Robustness of quantized control systems with mismatch between coder/decoder initializations ⋆

Tania Kameneva, Dragan Nešić

Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, 3010, Victoria, Australia

Abstract

This paper analyzes the stability of linear systems with quantized feedback in the presence of a mismatch between the initial conditions at the coder and decoder. Under the assumption of the prefect channel, we show that using the scheme proposed in [Liberzon, Nešić (2007)] it is possible to achieve global exponential stability of linear systems with quantized feedback when the coder and decoder are initialized at different initial conditions.

Key words: communication channel, quantization, controller design, stabilization

1 Introduction

The subject of this paper is the analysis of robustness of linear systems with quantized feedback with respect to a mismatch between the initial conditions at the coder and decoder. Control systems with quantized feedback are increasingly used in control practice due to advances in computer, sensor and actuator technologies, as well as our desire to decrease costs, simplify installation and maintenance. While a number of important results have been published on the topic of the quantized control systems, including [1], [3], [6], [7], [10], [8], to our best knowledge, none of them consider the issue of the mismatch between the initial conditions at the coder and decoder.

To simplify the presentation, we assume that the channel is perfect and concentrate on the robustness properties of the systems when the coder and decoder are initialized at different initial conditions. While the issues of the robustness with respect to the time-delays, data dropouts, bit-errors, corrupted signals and control, in general, over noisy channels were investigated [2], [9], [14], the robustness with respect to the computational errors at the coder and decoder did not receive much attention in the literature.

The device which at each instant of time maps the value of the plant output measurements into one of all possible symbols is called the coder (pre-processing device). The symbol generated by the coder is then transmitted through the channel to the receiver. At the reception, the decoder (after-processing device) generates an estimated value of the state from the received symbol. Note, that to deal with a finite capacity of the channel, we have to run two copies of the system on both sides of the digital channel. A common assumption made in the literature is that these two systems are initialized at the same initial condition and, hence, the issue of the discrepancy in the initialization at the coder and decoder is ignored. This may not be implementable in practice due to hardware imperfections. Even if the coder and decoder are initialized at the same value and the channel is perfect, since the coder and decoder dynamics evolve independently, the computational errors in the algorithm implemented at the coder and decoder can occur. At this point of time the mismatch in the coder and decoder takes place.

In other words, even if the channel is perfect and the internal coder/decoder factors are initialized at the same value, due to a finite precision of encoding and decoding schemes for transmitted information there might exist time such that these internal factors start to differ. We treat this time instant as the initial time when the mismatch occurs.

In this paper we investigate this phenomenon further. We explore the following question: does the system preserve stability properties when the coder and decoder are initialized at different initial conditions?

We explore the robustness with respect to the mismatch between the initial conditions at the coder and decoder

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E-mail address: {t.kameneva,d.nesic}@ee.unimelb.edu.au.

of the discrete linear time-invariant systems with quantized feedback. As a particular example of the quantizer-coder-decoder scheme, we analyze in detail the sampled-data hysteresis switching scheme proposed in [8]. We use this scheme as a representative example of other quantized control schemes, that have adaptive quantization as their main feature, that is: the quantizer's range and quantization error are changing adaptively depending on the quantized measurements of the plant.

Note, that in [8] the modified version of the hysteresis scheme, that we use in this paper was also introduced. That scheme was developed to handle disturbances, while here we consider systems with no disturbances. This scheme is known to lead to the various stability properties of the linear time-invariant systems when the coder and decoder are initialized at the same initial conditions [5], [8]. In the simulations we have observed, that when the mismatch between the coder and decoder initialization is sufficiently large, the system dynamics become unstable.

Under the assumption that the channel is perfect, using Liberzon and Nešić scheme, we analyze the robustness properties of the quantized control system with respect to the computational errors that occur due to the independent evaluation of the adaptive "scaling" factors at the coder and decoder. We give a quantitative measure on how much these adaptive "scaling" factors at the coder and decoder can differ so that the system preserves stability.

We show that if the channel is perfect and the coder and decoder are initialized at different initial conditions (but a bound on the mismatch holds), then using the time-sampled scheme introduced in [8], it is possible to adjust the parameters of the quantizer so that the systems is global exponential stable (GES) (refer to Definition 3 in Section 4). Our Theorem 1 in Section 4 shows that GES (in the sense of Definition 3) is possible when the channel is perfect, the parameters of the scheme are adjusted appropriately and the bound on the mismatch between the initial conditions at the coder and decoder holds. In other words, we show that the scheme has some intrinsic robustness properties with respect to small mismatches in the coder/decoder initialization. We believe that these results shed a light on the robustness properties of other quantized control scheme in the literature.

The remainder of the paper is organized as follows. In Section 2 we give definitions that are used in the sequel. The closed loop system, switching rules and protocol are given in Section 3. The main results are presented in Section 4. Section 5 offers the conclusions. The proofs are given in the appendix.

2 Notation and preliminaries

In this section we introduce some notation and give the definitions that will make the discussed concepts precise. In what follows, $|\cdot|$ denotes the Euclidean norm, $\|\cdot\|$ denotes the corresponding matrix induced norm. The infinity-norm of a sequence of vectors on a time-interval $[k_1, k_2]$ is denoted $\|z\|_{[k_1,k_2]} := \sup_{k\in[k_1,k_2]} |z_k|$.

A quantizer is a piecewise constant function $q : \mathbb{R}^n \to Q$, where Q is a finite subset of $\mathbb{R}^n$. We use the following assumption:

Assumption 1 *There exist strictly positive numbers $M_1 \geq M > \Delta > 0$, Δ_0 such that the following holds: 1. If $|z| \leq M$ then $|q(z) - z| \leq \Delta$; 2. If $|z| > M$ then $|q(z)| > M - \Delta$; 3. For all $|z| \leq \Delta_0$ we have that $q(z) = 0$; 4. $|q(z)| \leq M_1$ for all $z \in \mathbb{R}^n$.*

M is called the range of the quantizer; Δ is called the quantization error. The first condition gives a bound on the quantization error when the state is in the range of the quantizer, the second gives the possibility to detect saturation. The third condition is needed to preserve the origin as an equilibrium and, moreover, together with the forth condition it guarantees that there exists $L_q > 0$ such that $|q(z)| < L_q|z|\ \forall z \in \mathbb{R}^n$. The last conditions guarantees that the quantized values of z are globally bounded. Note, that for a sufficiently large M_1 without loss of generality we can assume that the following holds: $M_1 = L_q M$. We will use the following definitions:

Definition 1 *A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded.*

Definition 2 *A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be class $\mathcal{KL}$ if $\beta(\cdot, t)$ is continuous, strictly increasing and zero at zero for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.*

3 Closed-loop system

Consider the continuous-time linear system with a control input:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are respectively the state and control. The matrix A is nonzero and non-Hurwitz. Define $t_k = kT$ for $k = 0, 1, 2, \ldots$, where $T > 0$ is a given sampling period. We assume that $u(t) = cost. \forall t \in [kT, (k+1)T]$. We shortly denote $x(t_k) = x_k$, $u(t_k) = u_k$, $k = 0, 1, 2, \ldots$. The plant (1) induces the following discrete-time system which is more amenable to analysis:

$$x_{k+1} = \Phi x_k + \Gamma u_k, \quad x_0 \in \mathbb{R}^n, \tag{2}$$

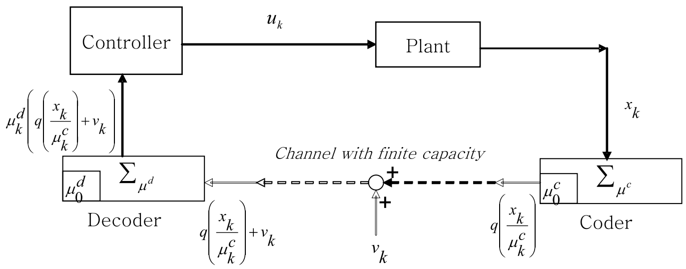

Fig. 1. A feedback system with a digital channel.

where $\Phi = e^{AT}$, $\Gamma = \int_0^T e^{As}B\, ds$. Note, that due to a finite capacity of the channel the state measurements are quantized into a finite subset of $\mathbb{R}^n$. We use the quantized measurements in the following form:

$$q_k^c := \mu_k^c q\left(\frac{x_k}{\mu_k^c}\right), \quad \mu_0^c > 0$$

on the coder side of the channel (see Figure 1) and

$$q_k^d := \mu_k^d\left(q\left(\frac{x_k}{\mu_k^c}\right) + v_k\right), \quad \mu_0^d > 0$$

on the decoder side of the communication channel. μ_k^c, μ_k^d are the adjustable parameters, called "zoom" variables, that are updated at discrete instants of time; μ_k^c and μ_k^d correspond to the coder and decoder dynamics respectively. The symbol $q\left(\frac{x_k}{\mu_k^c}\right)$ is send via the communication channel. At the reception, the decoder receives the symbol $q\left(\frac{x_k}{\mu_k^c}\right) + v_k$, which is, in general, not necessarily identical to the symbol which was send by the coder. The term v_k corresponds to a general noise in the channel, it can model the pure (propagation) time-delay, packets loss, bit-errors etc. To simplify the presentation, we do not combine the issues of the noisy channel with the issue of the robustness with respect to the mismatch in the coder/decoder initialization. This problem is outside the scope of this paper.

Assumption 2 *Assume that*

$$v_k \equiv 0 \quad \forall k \geq 0.$$

Assumption 2 guarantees that the channel is perfect: the data that the coder sends, the decoder receives without delay and without errors. The scheme of the discrete closed-loop system (2) is given in Figure 1.

To control the system (2) we use the quantized hybrid feedback that was introduced in [8]. We assume that (Φ, Γ) is stabilizable and let K be such that $\Phi + \Gamma K$ is Schur. Then the feedback is defined by the following equations:

$$u_k := \begin{cases} 0 & \text{if } \Omega_k^d = \Omega_{out} \\ K q_k^d & \text{if } \Omega_k^d = \Omega_{in}, \end{cases} \tag{3}$$

where the variable Ω_k^d determines the switching rules for the decoder. It can take only two strictly positive values Ω_{out} and Ω_{in}, that will be defined next. If $\Omega_k^d = \Omega_{out}$ we say that a zoom-out condition is triggered at the decoder at time k. If $\Omega_k^d = \Omega_{in}$ we say that a zoom-in condition is triggered at the decoder at time k. During the zoom-out stage of the decoder the system is running in an open loop: $u_k = 0$. During the zoom-in stage of the decoder the certainty equivalence feedback $u_k = K q_k^d$ is applied. The variable Ω_k^c determines the switching rules for the coder in the same manner as the variable Ω_k^d determines the switching rules for the decoder. The protocol dynamics is described by the following:

$$\mu_{k+1}^c := \begin{cases} \Omega_{out}\mu_k^c & \text{if } \Omega_k^c = \Omega_{out} \\ \Omega_{in}\mu_k^c & \text{if } \Omega_k^c = \Omega_{in} \end{cases} \quad \mu_0^c \in \mathbb{R}_{>0}. \tag{4}$$

$$\mu_{k+1}^d := \begin{cases} \Omega_{out}\mu_k^d & \text{if } \Omega_k^d = \Omega_{out} \\ \Omega_{in}\mu_k^d & \text{if } \Omega_k^d = \Omega_{in} \end{cases} \quad \mu_0^d \in \mathbb{R}_{>0}. \tag{5}$$

The adjustment policy for μ_k^c, μ_k^d can be thought of as implemented on both ends of the communication channel (at the coder and decoder) from some known initial values μ_0^c, μ_0^d.

Note, that in [8] it is assumed, that the coder and decoder are initialized at the same initial condition $\mu_0^c = \mu_0^d$, therefore $\mu_k^c = \mu_k^d$ for all $k = 0, 1, 2, \ldots$. We, on the other hand, assume that there may be a mismatch between the initial conditions at the coder and decoder and in our case, generally, $\mu_k^c \neq \mu_k^d$. In particular, we assume that the ratio (the mismatch) of the initial conditions at the coder and at the decoder is $r > 0$ (and not necessarily $r = 1$, as assumed in [8]):

$$r := \frac{\mu_0^c}{\mu_0^d}, \;\; \mu_0^c > 0, \;\; \mu_0^d > 0.$$

The adjustment policies for μ_k^c and μ_k^d are composed of two stages: a zoom-out stage and a zoom-in stage. During the zoom-out stage of the coder (respectively the decoder) the value of an adjustable parameter μ^c (respectively μ^d) is increased at the rate faster than the growth of $|x_k|$ until the state can be adequately measured. During the zoom-in stage of the coder (respectively the decoder) the value of an adjustable parameter μ^c (respectively μ^d) is decreased in such way as to drive the state to the origin. The hysteresis switching is used to switch between the zoom-in and zoom-out stages. It is described by the following:

$$\Omega_k^c := \begin{cases} \Omega_{out} & \text{if } |q_k^c| > l_{out}\mu_k^c \\ \Omega_{in} & \text{if } |q_k^c| < l_{in}\mu_k^c \\ \Omega_{k-1}^c & \text{if } |q_k^c| \in [l_{in}\mu_k^c, l_{out}\mu_k^c] \end{cases} \quad \Omega_0^c = \Omega_{out}. \quad (6)$$

$$\Omega_k^d := \begin{cases} \Omega_{out} & \text{if } |q_k^d| > l_{out}\mu_k^d \\ \Omega_{in} & \text{if } |q_k^d| < l_{in}\mu_k^d \\ \Omega_{k-1}^d & \text{if } |q_k^d| \in [l_{in}\mu_k^d, l_{out}\mu_k^d] \end{cases} \quad \Omega_0^d = \Omega_{out}. \quad (7)$$

where l_{out} and l_{in} are strictly positive numbers such that $l_{out} := M - \Delta$, $l_{in} := \Delta_M - \Delta$ and $\Delta_M > \Delta$, Δ_M will be defined later. Similarly to [8], we assume that the coder and decoder are initialized at the same synchronized stage:

Assumption 3 *Assume that*

$$\Omega_0^c = \Omega_0^d = \Omega_{out}.$$

Remark 1 *Note, that the evaluation of Ω_k^c and Ω_k^d do not require integration of any equation. The coder/decoder evaluators for Ω_k^c/Ω_k^d use the dynamic "look-up tables" (6)/(7) to set up the values of Ω_k^c/Ω_k^d based on the values of $q\left(\frac{x_k}{\mu_k^c}\right)$ and $q\left(\frac{x_k}{\mu_k^c}\right) + v_k$ respectively. Note, that due to Assumption 2, the values of both, Ω_k^c and Ω_k^d, depend on the same symbol $q\left(\frac{x_k}{\mu_k^c}\right)$. Therefore, if Ω_k^c and Ω_k^d initial stage is synchronized, then their synchronization is enforced at every time step.*

We assume that the coder and decoder evaluators for Ω_k^c and Ω_k^d are reliable, that there are no mistakes in the dynamic "look-up tables" for Ω_k^c/Ω_k^d at the coder and decoder.

Assumption 4 *Assume that*

$$\text{if for some } k \geq 0 \; v_k = 0 \text{ and } \Omega_k^c = \Omega_k^d,$$

$$\text{then } \Omega_{k+1}^c = \Omega_{k+1}^d.$$

Note, that if Assumptions 2 and 4 hold, the coder and decoder switching will be synchronized. Cancelling μ_k^c, μ_k^d in (6) and (7) we can conclude, that if the channel is perfect (Assumption 2 holds) and the coder/decoder evaluators for Ω_k^c/Ω_k^d are reliable (Assumption 4 holds), then the switching depends only on the value of $q\left(\frac{x_k}{\mu_k^c}\right)$. This can be interpreted as the fact that the switching is governed by the variable $\xi_k^c := \frac{x_k}{\mu_k^c}$ (see Remark below). Therefore, the coder and decoder switching conditions are the same. That is, the coder and decoder switching will be synchronized: if the coder is zooming-in, then the decoder is zooming-in; and vice versa.

Remark 2 *Consider the switching conditions for the coder. Note, that whenever $\left|\frac{x_k}{\mu_k^c}\right| < l_{in} - \Delta$ holds, $\left|\mu_k^c q\left(\frac{x_k}{\mu_k^c}\right)\right| < l_{in}\mu_k^c$ holds. Also, the zoom-out switching condition $\left|\mu_k^c q\left(\frac{x_k}{\mu_k^c}\right)\right| > l_{out}\mu_k^c$ implies that $\left|\frac{x_k}{\mu_k^c}\right| > l_{out} + \Delta$. The same observation holds for the decoder.*

Due to Assumption 2, in the sequel we treat the decoder quantized measurements as $q_k^d = \mu_k^d q\left(\frac{x_k}{\mu_k^c}\right)$.

Next we present a straightforward result (Proposition 1 below), that guarantees that if (i) the channel is perfect (Assumption 2 holds); (ii) the coder and decoder are initialized at the same synchronized stage (Assumption 3 holds); (iii) the coder/decoder evaluators for Ω_k^c/Ω_k^d are reliable (Assumption 4 holds); then the coder and decoder stage will be always synchronized.

Proposition 1 *Suppose Assumption 2 - 4 hold. Then*

$$\Omega_k^c = \Omega_k^d \quad \forall k \geq 0.$$

The proof of Proposition 1 is by induction and not presented here.

Remark 3 *Note, that the difference of Proposition 1 from Assumption 4 is that Assumption 4 guarantees the synchronized stage of coder and decoder only for one step ahead. Proposition 1, on the other hand, guarantees that if the coder and decoder stage is synchronized at some point of time, it will be synchronized for all future time. In other words, if Ω_0^c and Ω_0^d are synchronized at the first step, then the synchronization of Ω_k^c and Ω_k^d is enforced at each time step.*

Remark 4 *It is possible to show that bounds valid at sampling instants, can be extended for all time $t \in [t_k, t_{k+1}]$. We will analyze only the stability properties of the discrete-time system (2) with (3) - (7) induced by the sampled-data system (1). It was shown in [12] how to use the underlying discrete-time model to conclude appropriate stability properties of the sampled-data system.*

We introduce some notation. Due to Assumptions 2 - 3 the coder and decoder switching will be synchronized. We have that $\Omega_k^c = \Omega_k^d$ for all $k \geq 0$. We introduce $k_j \in \mathbb{N}$ such that

$$\Omega_k^c = \Omega_k^d = \Omega_{out} \text{ if } k \in [k_{2i}, k_{2i+1} - 1], \quad i = 0, 1, 2, \ldots, N$$
$$\Omega_k^c = \Omega_k^d = \Omega_{in} \text{ if } k \in [k_{2i+1}, k_{2i+2} - 1],$$

That is: k_{2i+1} is the time instant at which the coder and decoder switch from the zoom-out stage to the zoom-in stage; k_{2i+2} is the time instant at which the coder and decoder switch from the zoom-in stage to the zoom-out stage. We assume that $k_0 = 0$ and that the first interval is always the zoom-out. We will adjust the quantizer, coder, decoder and controller so that $N = 0$. In other words, the coder and decoder will zoom-out for $k \in [0, k_1 - 1]$ and zoom-in for all $k \geq k_1$.

To understand the operation of the plant (2) we need to consider two modes of the operation of the plant: **Mode 1.** The coder and decoder are zooming-out; **Mode 2.** The coder and decoder are zooming-in. The plant dynamics during each mode is considered in full details in Lemmas 1 and 2 in Section 4. Lemmas 1 and 2 show that if the quantizer, coder, decoder and controller are appropriately adjusted, then the following holds: *i* Mode 1 can happen only on the first zooming interval, after which the system switches to Mode 2; *ii* If Mode 2 happens then system stays in Mode 2 for all future time. The dynamics of the plant during Modes 1 and 2 is described below.

Mode 1: $k \in [0, k_1 - 1]$. The coder and decoder are zooming-out. During this mode the system dynamics is described by the following equations:

$$x_{k+1} = \Phi x_k, \quad x_0 \in \mathbb{R}^n, \tag{8}$$

$$\mu^c_{k+1} = \Omega_{out}\mu^c_k, \quad \mu^c_0 > 0,$$
$$\mu^d_{k+1} = \Omega_{out}\mu^d_k, \quad \mu^d_0 > 0.$$

The dynamics of ξ^c_k is described by the following equation:

$$\xi^c_{k+1} = \frac{1}{\Omega_{out}}\Phi\xi^c_k. \tag{9}$$

Note that during this mode the ratio $\frac{\mu^c_k}{\mu^d_k} = \frac{\Omega^k_{out}\mu^c_0}{\Omega^k_{out}\mu^d_0} = \frac{\mu^c_0}{\mu^d_0} = r$ stays constant for all $k \in [0, k_1]$. Our Lemma 1 show that $k_1 \leq \left\lfloor \frac{1}{\ln(\Omega_{out}/\|\Phi\|)} \ln\left(\frac{|\xi^c_0|}{l_{in} - \Delta}\right)\right\rfloor$.

Mode 2: $k \geq k_1$. The coder and decoder are zooming-in. During this mode the system dynamics is described by the following equations:

$$x_{k+1} = \Phi x_k + \Gamma K \mu^d_k q\left(\frac{x_k}{\mu^c_k}\right), \tag{10}$$

$$\mu^c_{k+1} = \Omega_{in}\mu^c_k,$$
$$\mu^d_{k+1} = \Omega_{in}\mu^d_k.$$

The dynamics of ξ^c_k is described by the following equation:

$$\xi^c_{k+1} = \frac{1}{\Omega_{in}}\Phi\xi^c_k + \frac{1}{\Omega_{in}}\Gamma K \frac{\mu^d_k}{\mu^c_k} q(\xi^c_k). \tag{11}$$

Note that during this mode the ratio $\frac{\mu^c_k}{\mu^d_k} = \frac{\Omega^{k-k_1}_{in}\mu^c_{k_1}}{\Omega^{k-k_1}_{in}\mu^d_{k_1}} = \frac{\mu^c_{k_1}}{\mu^d_{k_1}} = \frac{\mu^c_0}{\mu^d_0} = r$ stays constant for all $k \geq k_1$.

Adding and subtracting $\frac{1}{\Omega_{in}}\Gamma K \xi^c_k$, $\frac{1}{\Omega_{in}}\Gamma K q(\xi^c_k)$ terms to the equation (11), we can say, that during Mode 2 the system dynamics for ξ^c_k satisfies the following:

$$\xi^c_{k+1} = \frac{1}{\Omega_{in}}(\Phi + \Gamma K)\xi^c_k + \frac{1}{\Omega_{in}}\Gamma K \bar{\nu}_k, \tag{12}$$

where $\bar{\nu}_k = \nu^c_k + (\frac{1}{r} - 1)q(\xi^c_k)$ and $\nu^c_k = q(\xi^c_k) - \xi^c_k$.

Note that $|q(\xi^c_k)| \leq M_1 = L_q M$ by the forth condition of Assumption 1. Also during the zoom-in stage $|\xi^c_k| \leq l_{out} + \Delta = M$, and by the second condition of Assumption 1 we have that $|q(\xi^c_k) - \xi^c_k| \leq \Delta$. Therefore, we have that during the zoom-in stage $|\bar{\nu}_k|$ is bounded:

$$|\bar{\nu}_k| \leq |q(\xi^c_k) - \xi^c_k| + \left|\frac{1}{r} - 1\right| |q(\xi^c_k)| \leq \Delta + \left|\frac{1}{r} - 1\right| L_q M. \tag{13}$$

Also, when the initial conditions at the coder and at the decoder are the same ($\mu^c_0 = \mu^d_0$), the ξ^c_k dynamics satisfies (12) with $\bar{\nu}_k = \nu^c_k$ (since $r = \mu^c_0/\mu^d_0 = 1$ in this case).

Now we can state the following results, that are similar to Lemma III.2 and Corollary III.3 from [8]. The first result follows directly from [4], Example 3.4.

Corollary 1 *Suppose that $\Phi + \Gamma K$ is Schur. Then, there exists an $\Omega^*_{in} \in (0,1)$ such that for all $\Omega_{in} \in [\Omega^*_{in}, 1)$, $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur. Moreover, for any such Ω_{in}, there exist strictly positive K_1, λ, γ such that the solutions of the system (12) satisfy the following $\forall k \in [k_{2i+1}, k_{2i+2}]$:*

$$|\xi^c_k| \leq K_1 \exp(-\lambda(k - k_{2i+1}))|\xi^c_{k_{2i+1}}| + \gamma\|\bar{\nu}\|. \tag{14}$$

In particular, let $\kappa > 0$ and $\sigma \in (0,1)$ be such that[1] *$\|\frac{1}{\Omega^k_{in}}(\Phi + \Gamma K)^k\| \leq \kappa\sigma^k$ for all $k \geq 0$. Then, we can let*

$$K_1 = \kappa, \quad \lambda = -\ln(\sigma), \quad \gamma = \frac{\kappa\|\Gamma K\|}{\Omega_{in}(1 - \sigma)}. \tag{15}$$

Note, that if all conditions of Corollary 1 hold, then (12) is a sum of a stable first term (since $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur) and a bounded second term (due to (13) during the zoom-in stage).

Corollary 2 *Suppose*

$$\left|\frac{\mu^d_0 - \mu^c_0}{\mu^c_0}\right| < \frac{1}{\gamma L_q}. \tag{16}$$

[1] These numbers always exist since $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur

Let Ω_{in}, K_1, γ come from Corollary 1 and let strictly positive M and Δ be such that the following holds:

$$M > \frac{(2+K_1+\gamma)\Delta}{1-\gamma\left|\frac{1}{r}-1\right| L_q}. \tag{17}$$

Then there exists a $\Delta_M > 0$ with $\Delta_M - \Delta > 0$, such that whenever $|\xi^c_{k_{2i+1}}| \leq \Delta_M$ and $|\nu_k| \leq \Delta$ the following two properties hold for all $k \in [k_{2i+1}, k_{2i+2}]$:

$$|\xi^c_k| \leq M \tag{18}$$

and

$$|q(\xi^c_k)| \leq M - \Delta. \tag{19}$$

Proof of Corollary 2. First we show the necessity of the condition (16) for the proof. Suppose (17) holds. This condition can be re-written as

$$(2+K_1+\gamma)\Delta < M\left(1-\gamma\left|\frac{1}{r}-1\right| L_q\right). \tag{20}$$

Since K_1, γ, Δ, M are positive numbers, in order for (20) to hold, the following condition has to hold:

$$1-\gamma L_q\left|\frac{1}{r}-1\right| > 0, \tag{21}$$

which can be re-written as a condition (16).

Now we show that if all conditions of Corollary 2 hold, then for all $k \in [k_{2i+1}, k_{2i+2}]$ ξ^c_k and the quantized measurements of the state are in the range of the quantizer. Suppose all conditions of Corollary 2 hold. Then for all $k \in [k_{2i+1}, k_{2i+2}]$ the ξ^c_k dynamics satisfy the following:

$$\begin{aligned} |\xi^c_k| &\leq K_1 \exp(-\lambda(k-k_{2i+1}))|\xi^c_{k_{2i+1}}| + \gamma\|\bar{\nu}\| \\ &\leq K_1|l_{in} - \Delta| + \gamma(\Delta + \left|\frac{1}{r}-1\right| L_q M) \\ &\leq K_1\Delta_M + \gamma\Delta + \gamma\left|\frac{1}{r}-1\right| L_q M \\ &\leq M. \end{aligned} \tag{22}$$

The last inequality above comes from the following fact. Since (20) is a strict inequality, there exist Δ_M arbitrary close to Δ with $\Delta_M > \Delta$, such that the following holds:

$$\Delta + K_1\Delta_M + \gamma\Delta < M - M\gamma\left|\frac{1}{r}-1\right| L_q - \Delta.$$

Now we can write the following for all $k \in [k_{2i+1}, k_{2i+2}]$:

$$\begin{aligned} |q(\xi^c_k)| &= |q(\xi^c_k) - \xi^c_k + \xi^c_k| \\ &\leq \Delta + K_1\Delta_M + \gamma\Delta + \gamma\left|\frac{1}{r}-1\right| L_q M \\ &\leq M - M\gamma\left|\frac{1}{r}-1\right| L_q - \Delta + \gamma\left|\frac{1}{r}-1\right| L_q M \\ &\leq M - \Delta. \end{aligned}$$

This completes the proof. ■

4 Stability

In this section we present our main result, Theorem 1, that shows that if the channel is perfect (Assumption 2 holds), the coder and decoder are initialized at the same synchronized stage (Assumption 3 holds), the coder/decoder evaluators for Ω^c_k/Ω^d_k are reliable and the bound on the mismatch between the initial conditions at the coder and decoder holds, then it is possible to design the quantizer, coder/decoder and controller such that the closed-loop system (2) - (7) is stable.

Definition 3 *The system (2) is Globally Exponentially Stable [2] (GES) in x if for a fixed $\mu^c_0 > 0, \mu^d_0 > 0$ with $\mu^c_0/\mu^d_0 = r$ there exists $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathbb{R}$ and we have:*

$$|x_k| \leq \varphi(|x_0|) \ \ \forall k \geq 0 \tag{23}$$

and $|x_k| \to 0$ as $k \to \infty$ exponentially fast.

Remark 5 *Note that $\varphi(|x_0|)$ depends on μ^c_0 and μ^d_0.*

Definition 4 *The system $x_{k+1} = Ax_k + Dw_k$, $x_0 \in \mathbb{R}^n$, where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^l$ are respectively the state and the disturbance, is said to be Input-to-State Stable (ISS) with a linear gain $\tilde{\gamma} > 0$ if for every initial condition $x_0 \in \mathbb{R}$ and every bounded disturbance w there exist positive $\hat{K}, \hat{\lambda}$ such that we have:*

$$|x_k| \leq \hat{K}\exp(-\hat{\lambda}(k-k_0))|x_0| + \tilde{\gamma}\|w\| \quad \forall k > k_0.$$

The main contribution of our work is the following theorem, which shows that the system (2) with (3) - (7) is GES in the sense of our non-standard Definition 3 if the mismatch between μ^c_0 and μ^d_0 is sufficiently small.

Theorem 1 *Consider the system (2) with (3) - (7), when $\mu^c_0/\mu^d_0 = r > 0$. Let q be a quantizer fulfilling Assumption 1. Suppose Assumptions 2 - 4 hold and for a given sampling period $T > 0$ the pair (Φ, Γ) is stabilizable. Let*

(i) K be such that $\Phi + \Gamma K$ is Schur,

(ii) $\Omega_{in} \in (0,1)$ be such that $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur,

[2] Note that this is not widely used standard definition. Here we talk only about the stability of the state x of the plant, not μ. Also an overshoot may depend on the initial condition.

(iii) Ω_{out} *be such that* $\Omega_{out} > \|\Phi\|$*,*

(iv) $\left|\frac{\mu_0^d - \mu_0^c}{\mu_0^c}\right| < \frac{1}{\gamma L_q}$*, where* L_q *comes from Assumption 1 and* γ *is defined in (15),*

(v) M *and* Δ *in Assumption 1 be such that* $M > \frac{(2+K_1+\gamma)\Delta}{1-\gamma\left|\frac{1}{r}-1\right|L_q}$*, where* K_1*,* γ *are defined in (15),*

(vi) $l_{out} = M - \Delta$*,*

(vii) $l_{in} = \Delta_M - \Delta$*, where* Δ_M *comes from Corollary 2.*

Then, the system (2) is GES in x*.*

The proof of Theorem 1 is given in the appendix. The proof is a direct consequence of the fact that the system during Mode 2 behaves as a cascade of ISS $x-$subsystem and GES μ^c-, μ^d- subsystems.

Remark 6 *The first item of Theorem 1 requires, that the system is stabilizable with a certainty-equivalence controller; the second is a condition on how slow the* μ^c, μ^d*-subsystems have to be during the zoom-in stage; the third is a condition on how fast the* μ^c, μ^d*-subsystems have to be during the zoom-out stage; the forth is the bound on the mismatch between the initial conditions at the coder and decoder; the fifth is a condition on the data-rate of the channel; the sixth and seventh are the conditions on the switching parameters.*

Remark 7 *The fifth item of Theorem 1 (which is the condition (17) from Section 3) means that the range of the quantizer* M *has to be large enough compared to the quantization error* Δ *(i.e. the quantizer takes sufficiently many levels). Note that when the initial conditions at the coder and at the decoder are the same (*$\mu_0^c = \mu_0^d$*, i.e.* $r = 1$*), the condition on the data rate*

$$M > (2 + K_1 + \gamma)\Delta \tag{24}$$

for the system (12) with $\bar{\nu}_k = \nu_k^c$ *(which is the condition used in [8]) can be recovered from (17). On the other hand, since (24) is a strict inequality, whenever it holds, there exists* r *sufficiently close to one, such that the forth and the fifth items of Theorem 1 hold (conditions (16) and (17) from Section 3). Hence, this implies that the scheme, proposed in [8] has some intrinsic robustness properties with respect to the mismatch between the initialization at the coder and decoder.*

Remark 8 *Note that the forth condition of Theorem 1 shows a relationship between a ratio (mismatch) of the initial conditions at the coder and at the decoder* r*, the robustness measure (gain)* γ *of the plant and the quantizer characteristics* L_q*. It shows that for a fixed* L_q*, when the gain is large, the smaller mismatch can be tolerated. Also when the gain is small, the large mismatch can be tolerated. Note that without loss of generality we can assume that* $L_q = 1$*, since many quantizers satisfy this property.*

5 Conclusions

This paper is the first investigation of the problem of robustness of linear control systems with quantized control with respect to the computational errors at the coder and decoder. In this paper we analyze the stability of the quantized control systems when the data is transmitted via a perfect channel and the coder and decoder are initialized at different initial conditions. Using a trajectory-based scheme proposed in [8], under the assumption that the channel is perfect, we derived the bound on the mismatch between the initial conditions at the coder and decoder that can be tolerated in order to achieve GES. We believe that similar results can be proven for other quantized control schemes published in the literature. An interesting future research topic will be to analyze the robustness properties with respect to the initial coder/decoder mismatch of the nonlinear systems and systems with input disturbances.

Appendix

The Proof of Theorem 1 consists of Lemmas 1 and 2. These lemmas capture the dynamics of the system during two modes considered in the end of Section 3. The first lemma considers the plant dynamics during **Mode 1**. It claims that if the initial conditions are such that the zoom-out is triggered initially at both the coder and decoder, then both of them will switch to the zoom-in stage in the same finite time.

Lemma 1 *Consider the system (2) with (3) - (7). Suppose all conditions of Theorem 1 hold. Suppose the initial conditions are such that the zoom-out stage is triggered at both the coder and decoder* ***(Mode 1)****. Then there exists* $k_1 > 0$ *such that*

$$\Omega_{k_1}^c = \Omega_{k_1}^d = \Omega_{in}.$$

Moreover,

$$k_1 \leq \left\lfloor \frac{1}{\ln(\Omega_{out}/\|\Phi\|)} \ln\left(\frac{|\xi_0^c|}{l_{in} - \Delta}\right) \right\rfloor.$$

Proof of Lemma 1. Suppose that the initial conditions are such that the zoom-out stage is triggered at both the coder and decoder. The system dynamics for ξ_k^c during this mode (during Mode 1) satisfies the following for all $k \in [0, k_1 - 1]$:

$$|\xi_{k+1}^c| \leq \frac{1}{\Omega_{out}} \|\Phi\| |\xi_k^c| = \lambda |\xi_k^c|,$$

where $\lambda = \|\Phi\|/\Omega_{out} < 1$ since $\Omega_{out} > \|\Phi\|$ by design. We have that ξ_k^c is decreasing as $k \to \infty$ and there exists a time $k_1 > 0$ such that we have:

$$|\xi_{k_1}^c| \leq \lambda^{k_1}|\xi_0^c| < l_{in} - \Delta.$$

Hence the zoom-in stage is triggered at the coder and decoder at the time k_1, since $|\xi_{k_1}^c| < l_{in} - \Delta$ implies that $\left|\mu_{k_1}^c q(\xi_{k_1}^c)\right| < l_{in}\mu_{k_1}^c$ and $\left|\mu_{k_1}^d q(\xi_{k_1}^c)\right| < l_{in}\mu_{k_1}^d$ (see Remark 2). Also for a given $\lambda < 1$ there exists $a = \ln(1/\lambda)$ such that

$$\lambda^{k_1} = (\exp(-a))^{k_1} = \exp(-ak_1).$$

Then we can write:

$$k_1 \leq \left\lfloor -\frac{1}{a}\ln\left(\frac{l_{in}-\Delta}{|\xi_0^c|}\right)\right\rfloor = \left\lfloor \frac{1}{\ln(\Omega_{out}/\|\Phi\|)}\ln\left(\frac{|\xi_0^c|}{l_{in}-\Delta}\right)\right\rfloor.$$

From the above we can conclude the following: if $\mu_0^c/\mu_0^d = r > 0$ then there exists a time k_1 such that the zoom-in stage is triggered at both the coder and decoder at the same time k_1. Therefore, the plant will be in Mode 2. ■

The next lemma considers the plant dynamics during **Mode 2**. It claims that if the zoom-in stage is triggered at both the coder and decoder, then the coder and decoder will always stay in the zoom-in stage.

Lemma 2 *Consider the system (2) with (3) - (7). Suppose all conditions of Theorem 1 hold. Then*

$$\Omega_k^c = \Omega_k^d = \Omega_{in}$$

for all $k \geq k_1$, where k_1 comes from Lemma 1.

Proof of Lemma 2. Suppose the zoom-in stage is triggered at both the coder and decoder. The proof will be carried out by contradiction. Suppose there exists a time $k_2 > k_1$ such that $|q(\xi_{k_2})| > l_{out}$. That is: k_2 is the minimum time when the coder and decoder switch from the zoom-in stage to the zoom-out stage. Note that this implies that $|\xi_{k_2}| > l_{out} + \Delta$ (see Remark 2).

By Corollary 2 the ξ_k^c dynamics satisfies (22) during Mode 2 and we have that the following holds for all $k \in [k_1, k_2]$:

$$|\xi_k| \leq K_1\Delta_M + \gamma\Delta + \gamma\left|\frac{1}{r} - 1\right| L_q M \leq M = l_{out} + \Delta.$$

We have that $\forall k \in [k_1, k_2]$ $|\xi_k| \leq l_{out} + \Delta$, which contradicts to our assumption that $|\xi_{k_2}| > l_{out} + \Delta$. Hence, there does not exist a time k_2 such that $|q(\xi_{k_2})| > l_{out}$ (see Remark 2) and the zoom-out stage is triggered at the coder and decoder. We can conclude that once the coder and decoder switch to the zoom-in stage, if all conditions of Theorem 1 hold, then the coder and decoder will be zooming-in for all future time. ■

The proof of Theorem 1 below is a direct consequence of the fact that the system during Mode 2 behaves as a cascade of ISS $x-$subsystem and GES μ^c-, μ^d- subsystems.

Proof of Theorem 1. Suppose the zoom-in stage is triggered initially. By Lemma 2, the coder and decoder will be in the zoom-in stage for all future time $k \geq k_1$. Adding and subtracting some terms to (10), we can say that during Mode 2 the $x-$dynamics of the plant evolve according to the following for all $k \geq k_1$:

$$x_{k+1} = (\Phi+\Gamma K)x_k + \Gamma K\mu_k^d\left(q\left(\frac{x_k}{\mu_k^c}\right) - \frac{x_k}{\mu_k^c} + \frac{x_k}{\mu_k^c} - \frac{x_k}{\mu_k^d}\right)$$

Note, that $\mu_k^d < \mu_{k_1}^d$ $\forall k > k_1$ since μ_k^d is decreasing during the zoom-in stage. Also $\mu_{k_1}^d = \Omega_{out}^{k_1}\mu_0^d$, where $k_1 \leq \left\lfloor \frac{1}{\ln(\Omega_{out}/\|\Phi\|)}\ln\left(\frac{|\xi_0^c|}{l_{in}-\Delta}\right)\right\rfloor$.

Also since $\Omega_k^c = \Omega_k^d$ for all $k \geq 0$, we have that $\frac{\mu_0^c}{\mu_0^d} = \frac{\mu_k^c}{\mu_k^d} = r$, therefore we can write that $\frac{x_k}{\mu_k^d} = \frac{x_k}{\mu_k^c}r$. We have the following during Mode 2:

$$x_{k+1} = (\Phi+\Gamma K)x_k + \Gamma K\Omega_{out}^{k_1}\mu_0^d\left(q\left(\frac{x_k}{\mu_k^c}\right) - \frac{x_k}{\mu_k^c} + \frac{x_k}{\mu_k^c} - \frac{x_k}{\mu_k^c}r\right)$$

$$= (\Phi+\Gamma K)x_k + \Gamma K\Omega_{out}^{k_1}\mu_0^d\left(q\left(\frac{x_k}{\mu_k^c}\right) - \frac{x_k}{\mu_k^c} + (1-r)\frac{x_k}{\mu_k^c}\right). \tag{25}$$

During Mode 2 (during the zoom-in stage) $|\frac{x_k}{\mu_k^c}| \leq l_{out} + \Delta = M$, therefore we can use the second condition of Assumption 1 to conclude that $|\nu_k| = |q\left(\frac{x_k}{\mu_k^c}\right) - \frac{x_k}{\mu_k^c}| \leq \Delta$ and that $|1-r||\frac{x_k}{\mu_k^c}| \leq |1-r|M$. Therefore, (25) is a sum of a stable first term (since $\Phi + \Gamma K$ is Schur) and a bounded second term. Therefore, we can write for all $k \geq k_1$:

$$|x_k| \leq \bar{K}\exp(-\bar{\lambda}(k-k_1))|x_{k_1}| + \bar{\gamma}(\|\nu\| + |1-r|\|\xi^c\|).$$

In particular, let $\bar{\kappa} > 0$ and $\bar{\sigma} \in (0,1)$ be such that $\|(\Phi + \Gamma K)^k\| \leq \bar{\kappa}\bar{\sigma}^k$ for all $k \geq 0$. Then let

$$\bar{K} = \bar{\kappa}, \quad \bar{\lambda} = -\ln(\bar{\sigma}), \quad \bar{\gamma} = \frac{\bar{\kappa}\|\Gamma K\|\Omega_{out}^{k_1}\mu_0^d}{1-\bar{\sigma}},$$

where k_1 in the formulae for $\bar{\gamma}$ is a function of x_0 and μ_0^c. Therefore, the gain $\bar{\gamma}$ depends on x_0, μ_0^c, μ_0^d. We can say that the x-dynamics is ISS when ν_k, ξ_k^c are considered as inputs. Combining this fact and the fact that μ^c, μ^d dynamics are GES, we have a cascade of ISS x-subsystem and GES μ^c-, μ^d- subsystems. Hence, there exist $\tilde{K}, \tilde{\lambda}$ such that for all $k \geq 0$ we have:

$$|(x_k, \mu_k^c, \mu_k^d)| \leq \tilde{K}\exp(-\tilde{\lambda}k)|(x_0, \mu_0^c, \mu_0^d)|.$$

On the other hand, if the zoom-out stage is triggered initially, then by Lemma 1 for any x_0, μ_0^c, μ_0^d there exists a $k_1 := k_1(|x_0|, \mu_0^c)$, such that $|\frac{x_{k_1}}{\mu_{k_1}^c}| < l_{in} - \Delta$ and hence the zoom-in condition is triggered. Hence, for all $k \geq k_1 + 1$ we have

$$|(x_k, \mu_k^c, \mu_k^d)| \leq \tilde{K} \exp(-\tilde{\lambda}(k - k_1))|(x_{k_1}, \mu_{k_1}^c, \mu_{k_1}^d)|. \tag{26}$$

Moreover, for all $k \in [0, k_1]$ we have

$$|x_k| \leq |\Phi|^{k_1(|x_0|,\mu_0^c)}|x_0| := \gamma_1(|x_0|, \mu_0^c) \tag{27}$$

$$|\mu_k^c| \leq |\Omega_{out}|^{k_1(|x_0|,\mu_0^c)}|\mu_0^c| \leq \gamma_c(|x_0|, \mu_0^c) \tag{28}$$

$$|\mu_k^d| \leq |\Omega_{out}|^{k_1(|x_0|,\mu_0^c)}|\mu_0^d| \leq \gamma_d(|x_0|, \mu_0^c, \mu_0^d). \tag{29}$$

Combining bounds (26)-(29), we can write for all $k \geq 0$:

$$|x_k| \leq \exp(-\tilde{\lambda}k)\tilde{K} \exp(\tilde{\lambda}k_1(|x_0|, \mu_0^c)) \cdot$$

$$\sqrt{\gamma_1^2(|x_0|, \mu_0^c) + \gamma_c^2(|x_0|, \mu_0^c) + \gamma_d^2(|x_0|, \mu_0^c, \mu_0^d)},$$

which shows that x_k converges to zero exponentially. The proof would be complete if we had

$$\tilde{K} \exp(\tilde{\lambda}k_1(0, \mu_0^c)) \cdot$$

$$\sqrt{\gamma_1^2(|x_0|, \mu_0^c) + \gamma_c^2(|x_0|, \mu_0^c) + \gamma_d^2(|x_0|, \mu_0^c, \mu_0^d)} = 0$$

but this is not true since $\gamma_c(0, \mu_0^c) \neq 0$ and $\gamma_d(0, \mu_0^c, \mu_0^d) \neq 0$ for any $\mu^c > 0, \mu^d > 0$. In order to prove stability, we use Assumption 1 to prove that there exists a continuous and bounded $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\varphi(0, \mu^c, \mu^d) = 0$ so that (23) holds. With these properties, there is no loss of generality in taking $\varphi(\cdot, \mu^c, \mu^d) \in \mathcal{K}_\infty$ for any fixed $\mu^c, \mu^d > 0$ (just bound the original function with the $\mathcal{K}_\infty$ one). Let an arbitrary $\rho > 0$ be given and introduce

$$T^{k_1} := \max\left\{\left\lceil \ln\left(\frac{\rho|x_{k_1}|}{M}\right)(\ln(\Omega_{in}))^{-1}\right\rceil, 0\right\}.$$

Then we have for all $k \geq k_1 + T^{k_1}$ that

$$|x_k| \leq M\mu_k^c = M\Omega_{in}^{k-k_1}\mu_{k_1}^c \leq M\Omega_{in}^{T^{k_1}}\mu_{k_1}^c \leq \rho\mu_{k_1}^c|x_{k_1}|$$

$$=: \chi_c(|x_{k_1}|, \mu_{k_1}^c)$$

and

$$|x_k| \leq M\mu_k^d = M\Omega_{in}^{k-k_1}\mu_{k_1}^d \leq M\Omega_{in}^{T^{k_1}}\mu_{k_1}^d \leq \rho\mu_{k_1}^d|x_{k_1}|$$

$$=: \chi_d(|x_{k_1}|, \mu_{k_1}^d).$$

Due to the fact, that Assumption 1 guarantees that there exists an $L_q > 0$ such that $|q(z)| \leq L_q|z|$ for all z, for $k \in [k_1, k_1 + T^{k_1}]$ we can write:

$$|x_k| \leq (\|\Phi\| + \|\Gamma K\|L_q)^{T^{k_1}}|x_{k_1}| =: \chi(|x_{k_1}|).$$

Since $\|\Phi\| > 1$ and $\Omega_{in} < 1$, $\chi(0) = 0$ and $\chi(s)$ is bounded for all $s \geq 0$. Hence, we can bound it by $\chi_1 \in \mathcal{K}_\infty$. Finally, we define

$$\tilde{\varphi}(|x|, \mu^c, \mu^d) := \max\{\chi_1(|x|), \chi_c(|x|, \mu^c), \chi_d(|x|, \mu^d)\}.$$

We have that $\tilde{\varphi}(0, \mu^c, \mu^d) = 0$ and is increasing in all arguments. Hence, we can write that for all $k \geq k_1$:

$$|x_k| \leq \tilde{\varphi}(|x_{k_1}|, \mu_{k_1}^c, \mu_{k_1}^d) \leq \tilde{\varphi}(\gamma_1, \gamma_c, \gamma_d) =: \bar{\varphi}(|x_0|, \mu_0^c, \mu_0^d).$$

Note that $\bar{\varphi}(0, \mu_0^c, \mu_0^d) = 0$ for any $\mu^c, \mu^d > 0$. Finally, the conclusion (23) follows by noting that there exist a φ with the right properties such that

$$\varphi(s, \mu^c, \mu^d) \geq$$

$$\max\{\bar{\varphi}(s, \mu^c, \mu^d), \gamma_1(s, \mu^c), \gamma_c(s, \mu^c), \gamma_d(s, \mu^c, \mu^d)\}$$

for all s, μ^c, μ^d. ■

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