and hold on the environment. Importantly, this includes the comprehension of spaces that are of a highly abstract and ideal nature, e.g., infinite geometrical and mathematical spaces.

As indicated, the symbolic domestication of space can take a variety of cognitive forms depending on the types of symbolic media utilized. In the following sections, I focus on a cognitively extraordinarily constructive, yet at the same time restrictive form of the symbolic domestication of space. This form is the idealizing theoretic transposition of space into an inherently geometrical and mathematical and as such quantitatively exactly determined structure. According to Husserl's insightful phenomenological-philosophical account of the genesis and history of a mathematical-scientific notion of nature (1970), for this transposition to be historically accomplished, the human organism's spatial cognition had to be subjected to two subsequent processes of idealization. Such processes made it possible to theoretically conceive of and determine spatial entities independent of the embodied experience of concrete bodies. Apparently, the first idealization came to fruition in the geometrical thought of Ancient Greece, the second with the modern emergence of a scientific worldview around the sixteenth/seventeenth century in Europe. In what follows I will briefly review both forms of idealization, and develop on the epistemic change they entail for the ways in which space is theoretically posited and conceived.

2.1 The First Idealization of Space: The Institution of the Theoretic Science of Geometry

The Greek historian Herodotus famously reports that the first forms of an at least practically more or less systematized quantitative determination of space, and the first developed form of geometrical practice, consisted in the applied technique of surveying farmland in Ancient Egypt. We can only speculate today whether the technique of surveying farmland really was the first true form of applied geometry or not. Undeniably, however, is that the emergence of agricultural economies, with their complex systems of taxation tied to farmland, created the pressing need for a developed form of applied geometrical practice capable of producing reliable, precise results. Such developed form of geometrical practice (as well as that of a reliable form of mathematical practice allowing to determine the appropriate taxation sums) necessarily requires the development and use of a range of symbolic technologies, most importantly, that of writing.[4] Not surprisingly then, it has been claimed that the development of writing – and by extension that of a more advanced form of mathematics – was primarily a direct response to a newly emerging need for means allowing for the recording and reliably storing of considerable amounts of information (see Ritter 1995, p. 45). For obvious reasons, such needs ultimately only became pressing within the context of the emergence

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[4] On the beginnings of written forms of mathematics, which apparently emerged first of all in Egypt and Mesopotamia, see Joseph (2011, chapter 3 and 4) and Ritter (1995).

of hierarchically organized agricultural economies (see Schmandt-Besserat 1996, chapter 6), and more specifically, with the creation of agricultural surplus production that demands proper management and accounting practices (see Goody 1986). Telling in this overall context is that the Ancient Greek term 'geometry' itself makes direct reference to the practice of surveying farmland; literally translated, 'geometry' signifies measuring or measurement of land or earth.

That the sense of abstract, theoretical geometrical thinking must have first evolved out of, and thus developmentally presupposes, the domain of applied practice (e.g., the practice of surveying of farmland) is also stressed by Husserl. In his genetic-phenomenological account of the constitution of idealizing geometrical and mathematical conceptions of space and world, Husserl argues that the cultural practice of surveying, being a constructive “pregeometrical achievement”, provides a concrete “meaning-fundament for geometry” (1970, p. 49). That is to say, in Husserl’s view, the practice of surveying concrete empirical spatial bodies has thus been a condition necessary for the human mind arriving at a comprehensive notion of ideal geometrical spatial bodies, bodies which are the legitimate and legitimating objects for an abstract theoretical-scientific mode enquiry. Specifically, Husserl suggests that without the basic technique of measuring and comparing the magnitude of directly accessible spatial areas, there ultimately would have been no possibility for the human mind to establish a layer of meanings that allows for the “great invention of idealization” (Husserl 1970, p. 49). Generally speaking, idealization is a method that allows the human mind to theoretically posit and examine entities that exist independent of the confines of the human individual’s own, concrete, embodied intentionality, and of the particular bodies that are directly encountered in experiential cognition (see Husserl 1970, p. 301). The invention of the method of idealization is crucial for the “invention of the ideal world of geometry”, i.e., for the theoretic comprehension and systematic scientific construction of geometrical formations that are of a both ideal and objective nature (Husserl 1970, p. 49).

Following Husserl’s account, a first decisive step with regard to this theoretic achievement is reached when, within the applied context of the cultural practice of surveying, the proto-idealizing possibility is discovered to take certain, directly empirically accessible spatial forms as constant and definite, and as such standardized, measures (Husserl 1970, p. 28). Such discovery, facilitating a rudimentary metric determination of spaces, constitutes a first step toward the emancipation of geometrical thinking from its original ties with particular, directly empirically accessible spatial bodies. This is despite the fact the standardized measures preparing the way for the development of an idealizing geometrical thinking initially were, in all likelihood, directly modeled on, and remained attached to, concrete, tangible and spatially limited material things. Naturally, for practical reasons these were often things that were readily and constantly available in the human organism's encounter with its spatial environment. Most importantly, these were the constantly available parts of the human body. Note in this regard common measures the name of which makes reference to parts of the human body, e.g., 'foot'.[5]

However, even with the definition of standardized measures, geometry, as long as it remained integrated into a horizon of practical purposes and bound to the measurement of concrete, material bodies, did not yet require a general systematization and explanation of its rules, and as such did not yet attain the status of a science. Apparently, this general limitation applies to the form of geometrical and mathematical thinking found in the ancient, complex state organizations of both Egypt and Babylon, for example.[6] As it seems, in the overall process of the evolution and historical development of geometrical thinking, the idealization and systematization of geometry (and thus geometry's theoretic institution as a science) was only fully realized in the philosophical and mathematical traditions of Ancient Greece.

A substantive revision of the concept of measure epitomizes the momentous shift that occurred in Ancient Greek geometrical thinking, where this shift eventually led to the formulation of geometry as a theoretical science. Along with the development of a theoretical attitude, the measure, possibly as early as in Pythagorean thinking, is increasingly no longer conceived of as being tied to and becoming manifest in concrete and particular spatial bodies. For much of Ancient Greek thinking, it has been noted, the prevailing notion is rather that "a measure is not the things measured, but the measure (…) is the beginning (or principle) of number" (Heath 1981, p. 69). Thus liberated from its original ties with concrete, empirical nature, the measure can be theoretically conceived of and posited as an abstract and ideal norm allowing for the determination of all bodily nature. This is regardless of whether the body to be measured is actually existing and directly

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[5] For a brief discussion of the history, benefits and limitations of measures that were directly modeled on the human body see Kula (1986, chapter 5).

[6] This is despite the impressive mathematical and geometrical knowledge that was accumulated and applied in the institutions of both these states. In Egypt, it has been claimed, the domain of geometrical and mathematical techniques generally remained limited to the status of "merely applied arithmetic"; i.e., bound to practical applications that were relevant for the smooth functioning of the socio-economic apparatus (van der Waerden 1961, p. 31; but see Gillings 1972; Joseph 2011, chapter 3). Besides the aforementioned surveying and calculation of areas of farmland, elaborate geometrical and mathematical methods were, for example, used for structural and area calculations in the domain of architectural construction, as well as for a range of calculations necessary for taxation purposes (Kline 1972, p. 21). Similarly, it is commonly claimed that Babylonian geometry generally remained of an applied nature. The only notable exception is that in the context of the emergence of a more or less systematized form of astronomy in Babylonia, knowledge about the calculation of the radius and the area of circles existed that apparently did not serve any immediate practical purposes (Kline 1972, pp. 10-11).

empirically accessible or not. In this sense one may even regard the theoretical science of geometry, at least from a cognitive-ecological perspective that takes seriously the way cognitive processes occur and impact on real-world settings (see Hutchins 1995; Hutchins 2010), as a cognitively augmenting, artificial organ. This is because such science makes certain spaces theoretically accessibly and exactly determinable that the human mind, from within the basic domain of the embodied perception and experiential cognition of space, would have been unable to ever access.

The epistemically momentous emancipation of geometrical and mathematical conceptions from the directly accessible, concretely given spatial world has found its exemplary manifestation in the philosophical writings of Plato, as well as in the mathematical works of thinkers such as Euclid, Thales and Pythagoras, among others. In the Platonic framework, for instance, geometrical spatial forms are no longer conceived of as being derived from, and bound to, concrete material spatial bodies that can be directly experienced by the senses of the human body. Instead, these geometrical forms are understood as belonging to the realm of ideas. As is well known, within the Platonic framework, the realm of ideas is synonymous with a sphere of unchangeable, eternal, immaterial and hence empirically inaccessible truths. Due to the relative epistemic and ontological autonomy and priority that Plato generally attributes to the realm of ideas, these geometrical figures are thus understood as existing to some extent regardless of, and logically prior to, the concrete material manifestation in which they become physically tangible.[7] This in turn allows the theoretic human mind to legitimately posit and subsequently conceive of geometrical forms as ideal entities that exist always and everywhere the same. And by extension, with this shift toward a comprehensive idealizing understanding of geometrical forms, the goal of the theoretical construction of a logically coherent geometrical and mathematical system consisting of eternally and universally valid sentences, becomes an explicitly comprehensible and legitimate one.

Ultimately, however, it is only in Euclid's *Elements*, compiled around 300 BC, that the principles of geometry were first systematically incorporated, in written form, into one rigorous, logically coherent, axiomatic deductive framework. Such theoretic achievement is tantamount to the institution of geometry as science. The Euclidean systematization of geometry has as its constructive point of departure the axiomatic definition of ideal geometrical spatial figures such as point, line and surface. It is these ideal spatial figures, most of all that of the point, which establish a basis from

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[7] For instance, in his *Timaeus* Plato argues that the divine creator of the cosmos used ideal geometrical forms (lines, surfaces, and triangles) as the first grounds and principles to order the pre-existent world of chaos. In the *Republic*, the general priority Plato gives to the realm of ideas and ideal forms explicitly includes also that of number; both calculation and geometry are regarded as useful and spiritual exercises in so far as they concern the “numbers themselves, never permitting anyone to propose for discussion numbers attached to visible or tangible bodies” (525d5-6).

which, through the stringent use of deductive reasoning, the systematic construction of all sorts of geometrical formations can be achieved and their existence guaranteed. As is well known, Euclid begins his first book of the *Elements* with a definition of the spatial figure of the point. For Euclid, “a point is that of which there is not part” (Book I, Definition I). Importantly, the Euclidean notion of the point thus introduces (and subsequently takes for granted) a form of space that, in having no part, has neither body nor extension. The line, by comparison, is defined as “length without breadth” (Book I, Definition II). The surface, finally, is understood to be a plane “which has breadth and length alone” (Book I, Definition V). Clearly, what all these ideal geometrical figures have in common is that they are theoretically posited and conceived of as being truly disembodied: the Euclidean science of space is constructed on the notion of geometrical space being essentially composed of bodies that have no concrete, material existence.

In addition to its clearly defining ideal spatial figures, it is further of importance that the *Elements* also contain a theoretically developed and consistent formulation of geometrical space as extending infinitely in three dimensions. Central in this constructive accomplishment is first of all the notion of the ideal spatial figure of the line: for Euclid, “each in length finite line is extendable to an infinite one” (Book I, Postulate 2). In addition, Euclid also develops the theoretic notion of three-dimensional geometrical bodies (see Book XI, Definition I), i.e., of geometrical bodies that have the spatial dimensions of length, breadth and depth. Such bodies necessarily have to have surfaces, and these surfaces in turn must contain an infinite number of, in theory infinitely extendable, lines. Given that Euclidean space is conceived of as accommodating (ideal) three-dimensional bodies and lines extending infinitely in three directions, such space can thus be theoretically posited and legitimately conceived of as an ideal, infinite and three-dimensional geometrical totality. This theoretic achievement, it has been noted, is momentous. For concurrent with understanding of Euclidean space as an infinite ideal object, the system of Euclidean geometry, due to the rigorous, systematized and logically coherent nature of its own construction, can now itself be epistemologically and ontologically conceived of by the human mind as a “a totality formed of pure rationality” (Husserl 1970, p. 21). Henceforth, along with this notion of Euclidean space and geometry being intrinsically a rational totality, the theoretic mind can make claim for the possibility of a universally valid scientific determination of all actual and ideal spatial appearances. Ultimately, once the development of a systematic scientific notion of geometrical space has thus been accomplished by the theoretic mind, the geometrical constructions constituted, Husserl claims, thus attain a peculiar epistemic status that he refers to as “‘ideal’ objectivity” (Husserl 1970, p. 356). This is because the epistemic meanings conveyed by these idealized constructions remain for the most part unaffected by the contingencies of concrete history, and valid beyond the confines of concrete and particular, individually experienced situations.

Specifically, Husserl employs the notion of 'ideal objectivity' to take due account of the distinctive epistemic status of scientifically constructed and ideally conceived geometrical and mathematical objects. Objects of this type are distinctive, Husserl claims, in that, once scientifically constituted by the theoretic mind, they can be repeated innumerable times and yet remain all the time exactly the same objects. This aligns with Husserl's view that from the moment geometrical space is fully theoretically transposed into an axiomatically derived, transparent, logically coherent system comprising bodies that are ideal objects (as is paradigmatically the case in Euclidean geometry), this very system comes to anticipate the exact form and determinations of everything new that is from then on constructed on its basis (see Husserl 1970, p. 356). Overall, the geometrical science of space, once it truly has become a theoretic, rational science, according to Husserl thus makes accessible a cognitive domain, the meanings of which are valid and true regardless of particular spatial situations and historical contexts.

2.2 The Second Idealization of Space: The Mathematization of the Universe

Apparently, in spite of all its theoretic accomplishments, Greek geometry, including the Euclidean variety, still did not arrive at an overarching scientific notion of space where space is universally conceived of as being fully transposable into formal-mathematical idealities (Husserl 1970, p. 21). Two factors may have contributed to this. First, on the level of ontological conceptions, Ancient Greek thought, in spite of the emancipation of the idea of the measure it achieved, generally considered number and space as belonging to two heterogeneous ontological realms (Cassirer 1998, p. 8). Following from the idea of this ontological demarcation, the epistemic domains of spatial and mathematical cognition were thus generally regarded as being strictly distinct from one another. Second, Greek geometry, in spite of the Euclidean notion of infinite, three-dimensional space, did not yet develop a scientific worldview that aims and claims to master the infinite world, in its totality, by mathematical means.

It is commonly claimed that the systematic integration of the notion of infinite space into a scientific-mathematical worldview was only properly accomplished in the context of the so-called 'scientific revolution' occurring in Europe in the sixteenth and seventeenth centuries, a revolution of which the writings and conceptions of thinkers such as Galileo, Descartes and Newton are taken to be exemplary. During this period, an apparently astonishing progress in the overall evolution of human mathematical (and scientific) thinking and theoretic culture at large occurred. Following the account presented by the eminent historian of science Koyré (1957), this scientific revolution entailed both the undoing of a traditional cosmological worldview and the geometrical and mathematical rationalization of space. On the side of cosmology, a "conception of the world as a finite and well-ordered whole, in which the spatial structure embodied a hierarchy of perfection and value," is replaced by the conception of an "indefinite or even infinite universe no longer united by natural subordination, but unified only by the identity of its ultimate and basic components and laws" (Koyré 1957, p. viii). On the side of conceptualizations of space, by comparison, the geometrical-mathematical conception of space as "an essentially infinite and homogeneous extension" is in an overarching manner substituted for the classic Aristotelian notion of space, where this mathematical space is henceforth "considered as identical with the real space of the world" (Koyré 1957, p. viii). As a result of this general structural shift in the dominant theoretical understanding of space, the space of the world is thus increasingly equated in scientific thinking to an infinite, intrinsically mathematical totality. This totality, in all its properties, is of such a nature that it can be fully and exactly determined by mathematical-scientific means.

Famously, Galileo captured the momentous theoretical notion of an intrinsically mathematical nature with the illustrative metaphor of a 'Book of Nature'. The notion of a 'Book of Nature' epitomizes the view that the natural world itself, as an infinite whole, is essentially composed in mathematical symbols and geometric characters. Similarly, for Descartes, all nature, in being equated to spatial extension, and thus being considered to be exactly quantitatively determinable, is understood to exist in a way that is "comprised within the subject matter of pure mathematics" (Descartes 1988, p. 55). Apparently, this understanding also coincides with the development of a truly abstract, symbolic conception of number, as a result of which number is understood to be a concept that exists in its own right, i.e., as a form of reality *sui generis* (see also my brief discussion in section 3.1). Ultimately, in the Cartesian theoretic framework, the totality of spatial matter is thus essentially reduced to an ideal-theoretic construct. Specifically speaking, it is reduced to a pure concept of (symbolic-formal) mathematics.

The upshot of all this is that the natural world, in all its material and spatial manifestations, is understood to be exactly determinable, and completely decipherable, by mathematical means. In the light of this sweeping conceptual transposition brought about by the idealizing mathematization of the domain of natural phenomena, Husserl aptly characterizes Galileo as being "at once a discovering and concealing genius" (Husserl 1970, p. 52). This is because, on the one hand, the newly idealized world, understood on the basis of mathematical signs and geometrical figures, opens up new, almost unlimited epistemic possibilities of calculability and translatability to the enquiring human mind. On the other hand, however, the same epistemic transposition tends partially to obscure (the sense and cognitive status of) the structures of the pre-theoretic world as they are originally encountered by the individual human organism in its own embodied experiencing, acting and perceiving. In short, the idealizing mathematization of space and nature conceals the constructive cognitive dimensions of the individual human organism's environment as it exists as the direct, original, and constantly present spatial correlate of the organism's embodied activity.

2.3 Summary

Taking a long term evolutionary perspective on human theoretic culture and cognition, it may be said that with this second idealization accomplished, the general structural frame for further developments pertaining to the scientific geometrical and mathematical understanding of space had been prefigured to a significant extent.[8] In summary, the decisive stages in the genesis and development of the theoretic mind's scientific understanding of space can thus be sketched as follows.

Initially, the possibility of a geometrical determination of space was discovered from within a concrete, practical context of application, the surveying of farmland for instance. The next decisive stage is when geometry is liberated from the confines of its applied focus and elevated to the status of a theoretical science. Central for this epistemic achievement were the theoretic emancipation of the notion of the measure, the development of systematic axiomatic derivation, and the definition of ideal spatial forms. With this step taken, the object of this systematic geometrical science is no longer a concrete, perceivable spatial body, but, ultimately, an ideally conceived, three-dimensional, infinite form of spatiality. Another decisive stage is reached when this ideally conceived spatial infinity became fully incorporated into a mathematical-scientific worldview. This accomplishment coincides with a momentous transposition at the end of which idealized and mathematized space is taken to constitute the real and true space of the world – more 'real' than the space that is originally

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[8] This is, of course, not to say that there were no further important innovations and departures made in geometry and mathematical conceptions of space alike. Of significance in this regard are, e.g., the development of a sophisticated relational conception of space by Leibniz (which was later elaborated upon by Einstein), the formulation of Non-Euclidean geometries by Riemann and others, and the development of modern mathematical topological notions of space.

given through the senses of the human organism's own perceiving and moving body. Henceforth, with this momentous shift in worldviews, all spatial nature to be encountered in the natural world is theoretically understood to have the ideal and abstract mathematical domain of number as the only true norm and determination.

In a nutshell, the human theoretic accomplishment of arriving at a systemic scientific geometrical and mathematical understanding of space thus is attained by way of two interrelated conceptual shifts. First, there is the systematic emancipation of spatial thinking from the original, concrete and embodied dimensions of space and spatial cognition (and particular material bodies as they are directly given to the human individual in first-person perception). Second, and concurrent with the former process of emancipation, space thus emancipated is reconstituted by the theoretic mind, through processes of idealization and mathematization, as an ideal geometrical and ultimately mathematical object.

3. The External Dimensions of Theoretic Cognition: The Central Role of Symbolic Technologies

I have previously illustrated that on an epistemic level, the theoretic elimination of the (ontological) difference between number and world is central to the development of a modern mathematical-scientific understanding of space and world. With the modern theoretic accomplishment and legitimation of this elimination, the ostensibly 'real' since directly accessible, pre-scientific, primary and material world of embodied spatial experience is systematically substituted with a theoretically constructed world consisting of mathematical idealities (see Husserl 1970, pp. 48-49). In what follows, I aim to show that precisely this accomplishment of a theoretic constitution of a scientific notion of geometrical/mathematical space that Husserl explores in his genetic-phenomenological account, and underlying this constitution the structural emancipation of spatial cognition from its embodied roots, would have remained unthinkable without the innovative development and effective cognitive use of a range of external symbolic storage and representation technologies. In doing so, I argue against Husserl, who overall tends to hold the view that the theoretic "construction" of geometrical and mathematical space, and the form of intentionality constitutive of it, is primarily and essentially accomplished by an act "of 'pure' thinking" (Husserl 1970, p. 377) – a view that has been reaffirmed more recently by Tieszen, who claims that from a phenomenological perspective, number constructions are essentially accomplished by way of a mental process of intentions (Tieszen 1989, pp. 116-117). Instead, I claim and will illustrate in the following, the accomplishment of the theoretic mind and its underlying form of intentionality to achieve such idealizing construction successfully necessarily requires, from the very start, not only a primary, embodied cognitive access to the world, i.e., an embodied form of intentionality, but also specific types of concrete, material support. Specifically, it requires a range of material-symbolic tools that function both as a reliable cognitive aid and as the constructive medium for much of human theoretic thinking.[9]

3.1 Symbolic Technologies as Cognitive Tools: From Gesture to Notation

The view that external symbolic structures, i.e., structures that exist in durable form outside of the brains and bodies of living human organisms, are central to the theoretic construction of highly abstract, idealized notions of space aligns well with Donald's aforementioned account of theoretic culture. Central to this account is Donald’s observation that the development of complex theoretic ideas, from a certain stage onward, presupposes these ideas’ own, material embodiment in external symbolic representation and storage devices. These devices may include written texts, diagrams, pictures and also architectural constructions, among other things. Donald, in analogy to the brain's memories or engrams, refers to these external symbolic constructions as exograms (see Donald 1991, pp. 308-333). An exogram, Donald states, is “an external memory record of an idea” (1991, p. 314). Despite this terminological analogy, Donald correctly stresses that such exograms, in their

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[9] Interestingly, in his writings Husserl himself occasionally comes to question, at least tentatively, the notion that the theoretic construction of ideal-objective geometrical and mathematical formations is exclusively a product of ‘pure’ thinking. For instance, in his famous essay on the origins of geometrical thinking, Husserl notes that the constitution of theoretic formations of ideal objectivity, such as they can be found in their purest form in geometry and mathematics, as well as the “mental manipulation” of mathematical significations (Husserl 1970, p. 27), can only be accomplished by way of these formations’ symbolic embodiment (Husserl 1970, p. 358). Furthermore, such symbolic embodiment, Husserl notes, particularly if it involves a durable, material medium such as writing, is essential as it allows for the retaining of sense; including that of the original ideal constructs of human geometrical and mathematical thinking (Husserl 1970, p. 366). Despite these intriguing insights, however, Husserl ultimately falls short to give proper due to the constructive role that symbolic technologies play in the constitution of theoretic knowledge and that of a rational-scientific type of intentionality. This applies mainly in two respects. First, Husserl never came to fully recognize the extent to which symbolic technologies such as writing may fulfil a both creative and constructive cognitive function. Specifically, Husserl never fully admits that these technologies, and their material dimensions, are instrumental, indeed decisive, in bringing new domains of theoretic thinking into being. Second, in his discussions Husserl does not distinguish between different types of symbolic technologies (e.g., representations of geometrical figures, diagrams, different types of notations, etc.) and specify their respective contribution to the theoretic achievement of the constitution of ideal objectivity.

form, function and capacity, are not the same as the human organism's biological memory resources but – e.g., in terms of durability, capacity and retrievability of stored information – are vastly superior to their biological counterpart (Donald 1991, pp. 315-316; see for a discerning discussion stressing the differences between inner and external cognitive resources also Sutton 2010). Donald further claims that in the domain of theoretic cognition, the use of exograms is the more important the more complex, systematic and abstract the meanings of a cognitive construct are: in an advanced stage of theoretic culture, "the governing cognitive structures (…) exist mostly outside the individual mind, in external symbolic memory representations" (Donald 1991, p. 274). Ultimately, Donald goes as far as claiming that due to the essential externality of thinking that is central to and characteristic of the functioning of theoretic cognition, in the general context of the evolution of the theoretic culture, "until the right symbolic technology came along, certain kinds of thought simply could not be thought" (Donald 2001, p. 307).

Important in regard to this externalization of theoretic thinking is that in human cultures the material external storage of meanings and knowledge is in many instances synonymous with the constitution of durable, symbolic type of memory. Such type is distinctive not only in its symbolic dimensions, but also in that such memory exists in some sense within the environment and outside the limited confines and lifespan of individual human minds, brains, and bodies. The enduring, extrasomatic storage of memory, through material-symbolic means, in turn makes possible, at least theoretically, cognitive collaboration between a large number of human individuals and various groups over generations. Specifically, Donald claims that precisely the durable, external material-symbolic embodiment of ideas allows that "the products of thinking be frozen in time, held up to scrutiny at some future date, altered, and re-entered into storage, in a repetitive, iterative process of improvement" (Donald 1991, p. 316). Such stabilization and progressive elaboration of knowledge is indeed central to most of the more sophisticated and systematized accomplishments of theoretic cognition. Without any doubt, this also includes the theoretic constitution of abstract, idealized geometrical and mathematical notions of space.

However, as I will elaborate further, in the case of the constitution and progressive refinement of more abstract, theoretic constructs such as idealizing, scientific notions of space, material media and symbolic technologies may function not only as a durable memory device that allows the human mind to stabilize and retain ready-made products of thinking, but also more directly as a cognitively constructive medium. The claim even has been made by Donald that in terms of their own developmental formation, the majority of theoretic accomplishments of the human mind are in some sense from their "inception externally encoded" (Donald 1991, p. 274).

That is to say, many advances in theoretic knowledge were facilitated by those cognitive practices that necessarily involve a material-symbolic encoding, fixation, representation and manipulation of meanings. The natural questions – by and large unaddressed by Donald as they also have been by Husserl – then concern, first, the specific types of material-symbolic tools that are central to the formation of highly abstract, idealizing conceptions of geometrical and mathematical space, and second, the specific ways in which these tools contribute to facilitating such conceptions. To answer these questions, I will provide in the following, first, a brief overview of a range of cognitive tools aiding the human mind in its geometrical and mathematical thinking. Subsequently I will focus on exploring some of the cognitive and material-symbolic dimensions of techniques of formalization. These techniques, I argue, have been central to theoretic mind's accomplishment of arriving at an idealizing, scientifically effective, mathematical notion of space.

That the development of more abstract forms of human geometrical and mathematical thinking presupposes the development and utilization of complex, socio-culturally constituted systems of signs, and of material techniques for their recording and storage, is indeed not a novel insight. Take, for instance, Vygotsky's classic psychological studies into the development and history of the human organism's cognitive activity from the 1930s. In these studies, Vygotksy suggests that for the individual human mind to be able to go beyond the limitations of a "natural arithmetic" that is solely based on the "immediate perception of quantities", some sort of concrete external cognitive scaffolding is required (Vygotsky 1993, p. 131). Such scaffolding can take various, more or less effective forms. In line with Donald's (1991) distinction between three distinctively human cognitive stages previously discussed – the mimetic, the mythic and the theoretic – this scaffolding function can be fulfilled, as I develop upon below, by a) gestural means, b) speech, and c), in a most powerful fashion, by symbolic technological means that comprise the use of external material-symbolic storage and representation systems.[10] As I will illustrate, it is these latter systems that ultimately allow the individual human mind to engage in complex processes of formalization. It is these processes, in turn, that are crucial for both the theoretic constitution and effective operative use of highly abstract yet exact geometrical and mathematical notions of space.

A) *Gesture*

A range of more recent research has shown that the gestural activity of its own hands may indeed be effectively used by the human individual and its mind to assist in processes of numerical learning,

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[10] See for a related discussion of these and other external media assisting numerical thinking, e.g., De Cruz (2008).

counting and calculation (Goldin-Meadow, Cook & Mitchell 2009). This applies in particular from a developmental perspective, where gestures, even prior to the acquisition of language (Brissiaud 1992), apparently constitute the human infant's first model for the development of more complex numerical representations. The neuropsychologist Butterworth (1999), for instance, has comprehensively illustrated that the individual human organism's numerical and mathematical abilities, while being based in an innate core ability to represent small cardinal numbers (see also Butterworth 2005), also requires the development of fine manual motor skills for these abilities' further extension, e.g., through the use of the fingers of the human hand for counting (see also Brissiaud 1992). Of particular importance, it has been claimed by Butterworth and others, in regard to this extension of the capacity of numerical representation is the development of the capacity of the individual human organism to represent its own fingers mentally, something which is referred to by the term 'finger gnosia' (Butterworth 1999; see also Noël 2005). Recent experimental studies suggest further that even in educated adults, and in the case of abstract mental number representations, finger counting habits still play a significant, constructive role (Domahs, Moeller, Huber, Willmes & Nuerk 2010). Moreover, a prominent outcome of the various forms of scaffolding provided by the fingers of the human body assisting in the development of numerical cognition (and the performance of numerical tasks) may be the now almost universally used decimal numeral system, where the base number 'ten' directly corresponds to the number of fingers of both hands available for counting (Gibbs 2006, p. 105).[11]

B) *Speech*

Another manifest and effective cognitive scaffold for human numerical thinking is provided by speech. A range of developmental and cross-cultural psychological studies indicate that acquiring linguistic competence, in combination with a range of biologically grounded basic numerical cognitive capacities that seem particular to humans, generally assists in the acquisition of numerical skills that involve a high degree of exactness and relatively large quantities (Dehaene 1997; Frank, Everett, Fedorenko & Gibson 2008; Gordon 2004; Wynn 1990). Of specific importance in this regard

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[11] Notably, there also exists a considerable range of historical material documenting a cultural practice or art that involved the use of the fingers to assist in processes of counting and ultimately also calculation and computation (see for an instructive historical overview Menninger 1969, pp. 201-220). The existence of sophisticated finger-based counting and manual computation practices is particularly well documented for Antiquity (see the discussion by Williams & Williams 1995) as well as for Medieval times (Kusukawa 2001). Apparently, such manual techniques were still widely used relatively recently, e.g., among some traders in the Middle East (Menninger 1969, p. 201). Some of the historical examples of techniques of finger counting and manual computation were astonishingly developed; the Romans, for instance, were able to represent, and distinguish, numbers from 1-10.000 just by using the fingers (*digiti*) of both of their hands (Menninger 1969, p. 201).

is that in languages that possess a developed numerical vocabulary, the learning of number words and the practice of a more or less systematic naming of numbers, seems to precede and anticipate the conceptual understanding of exact numerical magnitudes (Wynn 1990; but then see Butterworth, Reeve, Reynolds, & Lloyd 2008). This applies in particular to greater magnitudes, the comprehension of which generally goes beyond the cognitive scope and focus of the human organism's direct, embodied intuition. This naming of numbers of all sorts of magnitudes, it has been comprehensively argued by mathematician and neuroscientist Dehaene, was made possible by the invention of number syntax (Dehaene 1997). The cognitive domain of number syntax is constituted through the construction of number words that systematically combine words for smaller numerical magnitudes to express larger sums, as in 'eighty-four' or 'sixty-seven', for instance. As such, number syntax not only allows "larger numerals to be expressed by combining several smaller ones", but also, ultimately made it both possible and relatively easy for the human organism to "express any number with perfect accuracy" (Dehaene 1997, pp. 94-95). All this suggests that the number words of a culture effectively fulfill the function of a sort of "cognitive technology" that allows "for representing, storing and manipulating the exact cardinalities of sets" (Frank, Everett, Fedorenko & Gibson 2008, p. 823; see also Fuson & Kwon 1992).

C) *Mathematical Notations*

Ultimately, however, it appears as if the transition of mathematical thinking toward a complex and highly abstract variety, i.e., of a variety that allows for the development and effective use of a scientific-theoretic mathematical notion of space, operationally requires the utilization of symbolic technological means, and maybe most importantly, that of developed systems of notation.[12] Such developed systems of notation are historically predated by material storage and recording techniques that involve the use of serially arranged and systematically distributed markings (similar to our present day tally marks). Archaeological evidence reveals that such techniques may date back as far as the Upper Paleolithic (see d'Errico 1998). Basic material recording techniques such as tally marks are already capable of functioning as an effective cognitive tool when it comes to solving some relatively simple numerical tasks, e.g., the counting a comparatively large number of items or the comparing of various numerical values (Norman 1991, pp. 32-33). However, in the case of more abstract, systematized notations, e.g., notations systems that make use of strictly symbolic

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[12] For a detailed and still unrivalled, comprehensive historical overview of mathematical notations systems see the classic work by Cajori (1952).

representations (for instance, the Hindu-Arabic system, see further below), the scope of cognitive functionality and constructiveness is significantly broader.

It has been proposed that generally such systems, in the more abstract domains of mathematical thinking and elsewhere, may not merely serve to represent (and as such conserve) information and to map pre-established conceptions, but they may also more 'actively' "constrain, anchor, structure and change people's cognitive behavior" (Zhang 1993, p. 775). Specifically, it has been claimed that in the cognitive domain of human numerical thinking, external symbolic number systems may even directly function in a cognitively constructive and augmenting manner as "conceptual systems" (Barton & Hamilton 1996, 818; see for a comprehensive argumentation that draws on a wide range of historical and psychological material, De Cruz & De Smedt, in press). The somewhat more controversial notion of external mathematical notation systems being conceptual systems is to signify that these external symbolic systems in some instances allow the individual mind to cognitively accomplish something novel that would have remained inaccessible without their use. This may, e.g., be the comprehension of the highly abstract yet exact notion of very large numerical magnitudes, which is considerably assisted through the use of sophisticated notational means.[13] Moreover, sophisticated notational means also allow the human mind to reliably compute such large magnitudes, e.g., through the use of notational place-value correspondence (see further below). Such computational efficiency cannot be achieved to the same degree through the use of other, individually embodied and cognitively more basic means such as intuition, finger counting, and even speech. In view of all this, it seems as if the speculative claim that without the innovative development and effective use of notational systems as an external cognitive tool, human mathematical thinking, at least for the most part, would have likely remained confined to notions of exact numerical quantities that can be concretely counted, and that are thus of a relatively limited nature (see Barton & Hamilton 1996, pp. 818-823; see also Vygotsky 1993, p. 131), could be a plausible one.

In any case, there are several reasons why mathematical notation systems are capable of functioning as a most effective cognitive aid for the individual, embodied mind. I will discuss in the following the two that I consider to be most important. First, mathematical notations, if effectively coupled with the individual mind and its body, allow constituting a distributed representational and

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[13] This is because sophisticated mathematical notations allow the human mind to quickly infer the exact magnitude of large sums from their corresponding numerical representation. For instance, the numerically literate individual is capable of almost immediately recognizing the exact value that the numeral 543 represents. This is not possible with more 'primitive' notational means – it would take considerable time, for instance, to count 543 tally marks and thus grasp the exact number represented.

cognitive system. Second, notations allow for visual-spatial cues that considerably alleviate the solving of complex mathematical tasks.

That mathematical notations allow for the constitution as cognitively powerful system of distributed representations has first been convincingly argued by psychologist Zhang and cognitive scientist Norman in an influential joint article (1995; see similarly also Zhang & Wang 2005). Such a distributed cognitive system, Zhang and Norman note, allows the embodied human agent and its mind to process the “information perceived from external representations”, i.e., from representations that take material form in the environment, and which can be accessed by means of perception, “and the information retrieved from internal representations in an interwoven, integrative, and dynamic manner” (Zhang & Norman 1995, pp. 279-280). Importantly, using both internal and external representations in an integrated manner may significantly extend the capacity of the mathematical mind: in most instances it is without doubt easier to solve a complex mathematical problem when one’s ‘internal’ thoughts are externalized and materially represented in written form, and once externalized in turn come to guide and inform the progress of one’s own ‘internal’ cognitive deliberations. As such, the claim can be made reasonably that in the case of mathematical cognition, such a system of distributed *representations* also constitutes a distributed *cognitive* system. A distributed cognitive system is a system in which heterogeneous, 'internal' and 'external' components are interwoven and, in a more or less coordinated manner, interact effectively in cognitive processes, ultimately allowing the cognitive capacity of the human mind to extend beyond the confines of the individual human brain and body (Hutchins 1995; see for succinct discussions also Kirsh 2006; Sutton 2006). In short, a cognitive system is a distributed system the external components of which are capable of actively shaping and augmenting the cognitive activity of the individual human mind.

It is significant in this regard that the external components of such a distributed cognitive system – the mathematical notations – provide the human mind with a concrete and more or less systematic spatial ordering and representation of concepts, ideas and information. This is important as such spatial representation allows for the direct tapping of the human organism’s visual-spatial intelligence for abstract cognitive tasks, e.g., in algebraic and other types of formal-symbolic thinking (Kirshner & Awtry 2004; Landy & Goldstone 2007). And it generally seems to be the case that the clearer and the more systematized the spatial organization of the notational means is, the more effective such notational means can be as a cognitive tool. For instance, in the case of the cognitively most effective visual-symbolic systems, the numerical symbols are systematically spatially arranged in a manner that significantly simplifies calculation and assists the human mind in its computations –

the Hindu-Arabic place-value notational system being an obvious example (see below for more detail). Interestingly, it even appears to be the case, at least in some instances, that specific visual-spatial symbolic storage and representation systems, in utilizing the apparently innate propensity of the human numerical mind to rely on 'internal' spatial schemas and representations, come to transform the spatial arrangement of these latter ‘internal’ representations.[14] The propensity of the human mind to use space as a medium for its numerical thinking has also been affirmed by a range of more recent psychological studies into the ontogenetic development of human numerical cognition. What these studies precisely demonstrate is that the human organism’s processing of numbers is closely linked with the development of visual-spatial skill, and the capacity to develop (imaginary or real) spatially orientated representations such as the mental number line (Dehaene, Bossini & Giraux 1993; Dehaene 1997; Fias & Fischer 2005; Fischer 2003). Overall, in view of this link, it may hence not come as a surprise that many historical innovations in the domain of notation systems have their grounds in a more “intelligent use of space” (see Kirsh 1995), directly “capitalizing on the ability of human vision to achieve complex spatial synthesis” (Donald 1991, p. 316).

A striking historical example of a symbolic technology whose innovative visual-spatial dimensions have come to directly expand the scope and efficiency of human numerical thinking is the Hindu-Arabic system of numeration. Notably, the numerical representations of this system are all arbitrary symbols. It is impossible, for instance, to infer at all visually from the material-spatial form of the numeral ‘8’ that the numerical value it represents is double that of the numeral ‘4’ (something which is generally possible when using tally marks and also in some, limited cases with Roman notations, see Norman 1991, pp. 32-33). Nonetheless, in terms of its overall cognitive productivity and effectiveness, this system nevertheless constitutes a significant improvement compared to that of other numeral systems including the Roman. This is primarily because the visual-spatial organization of the Hindu-Arabic notation system, in comparison to the Roman system, offers to the human individual and its mind a vastly improved, clear and consistent form of representational efficiency. This, it has been noted, has a direct performance enhancing effect on many of the human individual’s numerical cognitive processes (Dehaene 1997, pp. 98-100; Zhang & Norman 1995).

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[14] For instance, cross-cultural studies have shown the directionality of the prevailing writing systems shapes the orientational structure of the mathematical mind’s internal, spatial representations (Zebian 2005; also Dehaene 1997, pp. 81-83). Similarly, the repeated use of the abacus as a cognitive tool is capable of significantly informing the structure and cognitive function of one’s own, mental representations (Hatano, Miyake & Binks 1977; Stigler 1984).

It is fundamental, both in regard to their representational efficiency and reliable cognitive support function, that Hindu-Arabic numerals allow for the effective use of the positional place-value system. Without any doubt, such type of notation clearly exemplifies a most systematic, consistent and yet simple and transparent form of the use of concrete, visually accessible spatial relations and differentiations to support the computations of the individual human mind. A place-value notational system is a spatially organized visual-symbolic system where the place of a numeral corresponds with a particular numerical, in this case decimal value. As an effective place-value notation system, the Hindu-Arabic system allows the human mind to directly infer the magnitude of digits that make up numbers from their spatial position. It is immediately obvious to the mathematically trained individual that, for instance, in 15, the digit 1 stand for 10; whereas in 150 it stands for 100, and so forth. Importantly, this applies even when the actual digits that make up a number are identical, e.g., in the case of numbers such as 11, 222, or 5555. In addition to and related to the former, the correspondence of place and decimal value that is characteristic of the Hindu-Arabic notational system considerably assists the individual human mind in its numerical computations. This is because the spatial arrangement of Hindu-Arabic numerals makes it possible for the human mind to break up complex computational tasks into more easily digestible parts. For instance, it appears to be considerably easier for the averagely mathematically skilled human mind to compute, let's say, 134 x 7 than its equivalent in Roman numerals, CXXXIV x VII.[15]

3.2 The Technique of Formalization and its Material-Symbolic and Epistemic Dimensions

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[15] In may thus not come as a surprise to learn that the Romans, similarly to the Greeks who for the most part employed alphabetic letters as mathematical notations, commonly did not yet use their numerical representations as a means and medium for calculation. Instead, they resorted to the abacus to assist their numerical thinking, and thus to a systematically spatially organized cognitive tool, the functioning of which does not involve notations (see on this point Cajori 1952, vol. 1, chapter 2). In relation to the cognitive efficacy of the Hindu-Arabic notation system, it is likewise of importance to note that this system, in contrast to the Roman system, for instance, and also in contrast to other historical examples of notation systems that already used place-value notation such as the Babylonian, has a symbol notation for zero. The notational symbol zero came to fill the void in the positional place-value number system, where the void indicates an absent unit in a number comprising several digits. This in turn made it possible to unambiguously represent magnitudes (Dantzig 1954, pp. 30-31). In the numeral 3003, for instance, the first digit 3 unambiguously means 3 x 1000. By comparison, the traditional practice of leaving spaces blank to indicate the absent units (as in 3   3) is a likely source of confusion. Dantzig has argued that in addition to this representational efficacy, the invention of a symbol for "an empty class, a symbol for nothing", ultimately also made it possible for the theoretic mind to conceive of this symbolically denoted void as a number (Dantzig 1954, p. 31). This theoretic realization, he claims, in turn opened up the way for new, formerly unthinkable ways of calculation, and ultimately, for modern arithmetic thinking (see for a comprehensive argumentation Dantzig 1954).

Generally speaking, the operationally as well as cognitively most effective form of the use of material symbols in the human mind's mathematical deliberations necessarily involves the technique of formalization. The technique of formalization refers to use of artificial graphic representations and material symbols to describe all sorts of processes in a formal, that is, semantically reduced and decontextualized manner. The development and use of the technique of formalization requires, on the side of the theoretic mind, the realization that a set of material, artificial symbols can be effectively employed for cognitive purposes if they are used and related to one another in rigidly regulated and defined, formal-mechanical manner.

Krämer (1988, pp. 1-3) notes that generally, in order for the technique of formalization to function effectively, at least three conditions have to be met. First, the symbols employed have to be in their representational form stabilized and materially fixated, i.e., they have to be written down. Second, the symbols have to be used and related to one another in manner that makes it possible to describe processes schematically, i.e., through a procedure that is in principle repeatable indefinitely. Finally, the symbols employed, while being necessarily perceivable by the human organism's senses, have to be emptied from their intuitive content and associations. In short, the proper functioning of the technique of formalization requires, on the level of the visually appearing and perceivable symbolic content, a “formalization of graphic representation” (Goody 1986, p. 95). Taken together, it thus can be said that the practice of formalization entails the relative decontextualization, mechanization and externalization of human thought. Along with the use of the technique of formalization, human thinking detaches from immediate embodied experience and concrete situations in lieu of technical operations that have as their material medium an extrasomatic system of formal symbols.

In what follows I briefly discuss two historical types of symbolic technologies whose cognitive functioning is based upon processes of formalization. Such processes apply to both the technologies’ own, concrete graphic-representational form, as well as to the abstract semantic content conveyed by and through them. Of particular importance in this regard is that these symbolic technologies facilitate a range of innovative, cognitively efficacious combinatory processes. Interestingly, as I will discuss further, some of these combinatory, symbolic-technological innovations seem to directly align with some of the previously discussed, decisive epistemic shifts in the long-term development of a comprehensive mathematical-scientific understanding of space.

From Pictorial Symbolism to Lettered Diagrams

Generally speaking, the both graphically and semantically emptying and yet theoretically constructive process of formalization already began with the graphic representation of ideal geometrical figures. Take for example the Euclidean idealizing notion of the geometrical point referred to earlier. Taking seriously the primacy of embodied perception in the cognition of space, the original construction and cognitive comprehension of the geometrical point by the human mind constitutes indeed an astonishing achievement. The point, following Euclid's famous definition, is defined as that spatial form which has no parts. This entails that the point also has no material, perceivable presence in space. The immaterial essence of the geometrical point explains why the point’s construction cannot be achieved from within the domain of the human organism's primary mode of spatial cognition, which is the embodied process of spatial perception, alone. As Ströker (1987, p. 226) observes in this regard, due to the fact that the point's “genuine geometrical content is not at all perceivable in sensory intuition”, its meaning can only be constructively delineated symbolically, namely in abstraction “from intuition through a process of formalization”. Ströker (1987, pp. 194-200) mentions in this regard the development of the technique of a “pictorial symbolism” that, she claims, is characteristic of classical Euclidean geometry. The use of a pictorial symbolism facilitates the figurative representation of ideal geometrical figures and bodies in a concrete, visual-spatial medium. However, in a broader perspective, what was arguably most important for the theoretic development of Euclidean geometry, and crucial for the systematic development of the deductive structure of its reasoning, is, as Netz (1998; and comprehensively 1999) has illustrated, not a mere pictorial symbolism, but the development and the effective use of a novel, combinatory symbolic technology. This specific symbolic technology is the lettered diagram.

In a classic paper, psychologists Larkin and Simon (1987) suggested that external diagrammatic representations possess a computational efficacy that in many cases may exceed that of purely sentential representations such as a written text. This, they argue, is due to three interrelated features characterizing many diagrammatic representations. First, there is the capacity of diagrammatic representations to effectively group all the information together. Second, there is their capacity to organize and represent information spatially, in a plane. This spatial organization, thirdly, allows for relatively easy and salient perceptual inferences on the side of the perceiving human individual. While it can be said that all these features also apply to the diagrams the Greeks used for their theoretic geometrical disquisitions, what truly distinguishes these latter diagrams is the then innovative combination of abstract symbols – alphabetic letters – with pictorial representations of geometrical figures. In his *Elements*, for instance, Euclid, not only uses geometrical figures for illustrative purposes, but in addition also uses capital letters to denote and distinguish, for constructive reasons, geometrical points. Both the use of geometrical diagrammatic representation and the symbolic-technological integration of letters into processes of geometrical construction are of prime epistemic importance. This is first of all due to the fact that any geometrical point, in spite of its ideal and disembodied nature, can only be constructively delineated and effectively used in geometrical practice on the basis of its own concrete, visual-spatial, material-symbolic representation.

What is more, however, is that the combination of pictorial elements and letters allows, as Netz both comprehensively and convincingly illustrates (1999, chapter 1), for the creation of a number of powerful synergies shaping the deductive structure that is central for the construction of Euclid's geometrical system. Following Netz, these synergies become manifest and take effect on a number of planes; the arguably most important of these are the logical plane, the cognitive plane, and the semiotic plane. On the logical plane, Netz observes, the lettered diagram basically constitutes "a combination of the continuous (the diagram) and the discrete (the letters)" (Netz 1999, p. 67). Closely related to the former, on the cognitive plane, the lettered diagram effectively brings together material elements that facilitate the tapping of cognitive resources that have their basis in the human organism's visual-spatial sense (the actual diagram) and the tapping of those cognitive resources that, albeit also visual, are of a linguistic nature (mathematical language; in this case, letters). Finally, on the semiotic plane, Netz argues, the diagram combines iconic (diagram) and, understood in the Peircean sense, indexical features (letters) (Netz 1999, pp. 43-56). It is these precisely these combinations, most importantly that of written text and actual diagram, Netz claims, that make the lettered diagram both a powerful, visual and linguistic tool, and at the same time, a compelling "vehicle of logic" (1999, p. 33) that ultimately paves the way for the development of a systematized, logically coherent, theoretic account of geometry.

The Cartesian Coordinate System

Classical Greek geometrical thinking already was capable of effectively combining both pictorial and symbolic representational means, i.e., of diagrams and letters, to constructively arrive at a scientific, that is, systematized and logically coherent theoretic notion of geometrical space. Nonetheless, Greek geometrical thought did not yet possess the symbolic-technological means to translate all properties of geometrical spatial bodies into an abstract formal mathematical language; and to arrive at a truly *symbolic* notion of mathematics, where the letters used in mathematical practice are identified and specified as *symbols* (Netz 1999, p. 25). All this may explain why Ancient Greek thinking did not yet develop a theoretic understanding of spatial nature as essentially and universally mathematical, i.e., an understanding whereby number and space are ultimately considered to be identical.

The direct translation of the properties of geometrical spatial bodies into the purely formal-symbolic language of mathematics, and with it the possibility of a scientifically effective mathematization of spatial nature, only became possible with the modern algebraization of geometry. Central in regard to the algebraization of geometry is the development of Cartesian coordinate geometry (see also below). What characterizes Cartesian coordinate geometry, on the epistemic plane, is the way it allows to directly express geometrical properties, even complex ones, through algebraic equations. This is made possible, on the symbolic-technological plane, through a relatively simple yet ingenious procedure: within the Cartesian coordinate system, each spatial point "is labelled with an ordered triple of (directed) lengths; their relations can be determined by investigating quantitative relations between their labels; every line and surface can be defined as the locus of all points whose labels are related by a given equation" (Torretti 1978, p. 34). Ultimately, with the (Cartesian) algebraization of geometry, an understanding of algebra thus becomes conceivable, and practically possible, that considers algebra to be a universal language by means of which both geometrical and arithmetical problems can be solved in equal measure (Krämer 1988, p. 65). In the end, it has been observed, the Cartesian conception even goes a step further: from a certain stage onward, the possibility of algebraic calculability is thought to be the essential criterion for what counts as geometrical (and by extension spatial) nature in the first place (see Krämer 1989).

To clarify, on the symbolic-technological plane, in order to arrive at the understanding depicted above, the general transition from a direct, pictorial mode of representation to one of a symbolic nature does not suffice. Rather, what is specifically required is the development of an elaborate symbolic-technological apparatus that facilitates the effective combination of graphically reduced, formalized pictorial and symbolic-numerical components. Crucial in this regard is the development of the Cartesian coordinate system. This system is a result of Descartes' realization that geometrical figures and constructions can be directly mapped on a coordinate grid the axes of which are indexed with numerical symbols and, by extension, numerical values. It is this procedure which makes the spatial properties of these figures unambiguously translatable into numbers, and geometrical shapes of almost all sorts exactly expressible through algebraic equations accordingly. Once this idea has been put into practice, it is then possible to treat and solve problems of a geometrical nature, for example the geometrical construction of spatial curves, by way of purely algebraic means, i.e., through equations. In making complex geometrical properties directly translatable into algebraic language, the Cartesian coordinate system allows not only the solving of geometrical problems through algebraic means, but also makes it legitimately comprehensible for the theoretic human mind to directly and yet exactly conceive of all of spatial nature in the abstract, formal-symbolic terms of algebra. Viewed in this light, it is justified overall to regard the Cartesian coordinate system as an example of an innovative symbolic technology, the functioning of which directly translates to progress in the theoretic domain of mathematical knowledge. Central in this regard is that in the specific case of Cartesian coordinate geometry, the formalized-formalizing symbolic technological means employed function in a cognitively constructive manner that exceeds any mere representative, secondary function.

Given all the above, it may thus after all be no coincidence that, in Descartes' mathematical thinking, the innovative development of symbolic-technological means effectively allowing for the algebraization of geometry seems to be generally concurrent with the systematic development of a truly symbolic, abstract conception of number. Precisely such symbolic conception of number, at the least, appears to be central to the theoretic mind's achievement of arriving at the modern, scientific, i.e., abstract mathematical understanding of space and world. This, at least, has been convincingly argued by Klein (1968, chapter 12; Klein 1985). In such a symbolic conception, Klein notes, a number no longer represents something that has a concrete reference, e.g., a definite number of material bodies, but essentially constitutes a symbolic abstraction, where the number symbol signifies "the concept of the number as a multitude of units" (Klein 1985, pp. 62-63). As a consequence, the (symbolic) concept of number is thus understood to exist independent from any certain, countable objects, while at the same time, in algebraic practice, it is increasingly taken for granted in its own right, i.e., as something that has, as it were, its own, real existence (Klein 1985, p. 63). This transition to a symbolic, abstract conception of number may have been theoretically momentous for at least two reasons. First, this transition makes it plausible for the theoretic mind to conceptualize and mathematically think with non-intuitive numbers such as 'zero'. Second, this transition, in line with the Cartesian equation of space and number, makes it possible for theoretic mind to directly identify the matter of space with a mathematical entity that can only be truly conceived of and adequately represented abstract-symbolically. The upshot of all this is that, in the Cartesian theoretic framework, 'real' spatial nature is essentially identified with a symbolically generated mathematical construct, and thus, as it were, with a domain that one may refer to, following Klein (1985, p. 64), as possessing a cognitively peculiar dimension of "symbolic unreality".

To sum up, in the case of highly abstract and decontextualizing modes of theoretic thinking, the system of both graphically and semantically reduced formal symbols, in many instances, may function in some sense as the thinking individual's primary and constructive medium. As illustrated, this is particularly the case in the domain of theoretic thinking allowing for the development of abstract, scientific conceptions of spatial nature. My discussion has demonstrated that in such thinking, the symbolic technological means used for complex mathematical operations does not bear witness to a “process of thought that merely becomes subsequently fixed in a spatial medium; rather, from the very beginning the operation necessarily occurs through signs and only by way of signs” (Ströker 1987, p. 209). This dependence of thought on adequate material-symbolic means naturally entails that, in the long-term process of the progressive development of theoretic, idealized conceptions of space, symbolic technologies must have played and continue to play a central since constructive role: from a certain stage on, theoretic innovation to a significant extent must be driven by, not merely by the human mind and its brain but by its externalized symbolic operations, i.e., operations that involve the concrete, embodied manipulation of extrasomatic symbolic representations. In short, it indeed appears to be the case that idealizing geometrical and mathematical thought and its correlated type of intentionality, both in their functional and developmental dimensions, unavoidably have an external, material-symbolic dimension to it, the latter being central to this thought’s internal progression and perfection.

3.3 Summary:

In the preceding two sections I have illustrated that important theoretic accomplishments in the process that led to development of sophisticated, idealizing geometrical and mathematical conceptions of space (where space ultimately becomes posited as a universal ideal structure existing independent from perceivable and concrete spatial bodies) in many instances cannot be thought apart from the cultural development of specific material-symbolic technologies and their effective technical use as a cognitive tool and medium for theoretic thinking. It has been effectively shown that in regard to this theoretic process, symbolic technologies fulfill a cognitively central role in at least three ways.

First, the use of symbolic technologies allows for the concrete, material and durable fixation of ideal-abstract structures of meaning, and by extension that of elaborate theoretic constructs, outside of the confines of the human individual’s biological cognitive and memory systems, i.e., brain and body. Important in this context is that through such external, material-symbolic representation and storage of theoretic concepts and contents, the latter are placed in a domain where they can be continuously accessed by a theoretically unlimited number of individual, embodied minds, thus potentially turning theoretic thinking into a communal, collaborative endeavor. This is crucial, for it makes it possible to subject complex theoretic constructs to a sustained form of critical examination, and at the same time, to progressively building upon and systematically refining them. Such processes of elaboration and refinement can, at least potentially, continue indefinitely – as long as one exemplar of the material carriers used to store theoretic meanings survives, along with the skills that allow systematic reactivation of the stored meanings.

Second, in the domain of theoretic culture, symbolic technologies provide a powerful cognitive tool that aids the theoretic mind in its relative liberation from the cognitively limited scope of the individual, embodied mind and this latter mind’s intuitive conceptions of space and number. In doing so, these technologies pave the way for a thinking that allows posing and solving geometrical and mathematical problems that are of a highly abstract and complex variety. Of particular importance in regard to this functioning of symbolic technologies as cognitive tools is that such technologies, e.g., mathematical notations, facilitate the formation of distributed cognitive systems whose cognitive power and scope is significantly greater than that of the unaided human mind. Furthermore, and likewise crucial, symbolic technologies such as notations entail a spatial mode of representation, and in many instances make effectively use of space in a manner that allows for concrete visual-spatial cues that considerably assist the individual, embodied human mind in the solving of highly abstract and complex mathematical tasks and problems.

Third, symbolic technologies pave the way for formalizing modes of thinking. As shown, such modes of thinking are indispensable for the development and effective use of scientific notions of space. Formalized modes of thinking are facilitated by processes of formalization that take place simultaneously on the level of the concrete, visually appearing form of symbolic technologies, as well as on the level of content and concepts purveyed by the same technologies. I have further illustrated in my discussion that specific, combinatory symbolic technologies such as the Cartesian coordinate system, if used in a way that entails the effective utilization of the technique of formalization, may constructively function as a cognitive medium that ultimately makes it possible for the human mind to universally conceive of and exactly express spatial properties in abstract formal-symbolic mathematical terms.

Overall, as shown, the systematic 'internal' epistemic emancipation of human thinking from embodied cognition that is constitutive of theoretic, idealizing conceptions of space necessarily requires specific, durable, 'external' forms of the materialization of sense and, ultimately, a material-symbolic form of embodiment. Related to this, it can be concluded that the form of intentionality implicit to the constitution of ideal-objective, theoretic notions of geometrical and mathematical space, even if one takes the embodied, phenomenological mind as point of departure, is itself only constituted with and through the effective use of sophisticated cognitive tools that have their own, independent and concrete material existence.

4. Final Remarks: The Embodied Dimensions of the Theoretic Mind

I want to finish my discussion presented here with a brief account of the nature of and relations between the various types of embodiment that are intrinsic to the structure and functioning of the theoretic cognition of space, taking equally into account the phenomenological, material-symbolic and generally distributed dimensions of the theoretic mind. In order to do so I want to make use of a distinction between three cognitively efficacious types of embodiment that has been proposed by Sonesson (2007). These types are what Sonesson refers to as ‘primary’, ‘secondary’ and ‘tertiary embodiment’. The domain of primary embodiment is synonymous with the phenomenological body, i.e., with the cognitive agent’s own, living body, as it becomes manifest in and provides structure to this agent’s own, individual experiential and perceptual activity. By contrast, the domain of secondary embodiment comprises the intersubjective and by extension also the more general social relations and interactions between individual, embodied cognitive agents. Finally, the domain of tertiary embodiment comprises those inanimate material structures and artifacts that shape, participate in, support and constrain some of the enculturated and embodied cognitive agent’s thinking.

Now, the focus of my discussion obviously has been on the domain of tertiary embodiment. Specifically, it has been on illustrating that a range of specific, extrasomatic material-symbolic artifacts have been crucial to the process that has allowed the individual human organism and its mind to systematically overcome the cognitive limitations pertaining to its own, direct experiential cognition of space, and to arrive at highly abstract, idealizing geometrical and mathematical conceptions of space. Importantly, however, this does not mean that the theoretic mind does not require for its own development and proper functioning the phenomenological boy, i.e., the domain of primary embodiment. On the contrary, primary embodiment is a condition that is and continues to be necessary for the formation and functioning of the theoretic mind. This is mainly for two reasons.

First, on a basic level of cognitive activity, primary embodiment is that original, affective and constantly available cognitive means through which spatial objects are first of all accessed and grasped by the individual human mind. On this basis, it is only natural to assume that the same phenomenological body, ultimately, also provided the human mind with “the first model of those transpositions, equivalents and identifications which make space into an objective system” – a system that, one may add, lends itself to considerations of a geometrical and mathematical nature (Merleau-Ponty 1962, p. 142; see for related, comprehensive discussion Ströker 1987; and more recently, the somewhat controversial discussion presented by Lakoff & Núñez 2000; for a critical review of the latter see, e.g., Voorhees 2004). Second, primary embodiment, and thus a bodily form of intentionality, also plays a central role in process of the actual construction and use of externalized symbolic technologies. For instance, the human individual’s own, skillful manual practice historically was and to a significant degree remains central to the functioning of the technical process that allows materially transforming the environment in order to use the latter as storage of symbolic meanings. Similarly, the human individual’s visual sense is naturally essential for the possibility of reactivating symbolic meanings that are stored in visual-spatial information systems, outside of the individual human organism’s brain and body. Given the crucial role that symbolic technologies play in the domain of theoretic culture, it is perhaps justified to claim that if the (theoretic) mind cognitive processes were truly and completely disconnected from the human individual’s body and the direct access to the material environment that this body provides, there would have been no chance for this mind to ever arrive at a highly abstract and idealizing level of theoretic thinking, and by extension, at a scientific notion of geometrical and mathematical space.

Finally, without being able to go into great detail here, I also want to stress that both primary and tertiary embodiment require for their effective theoretic functioning the effective social mediation between individual and transindividual cognitive domains. In regard to this mediation, the domain of secondary embodiment naturally plays a central role. For instance, secondary embodiment, in its direct interpersonal manifestations, is generally indispensable for the pedagogical training and shaping of individual bodies and minds that effectively facilitating the transmission of relevant skills such as reading and writing from one human individual to the other. In the domain of theoretic cognition and elsewhere, such mechanism of transmission is a necessary condition not only for first of all stabilizing new knowledge in external symbolic storage devices but also for the possibility of repeatedly accessing such materially stored, symbolically encoded knowledge. Furthermore, it should also not be forgotten in this context that development of theoretic cognition, and ultimately that of modern scientific thinking, would have been impossible without the support of specialized pedagogical institutions allowing for “the systematic training of generations of students” in a number of cognitive skills that were deemed relevant by society, e.g., the use of specific symbolic technologies, according to a “systematic curriculum for the training of thought skills and habits” (Donald 1991, 346; see also Goody & Watt 1963).

To sum up, the domain of theoretic cognition, while relying on the development and effective use of symbolic technologies and external symbolic storage devices, also requires for its proper functioning the perceptual and motor activity of individual human bodies, and the same time, the intersubjective interlinking and institutionalized social organization of individual cognitive activity. It follows that the theoretic mind overall has to be recast as being both embodied and distributed, i.e., as an effectively organized cognitive network comprising a number of mutually interacting individual minds, their living bodies, and a range of extrasomatic symbolic storage and representation systems.

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