Author/s:
Maass, AI;Nesic, D;Postoyan, R;Dower, PM

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Observer design for non-linear networked control systems with persistently exciting protocols

Alejandro I. Maass, Dragan Nešić, Romain Postoyan, and Peter M. Dower

***Abstract*—We study the design of state observers for non-linear networked control systems (NCSs) affected by disturbances and measurement noise, via an emulation-like approach. That is, given an observer designed with a specific stability property in the absence of communication constraints, we implement it over a network and we provide sufficient conditions on the latter to preserve the stability property of the observer. In particular, we provide a bound on the maximum allowable transmission interval (MATI) that guarantees an input-to-state stability (ISS) property for the corresponding estimation error system. The stability analysis is trajectory-based, utilises small-gain arguments, and exploits a persistently exciting (PE) property of the scheduling protocols. This property is key in our analysis and allows us to obtain significantly larger MATI bounds in comparison to the ones found in the literature. Our results hold for a general class of NCSs, however, we show that these results are also applicable to NCSs implemented over a specific physical network called WirelessHART (WH). The latter is mainly characterised by its multi-hop structure, slotted communication cycles, and the possibility to simultaneously transmit over different frequencies. We show that our results can be further improved by taking into account the intrinsic structure of the WH–NCS model. That is, we explicitly exploit the model structure in our analysis to obtain an even tighter MATI bound that guarantees the same ISS property for the estimation error system. Finally, to illustrate our results, we present analysis and numerical simulations for a class of Lipschitz non-linear systems and high-gain observers.**

***Index Terms*—Networked control, Non-linear systems, Emulation, Observer design.**

I. Introduction

Networked control systems (NCSs) have attracted global interest due to the features they offer, such as, ease of installation, lower cost and great flexibility. Nevertheless, new challenges arise in NCSs given the intrinsic limitations that come with communication networks, e.g. time-varying sampling, scheduling, transmission delays, packet dropouts and quantisation, among others. Although many solutions have been proposed to deal with these constraints, significant theoretical challenges in their analysis and design remain, see e.g. [1], [2].

We restrict our attention to the design of state observers that receive both sensor and actuator data through a communication network. In this context, we consider that the network is subject to scheduling constraints and time-varying sampling, i.e. only a subset of sensors/actuators is allowed to send their data to the observer at time-varying transmission instants. The sporadic and partial availability of data, which are respectively characterised by the maximum allowable transmission interval (MATI) and the scheduling protocol, requires the development of appropriate observer design tools. Relevant work available in the literature on observer design is listed as follows. In [3], sufficient conditions for the existence of an observer-protocol pair are derived in terms of matrix inequalities for linear systems. Motivated by the observer design of non-linear sampled-data systems [4], the authors in [5] derive an observation structure that ensures the global asymptotic stability of the origin of the observation error, when sensor measurements are subject to network-induced constraints. A framework for the synthesis of observers for non-linear NCSs has been proposed in [6], via an emulaton-like approach, that encompasses the methods proposed in [4] and [5] as particular cases. Provided that the continuous-time observer is sufficiently robust to measurement errors, sufficient conditions in terms of MATI bounds are given to guarantee the global convergence of the observation error for various in-network processing and Lyapunov uniformly globally exponentially stable (UGES) protocols. Newer results on observer design of NCSs that rely on a Lyapunov-based analysis can be found in [7], in which the authors obtained a new MATI bound via an emulation procedure and a class of protocols that includes uniformly globally asymptotically stable (UGAS) protocols. The design of reduced-order observers via emulation in NCSs is studied in [8].

In this paper, we propose a framework for observer design in non-linear NCSs via an emulation-like approach. Within this framework, plants whose dynamics are affected by disturbances and measurement noise are considered, which are more general and induce additional technical difficulties compared with the aforementioned literature. In [6]–[8], the UGES/UGAS property imposed on the protocols was shown to be key in the convergence of the estimation error. However, this property is quite strong as it requires more information than is typically available (NCS state information). A less restrictive property on the protocol is the one called *persistence of excitation* (PE), which was introduced in [9] and translates into the existence of a fixed number of transmissions T within which all nodes of the NCS have transmitted. Whenever T is known, we say that the protocol is PE_T. This property turns out to be more natural in the context of real networks. In particular, PE is verified by many network technologies [9], such as Ethernet, IEEE 802.11, and IEEE 802.15.4 standard, including WirelessHART (WH) networks [10] discussed

Alejandro I. Maass, Dragan Nešić, and Peter M. Dower are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia (e-mail: alejandro.maass@unimelb.edu.au;dnesic@unimelb.edu.au;pdower@unimelb.edu.au).

Romain Postoyan is with the Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France (email: romain.postoyan@univ-lorraine.fr). His work is supported in part by ANR via grant HANDY, number ANR-18-CE40-0010.

further below. Under PE protocols and other reasonable conditions, we provide a bound on the MATI that guarantees an input-to-state stability property for the estimation error system. More importantly, we illustrate via examples that our proposed bound is less conservative than the bound in [6], which is mainly due to the PE property and the use of less conservative properties on the network-induced error dynamics.

In addition, we show that our framework can be used to study observer design of NCSs over WH networks. WH is a recent wireless communication protocol used in process automation. WH is a mesh network, which utilises field devices in a multi-hop fashion. Such devices act as buffers to forward data packets. Communications are precisely scheduled using time division multiple access, and using up to 15 available frequency channels for simultaneous transmissions. Existing results concerning estimation over WH can be found in e.g. [11], [12]. These works consider linear and discrete-time plant/observer models, together with equidistant transmission instants. Such assumptions may be hard to implement in real WH networks, where extra features need to be taken into consideration. NCSs implemented over WH, which we called WH–NCSs, have models that possess a very particular structure. In our previous work [10], we studied controller design of WH–NCSs over PE protocols. We showed that exploiting this WH network structure was key to obtaining less conservative MATI bounds in that context. In this current work, and inspired by [10], we use similar techniques in the context of observer design to show that exploitation of WH network structure yields less conservative MATI bounds guaranteeing an ISS property for the estimation error system. Moreover, we improve our previous results in [13], where we used UGES protocols and Lyapunov-based analysis in the observer design problem over WH–NCSs. Finally, we illustrate our results in the design of high gain observers for a class of Lipschitz non-linear systems.

The primary contributions of this paper are summarised as follows:

1) We extend [6] by considering non-linear systems with control inputs and measurement noise, and by imposing a less restrictive and more natural property on the protocols, namely PE (as opposed to UGES).
2) We derive an easily computable MATI bound that guarantees that an attendant estimation error system satisfies an ISS property, contingent upon the observer of interest satisfying a global asymptotic stability property in the absence of the network. This MATI bound is demonstrably less conservative than the bound proposed in [6].
3) We show that our results can be used to study WH–NCSs, which have a specific structure in their model. Such structure is exploited in our analysis to further reduce conservatism, yielding an even larger MATI bound. Moreover, we illustrate in examples that this bound is significantly larger than the bound proposed in [13].

The paper is organised as follows: Preliminaries are presented in Section II. The problem is stated in Section III. Section IV presents the overall NCS model and the results for such model are derived in Section V. We then study WH-NCSs in Section VI. In Section VII, we apply our results to a class of Lipschitz non-linear systems and illustrate this application via a numerical simulation. Lastly, we draw conclusions in Section VIII.

II. Preliminaries

A. Notation

Denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}^n$ the set of all real vectors with n components, and $\mathbb{R}^{m\times n}$ the set of all real matrices of dimension $m \times n$. Let $\mathcal{A}^n_{\geq 0}$ denote the set of all $n\times n$ matrices with non-negative entries, and let $\mathbb{R}^n_{\geq 0}$ denote the non-negative orthant of $\mathbb{R}^n$. Let $\mathbb{R}_{\geq 0} \doteq [0,\infty)$, $\mathbb{R}_{>0} \doteq (0,\infty)$, $\mathbb{Z}_{\geq 0} \doteq \{0,1,2,\dots\}$, and $\mathbb{N} \doteq \{1,2,3,\dots\}$. Given $a \in (0,\infty]$, a function $\alpha : [0,a) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ with $a = \infty$, and unbounded. For $a, b \in (0,\infty]$, a function $\gamma : [0,a) \times [0,b) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KK}$ if, for any $(s_1,s_2) \in [0,a)\times[0,b)$, $\gamma(s_1,\cdot)$ and $\gamma(\cdot,s_2)$ are of class $\mathcal{K}$. A continuous function $\beta : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each $t \geq 0$, and if $\beta(s,\cdot)$ is non-increasing and satisfies $\lim_{t\to\infty}\beta(s,t) = 0$ for each $s \geq 0$. Given $t \in \mathbb{R}$ and a piecewise continuous function $f : \mathbb{R} \to \mathbb{R}^n$, we use the notation $f(t^+) \doteq \lim_{s\to t, s>t} f(s)$. Given an initial time $t_0 \in \mathbb{R}_{\geq 0}$, the corresponding initial condition (or value) of a variable x is denoted by $x_0 = x(t_0)$. For simplicity, we use $(x,y) \doteq [x^T \;\; y^T]^T \in \mathbb{R}^{n+m}$, for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, we define $|x| \doteq \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$. The same notation is used to denote the induced 2-norm of a matrix. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalue of a real symmetric matrix, respectively. We use $I_n^N \doteq [I_n \;\cdots\; I_n]^N$ to denote the matrix $[I_n \;\cdots\; I_n] \in \mathbb{R}^{n\times Nn}$. We will often consider vectors of the form $\bar{x}$, where $x \in \mathbb{R}^n$ and $\bar{x} \doteq (|x_1|,\dots,|x_n|)^T$. For a matrix M, $\overline{M}$ denotes the absolute value of each entry of M. For a function $f : \mathbb{R} \to \mathbb{R}^n$, we define $\bar{f} : t \mapsto \overline{f(t)}$. The left-handed derivative of $f : \mathbb{R} \to \mathbb{R}^n$, if it exists, is denoted by $Df(t)$, i.e. $Df(t) \doteq \lim_{h\to 0, h<0} \frac{f(t+h)-f(t)}{h}$. We define the indicator function $\mathbb{1}_S : \mathbb{N} \to \{0,1\}$ by $\mathbb{1}_S(i) = 1$ if $i \in S$, and $\mathbb{1}_S(i) = 0$ if $i \notin S$. Given a (Lebesgue) measurable function $f : \mathbb{R} \to \mathbb{R}^n$, $\|f\|_{\mathcal{L}_p} \doteq \left(\int_{\mathbb{R}} |f(s)|^p \, ds\right)^{1/p}$, for $p \in \mathbb{N}$, $\|f\|_{\mathcal{L}_\infty} \doteq \operatorname{ess\,sup}_{t\in\mathbb{R}} |f(t)|$, and $\|f\|_{\mathcal{L}_\infty[a,b]} \doteq \operatorname{ess\,sup}_{t\in[a,b]} |f(t)|$. We say that $f \in \mathcal{L}_p$ for $p \in [1,\infty]$ whenever $\|f\|_{\mathcal{L}_p} < \infty$. Given $[a,b] \subset \mathbb{R}$, we use the notation $\|f\|_{\mathcal{L}_p[a,b]} \doteq \left(\int_{[a,b]} |f(s)|^p \, ds\right)^{1/p}$, to denote the $\mathcal{L}_p$ norm of f when restricted to the interval $[a,b]$. Given $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{R}^n$, a partial order $\preceq$ is given by $x \preceq y \iff x_i \leq y_i$, for all $i \in \{1,\dots,n\}$. An analogous partial order on elements of $\mathcal{A}^n_{\geq 0}$ is defined in the natural way, i.e. $A \preceq B \iff B - A \in \mathcal{A}^n_{\geq 0}$.

B. Underlying system and stability notions

Consider the jump-flow (hybrid system) model Σ,

$$\dot{x} = f(t,x,w), \; t \in [t_i, t_{i+1}], \tag{1a}$$

$$x(t_i^+) = h(i, x(t_i)), \tag{1b}$$

$$y = H(t,x), \tag{1c}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $w \in \mathbb{R}^{n_w}$ is an exogenous perturbation, $y \in \mathbb{R}^{n_y}$ is a prescribed output, $n_x, n_w, n_y \in \mathbb{N}$, and $\{t_i\}_{i\in\mathbb{N}}$ is a sequence of increasing time instants such that, for some $\tau \in \mathbb{R}$ and $\varepsilon > 0$, $\varepsilon \leq t_{i+1} - t_i \leq \tau < \infty$ for all $i \in \mathbb{N}$. Suppose Σ is initialised at (t_0, x_0). For a given w, we assume enough regularity on f and h to guarantee existence of the solution $x(\cdot) = x(\cdot, t_0, x_0, w)$ on the interval of interest, see e.g. [14]. By a solution we mean a (not necessarily unique) function $x(\cdot)$ such that $\dot{x}(t) = f(t, x(t), w(t))$ for almost all $t \in [t_i, t_{i+1}]$, satisfying (1b). A solution $x(t, t_0, x_0, w)$, $t \in [t_k, t_{k+1})$ can be constructed inductively by integrating (1a) from $(t_k, h(k, x(t_k)))$. This construction forgoes the discussion of the maximum interval of definition, for which we refer the reader to [15].

We now define the stability notions used throughout this paper.

Definition 1: Let $p \in \mathbb{N} \cup \{+\infty\}$ and $\gamma \geq 0$ be given. We say that Σ is $\mathcal{L}_p$ stable from w to y with gain γ if there exists $K \geq 0$ such that $\|y\|_{\mathcal{L}_p[t_0,t]} \leq K|x_0| + \gamma\|w\|_{\mathcal{L}_p[t_0,t]}$ for all $t \geq t_0 \geq 0$, $w \in \mathcal{L}_p[t_0, t]$ and $x_0 \in \mathbb{R}^{n_x}$. ■

Definition 2: We say that system Σ is input-to-output stable (IOS) from w to y if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $|y(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|w\|_{\mathcal{L}_\infty[t_0,t]})$ for all $t \geq t_0 \geq 0$, $w \in \mathcal{L}_\infty$ and $x_0 \in \mathbb{R}^{n_x}$. If $y = x$ then system Σ is input-to-state stable (ISS) w.r.t. w. If $\gamma(\cdot)$ is a linear function, then we say that the system Σ is IOS (ISS) with linear gain γ. ■

Definition 3: We say that system Σ is bounded-input bounded-state (BIBS) with input w if there exist $\alpha, \gamma \in \mathcal{K}$, such that $|x(t)| \leq \alpha(|x_0|) + \gamma(\|w\|_{\mathcal{L}_\infty[t_0,t]})$ for all $t \geq t_0 \geq 0$, $w \in \mathcal{L}_\infty$ and $x_0 \in \mathbb{R}^{n_x}$. ■

III. Problem statement

Consider a non-linear plant

$$\dot{x}_p = f(x_p, u, w), \quad y = g(x_p) + v, \tag{2}$$

where $x_p \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the external disturbance, $y \in \mathbb{R}^{n_y}$ is the plant output affected by the noise $v \in \mathbb{R}^{n_y}$, and $n_x, n_u, n_w, n_y \in \mathbb{N}$. To ease notation, we suppress the time-dependence of the variables in (2). The functions $u : \mathbb{R} \to \mathbb{R}^{n_u}$ and $v : \mathbb{R} \to \mathbb{R}^{n_v}$ are assumed to be Lebesgue measurable and differentiable almost everywhere. Moreover, these functions and their time-derivatives are assumed to have a finite $\mathcal{L}_\infty$ norm. We assume an observer has been designed for the above plant and has the form

$$\dot{z} = f_z(z, u, y - y_z), \ \ \tilde{x}_p = g_z(z), \ \ y_z = g(\tilde{x}_p), \tag{3}$$

where $z \in \mathbb{R}^{n_z}$ is the observer state, $n_z \in \mathbb{N}$, $\tilde{x}_p \in \mathbb{R}^{n_x}$ is the estimate of the state x_p, and $y_z \in \mathbb{R}^{n_y}$ is the output estimate. Notice that we allow the dimension of the observer to be bigger than the system dimension, hence covering immersion-based observers for instance. It is implicit in (3) that we measure u and y, whereas w and v are unmeasured. The stability property we will prove is natural for this setting.

We study the scenario depicted in Fig. 1, in which the plant communicates with the observer over a communication network. In particular, the control signal u is directly available to the plant and transmitted to the observer via the network. We denote the control signal received by the observer by $\hat{u}$. The noisy output y is also transmitted to the observer via the network. The output received by the observer is likewise denoted by $\hat{y}$. We emphasize that the observer has no longer access to (u, y) as in classical control theory, but to their networked versions $(\hat{u}, \hat{y})$.

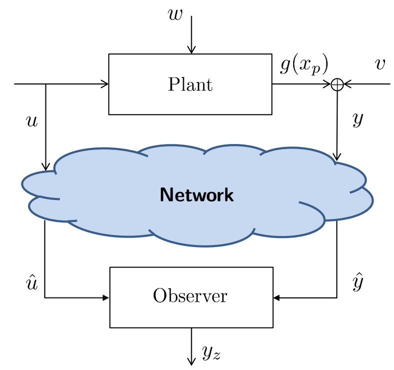

Fig. 1: Block diagram of our networked observer architecture.

Under the setup described above, our main objective is to provide conditions on the observer and the network, in particular on the scheduling protocols and the MATI, under which the state estimate $\tilde{x}_p$ (approximately) converges to the state of the plant x_p.

IV. NCS modelling

A. Network model

We consider a scenario where the sensors and actuators of the plant (2) are grouped into ℓ *nodes* (depending on their spatial location) which are connected to the network. At each transmission instant t_i, $i \in \mathbb{N}$, only one node is granted access to the network by the scheduling protocol. The transmission sequence $\{t_i\}_{i\in\mathbb{N}}$ is such that $\varepsilon \leq t_{i+1} - t_i \leq \tau$ for all $i \in \mathbb{N}$, where $\tau \in \mathbb{R}_{>0}$ is the MATI and ε is the lower bound on the minimum achievable transmission interval given by the hardware constraints (see, e.g. [10], [15], [16]). Notice that the transmission intervals $t_{i+1} - t_i$ may be time-varying and uncertain. In fact, the stability results we present later in this paper hold even if we do not exactly know the transmission interval at each time instant, as these results will be given in terms of the MATI. Moreover, the emulated observer does not need to know the exact value of the transmission instant to be implemented.

We next present the dynamics of the networked versions of y and u, that is $\hat{y}$ and $\hat{u}$, respectively. Let $(y, u) = (y_1, \dots, y_{n_y}, u_1, \dots, u_{n_u})$ and $(\hat{y}, \hat{u}) = (\hat{y}_1, \dots, \hat{y}_{n_y}, \hat{u}_1, \dots, \hat{u}_{n_u})$. Suppose that the node $j \in \{1, \dots, \ell\}$, $\ell \in \{1, \dots, n_y + n_u\}$, is selected by the protocol at time t_i, $i \in \mathbb{N}$, and say that the components y_{j_y} and u_{j_u} of y and u, respectively, are associated to node j, with $j_y \in \{1, \dots, n_y\}$ and $j_u \in \{1, \dots, n_u\}$. Then,

$$\hat{y}_{j_y}(t_i^+) = y_{j_y}(t_i), \quad \hat{u}_{j_u}(t_i^+) = u_{j_u}(t_i), \tag{4}$$

while for all other components of $\hat{y}$ and $\hat{u}$,

$$\hat{y}_{k_y}(t_i^+) = \hat{y}_{k_y}(t_i), \quad \hat{u}_{k_u}(t_i^+) = \hat{u}_{k_u}(t_i), \tag{5}$$

where $k_y \in \{1, \ldots, n_y\}$ and $k_u \in \{1, \ldots, n_u\}$ satisfy $k_y \neq j_y$ and $k_u \neq j_u$. This means that the components of $\hat{y}$ and $\hat{u}$ corresponding to the j–th node are updated and the other components are kept unchanged. Between transmission instants, $\hat{y}$ and $\hat{u}$ are generated according to the in-network processing implementation. For simplicity, we use zero-order hold devices which translates into $\dot{\hat{y}} = 0$ and $\dot{\hat{u}} = 0$ for $t \in [t_i, t_{i+1}]$ and $i \in \mathbb{N}$.

B. Observer implementation over the network

When the observer (3) is implemented over the network, it no longer receives (y, u) but $(\hat{y}, \hat{u})$, which is generated from the most recently transmitted measurement and control input as per Section IV-A. The dynamics of the observer now become

$$\dot{z} = f_z(z, \hat{u}, \hat{y} - \hat{y}_z). \tag{6}$$

Furthermore, we note that (6) does not depend on its output y_z, as in (3), but on $\hat{y}_z$, which is an artificially introduced networked version of y_z. The idea of using $\hat{y}_z$ instead of y_z was suggested in [7, Section VIII] and it allows stronger stability properties for the estimation error system to be established. A similar idea is proposed in [17] for the design of high-gain observers.

The variable $\hat{y}_z$ is constructed to evolve along the same vector field as $\hat{y}$ between two successive transmission instants, i.e. $\dot{\hat{y}}_z = 0$ for $t \in [t_i, t_{i+1}]$. Let $y_z = (y_{z,1}, \ldots, y_{z,n_y})$ and $\hat{y}_z = (\hat{y}_{z,1}, \ldots, \hat{y}_{z,n_y})$. At each transmission of a component of $\hat{y}$, say $\hat{y}_{j_y}$ with $j_y \in \{1, \ldots, n_y\}$, the corresponding component of $\hat{y}_z$, that is $\hat{y}_{z,j_y}$, is reset to y_{z,j_y}, that is

$$\hat{y}_{z,j_y}(t_i^+) = \begin{cases} y_{z,j_y}(t_i), & \text{if } \hat{y}_{j_y}(t_i^+) = y_{j_y}(t_i), \\ \hat{y}_{z,j_y}(t_i), & \text{otherwise.} \end{cases} \tag{7}$$

C. Scheduling protocols

For the sake of analysis, we introduce the network-induced errors on the plant output $e^y \doteq \hat{y} - y$, and the input $e^u \doteq \hat{u} - u$. To be consistent, we also define a network-induced error on the observer output $e^{y_z} \doteq \hat{y}_z - y_z$. Using these definitions, we rewrite (6) as

$$\dot{z} = f_z(z, u + e^u, y - y_z + e^y - e^{y_z}). \tag{8}$$

We can see that the dynamics of the observer are affected by $e^y - e^{y_z}$ and e^u, thus we define the overall network-induced error as $e \doteq (e^y - e^{y_z}, e^u)$. This error is useful to model the scheduling mechanism of the network. That is, considering (4), (5) and (7), we can model transmissions by the so-called *protocol equation* below

$$e(t_i^+) = (I - \Psi(i))e(t_i), \tag{9}$$

where Ψ is a time-varying matrix such that if the j–th node gets access to the network at time instant t_i, then the corresponding error component is reset to zero, i.e. $e_j(t_i^+) = 0$. For simplicity, we concentrate on static protocols, i.e. protocols in which Ψ is independent of e. This can be easily extended to cover dynamic protocols, where Ψ is allowed to depend on e, similarly to [9].

As foreshadowed in the introduction, we would like to implement scheduling protocols that are PE.

Assumption 1: The protocol (9) is said to be persistently exciting in T (PE_T) if there exists $T \in \mathbb{N}$ such that $\prod_{k=i}^{i+T-1}(I - \Psi(k)) = 0$, for every $i \in \mathbb{N}$. ■

This property means that there is a fixed (finite) number of transmissions T such that all nodes of the NCS have transmitted within T transmissions. It is shown in [9], for the controller design of NCSs, that protocols satisfying the PE_T property lead to the $\mathcal{L}_p$ stability of the closed-loop for high enough transmission rates. In this work, we use this property to ensure that the state estimate (approximately) converges to the state of the plant under high enough transmission rates.

We next include an example of a PE_T protocol that can be implemented in NCSs. A more detailed study of persistently exciting protocols is provided in [9], where more examples can be found. In physical networks such as WH, examples of PE_T protocols can be found in [10].

Example 1 (Round-Robin): Round-robin scheduling is employed in the token ring and token bus network protocols [18]. Nodes in the network are visited in a predetermined and cyclic manner [15]. For this scheduling protocol, we have that Ψ in (9) is defined as $\Psi(i) \doteq \text{diag}\{\delta_1(i)I_{n_1}, \ldots, \delta_\ell(i)I_{n_\ell}\}$, $i \in \mathbb{N}$, where I_{n_s} is the identity matrix of dimension $n_s \in \mathbb{N}$, $s \in \{1, \ldots, \ell\}$, with $\sum_{s=1}^{\ell} n_s = n_e \doteq n_y + n_u$, and $\delta_s(i) = 1$ if $i = s + \sigma\ell$, $\sigma \in \mathbb{Z}_{\geq 0}$, or $\delta_s(i) = 0$ otherwise. By following similar lines as in Lemma 11 in [10], it is straightforward to show that the round-robin protocol satisfies Assumption 1 with $T = \ell$. ■

D. Hybrid model

We are now in a position to present the hybrid model for the NCS in Fig. 1. We introduce for this purpose the estimation error $\chi \doteq x_p - \tilde{x}_p$, and $d \doteq (d_u, d_v)$, where d_u and d_v denote the time derivative of the input signal u and of the noise v, respectively. Then, by using (2), (3), (8), (9), we have that

$$\dot{\chi} = f_\chi(\chi, z, e, u, v, w), \quad \forall t \in [t_i, t_{i+1}], \tag{10a}$$
$$\dot{z} = f_z(\chi, z, e, u, v), \quad \forall t \in [t_i, t_{i+1}], \tag{10b}$$
$$\dot{e} = g_e(\chi, z, e, u, v, w, d), \quad \forall t \in [t_i, t_{i+1}], \tag{10c}$$
$$\chi(t_i^+) = \chi(t_i), \tag{10d}$$
$$z(t_i^+) = z(t_i), \tag{10e}$$
$$e(t_i^+) = (I - \Psi(i))e(t_i), \tag{10f}$$

where f_χ, f_z and g_e are defined in (11). Note that we are interested in different properties for the χ–system and the z–system. In particular, we want to prove a convergence property for the observation error χ, but only some well defined or bounded behaviour for all time is desired for the observer state z.

V. NCS STABILITY

In this section, we use the model (10) to obtain a bound on the MATI that guarantees an ISS stability property for the

$$f_\chi(\chi,z,e,u,v,w) \doteq f(\chi+g_z(z),u,w) - \frac{\partial g_z}{\partial z} f_z\left(z, u+e^u, g(\chi+g_z(z))+v-g(g_z(z))+e^y-e^{y_z}\right), \tag{11a}$$

$$f_z(\chi,z,e,u,v) \doteq f_z\left(z, u+e^u, g(\chi+g_z(z))+v-g(g_z(z))+e^y-e^{y_z}\right), \tag{11b}$$

$$g_e(\chi,z,e,u,v,w,d) \doteq \Bigg(-\frac{\partial g}{\partial x_p} f(\chi+g_z(z),u,w) - d_v + \frac{\partial g}{\partial \tilde{x}_p}\frac{\partial g_z}{\partial z} f_z(z,u+e^u, g(\chi+g_z(z))+v-g(g_z(z))+e^y-e^{y_z}), -d_u\Bigg). \tag{11c}$$

estimation error system. Before presenting the main results of this section, we list our assumptions.

Assumption 2: There exist a matrix $A \in \mathcal{A}^{n_e}_{\geq 0}$ and a continuous function $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_u+n_v} \to \mathbb{R}^{n_e}_{\geq 0}$ such that the error dynamics (10c) satisfy[1]

$$\bar{g}_e(\chi,z,e,u,v,w,d) \preceq A\bar{e} + \tilde{y}(\chi,z,u,v,w,d), \tag{12}$$

for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, e \in \mathbb{R}^{n_e}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}$, and $d \in \mathbb{R}^{n_u+n_v}$. ■

Assumption 2 is the vector analogue of the dissipation-type inequality used in Assumption 3 in [6]. A similar assumption was used in the context of controller design for NCSs in [9], [10]. We further assume the following.

Assumption 3: There exist $\gamma_2^\chi \in \mathbb{R}_{\geq 0}$ and $\sigma \in \mathcal{K}_\infty$ such that $\tilde{y}$ in (12) satisfies $|\tilde{y}(\chi,z,u,v,w,d)| \leq \gamma_2^\chi|\chi| + \sigma(|(u,v,w,d)|)$, for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}$, and $d \in \mathbb{R}^{n_u+n_v}$. ■

In addition, we assume the designed observer (3) ensures some global asymptotic stability property in the absence of the network, disturbances and noises.

Assumption 4: There exist $\beta_1 \in \mathcal{KL}, \gamma_1^e \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathcal{K}_\infty$ such that, for any $\chi_0 \in \mathbb{R}^{n_x}$ and $(e,v,w) \in \mathcal{L}_\infty$, solutions to (10a) satisfy

$$|\chi(t)| \leq \beta_1(|\chi_0|, t-t_0) + \gamma_1^e \|e\|_{\mathcal{L}_\infty[t_0,t]} + \mu\left(\|(v,w)\|_{\mathcal{L}_\infty[t_0,t]}\right), \tag{13}$$

for all $t \geq t_0 \geq 0$. ■

This type of ISS condition has already been used for observer design in sampled-data systems and NCSs [4], [5], respectively. A similar assumption can be found in the control of non-linear NCSs [10], [15].

Note that the bound on $\tilde{y}$ and χ in Assumptions 3 and 4, respectively, are independent of z. These assumptions hold for Lipschitz non-linear systems, and for high-gain observers, circle criterion observers, and those observers whose design is based on linear parameter-varying techniques, see e.g. [19]–[21]. We provide one example in Section VII. Assumptions 3 and 4 can be relaxed so that they also depend on z, as we later show in Theorem 2, at the expense of getting practical convergence w.r.t. MATI.

Given Assumptions 2–4, the stability of χ– and e–dynamics can be investigated separately from the system interconnection (10). Note that we are only interested in the stability of the aforementioned dynamics and the z–dynamics are not expected to converge. In that way, it is only left to assume that the observer dynamics behave nicely, i.e. no finite escape behaviour for bounded control inputs, noises, and errors.

Assumption 5: System (10b) is forward complete with input $(\chi,e,u,v,w) \in \mathcal{L}_\infty$ [22]. That is, there exist $\nu_1,\nu_2,\nu_3 \in \mathcal{K}$ and $c \in \mathbb{R}_{\geq 0}$ such that, for any $z_0 \in \mathbb{R}^{n_z}$ and $(\chi,e,u,v,w) \in \mathcal{L}_\infty$, $|z(t)| \leq \nu_1(t)+\nu_2(|z_0|)+\nu_3(\|(\chi,e,u,v,w)\|_{\mathcal{L}_\infty[t_0,t]})+c$, for all $t \geq t_0 \geq 0$. ■

Note that Assumption 5 can be written in terms of forward completeness of the plant, provided extra assumptions on g_z in (3) are asserted. In particular, recalling that $\chi = x_p - g_z(z)$, Assumption 5 implies forward completeness of the plant via Assumption 4 if g_z is continuous. Assumption 5 is also equivalent to the forward completeness of the plant if g_z is invertible.

We are now ready to state the results of this section. The main underlying idea is to consider system (10) as the interconnection of three subsystems in χ, z and e, and apply small-gain arguments to conclude a certain stability property for the overall system. The proofs can be found in the Appendix.

Proposition 1: Suppose Assumptions 1 and 2 hold. If MATI satisfies $\tau \in [\varepsilon, \tau_e^*)$, where $\tau_e^* = \ln(2)/(|A|T)$, then the system (10c), (10f) is IOS from $\tilde{y}$ to e, where $\tilde{y}$ is per Assumption 2, with linear gain

$$\tilde{\gamma}(\tau) = \frac{T\exp(|A|(T+1)\tau)(\exp(|A|\tau)-1)}{|A|\left(2-\exp(|A|T\tau)\right)}. \tag{14}$$

■

Theorem 1: Suppose Assumptions 1–5 hold. Let $\mathbf{x}$ be the solution of

$$\gamma_2^\chi\gamma_1^e T\mathbf{x}^{1+2/T} - \gamma_2^\chi\gamma_1^e T\mathbf{x}^{1+1/T} + |A|\mathbf{x} - 2|A| = 0, \tag{15}$$

with $\mathbf{x} \in [1,2]$, and define $\tau^* = \ln(\mathbf{x})/(|A|T)$. If $\tau \in [\varepsilon, \tau^*)$, then the following holds.

(i) There exist $\beta \in \mathcal{KL}, \eta_1 \in \mathcal{K}$ and $\eta_2 \in \mathcal{KK}$ such that, for all $(\chi_0, e_0) \in \mathbb{R}^{n_x+n_e}$ and $(u,v,w,d) \in \mathcal{L}_\infty$,

$$|(\chi(t),e(t))| \leq \beta(|(\chi_0,e_0)|, t-t_0) + \eta_1(\|(v,w)\|_{\mathcal{L}_\infty}) + \eta_2(\tau, \|(u,v,w,d)\|_{\mathcal{L}_\infty}) \tag{16}$$

holds for all $t \geq t_0 \geq 0$.

(ii) System (10) is forward complete with input $(u,v,w,d) \in \mathcal{L}_\infty$. ■

We can see from Theorem 1 that the estimation error χ and the network-induced error e converge to a ball centred at the origin and whose radius depends on the $\mathcal{L}_\infty$ norm of the input (u,v,w,d). We can see that χ and e do not a priori converge to the origin even when $w = v = 0$, since in this case we have that $|(\chi(t),e(t))| \leq \beta(|(\chi_0,e_0)|, t-t_0) + \eta_2(\tau, \|(u,d_u)\|_{\mathcal{L}_\infty})$.

[1] Recall that for any $x \in \mathbb{R}^n$, $\bar{x} \doteq (|x_1|,\ldots,|x_n|)^T$.

However, we can always reduce τ so that η_2 is small, and thus the effect of (u, d_u) is reduced. In the absence of inputs, i.e. $w = v = u = 0$, the estimation error indeed asymptotically converges to the origin.

The dependence on the control input u in (16) may look surprising a priori. This comes directly from Assumption 2 and the definition of g_e in (10), i.e. the network-induced error dynamics. As we already mentioned, this term can be made small by reducing MATI. However, this dependence can also be removed for certain particular cases, which our general setup covers. For instance, given a linear system and a full-state Luenberger observer, g_e would only depend on e^u and not u. Also, if the observer is collocated with the controller such that it has direct access to u, then g_e would not depend on u explicitly. On the other hand, the use of in-network processing algorithms such as the predictive-type or model-based holding functions [4]–[6] may overcome this issue, which is outside the scope of this paper.

In the following, we relax Assumptions 3 and 4, and state the corresponding stability result. That is, we replace Assumptions 3 and 4 with two new assumptions with weaker assertions. The proof follows similar steps as the proof of Theorem 1, and it is thus omitted.

Assumption 6: There exist $\gamma_2^\chi, \gamma_2^z \in \mathbb{R}_{\geq 0}$ and $\sigma \in \mathcal{K}_\infty$ such that $\tilde{y}$ in (12) satisfies $|\tilde{y}(\chi, z, u, v, w, d)| \leq \gamma_2^\chi|\chi| + \gamma_2^z|z| + \sigma(|(u, v, w, d)|)$, for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}$, and $d \in \mathbb{R}^{n_u+n_v}$. ■

Assumption 7: There exist $\beta_1 \in \mathcal{KL}$, $\gamma_1^e \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathcal{K}_\infty$ such that, for any $(\chi_0, z_0) \in \mathbb{R}^{n_x+n_z}$ and $(e, v, w) \in \mathcal{L}_\infty$, solutions to (10a) satisfy

$$|\chi(t)| \leq \beta_1(|(\chi_0, z_0)|, t - t_0) + \gamma_1^e\|e\|_{\mathcal{L}_\infty[t_0,t]} + \mu(\|(v, w)\|_{\mathcal{L}_\infty[t_0,t]}),$$

for all $t \geq t_0 \geq 0$. ■

By relaxing Assumptions 3 and 4, we can no longer analyse (χ, e)–dynamics separately from the whole system (10). That is, assuming forward completeness of z–dynamics in (10b) is not enough. However, it is sufficient to assume a boundedness property for the state of system (10b), which acts as a replacement of Assumption 5.

Assumption 8: There exist $\alpha_3 \in \mathcal{K}, \gamma_3^e \in \mathbb{R}_{\geq 0}$ and $\mu_z \in \mathcal{K}_\infty$ such that, for all $(\chi_0, z_0) \in \mathbb{R}^{n_x+n_z}$ and $(e, u, v, w) \in \mathcal{L}_\infty$, the following holds along solutions to (10b), $|z(t)| \leq \alpha_3(|(\chi_0, z_0)|) + \gamma_3^e\|e\|_{\mathcal{L}_\infty[t_0,t]} + \mu_z(\|(u, v, w)\|_{\mathcal{L}_\infty[t_0,t]})$, for all $t \geq t_0 \geq 0$. ■

Replacing Assumptions 3, 4, 5 with 6, 7, 8 subsequently yields the following replacement for Theorem 1.

Theorem 2: Suppose Assumptions 1, 2, 6, 7 and 8 hold. Let $\mathbf{x}$ be the solution of

$$(\gamma_2^\chi\gamma_1^e + \gamma_2^z\gamma_3^e)T\mathbf{x}^{1+2/T} - (\gamma_2^\chi\gamma_1^e + \gamma_2^z\gamma_3^e)T\mathbf{x}^{1+1/T} + |A|\mathbf{x} - 2|A| = 0, \quad (17)$$

with $\mathbf{x} \in [1, 2]$, and define $\tau^* = \ln(\mathbf{x})/(|A|T)$. If $\tau \in [\varepsilon, \tau^*)$, then the following holds.

(i) There exist $\beta \in \mathcal{KL}, \eta_1 \in \mathcal{K}$ and $\eta_2, \bar{\eta} \in \mathcal{KK}$ such that, for any $\Delta \in \mathbb{R}_{\geq 0}$, $(\chi_0, e_0, z_0) \in \mathbb{R}^{n_x+n_e+n_z}$ with $|(\chi_0, e_0, z_0)| < \Delta$, and $(u, v, w, d) \in \mathcal{L}_\infty$,

$$|(\chi(t), e(t))| \leq \beta(|(\chi_0, e_0, z_0)|, t - t_0) + \eta_1(\|(v, w)\|_{\mathcal{L}_\infty}) + \eta_2(\tau, \|(u, v, w, d)\|_{\mathcal{L}_\infty}) + \bar{\eta}(\tau, \Delta),$$

holds for all $t \geq t_0 \geq 0$.

(ii) System (10) is BIBS with (u, v, w, d) as input. ■

Theorem 2 is more general than Theorem 1. We can see that even if $w = v = u = 0$, the estimation error does not converge to the origin a priori, but to a ball centred at 0 with radius $\bar{\eta}(\tau, \Delta)$. However, for a given Δ, $\bar{\eta}(\tau, \Delta)$ can still be made small by reducing MATI. This implies a trade-off between the domain of attraction, transmission rate, and estimation error bound. That is, an increase in Δ and a decrease in $\bar{\eta}$ requires a decrease in τ.

Remark 1: Note that Assumptions 2–8 lead to global stability results in Theorems 1 and 2, hence why they can be strict. However, these can be further relaxed by following the same steps the authors in [23] made to extend the global results in [15] to semi-global practical results for controller design. The drawback of doing this extension is that the explicit formula to compute MATI in Theorems 1 and 2 would be lost. We could make a less general extension in which it is possible to relax the current assumptions to hold semi-globally or practically, in which case explicit MATI bounds would be possible to derive. However, this would induce technicalities that would obstruct the main message of this paper. In fact, as this is the first paper on observer design for non-linear systems under persistently exciting protocols, we present global results as a foundation for more general future work. ■

VI. Application to WirelessHART–NCSs

In this section, we present our results on observer design for WH–NCSs. We show that, exploiting the structure that WH provides, less conservative MATI bounds can be computed.

A. WH-NCS Model

Consider Fig. 2, in which the generic network of Fig. 1 has been replaced by a WH network (we refer the reader to [10], [24], [25] for a better understanding of WH and its features). We consider $\ell_y \in \mathbb{Z}_{\geq 0}$ field devices interconnected in the y-path and $\ell_u \in \mathbb{Z}_{\geq 0}$ in the u-path. We label the field devices as D_α^y and D_β^u, where $\alpha = 1, \ldots, \ell_y$ and $\beta = 1, \ldots, \ell_u$. For each field device, its inputs and outputs are depicted in Fig. 2. Note that the signals that actually reach the observer in Fig. 2, i.e. y_{ℓ_y} and u_{ℓ_u}, are denoted as $\hat{y}$ and $\hat{u}$, to be consistent with Fig. 1 and existent NCS literature. We model field devices as buffers, for which we introduce a buffer state variable, denoted by b_α^y and b_β^u for field devices in the y-path and u-path, respectively. In the following, we explain the reception and transmission behaviour of field devices, and we present the equations associated with this process.

Reception: Suppose a field device D_α^y receives a packet at time instant t_i. Then, D_α^y updates the content of its buffer via its input. During this process, the output of D_α^y remains unchanged. We write this as follows,

$$\dot{y}_\alpha(t) = 0, \quad t \in [t_i, t_{i+1}], \tag{18a}$$
$$\dot{b}_\alpha^y(t) = 0, \quad t \in [t_i, t_{i+1}], \tag{18b}$$
$$b_\alpha^y(t_i^+) = y_{\alpha-1}(t_i^+), \tag{18c}$$
$$y_\alpha(t_i^+) = y_\alpha(t_i), \tag{18d}$$

for all $\alpha = 1, \ldots, \ell_y$. Note that $y_0 \equiv y$ for $\alpha = 1$, i.e. device one samples the value of the plant output.

Transmission: Suppose a field device D_α^y is scheduled to transmit at time instant t_i. Here, D_α^y sends the content of its buffer through its output, and keeps it until a new packet is received. This can be written as follows,

$$\dot{y}_\alpha(t) = 0, \quad t \in [t_i, t_{i+1}], \tag{19a}$$
$$\dot{b}_\alpha^y(t) = 0, \quad t \in [t_i, t_{i+1}], \tag{19b}$$
$$y_\alpha(t_i^+) = b_\alpha^y(t_i), \tag{19c}$$
$$b_\alpha^y(t_i^+) = b_\alpha^y(t_i), \tag{19d}$$

for all $\alpha = 1, \ldots, \ell_y$.

As we did for the generic network in Section IV-D, we need to introduce appropriate network-induced errors for the WH case. That is, we introduce errors for the y-path and u-path, which we denote by $\zeta^y \in \mathbb{R}^{n_{\zeta^y}}$, $n_{\zeta^y} \doteq 2\ell_y n_y$, and $\zeta^u \in \mathbb{R}^{n_{\zeta^u}}$, $n_{\zeta^u} \doteq 2\ell_u n_u$, respectively.

$$\zeta^y \doteq \Big(b_1^y - y, b_2^y - y_1, \ldots, b_{\ell_y}^y - y_{\ell_y-1}, \\ y_1 - b_1^y, y_2 - b_2^y, \ldots, y_{\ell_y} - b_{\ell_y}^y\Big), \tag{20a}$$

$$\zeta^u \doteq \Big(b_1^u - u, b_2^u - u_1, \ldots, b_{\ell_u}^u - u_{\ell_u-1}, \\ u_1 - b_1^u, u_2 - b_2^u, \ldots, u_{\ell_u} - b_{\ell_u}^u\Big). \tag{20b}$$

The first $\ell_\star$ components of $\zeta^\star$, $\star \in \{y, u\}$, are related to the buffer update during reception. The remaining $\ell_\star$ components of $\zeta^\star$ are related to the transmission of such buffer value through their output. In particular, we will reset to zero these errors to model reception and transmission. This is a major difference with previous models of non-linear NCSs treated in Section IV or the literature [15], [26], [27], where the network-induced error for the plant output and input are given by $e^y = \hat{y} - y$ and $e^u = \hat{u} - u$, respectively (i.e. no specific network is considered, and buffer dynamics are ignored).

We also introduce an artificial variable $\hat{y}_z$. It evolves along the same vector field as $\hat{y}$ between two successive transmission instants, i.e. $\dot{\hat{y}}_z = 0$ for $t \in [t_i, t_{i+1}]$. At jumps, it evolves exactly as when y is sent through the network. Therefore, for analysis purposes, we introduce auxiliary field devices with buffer states $b_1^z, \ldots, b_{\ell_y}^z$ and outputs $y_{z,1}, \ldots, \hat{y}_z$. Then, we define the corresponding network induced error on the observer output

$$\zeta^{y_z} \doteq \Big(b_1^z - y_z, b_2^z - y_{z,1}, \ldots, b_{\ell_y}^z - y_{z,\ell_y-1}, \\ y_{z,1} - b_1^z, y_{z,2} - b_2^z, \ldots, \hat{y}_z - b_{\ell_y}^z\Big). \tag{21}$$

This will help to ensure that the update of $\hat{y}_z$ with y_z happens at the same time that $\hat{y}$ gets updated with y, so they can

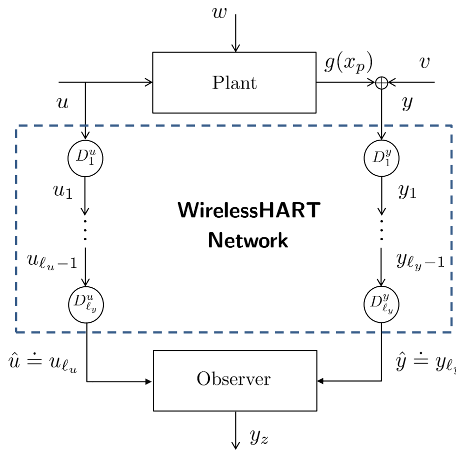

Fig. 2: NCS implemented over a WH network with ℓ_y field devices in the y-path and ℓ_u field devices in the u-path.

be properly compared through the observation error $y - y_z$. Hence, we artificially send y_z through a model of the y-path of the network at the same time that y is sent, to construct $\hat{y}_z$. The introduction of ζ^{y_z} is a key difference with Section IV-D (recall that e^{y_z}, the analogue of ζ^{y_z} in the generic model (10), is defined as $e^{y_z} \doteq \hat{y}_z - y_z$) and previous work on observer design [6], [7], [16], and it is used here as a consequence of how WH operates.

Using the definitions of ζ^y, ζ^u and ζ^{y_z} we can write

$$\dot{z} = f_z\big(z, u + \mathbf{I}_{n_u}^{2\ell_u} \cdot \zeta^u, y - y_z + \mathbf{I}_{n_y}^{2\ell_y} \cdot \zeta^e\big), \tag{22}$$

where $\zeta^e \doteq \zeta^y - \zeta^{y_z}$ corresponds to the network induced error on the observation error $y - y_z$. Note the difference between the network-induced errors appearing in (22) and in (8). The network-induced errors in (20) and (21) are useful to model scheduling in WH networks. Given that the dynamics of the observer in (22) are affected by ζ^e and ζ^u, we define $\varsigma \doteq (\zeta^e, \zeta^u)$, where $\varsigma \in \mathbb{R}^{n_\varsigma}$, $n_\varsigma \doteq n_{\zeta^y} + n_{\zeta^u}$.

We are now in a position to present the model for the overall WH–NCS of Fig. 2. By using (2), (18), (19), (20), (21) and (22), we present the hybrid model for the block diagram in Fig. 2,

$$\dot{\chi} = f_\chi(\chi, z, \varsigma, u, v, w), \quad \forall t \in [t_i, t_{i+1}], \tag{23a}$$
$$\dot{z} = f_z(\chi, z, \varsigma, u, v), \quad \forall t \in [t_i, t_{i+1}], \tag{23b}$$
$$\dot{\varsigma} = g_\varsigma(\chi, z, \varsigma, u, v, w, d), \quad \forall t \in [t_i, t_{i+1}], \tag{23c}$$
$$\chi(t_i^+) = \chi(t_i), \tag{23d}$$
$$z(t_i^+) = z(t_i), \tag{23e}$$
$$\varsigma(t_i^+) = (I - \Psi(i))\varsigma(t_i), \tag{23f}$$

where f_χ, f_z and g_ς are defined in (24).

As we already mentioned, the introduction of an appropriate network-induced error, namely ς, is required to have a model that covers the WH specifications. However, the protocol equation (23f) maintains the same form as in the generic model (10) (see (10f)). The difference is that now the matrix Ψ is defined differently as the scheduling protocols in WH are determined by the multi-hop nature of the network. That is, the matrix Ψ in the WH model (23) is fully determined by the

$$f_\chi(\chi, z, \varsigma, u, v, w) \doteq f(\chi + g_z(z), u, w) - \frac{\partial g_z}{\partial z} f_z\Big(z, u + \mathbf{I}_{n_u}^{2\ell_u} \cdot \zeta^u, g(\chi + g_z(z)) + v - g(g_z(z)) + \mathbf{I}_{n_y}^{2\ell_y} \cdot \zeta^e\Big), \quad (24a)$$

$$f_z(\chi, z, \varsigma, u, v) \doteq f_z\Big(z, u + \mathbf{I}_{n_u}^{2\ell_u} \cdot \zeta^u, g(\chi + g_z(z)) + v - g(g_z(z)) + \mathbf{I}_{n_y}^{2\ell_y} \cdot \zeta^e\Big), \quad (24b)$$

$$g_\varsigma(\chi, z, \varsigma, u, v, w, d) \doteq \big(g_{\varsigma,1}(\chi, z, \varsigma, u, v, w, d), 0, \ldots, 0, -d_u, 0, \ldots, 0\big), \quad (24c)$$

$$g_{\varsigma,1}(\chi, z, \varsigma, u, v, w, d) \doteq -\frac{\partial g}{\partial x_p} f(\chi + g_z(z), u, w) - d_v + \frac{\partial g}{\partial \tilde{x}_p}\frac{\partial g_z}{\partial z} f_z\Big(z, u + \mathbf{I}_{n_u}^{2\ell_u} \cdot \zeta^u, g(\chi + g_z(z)) + v - g(g_z(z)) + \mathbf{I}_{n_y}^{2\ell_y} \cdot \zeta^e\Big). \quad (24d)$$

scheduling of field devices in the communication frame. We next provide only one of the many scheduling protocols that can be implemented on WH , see [10], [25], [28] for more protocols.

Example 2: We consider a scheduling protocol that establishes a full-duplex communication link that uses two different frequency channels for measurements and actuation operations. In particular, we consider the communication frame given by Table I, where devices are scheduled in each frequency channel in a round-robin manner. We call this protocol Frequency Division Duplex Round Robin (FDD-RR) [10]. For this protocol, we have that (see (23f)) $\Psi(i) \doteq I - \mathcal{H}(i)$, where $\mathcal{H}(i) \doteq \text{diag}\{\mathcal{H}^y(i), \mathcal{H}^u(i)\}$, and

$$\mathcal{H}^\star(i) \doteq \begin{bmatrix} \Delta^\star(i) & 0 \\ I - \Delta^\star(i) & \Gamma^\star(i) \end{bmatrix},$$
$$\Delta^\star(i) \doteq \text{diag}\left\{\delta_1^\star(i) I_{n_\star}, \ldots, \delta_{\ell_\star}^\star(i) I_{n_\star}\right\},$$
$$\Gamma^\star(i) \doteq \begin{bmatrix} \gamma_1^\star(i) I_{n_\star} & & & 0 \\ (1-\gamma_1^\star(i)) I_{n_\star} & \gamma_2^\star(i) I_{n_\star} & & \\ & \ddots & \ddots & \\ 0 & & (1-\gamma_{\ell_\star - 1}^\star(i)) I_{n_\star} & \gamma_{\ell_\star}^\star(i) I_{n_\star} \end{bmatrix},$$

with $\star \in \{y, u\}$, and $\delta_\alpha^y(i) \doteq 1 - \mathbb{1}_{\mathcal{D}_\alpha^y}(i)$, $\delta_\beta^u(i) \doteq 1 - \mathbb{1}_{\mathcal{D}_\beta^u}(i)$, $\gamma_\alpha^y(i) \doteq 1 - \mathbb{1}_{\bar{\mathcal{D}}_\alpha^y}(i)$, and $\gamma_\beta^u(i) \doteq 1 - \mathbb{1}_{\bar{\mathcal{D}}_\beta^u}(i)$, where $\mathcal{D}_\alpha^y \doteq \{i \in \mathbb{N} : i = \alpha + (\ell_y + 1)\sigma, \sigma \in \mathbb{Z}_{\geq 0}\}$, $\mathcal{D}_\beta^u \doteq \{i \in \mathbb{N} : i = \beta + (\ell_u + 1)\sigma, \sigma \in \mathbb{Z}_{\geq 0}\}$, $\bar{\mathcal{D}}_\alpha^y \doteq \{i \in \mathbb{N} : i = \alpha + 1 + (\ell_y + 1)\sigma, \sigma \in \mathbb{Z}_{\geq 0}\}$, $\bar{\mathcal{D}}_\beta^y \doteq \{i \in \mathbb{N} : i = \beta + 1 + (\ell_u + 1)\sigma, \sigma \in \mathbb{Z}_{\geq 0}\}$ for $\alpha = 1, \ldots, \ell_y$ and $\beta = 1, \ldots, \ell_u$. It is possible to show that the FDD-RR protocol satisfies Assumption 1 with $T = \max\{2\ell_y + 1, 2\ell_u + 1\}$. ■

B. Stability results

In order to reveal the mathematical structure of the WH–NCS model, we re-arrange the error vector ς via a change of coordinates. That is, we define $\boldsymbol{\varsigma} \doteq \mathcal{T}\varsigma$, where $\mathcal{T}$ is given by

$$\mathcal{T} \doteq \begin{bmatrix} I_{n_y} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & I_{n_u} & 0 & \cdots & 0 \\ 0 & I_{n_y} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_y} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_{n_u} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_{n_u} \end{bmatrix}. \quad (25)$$

TABLE I: Superframe table for the FDD-RR protocol.

	t_1	t_2	$\cdots$	t_{ℓ_y}	t_{ℓ_y+1}
CH$_1$	$P \to D_1^y$	$D_1^y \to D_2^y$	$\cdots$	$D_{\ell_{y-1}}^y \to D_{\ell_y}^y$	$D_{\ell_y}^y \to C$
CH$_2$	$C \to D_1^u$	$D_1^u \to D_2^u$	$\cdots$	$D_{\ell_u}^u \to P$	

With this change of coordinates, we have that

$$\dot{\boldsymbol{\varsigma}} = \boldsymbol{g}_\varsigma(\chi, z, \varsigma, u, v, w, d) \doteq \mathcal{T} g_\varsigma(\chi, z, \varsigma, u, v, w, d) = \left(g_{\varsigma,1}(\chi, z, \varsigma, u, v, w, d), -d_u, 0, \ldots, 0\right), \quad (26)$$

where $g_{\varsigma,1}(\chi, z, \varsigma, u, v, w, d)$ has been defined in (24). This way of looking at the error is helpful for the stability analysis that follows. Note that the first two components of $\dot{\boldsymbol{\varsigma}}$ are non-zero while the rest are zero.

Similar to Section V, we assume the following.

Assumption 9: Let $L_{11} \in \mathcal{A}_{\geq 0}^{n_{\zeta_1^y} \times (n_{\zeta_1^y} + n_{\zeta_1^u})}$ and $L_{12} \doteq L_{11}\text{diag}\{I_{n_y}^{2\ell_y - 1}, I_{n_u}^{2\ell_u - 1}\}$. There exists a matrix $A \in \mathcal{A}_{\geq 0}^{n_\varsigma}$ of the form

$$A = \begin{bmatrix} L_{11} & L_{12} \\ 0 & 0 \end{bmatrix},$$

and a continuous function $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_u + n_v} \to \mathbb{R}_{\geq 0}^{n_\varsigma}$ such that the error dynamics (23c) satisfy

$$\bar{\boldsymbol{g}}_\varsigma(\chi, z, \boldsymbol{\varsigma}, u, v, w, d) \preceq A\bar{\boldsymbol{\varsigma}} + \tilde{y}(\chi, z, u, v, w, d), \quad (27)$$

for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, \boldsymbol{\varsigma} \in \mathbb{R}^{n_\varsigma}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}, d \in \mathbb{R}^{n_u + n_v}$. ■

Assumption 9 is the analogue of Assumption 2 for WH networks. However, in Assumption 2, the function g_e does not have the structure in (26) that follows directly from our WH–NCS model. Note that g_ς depends on sums of components of ς rather than the whole error like g_e does (recall definitions of g_e and g_ς in (11) and (24), respectively). In particular, it depends on $\zeta_1^e + \cdots + \zeta_{2\ell_y}^e$ and $\zeta_1^u + \cdots + \zeta_{2\ell_u}^u$, thus we assume a linear bound on each component of (26) as in Assumption 2, which leads to Assumption 9. That is, we assume there exist $L_1 \in \mathcal{A}_{\geq 0}^{n_{\zeta_1^y} \times n_{\zeta_1^y}}$, $L_2 \in \mathcal{A}_{\geq 0}^{n_{\zeta_1^y} \times n_{\zeta_1^u}}$ and $\tilde{y}_1 : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_v} \to \mathbb{R}_{\geq 0}^{n_{\zeta_1^{\bar{y}}}}$, such that $\bar{g}_{\varsigma,1}(\chi, z, \boldsymbol{\varsigma}, u, v, w, d) \leq L_1(\bar{\zeta}_1^e + \cdots + \bar{\zeta}_{2\ell_y}^e) + L_2(\bar{\zeta}_1^u + \cdots + \bar{\zeta}_{2\ell_u}^u) + \tilde{y}_1(\chi, z, u, v, w, d_v)$.

Then, given that $\varsigma = (\zeta_1^e, \zeta_1^u, \zeta_2^e, \ldots, \zeta_{2\ell_y}^e, \zeta_2^u, \ldots, \zeta_{2\ell_u}^u)$, we get

$$\bar{\boldsymbol{g}}_\varsigma \preceq \begin{bmatrix} L_1 & L_2 & L_1\mathbf{I}_{n_y}^{2\ell_y-1} & L_2\mathbf{I}_{n_u}^{2\ell_u-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \bar{\varsigma} + \begin{bmatrix} \tilde{y}_1 \\ \bar{d}_u \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By defining $L_{11} \doteq [L_1\ L_2]$, we get what is stated in Assumption 9. A similar assumption in the context of controller design for WH-NCSs can be found in [10].

In the following, we impose assumptions analogue to Assumptions 3–5 in WH–NCSs. Note that these are essentially the same, but the networked-induced error is now different, and also the dynamics in (23).

Assumption 10: There exist $\gamma_2^\chi \in \mathbb{R}_{\geq 0}$ and $\sigma \in \mathcal{K}_\infty$ such that $\tilde{y}$ in (27) satisfies $|\tilde{y}(\chi, z, u, v, w, d)| \leq \gamma_2^\chi|\chi| + \sigma(|(u, v, w, d)|)$, for all $\chi \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}$, and $d \in \mathbb{R}^{n_u+n_v}$. ■

Assumption 11: There exist $\beta_1 \in \mathcal{KL}, \gamma_1^\varsigma \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathcal{K}_\infty$ such that, for any $\chi_0 \in \mathbb{R}^{n_x}$ and $(\varsigma, v, w) \in \mathcal{L}_\infty$, solutions to (23a) satisfy

$$|\chi(t)| \leq \beta_1(|\chi_0|, t - t_0) + \gamma_1^\varsigma \|\varsigma\|_{\mathcal{L}_\infty[t_0,t]} + \mu\left(\|(v, w)\|_{\mathcal{L}_\infty[t_0,t]}\right), \quad (28)$$

for all $t \geq t_0 \geq 0$. ■

Assumption 12: System (23b) is forward complete with input $(\chi, \varsigma, w) \in \mathcal{L}_\infty$ [22]. That is, there exist $\nu_1, \nu_2, \nu_3 \in \mathcal{K}$ and $c \in \mathbb{R}_{\geq 0}$ such that, for any $z_0 \in \mathbb{R}^{n_z}$ and $(\chi, \varsigma, u, v, w) \in \mathcal{L}_\infty$, $|z(t)| \leq \nu_1(t) + \nu_2(|z_0|) + \nu_3(\|(\chi, \varsigma, u, v, w)\|_{\mathcal{L}_\infty[t_0,t]}) + c$, for all $t \geq t_0 \geq 0$. ■

We are now ready to state the main results of this section. These follow by mimicking the steps of the proof of Proposition 1 and Theorem 1, but instead of using Theorem 5.1 in [9], we use Proposition 1 in [10], which is tailored to WH.

Proposition 2: Suppose Assumptions 1 and 9 hold. If MATI satisfies $\tau \in [\varepsilon, \tau_\varsigma^*)$, where $\tau_\varsigma^* = \ln(1 + 1/\sqrt{\varrho})/(|L_{11}|T)$, $\varrho \doteq \max\{2\ell_y, 2\ell_u\}$, then the system (10c), (10f) is IOS from $\tilde{y}$ to ς, where $\tilde{y}$ is per Assumption 9, with linear gain

$$\tilde{\gamma}(\tau) = \frac{T\exp(|A|(T+1)\tau)(\exp(|A|\tau) - 1)}{|A|\left(1 - \sqrt{\varrho}\left(\exp(|L_{11}|T\tau) - 1\right)\right)}. \quad (29)$$

■

Theorem 3: Suppose Assumptions 1, 9–12 hold. Let $\mathbf{x}$ be the solution of

$$\gamma_2^\chi\gamma_1^\varsigma T\mathbf{x}^{1+2/T} - \gamma_2^\chi\gamma_1^\varsigma T\mathbf{x}^{1+1/T} + \sqrt{\varrho}|A|\mathbf{x}^{|L_{11}|/|A|} - (1 + \sqrt{\varrho})\,|A| = 0, \quad (30)$$

and define $\tau^* = \ln(\mathbf{x})/(|A|T)$. If $\tau \in [\varepsilon, \tau^*)$, then the following holds.

(i) There exist $\beta \in \mathcal{KL}, \eta_1 \in \mathcal{K}$ and $\eta_2 \in \mathcal{KK}$ such that, for all $(\chi_0, \varsigma_0) \in \mathbb{R}^{n_x+n_\varsigma}$ and $(u, v, w, d) \in \mathcal{L}_\infty$,

$$|(\chi(t), \varsigma(t))| \leq \beta(|(\chi_0, \varsigma_0)|, t - t_0) + \eta_1(\|(v, w)\|_{\mathcal{L}_\infty}) + \eta_2(\tau, \|(u, v, w, d)\|_{\mathcal{L}_\infty}) \quad (31)$$

holds for all $t \geq t_0 \geq 0$.

(ii) System (23) is forward complete with input $(u, v, w, d) \in \mathcal{L}_\infty$. ■

Remark 2: Note that the results in Theorems 1 and 2 hold for NCS models of the form (10), and the WH–NCS model (23) fits that model. Therefore, these results can be directly applied to WH–NCSs, but they do not explicitly exploit the mathematical structure (26) of the model (23). The same can be said for the results in [6], and we compare our results to it in the next section. Theorem 3 is the simile to Theorem 1 but tailored to the specific structure encountered in WH–NCSs, and thus the corresponding MATI bound not only depends on $|A|$, but also on $|L_{11}|$ (see (30)). Recall that L_{11} comes from Assumption 9, and note that $|L_{11}| \leq |A|$. It is not immediately clear how (30) behaves for different values of $|A|$ and $|L_{11}|$, however, we later illustrate with an example that, as expected, the bound in Theorem 3 is larger than the bound in Theorem 1. In a similar fashion, we can derive an analogue theorem to Theorem 2 that is better suited for WH–NCSs, but it is omitted for brevity. ■

VII. Case Study

In this section, we study Lipschitz non-linear systems as an example to illustrate our results. The observer design problem for Lipschitz non-linear systems have been studied actively in the literature, see e.g. [19], [20], [29]–[32], when the observer is directly connected to the plant. We now show how to apply our non-linear framework to this class of systems, in which the observer communicates with the plant via a WH network.

A. Analysis

We study the architecture in Fig. 2, where the plant and observer are Lipschitz non-linear systems communicating over a WH network. We want to obtain the MATI bound that ensures stability of the NCS via Theorem 3. To that end, we first show that Lipschitz non-linear systems verify the standing assumptions of Theorem 3 (i.e. Assumptions 1, 9–12). Then, we provide the required parameters to compute the MATI bound that ensures stability of the NCS. Consequently, we consider non-linear plants of the form

$$\dot{x}_p = A_p x_p + \pi(x_p) + w, \quad y = Cx_p + v, \quad (32)$$

where $x_p \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$, A_p and C are constant matrices of appropriate dimensions, and $\pi : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is globally Lipschitz with constant $\Pi \geq 0$. We design an observer of the form

$$\dot{z} = A_p z + \pi(z) + K(y - y_z), \quad y_z = Cz, \quad (33)$$

where $z = \tilde{x}_p \in \mathbb{R}^{n_x}$ and $K \in \mathbb{R}^{n_x \times n_y}$ is a matrix such that the following holds.

Assumption 13: There exist $K \in \mathbb{R}^{n_x \times n_y}$ and a real symmetric positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ such that for $V : \chi \mapsto \chi^T P\chi$, $\langle\nabla V(\chi), (A_p - KC)\chi + \pi(\chi + z) - \pi(z)\rangle \leq -cV(\chi)$ for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_x}$ and some $c \in \mathbb{R}_{>0}$. ■

Assumption 13 is a general way of writing that the observer is designed such that the estimation error converges to zero in absence of disturbances. There is a body of works for Lipschitz non-linear systems in which this assumption is satisfied for a large number of observer designs, see e.g. [19]–[21], [32].

We now implement the observer (33) over a WH network, in which transmissions are scheduled by the FDD-RR protocol in Example 2. As in Section VI, we have that, for this case study,

$$\begin{aligned} f_\chi(\chi,z,\varsigma,v,w) &= (A_p - KC)\chi + \pi(\chi+z) - \pi(z) \\ &\quad - K(\mathbf{I}_{n_y}^{2\ell_y}\cdot\varsigma + v) + w, \quad (34a)\\ f_z(\chi,z,\varsigma,v) &= A_p z + \pi(z) + K(\mathbf{I}_{n_y}^{2\ell_y}\cdot\varsigma + v) \\ &\quad + KC\chi, \quad (34b)\\ \boldsymbol{g}_\varsigma(\chi,z,\varsigma,v,w,d_v) &= \Big(-C(A_p-KC)\chi - Cw - d_v \\ &\quad - C(\pi(\chi+z)-\pi(z)) \\ &\quad + CK(\mathbf{I}_{n_y}^{2\ell_y}\cdot\varsigma + v), 0,\ldots,0\Big). \quad (34c)\end{aligned}$$

Note that $\boldsymbol{\varsigma} = \varsigma = \zeta^e$ in this case since system (32) has no control input. We then verify that Assumptions 1,9–11 hold in the proposition below.

Proposition 3: Consider system (23) with f_χ, f_p and g_ς as per (34), and the FDD-RR protocol is used to schedule field devices. Suppose Assumption 13 holds, then

(i) Assumption 1 holds with $T = \max\{2\ell_y+1, 2\ell_u+1\}$.

(ii) Assumption 9 holds with

$$A = \begin{bmatrix} \overline{CK} \cdots \overline{CK} \\ 0 \;\cdots\; 0 \\ \vdots \;\ddots\; \vdots \\ 0 \;\cdots\; 0 \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $\tilde{y}_1 \doteq -C(A_p-KC)\chi - C(\pi(\chi+z)-\pi(z)) + CKv - Cw - d_v$.

(iii) Assumption 10 holds with $\gamma_2^\chi = |C(A_p-KC)| + \Pi|C|$ and $\sigma(s) = \max\{|CK|, |C|, 1\}3s$.

(iv) Assumption 11 holds with

$$\begin{aligned}\beta_1(s,t) &= \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\exp(-ct/8)s,\\ \gamma_1^\varsigma &= \tfrac{4}{c\lambda_{\min}(P)}|PK\mathbf{I}_{n_y}^{2\ell_y}|,\\ \mu(s) &= \max\left\{\tfrac{4}{c\lambda_{\min}(P)}|PK|, \tfrac{4}{c\lambda_{\min}(P)}|P|\right\}2s,\end{aligned}$$

where P and c are as per Assumption 13. ■

Since π in (33) is globally Lipschitz, Assumption 12 always applies in view of Theorem 3.2 in [33]. Then, a direct consequence of Proposition 3 is that all conditions of Theorem 3 are satisfied, hence it can be directly applied. Moreover, all parameters needed to calculate the MATI bound in Theorem 3 are given in Proposition 3. This is formalised via the following corollary.

Corollary 1: Consider system (23) with f_χ, f_z and g_ς as per (34), and the FDD-RR is used to schedule field devices. Suppose Assumption 13 holds. Let $\mathbf{x}$ be the solution of (30) with $\gamma_2^\chi, \gamma_1^\varsigma, A$ and T as per Proposition 3, and define $\tau^* = \ln(\mathbf{x})/(|A|T)$. If MATI satisfies $\tau \in [\varepsilon, \tau^*)$, then (31) holds. ■

B. Numerical simulation

We now provide a physical example that belongs to the class of Lipschitz non-linear systems studied in Section VII-A, for which we numerically compute the MATI bounds using Corollary 1. To that end, consider a single-link rigid robot manipulator modelled via Lagrange mechanics by [34] $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = T_u + w$, where q is the joint variable vector, $M(q)$ is the inertia matrix, $C(q,\dot{q})$ is the Coriolis and centripetal matrix, $g(q)$ is the gravity vector, w denotes an external disturbance, and T_u is the input torque. Pick $M(q)=1$, $C(q,\dot{q})=1$, $g(q)=0.1\cos(q)$, $T_u=0$, and define $x_p \doteq (q,\dot{q})$ and $y \doteq q+v$, where v corresponds to the measurement noise. We can then re-write the dynamics as

$$\dot{x}_p = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ -0.1\cos(x_{p,1}) \end{bmatrix} + w, \quad (35a)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_p + v. \quad (35b)$$

We note that (35) has the form in (32), i.e. this robot manipulator belongs to the class of Lipschitz non-linear systems analysed in Section VII-A. Therefore, we can apply Corollary 1 to numerically compute the MATI bound that ensures stability. Note in particular that

$$A_p = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \pi(x_p) = \begin{bmatrix} 0 \\ -0.1\cos(x_{p,1}) \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

for which the Lipschitz constant is given by $\Pi = 0.1$. The robot communicates with an observer of the form (33) over a WH network with $\ell_y = 2$ and $\ell_u = 0$. To design this observer, we need to find K such that Assumption 13 holds. To do that, we resort to observer design tools in the literature for the class of non-linear plants (32), see e.g. [19], [32]. In particular, Assumption 13 holds if there exist matrices $P = P^T > 0$ and R of adequate dimensions so that the following LMI condition is satisfied [19], [32]

$$\begin{bmatrix} A_p^T P + PA_p - R^T C - C^T R + I_{n_p} & P \\ P & -(1/\Pi^2) I_{n_p} \end{bmatrix} < 0.$$

Then, the stabilising gain K is given by $K = P^{-1}R^T$. For the given A_p, C and Π above, the LMI gives

$$P = \begin{bmatrix} 46.99 & -2.01 \\ -2.01 & 48.15 \end{bmatrix}, R = \begin{bmatrix} 54.13 & 47.14 \end{bmatrix}, K = \begin{bmatrix} 1.19 \\ 1.03 \end{bmatrix}.$$

With the above, it is possible to find that c in Assumption 13 is given by $c = 886.36$. To compute the MATI bound for this example, we resort to Proposition 3 and Corollary 1. Specifically, we have that $T = 2\ell_y + 1 = 5$, $\gamma_2^\chi = 1.66$, $\gamma_1^\varsigma = 8.24$, and

$$A = \left[\begin{array}{c|ccc} 1.19 & 1.19 & 1.19 & 1.19 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We can now compute $\tau^*_{\text{cor.1}}$, which denotes the MATI bound given by Corollary 1. We also compute the MATI bound $\tau^*_{\text{thm.1}}$ given by Theorem 1, and the MATI bounds that can

TABLE II: MATI bound comparison for the robot manipulator.

	FDD-RR protocol
$\tau^*_{\text{cor.1}}$ in [ms]	10.74
$\tau^*_{\text{thm.1}}$ in [ms]	10.69
$\tau^*_{[13]}$ in [ms]	$1.77 \cdot 10^{-3}$
$\tau^*_{[6]}$ in [ms]	$2.13 \cdot 10^{-3}$
$\tau^*_{\text{cor.1}}$ vs. $\tau^*_{[13]}$	$\sim 6 \cdot 10^5\%$
$\tau^*_{\text{cor.1}}$ vs. $\tau^*_{[6]}$	$\sim 5 \cdot 10^5\%$

be found in the literature [6], [13], denoted by $\tau^*_{[6]}$ and $\tau^*_{[13]}$ respectively. With all the above, Table II is constructed. In order to have a clear comparison, we have included in the last three rows, the *percentage of improvement*[2] between the relevant bounds. Note that the MATI bounds in Table II do not explicitly depend on external disturbances. For instance, the MATI bounds $\tau^*_{\text{cor.1}}$ and $\tau^*_{\text{thm.1}}$ depend on the values of $\gamma_2^{\chi}, \gamma_1^{\varsigma}, A$ and T that we computed in Section VII-A for our case study. Further below, we simulate the response of the NCS for which external disturbances and noise are reflected in the estimation error. The following comments can be made from Table II.

1) Our recent work [13] was a first attempt to obtaining MATI bounds tailored to observer design in WH–NCSs. This bound $\tau^*_{[13]}$ is quite conservative in the context of WH, as we discuss further below.
2) As shown in [10], assuming a PE_T condition on the protocols allows for less conservative MATI bounds in the context of controller design. The same condition is used in this work in the context of observer design. We can see from Table II that our proposed bounds are significantly larger than both $\tau^*_{[6]}$ and $\tau^*_{[13]}$.
3) Note that $\tau^*_{[13]}$ and $\tau^*_{[6]}$ are equivalent to 574 Mbps and 477 Mbps, respectively (if we transmit packets of 127 bytes, which is the maximum packet length in WH). However, the maximum data rate allowed in WH networks is 250 kbps, meaning that $\tau^*_{[13]}$ and $\tau^*_{[6]}$ cannot be achieved in WH. On the contrary, our bound $\tau^*_{\text{cor.1}} = 10.74[ms]$ is equivalent to 95 kbps, which is achievable on current WH networks.
4) The bound $\tau^*_{\text{cor.1}}$ exploits the structure of the WH–NCS model to obtain an even tighter bound than $\tau^*_{\text{thm.1}}$. Similar to [10], the bounds get less conservative as the number of field devices increases.

Consider the following external disturbance $w(t) = (0.6\sin(40t), 0.3\sin(30t))$ and measurement noise $v(t) = 1.3\sin(80t)$, and assume transmissions in the network happen equidistantly, i.e. they are such that $t_{i+1} - t_i = \tau$ for all $i \in \mathbb{N}$ and some $\tau \in \mathbb{R}_{>0}$. We run simulations for 200 different initial conditions randomly distributed in a ball of radius 5. For each of these simulations we increased τ and find τ^*_{sim} such that the estimation error remains close to the origin for $\tau \in (0, \tau^*_{sim})$. We take the minimum of all the two hundred τ^*_{sim}, and we use it as a simulation-based estimate of the *real* MATI of the system. This value is equal to $23.1[ms]$. Our results on WH yield a theoretical bound of $\tau^*_{cor.1} = 10.74[ms]$, which is only about 2.15 times more conservative that the bound on MATI observed in simulations. Note that previous bounds such as $\tau^*_{[13]}$ are about 13000 times more conservative than the simulated bound. A simulation with initial conditions $\chi_0 = (1.46, 2.82)$ can be found in Fig. 3 for $\tau = 8[ms] < \tau^*_{\text{cor.1}}$. We see that the estimation error indeed converges to a neighbourhood of the origin, which is in agreement with Theorem 3.

[2]We define the percentage of improvement between $\tau_A > 0$ and $\tau_B > 0$, where $\tau_A \geq \tau_B$, as $100 \times (\tau_A - \tau_B)/\tau_B$.

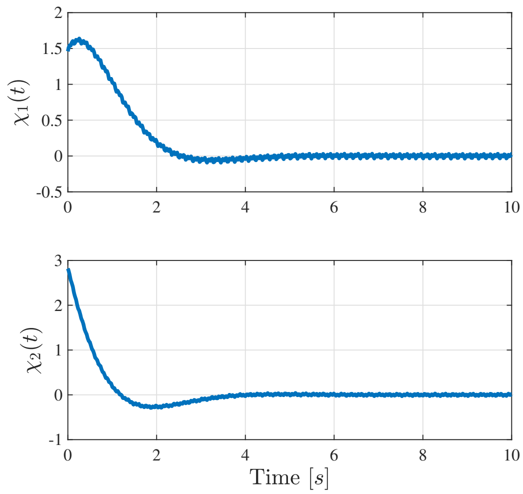

Fig. 3: Estimation error χ for a MATI satisfying the bound in Theorem 3.

VIII. Conclusions

We proposed an emulation approach for the observer design problem for non-linear NCSs affected by disturbances and measurement noise. In particular, we provided MATI bounds which ensure the convergence of the observation error under network-induced constraints. Our results exploit the persistence of excitation property of the implemented protocols, which allowed us to obtain larger MATI bounds compared with the literature, where more restrictive properties are used, such as UGES. We also showed that our approach is applicable to WH–NCSs, for which we obtain tailored results that exploit the intrinsic structure of WH. By doing so, we provided even tighter MATI bounds.

Appendix

Proof of Proposition 1

Let $e(t_0) = e_0 \in \mathbb{R}^{n_e}$, $\tilde{y} \in \mathcal{L}_\infty$ and $\tau \in [\varepsilon, \tau_e^*)$. Also let $[t_0, t_{\max})$ denote the maximum existence interval for system (10c), (10f), where $t_{\max} \in (t_0, \infty]$. Let $t \in [t_0, t_{\max})$ and recall $\tilde{y}$ from Assumption 2, which represents a disturbance term to e–dynamics. From the proof of Theorem 5.1 in [9], we can compute the contribution of the initial condition by setting the disturbance term $\tilde{y} = 0$, which gives $|\bar{e}(t)| \leq \exp(|A|T\tau)\lambda^m|\bar{e}(t_0)|$, for any $m \in \mathbb{N}$ such that $t \in (t_{mT-1}, t_{(m+1)T-1})$, where $\lambda \doteq \exp(|A|T\tau) - 1$, and T

comes from Assumption 1. Note that $0 < \lambda < 1$ for $\tau < \tau_e^*$. We also have that $t - t_0 \leq (m+1)T\tau$, then

$$\begin{aligned}|\bar{e}(t)| &\leq \frac{\exp(|A|T\tau)}{\lambda}\lambda^{m+1}|\bar{e}(t_0)| \\ &\leq \frac{\exp(|A|T\tau)}{\lambda}\lambda^{(t-t_0)/(T\tau)}|\bar{e}(t_0)| \\ &= \frac{\exp(|A|T\tau)}{\lambda}\exp\Big(-\frac{c}{T\tau}(t-t_0)\Big)|e_0|, \end{aligned} \tag{36}$$

where $c \doteq -\ln(\lambda) > 0$. In the last equality, we have used the fact that $e(t_0) = e_0$ and that $|\bar{x}| = |x|$ for any $x \in \mathbb{R}^n$. Now we set the initial condition $e_0 = 0$ and compute the contribution of the input $\tilde{y}$ by directly applying Theorem 5.1 in [9]. That is,

$$\|\bar{e}\|_{\mathcal{L}_\infty[t_0,t]} \leq \tilde{\gamma}(\tau)\|\tilde{y}\|_{\mathcal{L}_\infty[t_0,t]}. \tag{37}$$

By summing (36) and (37), we conclude that system (10c), (10f) is IOS from $\tilde{y}$ to e with linear gain $\tilde{\gamma}(\tau)$, completing the proof. ∎

Proof of Theorem 1

This proof employs a similar argument to that of Theorem 2.1 in [35] and Theorem 1 in [6]. For brevity, we omit some obvious intermediate algebraic steps.

(i) The proof of the first assertion follows via two steps. In the first step we prove that (χ, e)–system is BIBS with input w. In the second step we show convergence, i.e. the required ISS property (16).

Step 1 (BIBS Property) Let $(\chi_0, e_0, z_0) \in \mathbb{R}^{n_x+n_e+n_z}$, $(u, v, w, d) \in \mathcal{L}_\infty$ and $\tau \in [\varepsilon, \tau^*)$. Let $[t_0, t_{\max})$ denote the maximum existence interval for system (10), where $t_{\max} \in (t_0, \infty]$. Let $t \in [t_0, t_{\max})$. We first show that $1 - \tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e > 0$, which we need throughout the proof. Note that, in (14), $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(\tau) \to \infty$ as $\tau \to \tau_e^*$. Also note that $\tilde{\gamma}(\tau)$ in (14) is differentiable and monotonically increasing in τ for $\tau \in [\varepsilon, \tau_e^*]$ and thus, in view of the inverse function theorem [36], there exists a unique solution τ^* to $\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e = 1$, which satisfies $\tau^* < \tau_e^*$ (note that ε is sufficiently small so that $1/(\gamma_2^\chi\gamma_1^e)$ is always in the range of $\tilde{\gamma}(\tau)$ for $\tau \in [\varepsilon, \tau_e^*]$). That is, $\gamma_2^\chi\gamma_1^e T\exp(|A|(T+1)\tau)(\exp(|A|\tau)-1) - |A|\,(2-\exp(|A|T\tau)) = 0$, for which we define $\mathbf{x} \doteq \exp(|A|T\tau)$ and thus get (15) with $\mathbf{x} \in [1, 2]$. Then, by monotonicity of $\tilde{\gamma}(\tau)$, we have that $\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e < 1$ for any $\tau \in [\varepsilon, \tau^*)$.

By using Proposition 1 and Assumption 3, we have that

$$\begin{aligned}|e(t)| \leq \beta_2(|e_0|, t-t_0) + \tilde{\gamma}(\tau)\Big(&\gamma_2^\chi\|\chi\|_{\mathcal{L}_\infty[t_0,t]} \\ &+ \sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\Big),\end{aligned} \tag{38}$$

where $\tilde{\gamma}(\tau)$ is as per (14), which defines a class–$\mathcal{K}$ function on $[0, \tau^*)$. Therefore, in view of (13) and (38),

$$\begin{aligned}\|e\|_{\mathcal{L}_\infty[t_0,t]} &\leq \frac{\beta_2(|e_0|, 0)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e} + \frac{\tilde{\gamma}(\tau)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e} \\ &\quad\times\Big(\gamma_2^\chi\beta_1(|\chi_0|, 0) + \gamma_2^\chi\mu(\|(v,w)\|_{\mathcal{L}_\infty}) \\ &\quad+ \sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\Big) \\ &\doteq M_e(\tau, \chi_0, e_0, \|(u,v,w,d)\|_{\mathcal{L}_\infty[t_0,t]}).\end{aligned} \tag{39}$$

On the other hand, we have that

$$\begin{aligned}\|\chi(t)\|_{\mathcal{L}_\infty[t_0,t]} &\leq \beta_1(|\chi_0|, 0) + \gamma_1^e M_e + \mu(\|(u,v)\|_{\mathcal{L}_\infty}) \\ &\doteq M_\chi(\tau, \chi_0, e_0, \|(u,v,w,d)\|_{\mathcal{L}_\infty[t_0,t]})\end{aligned} \tag{40}$$

Note that we have omitted the argument of the function M_e for the purpose of clarity. In the following, we do the same for M_χ. It follows from the above that

$$\begin{aligned}|(\chi(t), e(t))| &\leq M_\chi + M_e \\ &\leq \alpha(|(\chi_0, e_0)|) + \mu(\|(v,w)\|_{\mathcal{L}_\infty}) \\ &\quad+ \vartheta_1(\tau)\mu(\|(v,w)\|_{\mathcal{L}_\infty}) \\ &\quad+ \vartheta_2(\tau)\sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty}),\end{aligned} \tag{41}$$

where $\alpha, \vartheta_1, \vartheta_2 \in \mathcal{K}$ are defined as $\alpha(s) = \beta_1(s,0) + \frac{(\gamma_1^e+1)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e}\beta_2(s,0) + \frac{(\gamma_1^e+1)\tilde{\gamma}(\tau)\gamma_2^\chi}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e}\beta_1(s,0)$, $\vartheta_1(\tau) = \frac{(\gamma_1^e+1)\tilde{\gamma}(\tau)\gamma_2^\chi}{1-\tilde{\gamma}(\tau)\gamma_2^\chi} + 1$, $\vartheta_2(\tau) = \frac{(\gamma_1^e+1)\tilde{\gamma}(\tau)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi}$. Therefore, (χ, e)–system is BIBS in view of Definition 3 and $t_{\max} = \infty$.

Step 2 (Convergence Property) For any $t_0 \leq t_{10} \leq t_{20} \leq t_{11} \leq t_{21}$, using time invariance and causality of the inequalities (13) and (38), we have

$$\begin{aligned}|\chi(t_{11})| &\leq \beta_1(|\chi(t_{10})|, t_{11}-t_{10}) + \gamma_1^e\|e\|_{\mathcal{L}_\infty[t_{10},t_{11}]} \\ &\quad+ \mu(\|(v,w)\|_{\mathcal{L}_\infty}),\end{aligned} \tag{42a}$$

$$\begin{aligned}|e(t_{21})| &\leq \beta_2(|e(t_{20})|, t_{21}-t_{20}) + \tilde{\gamma}(\tau) \\ &\quad\times\Big(\gamma_2^\chi\|\chi\|_{\mathcal{L}_\infty[t_{20},t_{21}]} + \sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\Big).\end{aligned} \tag{42b}$$

Let $t \in [t_0, \infty)$, and take $t_{10} = (t-t_0)/4 + t_0$, $t_{20} = (t-t_0)/2 + t_0$, $t_{21} = t$, and $t_{11} \in [(t-t_0)/2 + t_0, t]$. In view of (39), (40), and (42), we have that

$$\begin{aligned}|e(t)| &\leq \beta_2(M_e, (t-t_0)/2) + \tilde{\gamma}(\tau^*)\gamma_2^\chi \\ &\quad\times\beta_1(M_\chi, (t-t_0)/4) + \tilde{\gamma}(\tau) \\ &\quad\times\Big(\gamma_2^\chi\gamma_1^e\|e\|_{\mathcal{L}_\infty[(t-t_0)/4+t_0,\infty)} \\ &\quad+ \gamma_2^\chi\mu(\|(v,w)\|_{\mathcal{L}_\infty}) + \sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\Big).\end{aligned}$$

Recall that $\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e < 1$, since $\tau < \tau^*$. We now use Lemma A.1 in [35], with $z(t) = |e(t)|$, $\beta(s,t) = \beta_2(s, t/2) + \tilde{\gamma}(\tau^*)\gamma_2^\chi\beta_1(s, t/4)$, $s = M_e + M_\chi$, $\rho(s) = \tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e s$, $d = \tilde{\gamma}(\tau) \times \big(\gamma_2^\chi\mu(\|(v,w)\|_{\mathcal{L}_\infty}) + \sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\big)$, and $\mu = 1/4$, to show that there exist $\lambda_2 \in (1, \infty)$ and $\tilde{\beta}_2 \in \mathcal{KL}$ such that

$$\begin{aligned}|e(t)| &\leq \tilde{\beta}_2(M_e + M_\chi, t-t_0) + \frac{\lambda_2\tilde{\gamma}(\tau)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e} \\ &\quad\times\Big(\gamma_2^\chi\mu(\|(v,w)\|_{\mathcal{L}_\infty}) + \sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\Big).\end{aligned} \tag{43}$$

In a similar fashion, in view of (42), it can be shown that there exist $\tilde{\beta}_1 \in \mathcal{KL}$ and $\lambda_1 \in (1, \infty)$ such that

$$\begin{aligned}|\chi(t)| &\leq \tilde{\beta}_1(M_e + M_\chi, t-t_0) + \frac{\lambda_1}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e} \\ &\quad\times\Big(\mu(\|(v,w)\|_{\mathcal{L}_\infty}) + \tilde{\gamma}(\tau)\gamma_1^e\sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\Big).\end{aligned} \tag{44}$$

Define $\tilde{\beta} \doteq \tilde{\beta}_1 + \tilde{\beta}_2$. Combining (43) and (44), and noting that $1/(1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e) = 1 + \tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e/(1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e)$, it follows that

$$\begin{aligned}|(\chi(t),e(t))| &\leq \tilde{\beta}(M_e + M_\chi, t-t_0)\\ &+ \lambda_1\mu(\|(v,w)\|_{\mathcal{L}_\infty}) + \frac{\tilde{\gamma}(\tau)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e}\\ &\times \Big(\big(\lambda_2\gamma_2^\chi + \lambda_1\gamma_2^\chi\gamma_1^e\big)\mu(\|(v,w)\|_{\mathcal{L}_\infty})\\ &+ \big(\lambda_2 + \lambda_1\gamma_1^e\big)\sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty})\Big). \end{aligned}\tag{45}$$

According to (41), and using several times the fact that $\tilde{\beta}(a+b,t) \leq \tilde{\beta}(2a,t) + \tilde{\beta}(2b,t)$ for any $a,b,t \in \mathbb{R}_{\geq 0}$ since $\tilde{\beta}(\cdot,t) \in \mathcal{K}$, we have that

$$\begin{aligned}\tilde{\beta}(M_\chi + M_e, t-t_0) &\leq \tilde{\beta}\left(2\alpha(|(\chi_0,e_0)|), t-t_0\right)\\ &+ \tilde{\beta}\left(4\mu(\|(v,w)\|_{\mathcal{L}_\infty}),0\right) + \tilde{\beta}\big(4\vartheta_1(\tau)\mu(\|(v,w)\|_{\mathcal{L}_\infty})\\ &+ 4\vartheta_2(\tau)\sigma(\|(u,v,w,d)\|_{\mathcal{L}_\infty}),0\big).\end{aligned}\tag{46}$$

Consequently, in view of (45) and (46), we have that (16) holds with $\beta(s,t) = \tilde{\beta}(2\alpha(s),t)$, $\eta_1(s) = \lambda_1\mu(s) + \tilde{\beta}(4\mu(s),0)$, $\eta_2(\tau,s) = \frac{\tilde{\gamma}(\tau)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e}\big((\lambda_2\gamma_2^\chi + \lambda_1\gamma_2^\chi\gamma_1^e) \times \mu(s) + \big(\lambda_2 + \lambda_1\gamma_1^e\big)\sigma(s)\big) + \tilde{\beta}(4\vartheta_1(\tau)\mu(s) + 4\vartheta_2(\tau) \times \sigma(s),0)$, for $(s,t) \in \mathbb{R}^2_{\geq 0}$. Clearly $\beta \in \mathcal{KL}$ and $\eta_1, \eta_2(\tau,\cdot) \in \mathcal{K}$. It remains to show that $\eta_2(\cdot,s) \in \mathcal{K}$. Take $s \in \mathbb{R}_{\geq 0}$ and note that $\eta_2(\cdot,s)$ is continuous on $[0,\tau^*)$ and $\eta_2(0,s) = 0$ (since $\tilde{\gamma}$ is continuous on $[0,\tau^*)$ and $\tilde{\gamma}(0) = 0$). Furthermore, since $\tilde{\gamma}$ is strictly increasing on $[0,\tau^*)$, so is $\tau \mapsto \frac{\tilde{\gamma}(\tau)}{1-\tilde{\gamma}(\tau)\gamma_2^\chi\gamma_1^e}$, then we have that $\eta_2(\cdot,s) \in \mathcal{K}$, getting the desired result (16).

(ii) Using the forward completeness characterisation given in Corollary 2.3 in [22], proving that system (10) is forward complete is equivalent to proving that there exist $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3 \in \mathcal{K}$ and $\tilde{c} \in \mathbb{R}_{\geq 0}$ such that, for any $(\chi_0, z_0, e_0) \in \mathbb{R}^{n_x+n_z+n_e}$ and $(u,v,w,d) \in \mathcal{L}_\infty$, we have that $|(\chi(t),z(t),e(t))| \leq \tilde{\nu}_1(t) + \tilde{\nu}_2(|(\chi_0,z_0,e_0)|) + \tilde{\nu}_3(\|(u,v,w,d)\|_{\mathcal{L}_\infty}) + \tilde{c}$, for all $t \geq t_0 \geq 0$. From (16) and Assumption 5, we have that

$$\begin{aligned}|(\chi(t),z(t),e(t))| &\leq \nu_1(t) + \Big(\beta(|(\chi_0,e_0)|,0) + \nu_2(|z_0|)\\ &+ \nu_3(4\beta(|(\chi_0,e_0)|,0))\Big) + c + \Big(\eta_1(\|(v,w)\|_{\mathcal{L}_\infty})\\ &+ \nu_3(8\eta_1(\|(v,w)\|_{\mathcal{L}_\infty})) + \eta_2(\tau^*, \|(u,v,w,d)\|_{\mathcal{L}_\infty})\\ &+ \nu_3(2\|(u,v,w)\|_{\mathcal{L}_\infty}) + \nu_3(8\eta_2(\tau^*, \|(u,v,w,d)\|_{\mathcal{L}_\infty}))\Big).\end{aligned}$$

where clearly $\tilde{\nu}_1(t) = \nu_1(t), \tilde{\nu}_2(s) = \beta(s,0) + \nu_2(s) + \nu_3(4\beta(s,0)), \tilde{\nu}_3(s) = \eta_1(s) + \nu_3(8\eta_1(s)) + \eta_2(\tau^*,s) + \nu_3(8\eta_2(\tau^*,s)) + \nu_3(2s)$ and $\tilde{c} = c$, proving the result. ■

PROOF OF PROPOSITION 3

(i) This part follows directly from the proof of Lemma 11 in [10], and the definition of the FDD-RR protocol in Example 2.

(ii) By using (34), we have that $\bar{\boldsymbol{g}}_\varsigma \preceq \Big(CK\mathbf{I}^{2\ell_y}_{n_y} \cdot \varsigma, 0, \ldots, 0\Big) + \big(\overline{\tilde{y}_1}, 0, \ldots, 0\big)$, and the result follows from the definition of $\mathbf{I}^{2\ell_y}_{n_y}$.

(iii) Taking the euclidean norm of $\tilde{y}$ in assertion *(ii)*, using the triangle inequality and the globally Lipschitz property of π, we have that $|\tilde{y}(\chi,z,v,w,d_v)| \leq \big(|C(A-KC)| + \Pi|C|\big)|\chi| + \max\{|CK|,|C|,1\}(|v|+|w|+|d_v|)$, completing the proof of assertion *(iii)*.

(iv) Under Assumption 13, we have that along solutions to $\dot{\chi} = f_\chi(\chi,z,\varsigma,v,w)$,

$$\begin{aligned}\dot{V} &\leq -cV + 2|\chi||PK\mathbf{I}^{2\ell_y}_{n_y}||\boldsymbol{\varsigma}| + 2|\chi||PK||v| + 2|\chi||P||w|\\ &\overset{(a)}{\leq} -\frac{c}{4}V + \frac{4\big|PK\mathbf{I}^{2\ell_y}_{n_y}\big|^2}{c\lambda_{\min}(P)}|\boldsymbol{\varsigma}|^2 + \frac{4|PK|^2}{c\lambda_{\min}(P)}|v|^2\\ &\quad + \frac{4|P|^2}{c\lambda_{\min}(P)}|w|^2,\end{aligned}$$

where (a) follows from $\lambda_{\min}(P)|\chi|^2 \leq V(\chi) \leq \lambda_{\max}(P)|\chi|^2$, and from using the fact that $2ab \leq (c/4)a^2 + (4/c)b^2$. By invoking the comparison principle (see Lemma 3.4 in [33]), we have that

$$\begin{aligned}V(t) &\leq \exp\left(-\tfrac{c}{4}(t-t_0)\right)V(t_0)\\ &+ \tfrac{4}{c}\left(\tfrac{4|PK\mathbf{I}^{2\ell_y}_{n_y}|^2}{c\lambda_{\min}(P)}\|\boldsymbol{\varsigma}\|^2_{\mathcal{L}_\infty[t_0,t]} + \tfrac{4|PK|^2}{c\lambda_{\min}(P)}\|v\|^2_{\mathcal{L}_\infty[t_0,t]}\right.\\ &\left.+\tfrac{4|P|^2}{c\lambda_{\min}(P)}\|w\|^2_{\mathcal{L}_\infty[t_0,t]}\right).\end{aligned}$$

Therefore, $|\chi(t)| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\exp\left(-\frac{c}{8}(t-t_0)\right)|\chi(t_0)| + \frac{4\left|PK\mathbf{I}^{2\ell_y}_{n_y}\right|}{c\lambda_{\min}(P)}\|\boldsymbol{\varsigma}\|_{\mathcal{L}_\infty[t_0,t]} + \frac{4|PK|}{c\lambda_{\min}(P)}\|v\|_{\mathcal{L}_\infty[t_0,t]} + \frac{4|P|}{c\lambda_{\min}(P)}\|w\|_{\mathcal{L}_\infty[t_0,t]}$, which completes the proof. ■

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Alejandro I. Maass is a research fellow in the Department of Electrical and Electronic Engineering at the University of Melbourne, Australia. He received an electronic engineering degree and an M.Sc. degree in electronic engineering from the Universidad Técnica Federico Santa María (UTFSM), Chile, in 2013. In 2014, he was a research assistant in the Department of Electronic Engineering at UTFSM. He subsequently completed his Ph.D. degree in electrical and electronic engineering with the University of Melbourne in 2019. He was awarded a doctoral scholarship (2015) from the Chilean government agency CONICYT. His main research interests include networked control systems (NCSs), optimal estimation and control with communication constraints, and non-linear hybrid control systems with applications in wireless NCSs.

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Dragan Nešić (S'96–M'02–SM'02–F'08) is a Professor in the Department of Electrical and Electronic Engineering (DEEE) at The University of Melbourne, Australia. He received his BE degree in Mechanical Engineering from The University of Belgrade, Yugoslavia in 1990, and his Ph.D. degree from Systems Engineering, RSISE, Australian National University, Canberra, Australia in 1997. His research interests include networked control systems, reset systems, extremum seeking control, hybrid control systems, and so on. He was a co-recipient of the George S. Axelby Outstanding Paper Award (2018). He was also awarded a Humboldt Research Fellowship (2003) by the Alexander von Humboldt Foundation, an Australian Professorial Fellowship (2004-2009) and Future Fellowship (2010-2014) by the Australian Research Council.

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Romain Postoyan received the "Ingénieur" degree in Electrical and Control Engineering from ENSEEIHT (France) in 2005. He obtained the M.Sc. by Research in Control Theory & Application from Coventry University (United Kingdom) in 2006 and the Ph.D. in Control Theory from Université Paris-Sud (France) in 2009. In 2010, he was a research assistant at the University of Melbourne (Australia). Since 2011, he is a CNRS researcher at the "Centre de Recherche en Automatique de Nancy" (France). He received the 'Habilitation à Diriger des Recherches (HDR)" in 2019 from Université de Lorraine (Nancy, France). He serves/served as an associate editor for the journals: Automatica, IEEE Control Systems Letters and IMA Journal of Mathematical Control and Information.

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Peter M. Dower (M'98) is an Associate Professor with the Department of Electrical and Electronic Engineering at the University of Melbourne, Australia. Peter received a computer engineering degree (with hons., university medal) from the University of Newcastle, Australia, and a systems engineering PhD from the Australian National University. He subsequently was a post-doctoral fellow in the Department of Mathematics at the University of California at San Diego (UCSD). His research expertise is in the area of optimal control and robust stability analysis for nonlinear dynamical systems, with an emphasis on dynamic programming, Hamilton-Jacobi-Bellman PDEs, idempotent analysis, convex analysis, semigroup theory, and computational methods. He was a co-recipient, with W.M. McEneaney (UCSD), of the 2017 SIAG/CST Best SICON Paper Prize for the development of an optimal control based fundamental solution to the gravitational N-body problem. His research is supported by grants from the US Air Force Office of Scientific Research (AFOSR), through its Dynamics & Control Program.