Stability Analysis of Networked Control Systems with Direct-Feedthrough Terms: Part II – The Linear Case

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Abstract— **In this paper, we consider networked control systems (NCSs) composed of a linear plant and a linear controller with direct-feedthrough terms, i.e., terms that directly connect the plant's input and output from/to the controller with each other and the controller's input and output from/to the plant with each other. The presence of such direct-feedthrough terms generates non-trivial difficulties in terms of the modeling and the analysis of NCSs. In particular, a novel stability analysis is required to address standard scheduling protocols such as the sampled-data (SD) and try-once-discard (TOD) protocols. Hereto, we will take a renewed look at the concept of uniformly globally exponentially stable (UGES) scheduling protocols for these standard scheduling protocols as used in literature, such that the direct-feedthrough terms can be incorporated in the system configuration. The application of our results are illustrated using the benchmark example of a batch reactor.**

I. Introduction

In many applications, including manufacturing plants, vehicles, and aircraft, communication is needed for the exchange of information and control signals between spatially distributed system components, such as supervisory computers, controllers, and intelligent input-output (I/O) devices. When sensor and actuator data is communicated over a shared (wired or wireless) packet-based communication network, the system is called a *networked control system* (NCS). Such NCSs have received considerable attention in recent years [1]–[3]. This interest is motivated by the many advantages their flexible architectures offers, such as reduced installation and maintenance costs, when compared to conventional control systems in which sensor and actuation data is transmitted over dedicated point-to-point (wired) links, see, e.g., [4]. Additionally, wireless communication is able to overcome the physical limitations of employing wired links, which is very appealing in, for instance, intelligent transportation, see, e.g., [5].

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On the other side, the usage of packet-based networked communication comes also with the inevitable network-induced imperfections, such as varying delays, dropouts, varying transmission intervals, and so on. Sensor and actuator data need to be quantized and cannot be transmitted continuously, but only at discrete time instants. Moreover, as the communication network is often shared by multiple sensors and actuators, there is a need for so-called scheduling protocols, which govern the access of the nodes to the network. To deal with all these networked-induced phenomena, several modeling frameworks have been developed. In this paper we are particularly interested in the framework where the NCS is modeled as a hybrid inclusion. Based on this hybrid framework, conditions to guarantee overall stability or $\mathcal{L}_p$-gain performance of the NCS have been derived, see [6]–[9]. In addition to this general setup, many extensions can be found in [10]–[13], and the references therein.

Unfortunately however, it appears that the aforementioned results of [6]–[9] always avoided the inclusion of so-called direct-feedthrough terms, i.e., terms that directly connect the plant's input and output from/to the controller with each other and the controller's input and output from/to the plant with each other, when *both* actuator and sensor signals are transmitted over the communication network. This is because these feedthrough terms lead to non-trivial difficulties in terms of modeling and analysis, a phenomenon that we will refer to as the *direct-feedthrough problem* in this paper. In particular, the presence of the feedthrough terms significantly modifies the model of the networked-induced error at updates of the networked values, and, as such, requires a novel (stability) analysis to address standard scheduling protocols such as the sampled data (SD), try-once-discard (TOD), and round-robin (RR) protocols. As a result, the classes of NCSs that can be analyzed using the results of [6]–[9] cannot handle at present classical controllers such as, for instance, Proportional-Integral(-Derivative) (PI(D)) controllers.

Given the importance of PI(D) control and other control/plant structures with feedthrough terms, in [14] this direct-feedthrough problem was first addressed by showing that, for the case where *only* the controller contained feedthrough terms, stability of nonlinear NCSs could still be guaranteed when using standard scheduling protocols such as RR and TOD. In this paper, we will study linear NCSs with direct-feedthrough terms in *both* the plant as well as the controller, which introduces additional difficulties as we will see below. Therefore, as a starting point, we focus in this work on *linear* NCSs for which, based on [6]–[8], we will briefly provide a recap on the stability analysis results concerning uniform global exponential stability (UGES), however now slightly altered to take into account the presence of the direct-feedthrough terms. In addition, and more importantly, we will also revisit the concept of UGES scheduling protocols as introduced in [7] for the standard SD, TOD, and RR scheduling protocols and show that, under certain conditions, the direct-feedthrough terms can be incorporated in the NCS configuration and the stability analysis for UGES of [6]–[8] can still be applied. To illustrate the usefulness of our results we apply them to the benchmark numerical example of a batch reactor.

The remainder of this paper is organized as follows. After presenting the necessary notations in Section II, the class of systems considered in this paper is described in Section III. In Section IV, we provide conditions for the NCS *with* direct-feedthrough terms such that UGES is guaranteed and in Section V we take a look at some well-known scheduling protocols and prove that they satisfy these conditions for stability, even in the presence of the direct-feedthrough terms. Finally, in Section VI, the numerical example illustrating the applications of our results is provided, and in Section VII some concluding remarks are given.

II. Preliminaries

The sets of non-negative integers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, and the set of non-negative real numbers by $\mathbb{R}_{\geq 0}$. The notation $v \in \mathbb{R}^{\bullet}$ will denote real valued, finite vectors whose size is either clear from context or not relevant to the discussion. For vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^{\bullet}$, we denote by $(v_1, v_2, \dots, v_n)$ the vector $\begin{bmatrix} v_1^\top & v_2^\top & \dots & v_n^\top \end{bmatrix}^\top$, and by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the usual inner product, respectively. Moreover, we use the notation $r^+ = r(t^+) = \lim_{\tau \downarrow t} r(\tau)$ where r is any left-continuous mapping from $\mathbb{R} \to \mathbb{R}^{\bullet}$. The n by n identity and zero matrices are denoted by I_n and 0_n, respectively. When the dimensions are clear from the context, these notations are simplified to I and 0. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed s, the mapping $r \mapsto \beta(r, s)$ belongs to class $\mathcal{K}$ and for each fixed r, the mapping $s \mapsto \beta(r, s)$ is decreasing and $\beta(r, s) \to 0$ when $s \to \infty$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KLL}$ if, for each $r \geq 0$, both $\gamma(\cdot, \cdot, r)$ and $\gamma(\cdot, r, \cdot)$ belong to class $\mathcal{KL}$. A function $f : \mathbb{R}^{\bullet} \to \mathbb{R}^{\bullet}$ is said to be locally Lipschitz continuous if for each $x_0 \in \mathbb{R}^{\bullet}$ there exists constants $\delta > 0$ and $L > 0$ such that for all $x \in \mathbb{R}^{\bullet}$ we have that $|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| \leq L|x - x_0|$.

III. System description: The NCS model

In this section, the considered class of systems is introduced. In particular, we will give a quick recap on the networked control configuration of the NCS and the working principle of the communication network itself. Then, we will model the overall system as a hybrid system, however, we will see that, as a result of the presence of the feedthrough terms, this will require some particular care.

A. *Networked control configuration*

Consider the NCS as shown in Fig. 1, where the continuous-time plant $\mathcal{P}$ is given by

$$\mathcal{P} : \begin{bmatrix} \dot{x}_p \\ y \end{bmatrix} = \begin{bmatrix} A_{\mathbf{P}} & B_{\mathbf{P}} \\ C_{\mathbf{P}} & C_{\mathbf{Y}} \end{bmatrix} \begin{bmatrix} x_p \\ \hat{u} \end{bmatrix} \tag{1}$$

with the initial condition $x_p(0) = x_{0,p} \in \mathbb{R}^{m_{x_p}}$ and where $x_p \in \mathbb{R}^{m_{x_p}}$ denotes the state, $\hat{u} \in \mathbb{R}^{m_u}$ the most recently received control input, and $y \in \mathbb{R}^{m_y}$ the output to the controller.

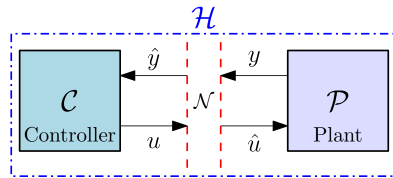


Fig. 1. The NCS setup as described in [6]–[8]. The plant $\mathcal{P}$ has its own network $\mathcal{N}$ to communicate with its controller $\mathcal{C}$. The overall "networked" (hybrid) system $\mathcal{H}$ is the combination of the plant $\mathcal{P}$, its controller $\mathcal{C}$, and its network $\mathcal{N}$.

As shown in Fig. 1, the plant $\mathcal{P}$ is controlled by its own controller $\mathcal{C}$, which communicate with each other via the communication network $\mathcal{N}$. The controller $\mathcal{C}$ is described by

$$\mathcal{C} : \begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_{\mathbf{C}} & B_{\mathbf{C}} \\ C_{\mathbf{C}} & C_{\mathbf{U}} \end{bmatrix} \begin{bmatrix} x_c \\ \hat{y} \end{bmatrix}, \tag{2}$$

where $x_c \in \mathbb{R}^{m_c}$ denotes the controller state, $\hat{y} \in \mathbb{R}^{m_y}$ the most recently received output measurement of the plant, and $u \in \mathbb{R}^{m_u}$ the controller output.

Note now that the difference between the setups in [6]–[12] and (1)-(2) is given by the matrices $C_{\mathbf{U}}$ and $C_{\mathbf{Y}}$ from which at least one of them is nonzero while *both* the control input u as well as the output to the controller y are transmitted over the communication network $\mathcal{N}$.

To complete the description of the NCS setup, it has to be explained how the communication network $\mathcal{N}$ operates. This network $\mathcal{N}$ has a collection of sampling/transmission times t_j, $j \in \mathbb{N}$, which satisfy $0 \leq t_1 < t_2 < \dots$. In the considered setup, similar to [6]–[11], it is assumed that the transmission times satisfy

$$\delta \leq t_{j+1} - t_j \leq \tau_{mati}$$

for all $j \in \mathbb{N}$, where $\delta > 0$ is a certain (arbitrarily small) constant to prevent Zeno behavior and τ_{mati} denotes the *maximally allowable transmission interval* (MATI). For the NCS, (parts of) the output y and input u are sampled and transmitted over the network $\mathcal{N}$ to the controller $\mathcal{C}$ and/or plant $\mathcal{P}$, respectively, at such a transmission time t_j. The network $\mathcal{N}$ might be subdivided in several (sensor and/or actuator) nodes, where each node corresponds to a subset of the entries $y/\hat{y}$ and/or $u/\hat{u}$. A scheduling protocol determines which of the nodes in the network is granted access to the network at a transmission time. After a node is granted access to the network, it collects and transmits the values of the corresponding entries in $y(t_j)$ and $u(t_j)$, which results in an update according to

$$\begin{aligned}\hat{y}\left(t_j^+\right) &= y\left(t_j\right) + h_y\left(j, e\left(t_j\right)\right)\\ \hat{u}\left(t_j^+\right) &= u\left(t_j\right) + h_u\left(j, e\left(t_j\right)\right),\end{aligned} \tag{3}$$

where the functions $h := (h_y, h_u)$ models the (local) network protocol and where e denotes the network-induced error defined by

$$e := \begin{bmatrix} e_y \\ e_u \end{bmatrix} = \begin{bmatrix} \hat{y} - y \\ \hat{u} - u \end{bmatrix}. \tag{4}$$

Finally, it is assumed that $\hat{y}$ and $\hat{u}$ are constant in between two successive transmissions, i.e., we assume a zero-order-hold (ZOH) fashion.

Note that, from this point forward, we will use the shorthand notations $\hat{y}^+ = \hat{y}\left(t_j^+\right)$, $\hat{u}^+ = \hat{u}\left(t_j^+\right)$, $y = y(t_j)$, $\hat{y} = \hat{y}(t_j)$, $u = u(t_j)$, $\hat{u} = \hat{u}(t_j)$, $x_p = x_p(t_j)$, $x_c = x_c(t_j)$, $e^+ = e\left(t_j^+\right)$, and $e = e(t_j)$ in all of the equations.

B. Updating the network-induced error e

Because of the presence of the direct-feedthrough terms $C_{\mathbf{U}}$ and $C_{\mathbf{Y}}$ in the networked interconnection, we have that u and y depend on the networked values $\hat{y}$ and $\hat{u}$, respectively. As a result, an update of $\hat{y}$ and $\hat{u}$ also results in a change of the values of y and u, i.e., we have that

$$y^+ = C_{\mathbf{P}} x_p + C_{\mathbf{Y}} \hat{u}^+ \quad \text{and} \quad u^+ = C_{\mathbf{C}} x_c + C_{\mathbf{U}} \hat{y}^+. \tag{5}$$

As a consequence, we encounter some non-trivial difficulties regarding the modeling of the update equation for the networked-induced error e. To put this into more context, consider the following analysis. Using the expressions of (1)-(2) and (4) it can be obtained that the errors e_y and e_u themselves are given by

$$\begin{aligned} e_y &= \hat{y} - y = \hat{y} - C_{\mathbf{P}} x_p - C_{\mathbf{Y}} \hat{u}\\ e_u &= \hat{u} - u = \hat{u} - C_{\mathbf{C}} x_c - C_{\mathbf{U}} \hat{y}.\end{aligned} \tag{6}$$

Consider now the situation that we have an update of our networked values at transmission time t_j, $j \in \mathbb{N}$, according to (3), i.e.,

$$\begin{aligned}\hat{y}^+ &= y + h_y\left(j, e\right) = C_{\mathbf{P}} x_p + C_{\mathbf{Y}} \hat{u} + h_y\left(j, e\right)\\ \hat{u}^+ &= u + h_u\left(j, e\right) = C_{\mathbf{C}} x_c + C_{\mathbf{U}} \hat{y} + h_u\left(j, e\right).\end{aligned} \tag{7}$$

By using (5)-(7), we derive that this update of the networked values leads to the error being updated according to

$$\begin{aligned} e_y^+ &= \hat{y}^+ - C_{\mathbf{P}} x_p - C_{\mathbf{Y}} \hat{u}^+\\ &= C_{\mathbf{Y}} \hat{u} + h_y(j,e) - C_{\mathbf{Y}} C_{\mathbf{C}} x_c - C_{\mathbf{Y}} C_{\mathbf{U}} \hat{y} - C_{\mathbf{Y}} h_u(j,e)\\ &= h_y(j,e) - C_{\mathbf{Y}} h_u(j,e) + C_{\mathbf{Y}} e_u\\ e_u^+ &= \hat{u}^+ - C_{\mathbf{C}} x_c - C_{\mathbf{U}} \hat{y}^+\\ &= C_{\mathbf{U}} \hat{y} + h_u(j,e) - C_{\mathbf{U}} C_{\mathbf{P}} x_p - C_{\mathbf{U}} C_{\mathbf{Y}} \hat{u} + C_{\mathbf{U}} h_y(j,e)\\ &= h_u(j,e) - C_{\mathbf{U}} h_y(j,e) + C_{\mathbf{U}} e_y.\end{aligned}$$

Hence, we have that, in general, the update equation of the error e can be described by using an update function $h_{df}: \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$, i.e,

$$e^+ = \underbrace{h\left(j, e\right) + \begin{bmatrix} 0_{m_y} & C_{\mathbf{Y}} \\ C_{\mathbf{U}} & 0_{m_u} \end{bmatrix} \left(e - h\left(j, e\right)\right)}_{:= h_{df}\left(h\left(j,e\right), e\right)}. \tag{8}$$

From this result it thus follows that it is not straightforward to describe the update of the error e similar to the situation *without* direct-feedthrough terms, i.e., $C_{\mathbf{Y}} = 0$ and $C_{\mathbf{U}} = 0$, as described in [6]–[12], which resulted in

$$e^+ = h\left(j, e\right). \tag{9}$$

More precisely, we can conclude that the update property of (9) as studied in many previous papers, see [6]–[12], can a priori be lost because of the perturbative term described by the update function h_{df} induced by the feedthrough terms $C_{\mathbf{Y}}$ and $C_{\mathbf{U}}$. As such, a careful analysis is needed for the considered NCSs described by (1)-(3). However, we would still like to base our analysis on previous results in literature. Therefore, similar to [6]–[12], we will first rewrite the modeling setup in the form of a hybrid system [15].

C. A hybrid modeling framework

Based on the above setup, the triple $(\mathcal{P}, \mathcal{C}, \mathcal{N})$ can be rewritten into the format of a hybrid system $\mathcal{H}$, as described in [6]–[8], where each jump of the hybrid system corresponds to an update of the networked values according to (3). To do so, we need to eliminate the control variables $u/\hat{u}$ and $y/\hat{y}$ from the state dynamics, or in other words, we need to express the networked values in terms of the state $x := (x_p, x_c)$ and the error e. Based on (1), (2), and (4), we have that

$$\begin{aligned}\hat{u} &= u + e_u = C_{\mathbf{C}} x_c + C_{\mathbf{U}} \hat{y} + e_u = C_{\mathbf{C}} x_c + C_{\mathbf{U}} \left(y + e_y\right) + e_u\\ &= C_{\mathbf{C}} x_c + C_{\mathbf{U}} \left(C_{\mathbf{P}} x_p + C_{\mathbf{Y}} \hat{u} + e_y\right) + e_u\\ &= C_{\mathbf{C}} x_c + C_{\mathbf{U}} \left(C_{\mathbf{P}} x_p + C_{\mathbf{Y}} \left(C_{\mathbf{C}} x_c + C_{\mathbf{U}} \hat{y} + e_u\right) + e_y\right) + e_u\\ &= \left(I + C_{\mathbf{U}} C_{\mathbf{Y}}\right)\left(C_{\mathbf{C}} x_c + e_u\right) + C_{\mathbf{U}} \left(C_{\mathbf{P}} x_p + e_y + C_{\mathbf{Y}} C_{\mathbf{U}} \hat{y}\right)\\ &= \left(I + C_{\mathbf{U}} C_{\mathbf{Y}}\right)\left(C_{\mathbf{C}} x_c + e_u + C_{\mathbf{U}} C_{\mathbf{P}} x_p + C_{\mathbf{U}} e_y\right)\\ &\quad + \left(C_{\mathbf{U}} C_{\mathbf{Y}}\right)^2 \hat{u}\\ &= \ldots\\ &= \Big(I + C_{\mathbf{U}} C_{\mathbf{Y}} + \left(C_{\mathbf{U}} C_{\mathbf{Y}}\right)^2 + \left(C_{\mathbf{U}} C_{\mathbf{Y}}\right)^3 + \ldots\Big)\\ &\qquad \cdot \left(C_{\mathbf{C}} x_c + C_{\mathbf{U}} C_{\mathbf{P}} x_p + C_{\mathbf{U}} e_y + e_u\right),\end{aligned}$$

and similarly that

$$\begin{aligned}\hat{y} &= \Big(I + C_{\mathbf{Y}} C_{\mathbf{U}} + \left(C_{\mathbf{Y}} C_{\mathbf{U}}\right)^2 + \left(C_{\mathbf{Y}} C_{\mathbf{U}}\right)^3 + \ldots\Big)\\ &\qquad \cdot \left(C_{\mathbf{P}} x_p + C_{\mathbf{Y}} C_{\mathbf{C}} x_c + C_{\mathbf{Y}} e_u + e_y\right).\end{aligned}$$

Hence, using Neumann series, see, e.g., [16], we can rewrite these expressions as

$$\begin{aligned}\hat{u} &= \left(I_{m_u} - C_{\mathbf{U}} C_{\mathbf{Y}}\right)^{-1}\left(C_{\mathbf{C}} x_c + C_{\mathbf{U}} C_{\mathbf{P}} x_p + C_{\mathbf{U}} e_y + e_u\right)\\ \hat{y} &= \left(I_{m_y} - C_{\mathbf{Y}} C_{\mathbf{U}}\right)^{-1}\left(C_{\mathbf{P}} x_p + C_{\mathbf{Y}} C_{\mathbf{C}} x_c + C_{\mathbf{Y}} e_u + e_y\right),\end{aligned} \tag{10}$$

provided that the inverses as in (10) exist and

$$\lim_{n \to \infty} \left(C_{\mathbf{U}} C_{\mathbf{Y}}\right)^n = 0 \quad \text{and} \quad \lim_{n \to \infty} \left(C_{\mathbf{Y}} C_{\mathbf{U}}\right)^n = 0.$$

Hence, to construct our hybrid model, we need the following assumption on the interconnection itself.

Assumption 1: For the NCS described by (1)-(3) the interconnection is well-posed in the sense that it holds that

$$\max_i \left|\lambda_i\left(C_{\mathbf{U}} C_{\mathbf{Y}}\right)\right| < 1, \tag{11}$$

or, equivalent, $\min\left(\left|C_{\mathbf{U}} C_{\mathbf{Y}}\right|, \left|C_{\mathbf{Y}} C_{\mathbf{U}}\right|\right) < 1$, where λ_i denotes the i-th eigenvalue. ■

$$\mathbf{A} = \begin{bmatrix} A_{\mathbf{P}} + B_{\mathbf{P}}\left(I_{m_u} - C_{\mathbf{U}}C_{\mathbf{Y}}\right)^{-1}C_{\mathbf{U}}C_{\mathbf{P}} & B_{\mathbf{P}}\left(I_{m_u} - C_{\mathbf{U}}C_{\mathbf{Y}}\right)^{-1}C_{\mathbf{C}} \\ B_{\mathbf{C}}\left(I_{m_y} - C_{\mathbf{Y}}C_{\mathbf{U}}\right)^{-1}C_{\mathbf{P}} & A_{\mathbf{C}} + B_{\mathbf{C}}\left(I_{m_y} - C_{\mathbf{Y}}C_{\mathbf{U}}\right)^{-1}C_{\mathbf{Y}}C_{\mathbf{C}} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -C_{\mathbf{P}} & 0 \\ 0 & -C_{\mathbf{C}} \end{bmatrix}\mathbf{A},$$
$$\mathbf{E} = \begin{bmatrix} B_{\mathbf{P}}\left(I_{m_u} - C_{\mathbf{U}}C_{\mathbf{Y}}\right)^{-1}C_{\mathbf{U}} & B_{\mathbf{P}}\left(I_{m_u} - C_{\mathbf{U}}C_{\mathbf{Y}}\right)^{-1} \\ B_{\mathbf{C}}\left(I_{m_y} - C_{\mathbf{Y}}C_{\mathbf{U}}\right)^{-1} & B_{\mathbf{C}}\left(I_{m_y} - C_{\mathbf{Y}}C_{\mathbf{U}}\right)^{-1}C_{\mathbf{Y}} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -C_{\mathbf{P}} & 0 \\ 0 & -C_{\mathbf{C}} \end{bmatrix}\mathbf{E}. \tag{13}$$

Assumption 1 can be related to a small-gain type of condition for feedback systems between the $\hat{u}$- and $\hat{y}$-systems in (10). Note that Assumption 1 is always satisfied when only the plant or controller has feedthrough terms, see also Remark 2.

Now, by using (10) we can eliminate the control variables in the state dynamics and, by using the ZOH assumption, we can also derive an expression for the error dynamics by directly using (1), (2), and (6), which yields

$$\begin{bmatrix} \dot{e}_y \\ \dot{e}_u \end{bmatrix} = \begin{bmatrix} -C_{\mathbf{P}}\dot{x}_p \\ -C_{\mathbf{C}}\dot{x}_c \end{bmatrix}.$$

As a result, similar to [6]–[12], by introducing the timer $\tau \in \mathbb{R}_{\geq 0}$, which is used to generate jumps in the hybrid model that correspond to data transmissions, and the counter $\kappa \in \mathbb{N}$, which counts the amount of transmissions and is needed to implement certain scheduling protocols, we obtain for the triple $(\mathcal{P}, \mathcal{C}, \mathcal{N})$ the following hybrid model

$$\mathcal{H}: \begin{cases} \left.\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{E}e \\ \dot{e} &= \mathbf{C}x + \mathbf{F}e \\ \dot{\tau} &= 1 \\ \dot{\kappa} &= 0 \end{aligned}\right\} & \text{when } \tau \in [0, \tau_{mati}] \\ \left.\begin{aligned} x^+ &= x \\ e^+ &= h(\kappa, e) + h_{df}(h(\kappa, e), e) \\ \tau^+ &= 0 \\ \kappa^+ &= \kappa + 1 \end{aligned}\right\} & \text{when } \tau \in [\delta, \infty) \end{cases} \tag{12}$$

where the various matrices are given by (13) and with the new state of the hybrid system $\xi := (x, e, \tau, \kappa) \in \mathbb{X} := \mathbb{R}^{m_x} \times \mathbb{R}^{m_e} \times \mathbb{R}_{\geq 0} \times \mathbb{N}$. Using this hybrid modeling framework, stability in the sense of UGES for the NCS can now be analyzed.

IV. Stability analysis

In this section, we analyze the stability of the system described by (12) as defined next.

Definition 1: For the overall system $\mathcal{H}$ that satisfies Assumption 1 given by (12), the set

$$\mathcal{E} = \{\xi \in \mathbb{X} \mid x = 0 \wedge e = 0\} \tag{14}$$

is said to be *uniformly globally exponentially stable* (UGES) if there exists a function $\beta \in \mathcal{KLL}$, which can be taken of the form $\beta(r, t, j) = Mr\exp(-\rho(t + j))$ for some $M \geq 0$ and $\rho > 0$, such that for any initial condition $\xi(0, 0) \in \mathbb{X}$, all corresponding maximal solutions ξ are complete and satisfy for all $(t, j) \in \operatorname{dom}\xi$

$$\left|(x(t, j), e(t, j))\right| \leq \beta\left(\left|(x(0, 0), e(0, 0))\right|, t, j\right). \quad \blacksquare$$

Remark 1: Note that in Definition 1 we used the solution concept as introduced in [15] for describing the NCS in terms of the hybrid system (12). For more information and a detailed analysis the interested reader is referred to [15].

We will now provide a brief recap of the stability analysis as introduced in [7], [8], although slightly altered to incorporate the presence of the direct-feedthrough terms. In particular, we will derive LMI-based conditions that guarantee UGES of the set (14). However, to do so, the way of viewing the scheduling protocols first needs to be re-examined.

A. UGES scheduling protocols

One of the most important aspects in the analysis approach of [7] is the introduction of the concept of UGES scheduling protocols. Hereto, the update equation of the error is modeled as a discrete-time system of the form

$$e(i + 1) = p_f(i, e(i)) \tag{15}$$

induced by a certain function $p_f : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$. Hence, when there are no direct-feedthrough terms, i.e., $C_{\mathbf{Y}} = 0$ and $C_{\mathbf{U}} = 0$, and we thus have that the update of the error is described by (9), then the function p_f is given by the scheduling protocol function h itself, i.e., for all $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$

$$p_f(i, e) = h(i, e). \tag{16}$$

Similarly, when the direct-feedthrough terms are present in the interconnection, we have that the update of the error is given by (8) and, hence, for all $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$

$$p_f(i, e) = h(i, e) + h_{df}(h(i, e), e). \tag{17}$$

Consider now the following definition.

Definition 2: Let $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ be given and suppose that there exist constants $\lambda \in (0, 1)$ and $\underline{\alpha}_W^c, \overline{\alpha}_W^c > 0$ such that the following conditions hold for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{m_e}$:

$$\underline{\alpha}_W^c|e| \leq W(i, e) \leq \overline{\alpha}_W^c|e| \tag{18a}$$
$$W(i + 1, p_f(i, e)) \leq \lambda W(i, e). \tag{18b}$$

Then the discrete-time system of (15) is said to be UGES with Lyapunov function W. $\blacksquare$

When Definition 2 is satisfied by the discrete-time system (15) with (16), as was the case in [7], then it is even said that the scheduling protocol function h is UGES with Lyapunov function W. As shown in [7], various scheduling protocols exist in this case that satisfy this definition of UGES scheduling protocols, including the try-once-discard (TOD), sampled-data (SD), and round-robin (RR) protocols.

However, as we are in this work dealing with an update of the error according to (8) rather than (9), this concept of UGES scheduling protocols needs to be slightly altered. In particular, we need to take into account that in this paper, as a result of the update function h_{df}, the update of the error e depends on the system matrices $C_{\mathbf{Y}}$ and $C_{\mathbf{U}}$, while in [7] (15) did not depend on the controller/plant parameters at all. Hence, for a given scheduling protocol, the corresponding system (15) with (17) might be UGES in some cases, while in others it is not. As such, consider the following definition.

Definition 3: The scheduling protocol modeled by the scheduling protocol function $h : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$ is said to be UGES with Lyapunov function W for the given system of (1)-(3) when the discrete-time system of (15) with (17) is UGES according to Definition 2 with Lyapunov function W. ∎

In Section V, we will show that various well-known scheduling protocols from [6], [7] are also UGES according to Definition 3 under appropriate conditions on the direct-feedthrough matrices $C_{\mathbf{U}}$ and $C_{\mathbf{Y}}$.

B. Lyapunov-based conditions for UGES

We now recapitulate the main results of [7], [8] concerning UGES for the NCS given by (1)-(3) and modeled by (12), while also taking Definition 3 into account. In particular, we formulate conditions guaranteeing stability in the sense of UGES for the NCS with direct-feedthrough terms modeled by (12).

Theorem 1: Consider the system $\mathcal{H}$ of (12) that satisfies Assumption 1. Assume there exist a function $W : \mathbb{N}\times\mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz function $V : \mathbb{R}^{m_x} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{m_x} \to \mathbb{R}$, strictly positive real numbers $\underline{\alpha}_W^c$, $\overline{\alpha}_W^c$, $\underline{\alpha}_V^c$, $\overline{\alpha}_V^c$, $\lambda \in (0,1)$, and $0 < \varepsilon < \gamma$ such that

- the scheduling protocol function $h : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$ is UGES according to Definition 3 with Lyapunov function W and the constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c$, and λ
- for all $\kappa \in \mathbb{N}$, $x \in \mathbb{R}^{m_x}$, and for almost all $e \in \mathbb{R}^{m_e}$ it holds that

$$\left\langle \frac{\partial W(\kappa, e)}{\partial e}, \mathbf{C}x + \mathbf{F}e \right\rangle \leq L W(\kappa, e) + H(x), \tag{19}$$

- for all $x \in \mathbb{R}^{m_x}$ it holds that $\underline{\alpha}_V^c |x|^2 \leq V(x) \leq \overline{\alpha}_V^c |x|^2$,
- and for all $e \in \mathbb{R}^{m_e}$, $\kappa \in \mathbb{N}$, and almost all $x \in \mathbb{R}^{m_x}$

$$\begin{aligned}\langle \nabla V(x), \mathbf{A}x + \mathbf{E}e\rangle \leq -\varepsilon^2 |x|^2 - H^2(x) \\ + [\gamma^2 - \varepsilon^2] W^2(\kappa, e)).\end{aligned} \tag{20}$$

If now τ_{mati} satisfies the bound

$$\tau_{mati} \leq \begin{cases} \frac{1}{Lr} \arctan\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma > L \\ \frac{1}{L}\frac{1-\lambda}{1+\lambda} & \gamma = L \\ \frac{1}{Lr} \operatorname{arctanh}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma < L \end{cases} \tag{21}$$

with $r = \sqrt{\left|\left(\frac{\gamma}{L}\right)^2 - 1\right|}$, then the set $\mathcal{E}$ of (14) is UGES. ∎

The proof is given in [8]. Theorem 1 provides tractable conditions to guaranteed UGES of the overall system. However, it might not be straightforward to construct the appropriate functions. Therefore, it will be shown that this analysis can be described in terms of a linear matrix inequality (LMI) optimization problem.

C. LMI-based condition for UGES

As shown before in, for instance, [10]–[13], to systematically verify the conditions of Theorem 1 it is possible to reformulate the presented conditions into LMI-based conditions. Consider hereto condition (19) and assume now that for almost all $e \in \mathbb{R}^{m_e}$ and all $\kappa \in \mathbb{N}$ it holds that

$$\left|\frac{\partial W(\kappa, e)}{\partial e}\right| \leq M \tag{22}$$

for some constant $M > 0$, see also Section V. Moreover, when using (12), it follows for the error dynamic that

$$|\dot{e}| \leq |\mathbf{C}x| + |\mathbf{F}e|$$

and hence, in (19), $L = M{\underline{\alpha}_W^c}^{-1}|\mathbf{F}|$ and $H(x) = M|\mathbf{C}x|$. Using this result in (20), it can be directly obtained that for the right-hand side it holds that

$$-\varepsilon^2 |x|^2 + {\underline{\alpha}_W^c}^2 \left[\gamma^2 - \varepsilon^2\right] |e|^2 - H^2(x) = \begin{bmatrix} x \\ e \end{bmatrix}^\top J \begin{bmatrix} x \\ e \end{bmatrix} \tag{23}$$

with

$$J := \begin{bmatrix} -\varepsilon^2 I_{m_{x_p}+m_{x_c}} - M^2 \mathbf{C}^\top \mathbf{C} & 0 \\ 0 & {\underline{\alpha}_W^c}^2 \left[\gamma^2 - \varepsilon^2\right] I_{m_y+m_u} \end{bmatrix}.$$

To now arrive at an LMI-based condition which guarantees UGES, also the left-hand side of (20) needs to be evaluated. Therefore, we take $V(x) = x^\top X_{\mathbf{T}} x$ with $X_{\mathbf{T}}$ being a symmetric positive definite matrix of size $m_x \times m_x$. Hence, we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V(x) &= \langle \nabla V(x), \mathbf{A}x + \mathbf{E}e\rangle \\ &= x^\top \left(\mathbf{A}^\top X_{\mathbf{T}} + X_{\mathbf{T}}\mathbf{A}\right) x + x^\top X_{\mathbf{T}}\mathbf{E}e + e^\top \mathbf{E}^\top X_{\mathbf{T}} x.\end{aligned} \tag{24}$$

Combining now the results of (23) and (24), Theorem 1 can be rewritten into LMI conditions.

Theorem 2: Consider the system $\mathcal{H}$ of (12) that satisfies Assumption 1. Assume there exist a function $W : \mathbb{N}\times\mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a matrix $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$, and strictly positive real numbers $\underline{\alpha}_W^c$, $\overline{\alpha}_W^c$, $\lambda \in (0,1)$, and $0 < \varepsilon < \gamma$ such that

- the scheduling protocol function $h : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$ is UGES according to Definition 3 with Lyapunov function W and the constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c$, and λ
- for all $\kappa \in \mathbb{N}$, and for almost all $e \in \mathbb{R}^{m_e}$ (22) holds
- $\begin{bmatrix} \mathbf{A}^\top X_{\mathbf{T}} + X_{\mathbf{T}}\mathbf{A} + \varepsilon^2 I_{m_x} + M^2 \mathbf{C}^\top \mathbf{C} & X_{\mathbf{T}}\mathbf{E} \\ \mathbf{E}^\top X_{\mathbf{T}} & -{\underline{\alpha}_W^c}^2 \left[\gamma^2 - \varepsilon^2\right] I_{m_e} \end{bmatrix} \leq 0.$ (25)

If now τ_{mati} satisfies (21) then the set $\mathcal{E}$ of (14) is UGES. ∎

Since γ is the only free variable for the computation of the bound for τ_{mati} as λ follows from the scheduling protocol, τ_{mati} can be maximized by means of minimizing γ subjected to $0 < \varepsilon < \gamma$ and the LMI of (25).

In order to use Theorem 2, and, in particular, the LMI (25), to verify stability of the linear hybrid system given by (12), it is necessary to have a scheduling protocol function h that is UGES according to Definition 3 such that (18) and (22) are satisfied. In the next section, this problem is analyzed.

V. UGES scheduling protocols for the NCS with direct-feedthrough terms

In this section,we will only focus on the most well-known scheduling protocols for illustrative purposes and show under which conditions those protocols are UGES according to Definition 3, although we envision that many other protocols are UGES in the sense of Definition 3 as well.

A. Sampled-data protocol

First we consider the sampled-data (SD) protocol, see, e.g., [6] or [7], which is modeled for $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$ by

$$h(i,e) = 0. \tag{26}$$

From the results obtained in Section III-B, we now know that the function p_f in (15) corresponding to the NCS with direct-feedthrough terms of (1)-(3) is given by (17), which results for the SD protocol in

$$p_f(i,e) = C_{\mathbf{YU}} e, \text{ with } C_{\mathbf{YU}} := \begin{bmatrix} 0_{m_y} & C_{\mathbf{Y}} \\ C_{\mathbf{U}} & 0_{m_u} \end{bmatrix}. \tag{27}$$

Hence, UGES of the discrete-time system (15) in the sense of Definition 3 follows now directly when we have that for all eigenvalues λ_i of $C_{\mathbf{YU}}$ it holds that

$$\left|\lambda_i(C_{\mathbf{YU}})\right| < 1. \tag{28}$$

As such, for the SD protocol, we can now present the following result (the proof is given in the Appendix.).

Proposition 1: The SD protocol, modeled by the scheduling protocol function (26), is UGES according to Definition 3 with Lyapunov function $W(e) = \sqrt{e^\top P e}$ for the NCS given by (1)-(3), where the real symmetric matrix $P \succ 0$ can be computed by solving

$$C_{\mathbf{YU}}^\top P C_{\mathbf{YU}} - \rho P \preceq 0 \tag{29}$$

for a certain constant $\rho \in (0,1)$, if Assumption 1 is satisfied. Moreover, (18a) is then satisfied for $\underline{\alpha}_W^c = \sqrt{\lambda_{\min}(P)}$ and $\overline{\alpha}_W^c = \sqrt{\lambda_{\max}(P)}$, (18b) for $\lambda = \sqrt{\rho}$, and (22) for $M = \sqrt{\lambda_{\max}(P)}$, where $\lambda_{\min}(P)/\lambda_{\max}(P)$ denote the smallest/largest eigenvalue of P. ■

Remark 2: From this analysis it can thus be concluded that the condition (11) implies that when either $C_{\mathbf{Y}}$ or $C_{\mathbf{U}}$ is absent in the dynamics, i.e., $C_{\mathbf{Y}} C_{\mathbf{U}} = 0$, the other-ones matrix norm can grow arbitrarily large and still the SD protocol will ensure UGES of (15).

B. Try-once-discard protocol

Secondly, we consider the TOD protocol as introduced in [6]. As such, we have the situation that there are l nodes competing for access to the network and, hence, the error vector can be partitioned as $e = (e_1, e_2, \ldots, e_l)$. The scheduling protocol function h for the TOD protocol is for $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$ given by [7]

$$h(i,e) = (I - \Psi(e))\, e \tag{30}$$

with $\Psi(e) = \operatorname{diag}\{\psi_1(e) I_{m_1}, \psi_2(e) I_{m_2}, \ldots, \psi_l(e) I_{m_l}\}$, where I_{m_j} are identity matrices of dimension m_j with $\sum_{j=1}^{l} m_j = m_y + m_u = m_e$ and

$$\psi_j(e) = \begin{cases} 1, & \text{if } j = \min\left(\arg\max_j |e_j|\right) \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

As a result of this modeling framework, we can state the following result for the TOD protocol (the proof is given in the appendix).

Proposition 2: The TOD protocol, modeled by the scheduling protocol function of (30) with (31), is UGES according to Definition 3 with Lyapunov function $W(e) = \sqrt{e^\top P e}$ for the NCS given by (1)-(3) if there exist a matrix $P \succ 0$, a constant $\rho \in (0,1)$, and nonnegative constants β_j^k, $l \in \bar{l}$ and $j \in \bar{l} \setminus \{k\}$, such that

$$A_k^\top P A_k - \rho P + \sum_{j=1, j\neq k}^{l} \beta_j^k Q_{kj} \preceq 0 \tag{32}$$

holds for all for all $k \in \bar{l}$ with the matrices $Q_{kj} := \operatorname{diag}\{0_{m_1}, \ldots, 0_{m_{k-1}}, I_{m_k}, 0_{m_{k+1}}, \ldots, 0_{m_{j-1}}, -I_{m_j}, 0_{m_{j+1}}, \ldots, 0_{m_l}\}$ and where the matrix A_k is given in (37). Moreover, (18a) is then satisfied for $\underline{\alpha}_W^c = \sqrt{\lambda_{\min}(P)}$ and $\overline{\alpha}_W^c = \sqrt{\lambda_{\max}(P)}$, (18b) for $\lambda = \sqrt{\rho}$, and (22) for $M = \sqrt{\lambda_{\max}(P)}$. ■

Remark 3: Note that the SD protocol is a special case of the TOD protocol. Indeed, with $l = 1$ we have that in (37) $A_k = C_{\mathbf{YU}}$ for $k = 1$, implying that for this special case (32) indeed simplifies to (29).

C. Round-robin protocol

Finally, we consider the round-robin (RR) protocol, for which the scheduling protocol function is given for $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$ by

$$h(i,e) = (I - \Delta(i))e \tag{33}$$

where $\Delta(i) = \operatorname{diag}\{\Delta_1(i), \Delta_2(i), \ldots, \Delta_l(i)\}$. The square matrices $\Delta_k(i)$ have the dimension m_k, with $\sum_{k=1}^{l} m_k = m_e$ and $\Delta_k(i) = \delta_k(i) I_{m_k}$, where I_{m_k} are identity matrices of dimension m_k and

$$\delta_k(i) = \begin{cases} 1, & \text{if } i = k + jl,\ j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

As such, we can compose the following result for the RR protocol (the proof is given in the Appendix).

Proposition 3: The RR protocol, modeled by the scheduling protocol function of (33) with (34), is UGES according to Definition 3 with Lyapunov function $W(e(i)) = \sqrt{e^\top(i) P(i) e(i)}$ for the NCS given by (1)-(3) if there exist an l-periodic scalar $\rho(k) = \rho(k+l) \in (0,1)$ and an l-periodic matrix $P(k) = P(k+l)$ with $P(k) \succ 0$ such that

$$A_k^\top P(k+1) A_k - \rho(k) P(k) \preceq 0 \tag{35}$$

holds for all for all $k \in \bar{l}$, where the matrix A_k is given in (37). Moreover, (18a) is then satisfied for $\underline{\alpha}_W^c = \min_k \sqrt{\lambda_{\min}(P(k))}$ and $\overline{\alpha}_W^c = \max_k \sqrt{\lambda_{\max}(P(k))}$, (18b) for $\lambda = \max_k \sqrt{\rho(k)}$, and (22) for $M = \max_k \sqrt{\lambda_{\max}(P(k))}$. ■

Remark 4: Note that the SD protocol is also a special case of the RR protocol. Indeed, similarly to Remark 3, we have that, for $l = 1$, (35) indeed simplifies to (29).

VI. Numerical example

Consider the benchmark example of the unstable batch reactor, see [6]–[9]. The linearized model of an unstable batch reactor and its controller is a two-input-two-output NCS system that can be captured by the model of (1)-(2) where

$$A_P = \begin{pmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{pmatrix},$$

$$B_P = \begin{pmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{pmatrix}, \quad C_P = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B_C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_C = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}, \quad C_{\mathbf{U}} = \begin{pmatrix} 0 & -2 \\ 5 & 0 \end{pmatrix},$$

$A_C = 0$, and $C_{\mathbf{Y}} = 0$, see, e.g., [6], [7]. Since the matrix $C_{\mathbf{U}} \neq 0$ we indeed have direct-feedthrough terms. In [6]–[9] it was always assumed that *only* the outputs y to the controller were transmitted via a communication network, implying that an update of the error occurred according to $e^+ = e_y^+ = h_y(j, e)$, which is of the form (9). Hence, despite the presence of the direct-feedthrough terms, the stability analysis as presented in [7], [8] could still be applied. Using the SD protocol with $\lambda = \frac{1}{2}\sqrt{2}$, and $M = \overline{\alpha}_W^c = 1$, this resulted in a bound on the MATI $\tau_{mati} \leq 0.0108$ guaranteeing UGES of the set $\mathcal{E}$ of (14) for the overall system $\mathcal{H}$ of (12).

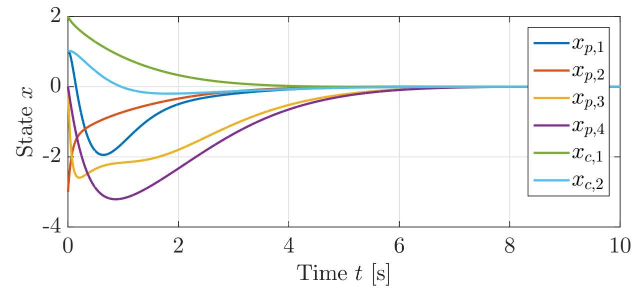
In this paper, we assume that *both* the output y to the controller as well as the controller output u itself are transmitted over a communication network. As a result, the network-induced error is now updated according to (8), implying that the stability analysis of [7], [8] can no longer be applied and, hence, the results as presented in this work should be used to guarantee UGES of the set $\mathcal{E}$ in (14). Note hereby that the setup of the NCS indeed satisfies Assumption 1 since $C_{\mathbf{Y}} = 0$ and indeed Theorem 1 and 2 can be applied.

We again use the SD protocol, which is a valid choice as Proposition 1 is satisfied. Using the given matrices $C_{\mathbf{Y}}$ and $C_{\mathbf{U}}$, it is possible to compute the constants $\underline{\alpha}_W^c$, $\overline{\alpha}_W^c$, λ, and M such that the conditions of (18) and (22) are satisfied by using (29), which gives $\rho = 0.5$ and

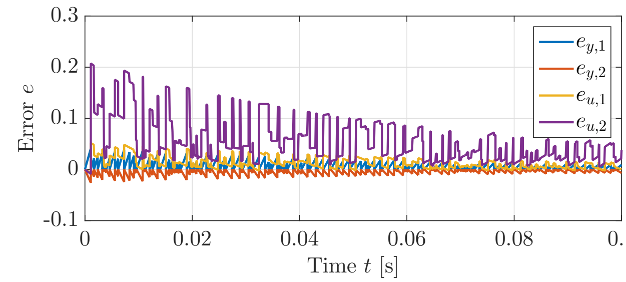
$$P = \begin{pmatrix} 3.8678 & -3.6105{\cdot}10^{-2} & 1.0011{\cdot}10^{-3} & 1.1922{\cdot}10^{-3} \\ -3.6105{\cdot}10^{-2} & 3.4502 & 1.3669{\cdot}10^{-3} & 1.2047{\cdot}10^{-3} \\ 1.0011{\cdot}10^{-3} & 1.3669{\cdot}10^{-3} & 3.1348{\cdot}10^{-1} & 1.8706{\cdot}10^{-3} \\ 1.1922{\cdot}10^{-3} & 1.2047{\cdot}10^{-3} & 1.8706{\cdot}10^{-3} & 5.7145{\cdot}10^{-2} \end{pmatrix}$$

with $\lambda_{\min}(P) = 5.7130{\cdot}10^{-2}$ and $\lambda_{\max}(P) = 3.8709$. Hence, $\underline{\alpha}_W^c = 0.23902$, $\overline{\alpha}_W^c = M = 1.9675$, and $\lambda = \frac{1}{2}\sqrt{2}$. Now by using Theorem 2 the τ_{mati} bound can be computed by minimizing γ, as indicated at the end of Section IV-C, such that UGES of the set $\mathcal{E}$ is guaranteed. We can thus find that $L = 147.10$, $\gamma = 159.14$, and $\tau_{mati} \leq 1.1201 \cdot 10^{-3}$.

In Fig. 2 the results of a simulation for the numerical example of the batch reactor are shown for $\tau_{mati} = 1.1201 \cdot 10^{-3}$, $\delta = 1 \cdot 10^{-4}$, and initial condition $x = \begin{bmatrix} 1 & -3 & 0 & 0 & 2 & 1 \end{bmatrix}^\top$. As can be seen from Fig. 2(a), the overall system is indeed (exponentially) stable (all the states converge to zero), even with the error increasing at some transmission/inter-event times, see Fig. 2(b).



(a) The evolution of the states x.



(b) The evolution of the error e in the time frame $0 \leq t \leq 0.1$.

Fig. 2. Simulation results for the numerical example of the batch reactor with $\tau_{mati} = 1.1201 \cdot 10^{-3}$.

VII. Concluding remarks

In this paper we considered the so-called direct-feedthrough problem for linear networked control systems. We have shown that the conditions proposed in [7] and [8], which ensure a uniform global exponential stability property, can still be used under suitable assumptions on the direct-feedthrough matrices. Hereto, the concept of UGES scheduling protocols as introduced in [7] had to be adapted for the influence of the direct-feedthrough terms on the networked-induced error. Based on this new concept, we have shown that the well-known sampled-data (SD), try-once-discard (TOD), and round-robin (RR) protocols still can be used in the NCS setup with direct-feedthrough terms under suitable assumptions. Finally, we have illustrated the application of our new results by means of the benchmark example of the batch reactor.

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APPENDIX

Proof of Proposition 1: Since (11) and (28) are equivalent conditions, it follows directly from standard results on linear discrete-time systems that, when Assumption 1 is satisfied for the NCS of (1)-(3), the discrete-time system (15) with (27) is indeed UGES according to Definition 2 with the Lyapunov function $W(i,e) = \sqrt{e^\top P e}$ that satisfies (29), $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$.

To shown that this Lyapunov function W also satisfies (22), we will make use of the eigenvalue decomposition of the matrix P. Since P is a real symmetric matrix, its eigenvalues are real and the corresponding eigenvectors can be chosen such that they are orthogonal to each other. Hence, the matrix P can be decomposed by means of $P = Q\Lambda Q^\top$, where Q is an orthogonal matrix, and Λ is a diagonal matrix whose entries are the eigenvalues of P. As a result, we now have that

$$\begin{aligned} W(i,e) &= \sqrt{e^\top P e} = \sqrt{e^\top Q \Lambda Q^\top e} \\ &= \sqrt{v^\top \Lambda v} = \sqrt{\sum_{j=1}^{m_e} \lambda_j(P) \left|v_j\right|^2} \end{aligned}$$

with $v := Q^\top e$ and $\lambda_j(P)$ being the j-th eigenvalue of P. When it is now used that

$$\begin{aligned} \frac{\partial W^2(i,e)}{\partial e} &= 2W(i,e)\frac{\partial W(i,e)}{\partial e} \\ &= 2\left(\lambda_1(P)v_1, \ldots, \lambda_{m_e}(P)v_{m_e}\right)^\top \end{aligned}$$

we obtain that

$$\left|\frac{\partial W(i,e)}{\partial e}\right| = \frac{\sqrt{\sum_j \lambda_j^2(P) v_j^2}}{\sqrt{\sum_j \lambda_j(P) v_j^2}} \leq \sqrt{\max_j \lambda_j(P)} = \sqrt{\lambda_{\max}(P)}$$

and, hence, condition (22) is satisfied for $M = \sqrt{\lambda_{\max}(P)}$. This completes the proof. ∎

Proof of Proposition 2: Suppose that, without loss of generality,

$$|e_k| = \max_{j \in \bar{l}} |e_j|$$

for some $k \in \bar{l}$ with $\bar{l} := \{1, 2, \ldots, l\}$, or, stated differently,

$$|e_k|^2 - |e_j|^2 \geq 0, \quad \text{for all } j \neq k. \tag{36}$$

Hence, according to (30) we have that

$$h(i,e) = \operatorname{diag}\{I_{m_1}, I_{m_2}, \ldots, I_{m_{k-1}}, 0_{m_k}, I_{m_{k+1}}, \ldots, I_{m_l}\}e.$$

Based on (36), we also introduce for each $k \in \bar{l}$ the set

$$\mathcal{C}_k := \left\{e \in \mathbb{R}^{m_e} \mid e^\top Q_{kj} e \geq 0,\ j \in \bar{l}\backslash\{k\}\right\}.$$

By combining (30) and (36), the discrete-time system of (15) can be described as

$$\begin{aligned} &e(i+1) = (I - \Psi(e(i)))\,e(i) + C_{\mathbf{YU}}\Psi(e(i))\,e(i) \\ &= \operatorname{diag}\{I_{m_1}, \ldots, I_{m_{k-1}}, 0_{m_k}, I_{m_{k+1}}, \ldots, I_{m_l}\}\,e(i) \\ &+ C_{\mathbf{YU}} \operatorname{diag}\{0_{m_1}, \ldots, 0_{m_{k-1}}, I_{m_k}, 0_{m_{k+1}}, \ldots, 0_{m_l}\}\,e(i) \\ &=: A_k e(i) \end{aligned} \tag{37}$$

when $e(i) \in \mathcal{C}_k$ for some $k \in \bar{l}$, where the matrix $C_{\mathbf{YU}}$ is as in (27). Note now that (37) describes a *piecewise linear discrete-time system*, implying that its stability can be determined by solving a set of LMIs, see, e.g., [17]. In particular, we propose to use the S-procedure [17]. Consider hereto again the Lyapunov function $W(i,e) = \sqrt{e^\top P e}$ with P being a symmetric positive definite matrix, similar as in Section V-A. This leads directly to (32) and therefore completes the proof. ∎

Proof of Proposition 3: As a result of (34), it follows that the discrete-time system of (15) becomes a *periodic linear discrete-time system* of the form $e(i+1) = A_k e(i)$ with the matrix A_k as in (37) and where $k = i \bmod l$. Hence, the stability of the RR protocol can be analyzed in a similar manner as for the TOD protocol, however, now by using the periodic Lyapunov Lemma, see, e.g., [18], [19], which directly results in the LMI-based necessary and sufficient conditions of (35). This completes the proof.