Output Feedback Event-Triggered Control

Chapter · June 2016

DOI: 10.1007/978-3-319-32372-5_7

Mahmoud Abdelrahim

Assiut University

Romain Postoyan

French National Centre for Scientific Research

Jamal Daafouz

University of Lorraine

D. Nesic

University of Melbourne

Output feedback event-triggered control

Mahmoud Abdelrahim, Romain Postoyan, Jamal Daafouz and Dragan Nešić

Abstract Event-triggered control has been proposed as an alternative implementation to conventional time-triggered approach in order to reduce the amount of transmissions. The idea is to adapt transmissions to the state of the plant such that the loop is closed only when it is needed according to the stability or/and the performance requirements. Most of the existing event-triggered control strategies assume that the full state measurement is available. Unfortunately, this assumption is often not satisfied in practice. There is therefore a strong need for appropriate tools in the context of output feedback control. Most existing works on this topic focus on linear systems. The objective of this chapter is to first summarize our recent results on the case where the plant dynamics is nonlinear. The approach we follow is emulation as we first design a stabilizing output feedback law in the absence of sampling then we consider the network and we synthesize the event-triggering condition. The latter combines techniques from event-triggered and time-triggered control. The results are then proved to be applicable to linear time-invariant (LTI) systems as a particular case. We then use these results as a starting point to elaborate a co-design method, which allows us to jointly construct the feedback law and the triggering condition for LTI systems where the problem is formulated in terms of linear matrix inequalities (LMI). We then exploit the flexibility of the method to maximize the guaranteed minimum amount of time between two transmissions. The results are illustrated on physical and numerical examples.

M. Abdelrahim, R. Postoyan and J. Daafouz
Université de Lorraine, CRAN, UMR 7039 and CNRS, CRAN, UMR 7039, France, e-mail: `{othmanab1,romain.postoyan,jamal.daafouz}@univ-lorraine.fr`. Their work is partially supported by the ANR under the grant COMPACS (ANR-13-BS03-0004-02).

D. Nešić
Department of Electrical and Electronic Engineering, the University of Melbourne, Parkville, VIC 3010, Australia, e-mail: `dnesic@unimelb.edu.au`. His work is supported by the Australian Research Council under the Discovery Projects and Future Fellowship schemes.

1 Introduction

In many control applications nowadays, the plant and the controller communicate with each other via a shared communication network. This architecture is referred to as *networked control systems* (NCS) and offers great benefits compared to the conventional point-to-point connection in terms of lighter wiring, lower installation costs, greater abilities for diagnosis, flexible reconfiguration and ease of maintenance. A major challenge in NCS is to achieve the control objectives despite the communication constraints induced by the network (like time-varying sampling, delay, packet dropout, etc.). Since the network may be shared by other applications, it is desirable in practice to reduce the usage of the network.

In conventional setups, data transmissions are time-driven and two successive transmission instants are constrained to be less than a fixed constant, called the *maximum allowable transmission interval* (MATI) (see *e.g.* [26], [16]). Although this strategy is appealing from the analysis and the implementation point of views, it is not obvious that time-triggering is always appropriate for NCS. Indeed, the same amount of transmissions per unit of time is generated under this paradigm, even when transmissions are not necessary in view of the control objectives. To overcome this shortcoming, event-triggered control has been proposed as an alternative [6], [7].

Event-triggered control is an implementation technique in which the transmission instants are defined based on a state-dependent criterion. The idea is to adapt the amount of transmissions according to the system state such that the feedback loop is closed only when it is needed in view of the stability and/or performance requirements. This may significantly reduce the amount of transmissions compared to the time-triggering paradigm, see *e.g.* [14, 20, 24] and the references therein. A fundamental issue in event-triggered implementation is to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. The existence of such a lower bound on the inter-transmission times is not only useful to prove stability but this requirement is also essential to prevent the occurrence of Zeno phenomenon, *i.e.* to avoid the generation of an infinite number of transmissions in a finite time, as well as to respect the hardware constraints.

Most of the existing results on event-triggered control assume that the full state measurement is available and can be used for feedback, see *e.g.* [14, 15, 21]. This is not realistic in many applications since in practice we often have access to an output of the plant and not to the full state. It appears that the design of output feedback event-triggered controllers is much more challenging, in particular because it is more difficult to ensure the existence of a minimum amount of time between two control input updates compared to the state feedback case, see [8]. Few results in the literature have addressed this problem and mostly for linear systems. To the best of our knowledge, this problem has been first investigated in [12] and then in *e.g.* [8, 9, 18, 25, 28] for LTI systems and only in [27] for nonlinear systems.

The purpose of this chapter is to explain how to synthesize stabilizing output feedback event-triggered controllers for a class of nonlinear systems. We first design the event-triggered controllers by emulation, *i.e.* a stabilizing feedback law is first constructed in the absence of network and then the triggering condition is synthesized to preserve stability. The design objectives are to guarantee a global asymptotic stability property and to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. The proposed strategy combines the event-triggering condition of [24] adapted to output measurements and the results on time-driven sampled-data systems in [17]. Indeed, the event-triggering condition is only (continuously) evaluated after T units of times have elapsed since the last transmission, where T corresponds to the MATI given by [17]. This two-step procedure is justified by the fact that the adaption of the event-triggering condition of [24] to output feedback on its own can lead to the Zeno phenomenon. It has to be noted that the event-triggering mechanism that we propose is different from the periodic event-triggered control paradigm, see *e.g.* [13], [19], where the triggering condition is verified only at some periodic sampling instants. In our case, the triggering mechanism is *continuously* evaluated, once T units of time have elapsed since the last transmission. For LTI systems, the required conditions are reformulated as an LMI which is shown to be always feasible for LTI systems that are stabilizable and detectable. To further reduce transmissions, we start from the results obtained by emulation to develop an LMI-based co-design procedure to simultaneously design the output feedback law and the event-triggering condition for LTI systems. We have then exploited these LMIs to enlarge the guaranteed minimum inter-transmission time. The results are demonstrated on illustrative examples. This chapter summarizes our results in [1–4], where the interested reader will find the proofs as well as additional results and examples.

2 Emulation design for nonlinear systems

In this section, we explain how to synthesize stabilizing output feedback event-triggered controllers for a class of nonlinear systems by emulation. After deriving the hybrid model of the closed-loop system, we recall on an illustrative example taken from [8] the main issue with output feedback event-triggered controllers which does not allow for straightforward extension of the existing results on state feedback control. Then, we present the technical assumptions we impose on the nonlinear system and we introduce the triggering condition. We then state the main stability result. Finally, we demonstrate the technique on a single link robot arm model.

2.1 Hybrid model

Consider the nonlinear plant model

$$\dot{x}_p = f_p(x_p, u), \qquad y = g_p(x_p), \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output of the plant. We first ignore the communication constraints and we focus on general dynamic controllers of the form

$$\dot{x}_c = f_c(x_c, y), \qquad u = g_c(x_c, y), \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state. We emphasize that the x_c-system is not necessarily an observer. Moreover, (2) captures static feedback laws as a particular case by setting $u = g_c(y)$. We follow an emulation approach in this section. Hence, we assume that the controller (2) renders the origin of system (1) globally asymptotically stable in the absence of network. Afterwards, we take into account the communication constraints in the sense that the plant output and the control input are sent only at transmission instants $t_i, i \in \mathbb{Z}_{\geq 0}$. We are interested in an event-triggered implementation in the sense that the sequence of transmission instants is determined by a criterion based on the output measurement, see Figure 1. At each transmission

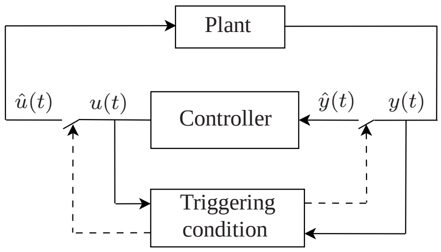

Fig. 1 Event-triggered control schematic [8]

instant, the plant output is sent to the controller which computes a new control input that is instantaneously transmitted to the plant. We assume that this process is performed in a synchronous manner and we ignore the computation times and the possible transmission delays. In that way, we obtain

$$\begin{array}{lll} \dot{x}_p = f_p(x_p, \hat{u}), & \dot{x}_c = f_c(x_c, \hat{y}) & t \in [t_i, t_{i+1}] \\ \dot{\hat{y}} = 0, & \dot{\hat{u}} = 0 & t \in [t_i, t_{i+1}] \\ \hat{y}(t_i^+) = y(t_i), & \hat{u}(t_i^+) = u(t_i), & u = g_c(x_c, \hat{y}), \end{array}$$

where $\hat{y}$ and $\hat{u}$ respectively denote the last transmitted values of the plant output and the control input. We assume that zero-order-hold devices are used to generate the sampled values $\hat{y}$ and $\hat{u}$, which leads to $\dot{\hat{y}} = 0$ and $\dot{\hat{u}} = 0$. We introduce the network-induced error $e := (e_y, e_u) \in \mathbb{R}^{n_e}$, where $e_y := \hat{y} - y$ and $e_u := \hat{u} - u$ which are reset to 0 at each transmission instant. We notice that the closed-loop system is a hybrid dynamical model since it combines continuous-time evolutions (the plant and the controller dynamics) and discrete phenomena (transmissions). We model the event-triggered control system using the hybrid formalism of [10] as in *e.g.* [8, 9, 20, 23] for which a jump corresponds to a transmission. In that way, the system is modeled as

$$\begin{pmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} f(x,e) \\ g(x,e) \\ 1 \end{pmatrix} \quad (x,e,\tau) \in C, \qquad \begin{pmatrix} x^+ \\ e^+ \\ \tau^+ \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad (x,e,\tau) \in D, \quad (3)$$

where $x := (x_p, x_c) \in \mathbb{R}^{n_x}$ and $\tau \in \mathbb{R}_{\geq 0}$ is a clock variable which describes the time elapsed since the last jump, $f(x,e) = (f_p(x_p, g_c(x_c, y+e_y)+e_u),\ f_c(x_c, y+e_y))$ and $g(x,e) = (-\frac{\partial}{\partial x_p} g_p(x_p) f_p(x_p, g_c(x_c, y+e_y)+e_u), -\frac{\partial}{\partial x_c} g_c(x_c, y+e_y) f_c(x_c, y+e_y))$.

The flow and the jump sets of (3) are defined according to the triggering condition we will define. As long as the triggering condition is not violated, the system flows on C and a jump occurs when the state enters in D. When $(x,e,\tau) \in C \cap D$, the solution may flow only if flowing keeps (x,e,τ) in C, otherwise the system experiences a jump. The functions f and g are assumed to be continuous and the sets C and D will be closed (which ensure that system (3) is well-posed, see Chapter 6 in [10]). We briefly recall some basic notions related to the hybrid formalism of [10].

The solutions to system (3) are defined on so-called hybrid time domains. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact hybrid time domain* if $E = \cup([t_j, t_{j+1}], j)$ for $j \in \{0, ..., J-1\}$ and for some finite sequence of times $0 = t_0 \leq t_1 \leq ... \leq t_J$, and it is a *hybrid time domain* if for all $(T,J) \in E, E \cap ([0,T] \times \{0,1,...,J\})$ is a compact hybrid time domain. A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{Z}_{\geq 0}, t \mapsto \phi(t,j)$ is locally absolutely continuous on $I^j := \{t : (t,j) \in E\}$. A hybrid arc ϕ is a solution to system (3) if: (i) $\phi(0,0) \in C \cup D$; (ii) for any $j \in \mathbb{Z}_{\geq 0}$, $\phi(t,j) \in C$ and $\dot{\phi}(t,j) = F(\phi(t,j))$ for almost all $t \in I^j$; (iii) for every $(t,j) \in \operatorname{dom}\phi$ such that $(t,j+1) \in \operatorname{dom}\phi$, $\phi(t,j) \in D$ and $\phi(t,j+1) = G(\phi(t,j))$. A solution ϕ to system (3) is *maximal* if it cannot be extended, *complete* if its domain, $\operatorname{dom}\phi$, is unbounded, and it is *Zeno* if it is complete and $\sup_t \operatorname{dom}\phi < \infty$.

2.2 Counter example [8]

Before presenting our results, we first explain the issue with output-based event-triggered controllers which prevents the direct extension of state feedback results. To clarify the problem, we recall the numerical example in [8] where an LTI plant model is given by

$$\dot{x}_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = [-1 \;\; 4]\, x_p, \quad (4)$$

where $x \in \mathbb{R}^2$ is the plant state, $u \in \mathbb{R}$ is the control input and $y \in \mathbb{R}$ is the output of the plant. Let us first consider the state feedback case as in [24]. In the absence of network, the closed-loop system can be stabilized by the state feedback controller $u = [1 \quad -4]x_p$. By taking into account the effect of the network, we define the network-induced error as $e(t) = x(t_i) - x(t)$ for almost all $t \in [t_i, t_{i+1}]$. It has been shown in [24] that the triggering condition

$$|e| \leq \sigma |x| \tag{5}$$

for some sufficiently small $\sigma > 0$, guarantees an asymptotic stability property for the closed-loop while the inter-transmission times are lower bounded by a strictly positive lower bound, under some conditions as illustrated in Figures 2, 3.

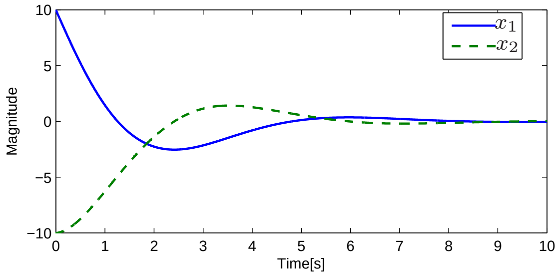

Fig. 2 State trajectories of the plant.

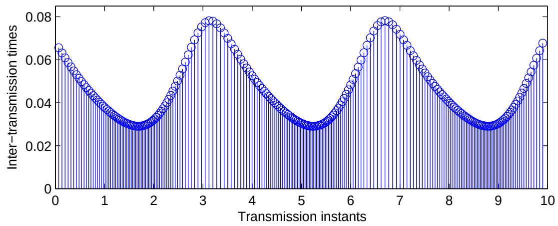

Fig. 3 Inter-transmission times.

We recall now the output feedback case in [8] where system (4) is stabilized by the following dynamic controller

$$\dot{x}_c = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y, \qquad u = [1 \quad -4]\, x_c, \tag{6}$$

where $x_c \in \mathbb{R}^2$ is the state of the dynamic controller. Define the network-induced error as $e_y(t) = y(t_i) - y(t)$ for almost all $t \in [t_i, t_{i+1}]$. The straightforward extension of the triggering condition (5) yields

$$|e_y| \leq \sigma |y|. \tag{7}$$

Unfortunately, this triggering rule is not suitable since when $y = 0$, an infinite number of jumps occurs for any value of $x_p \neq 0$. This situation is shown in Figure

4 where we note that the transmission instants accumulate at $t = 1.7674$. In [8], this issue was overcome by adding a constant to the triggering condition which leads to

$$|e_y| \leq \sigma|y| + \varepsilon \tag{8}$$

for some $\varepsilon > 0$, from which a practical stability property is derived, *i.e.* the state trajectory converges to a neighbourhood to the origin whose 'size' depends on the parameter ε.

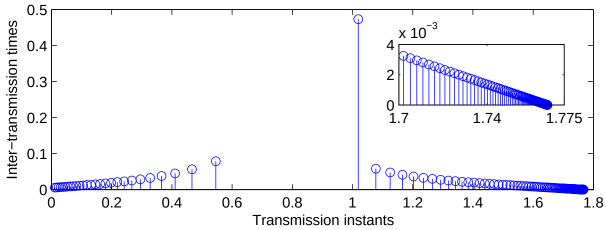

Fig. 4 Inter-transmission times with a zoom-in of the last transmissions.

In this chapter, we aim to design the flow and the jump sets of system (3), *i.e.* the triggering condition, such that a global asymptotic stability property is guaranteed and the number of transmissions is reduced, while ensuring the existence of a strictly positive lower bound on the inter-transmission times.

2.3 Stability results

We first make the following assumption on system (3), which is inspired by [17].

Assumption 1 *There exist locally Lipschitz positive definite functions $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ and $W : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, real numbers $L \geq 0$, $\gamma > 0$, $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$[1] and continuous, positive definite functions $\delta : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$ and $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^{n_x}$*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|), \tag{9}$$

for all $e \in \mathbb{R}^{n_e}$ and almost all $x \in \mathbb{R}^{n_x}$

$$\langle \nabla V(x), f(x,e) \rangle \leq -\alpha(|x|) - H^2(x) - \delta(y) + \gamma^2 W^2(e) \tag{10}$$

and for all $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$

[1] A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is zero at zero, strictly increasing, and it is of class $\mathcal{K}_\infty$ if in addition $\gamma(s) \to \infty$ as $s \to \infty$.

$$\langle \nabla W(e), g(x,e) \rangle \leq LW(e) + H(x). \tag{11}$$

□

Conditions (9)-(10) imply that the system $\dot{x} = f(x,e)$ is $\mathcal{L}_2$-gain stable from W to $(H, \sqrt{\delta})$. This property can be analysed by investigating the robustness property of the closed-loop system (1)-(2) with respect to input and/or output measurement errors in the absence of sampling. Note that, since W is positive definite and continuous (since it is locally Lipschitz), there exists $\chi \in \mathcal{K}_\infty$ such that $W(e) \leq \chi(|e|)$ (according to Lemma 4.3 in [11]) and hence (9), (10) imply that the system $\dot{x} = f(x,e)$ is input-to-state stable (ISS). We also assume an exponential growth condition of the e-system on flows in (11) which is similar to the one used in [17].

Under Assumption 1, the adaptation of the idea of [24] leads to a triggering condition of the form

$$\gamma^2 W^2(e) \leq \delta(y). \tag{12}$$

The problem is that Zeno phenomenon may occur with this type of triggering conditions as explained in Section 2.2. We propose instead to evaluate the event-triggering condition only after T units have elapsed since the last transmission, where T corresponds to the MATI given by [17]. We thus redesign the triggering condition as follows

$$\gamma^2 W^2(e) \leq \delta(y) \text{ or } \tau \in [0,T], \tag{13}$$

where we recall that $\tau \in \mathbb{R}_{\geq 0}$ is the clock variable introduced in (3). Consequently, the flow and the jump sets of system (3) are

$$\begin{aligned} C &= \Big\{ (x,e,\tau) : \gamma^2 W^2(e) \leq \delta(y) \text{ or } \tau \in [0,T] \Big\} \\ D &= \Big\{ (x,e,\tau) : \Big(\gamma^2 W^2(e) = \delta(y) \text{ and } \tau \geq T \Big) \text{ or } \Big(\gamma^2 W^2(e) \geq \delta(y) \text{ and } \tau = T \Big) \Big\}. \end{aligned} \tag{14}$$

Hence, the inter-jump times are uniformly lower bounded by T. This constant is selected such that $T < \mathcal{T}(\gamma, L)$, where

$$\mathcal{T}(\gamma, L) := \begin{cases} \frac{1}{Lr} \arctan(r) & \gamma > L \\ \frac{1}{L} & \gamma = L \\ \frac{1}{Lr} \operatorname{arctanh}(r) & \gamma < L \end{cases} \tag{15}$$

with $r := \sqrt{\left| (\frac{\gamma}{L})^2 - 1 \right|}$ and L, γ come from Assumption 1 as in [17]. We are ready to state the main result.

Theorem 1. *Suppose that Assumption 1 holds and consider system (3) with the flow and the jump sets (14), where the constant T is such that $T \in (0, \mathcal{T}(\gamma, L))$. There exists*[2] $\beta \in \mathcal{KL}$ *such that any solution $\phi = (\phi_x, \phi_e, \phi_\tau)$ with $(\phi_x(0,0), \phi_e(0,0)) \in \mathbb{R}^{n_x + n_e}$ satisfies*

[2] A continuous function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_{\geq 0}$, $\gamma(.,t)$ is of class $\mathcal{K}$, and, for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s,.)$ is decreasing to zero.

$$|\phi_x(t,j)| \leq \beta(|(\phi_x(0,0),\phi_e(0,0))|, t+j) \quad \forall (t,j) \in dom\, \phi, \tag{16}$$

furthermore, if ϕ is maximal, then it is complete. □

Property (16) indicates that the state trajectory of the x-system asymptotically converges to the origin while the completeness property implies that the hybrid time domains of the maximal solutions to (3) are unbounded. That is equivalent to forward completeness for continuous-time ordinary differential equations [5].

2.4 Illustrative example

Consider the dynamics of a single-link robot arm

$$\dot{x}_{p1} = x_{p2}, \qquad \dot{x}_{p2} = -\sin(x_{p1}) + u, \qquad y = x_{p1}, \tag{17}$$

where x_{p1} denotes the angle, x_{p2} the rotational velocity and u the input torque. The system can be written as

$$\dot{x}_p = Ax_p + Bu - \phi(y), \qquad y = Cx_p, \tag{18}$$

where $x_p = (x_{p1}, x_{p2})$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$, $\phi(y) = \begin{bmatrix} 0 \\ \sin(y) \end{bmatrix}$. In order to stabilize system (19), we first construct a state feedback controller of the form $u = Kx_p + B^T\phi(y)$. Hence, system (17) reduces to

$$\dot{x}_p = (A + BK)x_p, \qquad y = Cx_p. \tag{19}$$

We design the gain K such that the eigenvalues of the closed loop system (19) are $(-1, -2)$ (which is possible since the pair (A, B) is controllable). Hence, the gain K is selected to be $K = [-2 \quad -3]$. Next, since only the measurement of y is available, we construct a state-observer of the following form

$$\begin{aligned} \dot{x}_c &= Ax_c + Bu - \phi(y) + M(y - Cx_c) \\ &= (A - MC)x_c + Bu - \phi(y) + My, \end{aligned} \tag{20}$$

where $x_c \in \mathbb{R}^2$ is the estimated state and M is the observer gain matrix. We design the gain matrix M such that the eigenvalues of $(A - MC)$ are $(-5, -6)$ (which is possible since the pair (A, C) is observable). Thus, the observer gain is selected to be $M = [11 \quad 30]^T$. As a result, the closed-loop system in the absence of sampling is given by

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu - \phi(y), & y &= Cx_p \\ \dot{x}_c &= (A - MC)x_c + Bu - \phi(y) + My, & u &= Kx_c + B^T\phi(y). \end{aligned} \tag{21}$$

We now take into account the effect of the network. We consider the scenario where the controller receives the output measurements only at transmission instants $t_i, i \in \mathbb{Z}_{\geq 0}$ while the controller is directly connected to the plant actuators. We design a triggering condition of the form (13). As a consequence, the network-induced error is $e = e_y = \hat{y} - y$ and we obtain, for almost all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} \dot{x}_p &= Ax_p + B\Big(Kx_c + B^T\phi(\hat{y})\Big) - \phi(y) \\ \dot{x}_c &= (A - MC + BK)x_c + MCx_p + Me_y. \end{aligned} \tag{22}$$

Let $x = (x_p, x_c)$. Then, system (22) can be written as follows

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & BK \\ MC & A - MC + BK \end{pmatrix} \begin{pmatrix} x_p \\ x_c \end{pmatrix} + \begin{pmatrix} 0 \\ M \end{pmatrix} e + \begin{pmatrix} \phi(y+e) - \phi(y) \\ 0 \end{pmatrix} \\ &=: \mathcal{A}x + \mathcal{B}e + \psi(y, e). \end{aligned} \tag{23}$$

Since $e = \hat{y} - y$ and in view of (17), we have $\dot{e} = -\dot{y} = -x_{p2}$. Hence, the functions f, g in (3) are $f(x, e) = \mathcal{A}x + \mathcal{B}e + \psi(y, e)$ and $g(x, e) = -x_{p2}$. By taking $W(e) = |e|$ and $V(x) = x^T Px$, all conditions in Assumption 1 are satisfied. We obtain the numerical values $L = 0, \gamma = 26.5333$, which give, in view of (15), $\mathcal{T} = 0.0592$. We take $T = 0.059$. Figure 5 shows that the plant and the estimated state asymptotically converge to the origin as expected. The generated inter-transmission times are shown in Figure 6 where we can observe the interaction between the time-triggered and the event-triggered criteria. We ran simulations for 200 randomly distributed initial conditions such that $|(x(0, 0), e(0, 0))| \leq 100$ and $\tau(0, 0) = 0$. The obtained minimum and average inter-transmission times, respectively denoted as $\tau_{\min}$ and τ_{avg} are $\tau_{\min} = 0.059$ and $\tau_{\text{avg}} = 0.0625$. The constant τ_{avg} serves as a measure of the amount of transmissions (the bigger τ_{avg}, the less transmissions). Figure 7 presents the inter-transmission times with the triggering condition $\gamma^2 W^2(e) \leq \delta(y)$ without enforcing a constant time T between transmissions (*i.e.* $T = 0$ in (13), (14)). We note that Zeno phenomenon occurs in this case as discussed in Section 2.2.

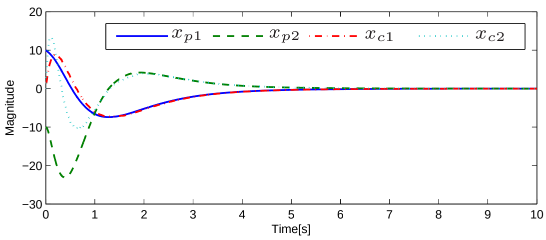

Fig. 5 Actual and estimated states of the plant.

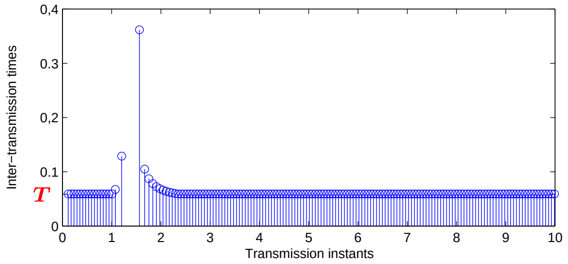

Fig. 6 Inter-transmission times.

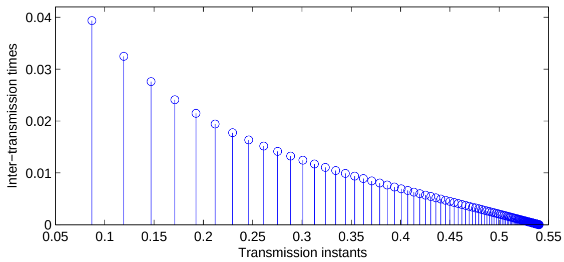

Fig. 7 Inter-transmission times with [24].

2.5 Application to LTI systems

We now focus on the particular case of linear systems. We formulate the required conditions in Assumption 1 as an LMI constraint. Consider the LTI plant model

$$\dot{x}_p = A_p x_p + B_p u, \qquad y = C_p x_p, \tag{24}$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and A_p, B_p, C_p are matrices of appropriate dimensions. We design the following dynamic controller to stabilize (24) in the absence of sampling

$$\dot{x}_c = A_c x_c + B_c y, \qquad u = C_c x_c + D_c y, \tag{25}$$

where $x_c \in \mathbb{R}^{n_c}$ and A_c, B_c, C_c, D_c are matrices of appropriate dimensions. Afterwards, we take into account the communication constraints. Then, the hybrid model (3) is

$$\begin{pmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_1 x + \mathcal{B}_1 e \\ \mathcal{A}_2 x + \mathcal{B}_2 e \\ 1 \end{pmatrix} \quad (x,e,\tau) \in C, \qquad \begin{pmatrix} x^+ \\ e^+ \\ \tau^+ \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad (x,e,\tau) \in D, \tag{26}$$

where $\mathcal{A}_1 := \begin{pmatrix} A_p+B_pD_cC_p & B_pC_c \\ B_cC_p & A_c \end{pmatrix}$, $\mathcal{B}_1 := \begin{pmatrix} B_pD_c & B_p \\ B_c & 0 \end{pmatrix}$,
$\mathcal{A}_2 := \begin{pmatrix} -C_p(A_p+B_pD_cC_p) & -C_pB_pC_c \\ -C_cB_cC_p & -C_cA_c \end{pmatrix}$ and $\mathcal{B}_2 := \begin{pmatrix} -C_pB_pD_c & -C_pB_p \\ -C_cB_c & 0 \end{pmatrix}$.
We obtain the following result.

Proposition 1. *Consider system (26). Suppose that there exist $\varepsilon_1, \varepsilon_2, \mu > 0$ and a positive definite symmetric real matrix P such that*

$$\begin{pmatrix} \mathcal{A}_1^T P + P\mathcal{A}_1 + \varepsilon_1 \mathbb{I}_{n_x} + \mathcal{A}_2^T\mathcal{A}_2 + \varepsilon_2 \overline{C}_p^T \overline{C}_p & P\mathcal{B}_1 \\ \mathcal{B}_1^T P & -\mu \mathbb{I}_{n_e} \end{pmatrix} \leq 0, \tag{27}$$

where $\overline{C}_p = [C_p \quad 0]$ and 0 represents the matrix of zeros of size $n_y \times n_c$. Then Assumption 1 globally holds with $V(x) = x^T P x$, $\underline{\alpha}(s) = \lambda_{\min}(P)s^2$, $\overline{\alpha}(s) = \lambda_{\max}(P)s^2$, $W(e) = |e|$, $H(x) = |\mathcal{A}_2 x|$, $L = |\mathcal{B}_2|$, $\gamma = \sqrt{\mu}$, $\alpha(s) = \varepsilon_2 s^2$ and $\delta(y) = \varepsilon_1 |y|^2$, for $s \geq 0$. □

Note that, in view of (14) and Proposition 1, the flow and the jump sets become

$$\begin{aligned} C &= \Big\{(x,e,\tau) : \gamma^2|e|^2 \leq \varepsilon_1|y|^2 \text{ or } \tau \in [0,T]\Big\} \\ D &= \Big\{(x,e,\tau) : \Big(\gamma^2|e|^2 = \varepsilon_1|y|^2 \text{ and } \tau \geq T\Big) \text{ or } \Big(\gamma^2|e|^2 \geq \varepsilon_1|y|^2 \text{ and } \tau = T\Big)\Big\}. \end{aligned} \tag{28}$$

3 Co-design for LTI systems

In Section 2, we have assumed that the feedback control law was known in the absence of network, then we synthesized the triggering condition. This sequential order of design may prevent an efficient usage of the computation and communication resources as we are restricted by the initial choice of the feedback law. To overcome this limitation, in this section, we use the triggering condition designed in Section 2.5 for LTI systems as a starting point to simultaneously design the event-triggering condition and the feedback law.

Consider the LTI plant model (24) and the dynamic controller (25). For the sake of simplicity, we design the dynamic controller (25) with $D_c = 0$ and we obtain the hybrid model (26). Our objective is to design the dynamic controller (25) and the flow and the jump sets (28) of the hybrid system (26) such that the conclusions of Theorem 1 hold. The idea is to start from LMI (27) to establish an LMI-based co-design procedure of both the flow and the jump sets (28) and the dynamic controller (25). It is important to note that the derivation of LMI for co-design from (27) is not trivial as the nonlinear term $\mathcal{A}_2^T\mathcal{A}_2$ depends on the controller matrices. This term does not appear in the classical output feedback design problems and cannot be directly handled by congruence transformations like in standard output feedback design problems [22].

3.1 Analytical result

The following theorem reduces the co-design problem of the output feedback law (25) and the parameters of the flow and the jump sets (28) to the solution of LMI. We use boldface symbols to emphasize the LMI decision variables.

Theorem 2. *Consider system (26) with the flow and the jump sets (28). Suppose that there exist symmetric positive definite real matrices* $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n_p \times n_p}$*, real matrices* $\boldsymbol{M} \in \mathbb{R}^{n_p \times n_p}, \boldsymbol{Z} \in \mathbb{R}^{n_p \times n_y}, \boldsymbol{N} \in \mathbb{R}^{n_u \times n_p}$ *and* $\boldsymbol{\varepsilon}, \boldsymbol{\mu} > 0$ *such that*[3]

$$\begin{pmatrix} \Sigma(\boldsymbol{Y}A_p + \boldsymbol{Z}C_p) & \star & \star & \star & \star & \star & \star \\ A_p + \boldsymbol{M}^T & \Sigma(A_p\boldsymbol{X} + B_p\boldsymbol{N}) & \star & \star & \star & \star & \star \\ \boldsymbol{Z}^T & 0 & -\boldsymbol{\mu}\mathbb{I}_{n_y} & \star & \star & \star & \star \\ B_p^T\boldsymbol{Y} & B_p^T & 0 & -\boldsymbol{\mu}\mathbb{I}_{n_u} & \star & \star & \star \\ \boldsymbol{Y}A_p + \boldsymbol{Z}C_p & \boldsymbol{M} & 0 & 0 & -\boldsymbol{Y} & \star & \star \\ A_p & A_p\boldsymbol{X} + B_p\boldsymbol{N} & 0 & 0 & -\mathbb{I}_{n_p} & -\boldsymbol{X} & \star \\ C_p & C_p\boldsymbol{X} & 0 & 0 & 0 & 0 & -\boldsymbol{\varepsilon}\mathbb{I}_{n_y} \end{pmatrix} < 0 \tag{29}$$

$$\begin{pmatrix} -\mathbb{I}_{n_y} & \star & \star & \star \\ 0 & -\mathbb{I}_{n_u} & \star & \star \\ -C_p^T & 0 & -\boldsymbol{Y} & \star \\ -\boldsymbol{X}C_p^T & -\boldsymbol{N}^T & -\mathbb{I}_{n_p} & -\boldsymbol{X} \end{pmatrix} < 0. \tag{30}$$

Take $\gamma = \sqrt{\boldsymbol{\mu}}$*,* $L = |\mathcal{B}_2|$*,* $\varepsilon_1 = \boldsymbol{\varepsilon}^{-1}$ *and*

$$\begin{aligned} A_c &= V^{-1}(\boldsymbol{M} - \boldsymbol{Y}A_p\boldsymbol{X} - \boldsymbol{Y}B_p\boldsymbol{N} - \boldsymbol{Z}C_p\boldsymbol{X})U^{-T} \\ B_c &= V^{-1}\boldsymbol{Z}, \qquad C_c = \boldsymbol{N}U^{-T}, \end{aligned} \tag{31}$$

where $U, V \in \mathbb{R}^{n_p \times n_p}$ *are any square and invertible matrices such that*[4] $UV^T = \mathbb{I}_{n_p} - \boldsymbol{XY}$*. Then, there exists* $\chi \in \mathcal{KL}$ *such that any solution* $\phi = (\phi_x, \phi_e, \phi_\tau)$ *satisfies*

$$|\phi_x(t,j)| \leq \chi(|(\phi_x(0,0), \phi_e(0,0))|, t+j) \quad \forall (t,j) \in \mathit{dom}\,\phi \tag{32}$$

and, if ϕ *is maximal, it is also complete.* □

---

[3] The symbol $\star$ denotes symmetric blocks while $\Sigma(.)$ stands for $(.) + (.)^T$.

[4] In view of the Schur complement of LMI (30), we deduce that $\begin{pmatrix} \boldsymbol{Y} & \mathbb{I}_{n_p} \\ \mathbb{I}_{n_p} & \boldsymbol{X} \end{pmatrix} > 0$ which implies that $\boldsymbol{X} - \boldsymbol{Y}^{-1} > 0$ and thus, $\mathbb{I}_{n_p} - \boldsymbol{XY}$ is nonsingular. Hence, the existence of nonsingular matrices U, V is always ensured.

We note that by solving the LMI (29), (30), which are computationally tractable, we obtain the feedback law, see (31), and the parameters of the triggering condition $\gamma^2|e|^2 \leq \varepsilon_1|y|^2$ or $\tau \in [0,T]$. We note also that the nonstandard term $\mathcal{A}_2^T\mathcal{A}_2$ in (27) is the reason why the constructed LMI (29) differs from the classical one and why the additional convex constraint (30) is needed in Theorem 2.

3.2 Optimization

Although the existence of strictly positive lower bound on the inter-transmission times is guaranteed by different techniques in the literature, the available expressions are often subject to some conservatism. It is therefore unclear whether the event-triggered controller has a dwell-time which is compatible with the hardware limitations. We investigate in this section how to employ the LMI conditions (29), (30) to maximize the guaranteed minimum inter-transmission time in order to increase the implementability of the event-triggered controller. We first state the following lemma to motivate our approach.

Lemma 1. *Let $\mathcal{S}$ be the set of solutions to system (26), (28). It holds that*

$$T = \inf_{\phi\in\mathcal{S}}\{t'-t : \exists j \in \mathbb{Z}_{>0},\ (t,j),(t,j+1),(t',j+1),(t',j+2) \in \mathit{dom}\,\phi\}. \quad (33)$$

□

Lemma 1 implies that the lower bound T on the inter-transmission times guaranteed by (28) corresponds to the actual minimum inter-transmission time as defined by the right-hand side of (33). Hence, by maximizing T, we enlarge the minimum inter-transmission time.

To maximize T, we will maximize $\mathcal{T}(\gamma,L)$ in (15). We see that $\mathcal{T}$ increases as γ and L decrease. Hence, our objective is to minimize γ and L. Since γ corresponds to $\sqrt{\mu}$ and μ enters linearly in the LMI (29), we can directly minimize γ under the LMI (29), (30). The minimization of L, on the other hand, requires more attention. We recall that $L = |\mathcal{B}_2| = \sqrt{\lambda_{\max}(\mathcal{B}_2^T\mathcal{B}_2)}$, where

$$\mathcal{B}_2^T\mathcal{B}_2 = \begin{pmatrix} B_c^TC_c^TC_cB_c & 0 \\ 0 & B_p^TC_p^TC_pB_p \end{pmatrix} \quad (34)$$

hence,

$$L = \max\left(\sqrt{\lambda_{\max}(B_c^TC_c^TC_cB_c)}, \sqrt{\lambda_{\max}(B_p^TC_p^TC_pB_p)}\right). \quad (35)$$

Therefore, L can be minimized up to $\sqrt{\lambda_{\max}(B_p^TC_p^TC_pB_p)}$ which is fixed as it only depends on the plant matrices. In view of (31), we have that

$$B_c^T C_c^T C_c B_c = \boldsymbol{Z}^T V^{-T} U^{-1} \boldsymbol{N}^T \boldsymbol{N} U^{-T} V^{-1} \boldsymbol{Z}. \tag{36}$$

Thus, L depends nonlinearly on the LMI variables $\boldsymbol{N}$ and $\boldsymbol{Z}$ and it can a priori not be directly minimized. To overcome this issue, we impose the following upper bound

$$B_c^T C_c^T C_c B_c < \alpha\beta\mathbb{I}_{n_y} \tag{37}$$

for some $\alpha, \beta > 0$. As a result, minimizing α and β may help to minimize L. We translate inequality (37) into an LMI and we state the following claim.

Claim. Assume that LMI (29), (30) are verified. Then, there exist $\boldsymbol{\alpha}, \boldsymbol{\beta} > 0$ such that

$$\begin{pmatrix} \boldsymbol{\alpha}\mathbb{I}_{n_y} & \star & \star & \star \\ 0 & \boldsymbol{\beta}\mathbb{I}_{n_u} & \star & \star \\ 0 & \boldsymbol{N}^T & \boldsymbol{X} & \star \\ \boldsymbol{Z} & 0 & \mathbb{I}_{n_p} & \boldsymbol{Y} \end{pmatrix} > 0 \tag{38}$$

which implies that inequality (37) holds. □

We note that (38) does not introduce additional constraints on system (26) compared to (29), (30). This comes from the fact that there always exist $\alpha, \beta > 0$ (eventually large) such that (38) holds, in view of Schur complement of (38).

In conclusion, we formulate the problem as a multiobjective optimization problem as we want to minimize μ, α, β under the constraint (29), (30) and (38). Several approaches have been proposed in the literature to handle such problems. We choose the weighted sum strategy among others and we formulate the LMI optimization problem as follows

$$\begin{gathered} \min \lambda_1\mu + \lambda_2\alpha + \lambda_3\beta \\ \text{subject to (29), (30), (38)} \end{gathered} \tag{39}$$

for some weights $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

3.3 Illustrative example

Consider the LTI plant model

$$\dot{x}_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -5 & -6 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 10 \\ -50 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 \; 0 \; 0 \end{bmatrix} x_p. \tag{40}$$

First, we solve the optimization problem (39) to seek for the largest possible lower bound on the inter-transmission times. We set $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and we obtain $T = 0.005$. Table 1 gives the minimum and the average inter-sampling times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$. We observe from the corresponding entries in Table 1 that $\tau_{\min} = \tau_{\text{avg}}$ which implies that generated transmission instants are periodic. This may be explained by the fact that the output-dependent part in (28), *i.e.* $\mu|e|^2 \leq \varepsilon^{-1}|y|^2$, is 'quickly' violated. To avoid that phenomenon, we optimize the parameters μ, ε such that the rule is violated after the longest possible time since the last transmission instant, see [4] for more detailed explanations. Thus, the problem can be formulated as follows

$$\begin{array}{c} \min \lambda_1\mu + \lambda_2\alpha + \lambda_3\beta + \lambda_4\varepsilon \\ \text{subject to (29), (30), (38)} \end{array} \tag{41}$$

for some weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

	Guaranteed dwell-time	$\tau_{\min}$	τ_{avg}
Optimization problem (39) $\lambda_1 = \lambda_2 = \lambda_3 = 1$	0.0049	0.0049	0.0049
Optimization problem (41) $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 10^4$	0.0047	0.0047	0.0052
Emulation (27)	7.3389 $\times 10^{-6}$	7.3389 $\times 10^{-6}$	8.5396 $\times 10^{-4}$

Table 1 Minimum and average inter-transmission times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$ for a simulation time of 10s.

By playing with the weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we found that the best tradeoff between $\tau_{\min}$ and τ_{avg} is obtained with $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 10^4$, as shown in Table 1. Note that when we consider the same dynamic controller as in the case (41) and we design the triggering condition by emulation according to (27), we obtain the results shown in the last row of Table 1. We note that the co-design procedure yields larger lower bound on the inter-transmission times as well as larger average inter-transmission time, *i.e.* less amount of transmissions. Figures 8, 9 respectively shows the state trajectory and a close up of the inter-transmission times generated by the event-triggered controller obtained by (41).

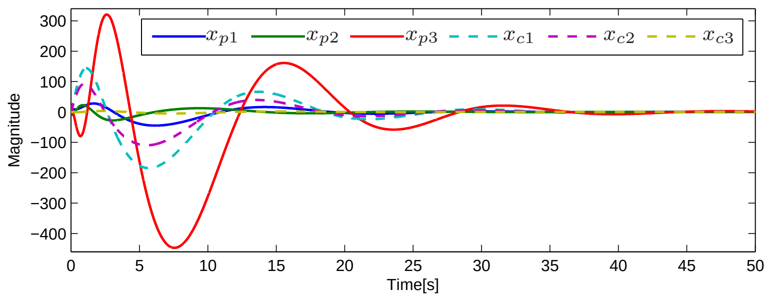

Fig. 8 Inter-transmission times.

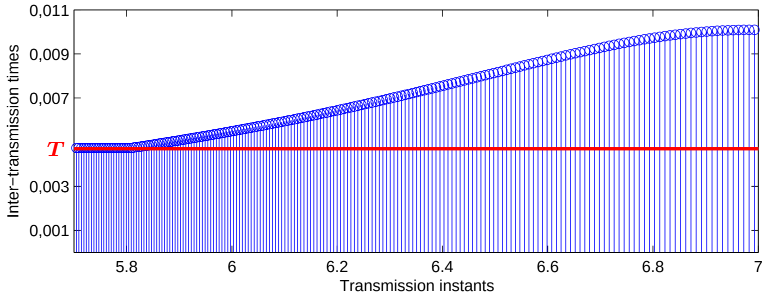

Fig. 9 Inter-transmission times.

4 Conclusion

In this chapter, we have first developed output feedback event-triggered controllers to stabilize a class of nonlinear systems by following the emulation design approach. In emulation, the sequential order of the synthesis of event-triggered controllers restricts us with the initial choice of the stabilizing feedback law. One solution to increase the design flexibility is to simultaneously establish the feedback law and the event-triggering condition so that the stabilization and the communication constraints are handled at the same time. For this purpose, we have proposed an LMI-based co-design algorithm for LTI systems and we have then discussed how the resulted LMI can be exploited to optimize the parameters of the event-triggering condition. Future work will focus on the robustness of the triggering mechanism with respect to measurement noise and to consider some performance requirements.

References

1. Abdelrahim M (2014) Output feedback event-triggered control. PhD thesis, Centre de Recherche en Automatique de Nancy (CRAN), Université de Lorraine, 2014.
2. Abdelrahim M, Postoyan R, Daafouz J, Nešić D (2014) Stabilization of nonlinear systems using event-triggered output feedback laws. Proceedings of the 21th International Symposium on Mathematics Theory of Networks and Systems, Groningen, The Netherlands
3. Abdelrahim M, Postoyan R, Daafouz J, Nešić D (2014) Stabilization of nonlinear systems using event-triggered output feedback laws. IEEE Transactions on Automatic Control, submitted
4. Abdelrahim M, Postoyan R, Daafouz J, Nešić D (2014) Co-design of output feedback laws and event-triggering conditions for linear systems. Proceedings of the IEEE Conference on Decision and Control, Los Angeles, U.S.A. submitted
5. Angeli A, Sontag E.D. (1999) Forward completeness, unboundedness observability, and their Lyapunov characterizations. Systems Control Letters 38:209–217
6. Årzén K E (1999) A Simple event-based PID controller. Proceedings of the 14th IFAC World Congress, Beijing, China 423–428

7. Åström K J, Bernhardsson B M (1999) Comparison of periodic and event based sampling for first order stochastic systems. Proceedings of the 14th IFAC World Congress, Beijing, China 301–306
8. Donkers M C F, Heemels W P M H (2012) Output-based event-triggered control with Guaranteed $\mathcal{L}_\infty$-gain and improved and decentralised event-triggering. IEEE Transactions on Automatic Control 57:1362–1376
9. Forni F, Galeani S, Nešić D, Zaccarian L (2014) Event-triggered transmission for linear control over communication channels. Automatica 50:490–498
10. Goebel R, Sanfelice R G, Teel A R (2012) Hybrid Dynamical Systems: Modeling, Stability, and Robustness. Princeton University Press
11. Khalil H K (2002) Nonlinear Systems. 3rd edn, Prentice Hall
12. Kofman E, Braslavsky J H (2006) Level crossing sampling in feedback stabilization under data-rate constraints. Proceedings of the IEEE Conference on Decision and Control, San Diego, U.S.A. 4423–4428
13. Heemels W P M H, Donkers M C F, Teel A R (2013) Periodic Event-Triggered Control for Linear Systems. IEEE Transactions on Automatic Control 58:847–861
14. Heemels W P M H, Johansson K H, Tabuada P (2012) An Introduction to Event-triggered and Self-triggered Control. Proceedings of the IEEE Conference on Decision and Control, Hawaii, U.S.A. 3270–3285
15. Liu Q, Wang Z, He X, Zhou Z.H. (2014) A survey of event-based strategies on control and estimation. Systems Science & Control Engineering 2:90–97
16. Nešić D, Teel A (2004) Input-output stability properties of networked control systems. IEEE Transactions on Automatic Control 49:1650–1667
17. Nešić D, Teel A R, Carnevale D (2009) Explicit Computation of the Sampling Period in Emulation of Controllers for Nonlinear Sampled-Data Systems. IEEE Transactions on Automatic Control 54:619–624
18. Peng C, Han Q (2013) Output-based event-triggered $\mathcal{H}_\infty$ control for sampled-data control systems with nonuniform sampling. Proceedings of the American Control Conference, Washington, U.S.A. 1727–1732
19. Postoyan R, Anta A, Heemels W P M H, Tabuada P, Nešić D (2013) Periodic event-triggered control for nonlinear systems. Proceedings of the IEEE Conference on Decision and Control, Florence, Italy 7397–7402
20. Postoyan R, Anta A, Nešić D, Tabuada P (2011) A unifying Lyapunov-based framework for the event-triggered control of nonlinear systems. Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Orlando, U.S.A. 2559–2564
21. Postoyan R, Tabuada P, Nešić D, Anta A (2011) Event-triggered and self-triggered stabilization of distributed networked control systems. Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Orlando, U.S.A. 2565–2570
22. Scherer C, Gahinet P, Chilali M (1997) Multiobjective output-feedback control via LMI optimization. IEEE Transactions on Automatic Control 42:896–911
23. Seuret A, Prieur C, Marchand N (2013) Stability of non-linear systems by means of event-triggered sampling algorithms. IMA Mathematical Control & Information to appear
24. Tabuada P (2007) Event-triggered real-time scheduling of stabilizing control tasks. IEEE Transactions on Automatic Control 52:1680–1685
25. Tallapragada P, Chopra N (2012) Event-triggered decentralized dynamic output feedback control for LTI systems. Estimation and Control of Networked Systems 3:31–36
26. Walsh G C, Beldiman O, Bushnell L G (2001) Asymptotic behavior of nonlinear networked control systems. IEEE Transactions on Control System Technology 46:1093–1097
27. Yu H, Antsaklis P J (2012) Event-Triggered Output Feedback Control for Networked Control Systems Using Passivity: Achieving $\mathcal{L}_2$ Stability in the Presence of Communication Delays and Signal quantization. Automatica 49:30–38
28. Zhang X, Han Q (2013) Event-based dynamic output feedback control for networked control systems. Proceedings of the American Control Conference, Washington, U.S.A. 3008–3013