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Analytic Synchronization Conditions for a Network of Wilson and Cowan Oscillators

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Abstract— **We investigate the problem of synchronization in a network of homogeneous Wilson-Cowan oscillators with diffusive coupling. Such networks can be used to model the behavior of populations of neurons in cortical tissue, referred to as neural mass models. A new approach is proposed to address local synchronization for these types of neural mass models. By exploiting the linearized model around a limit cycle, we analyze synchronization within a network for weak, intermediate, and strong coupling. We use two-time scale averaging and the Chetaev theorem to analytically check the absence or presence of synchronization in the network with weak coupling. We also utilize the Chetaev theorem to analytically prove synchronization death in a network with strong coupling. For intermediate coupling, we use a recently proposed numerical approach to prove synchronization in the network. Simulation results confirm and illustrate our results.**

I. Introduction

Synchronization is a ubiquitous phenomenon observed in diverse networks of interconnected agents that arise in physics, biology, social networks, and many others. Synchronization occurs when the agents' outputs converge to the same behavior. Since synchronization is a particular type of stability, some of the existing approaches to stability analysis of dynamical systems can be applied to study synchronization ([1], [2]).

Methods for the study of synchronization can be categorized into two groups: global and local. Global synchronization has been addressed by a range of different methods, such as passivity [3], dissipativity [2], and semi-passivity [4]. In these approaches, all agents in the underlying network are required to satisfy a specific dissipation or passivity property; however, in some applications, it may be challenging to demonstrate that these conditions are satisfied. If the conditions are not satisfied, local approaches are a useful alternative. Recently, Shafi et al. [5] proposed a framework to study synchronization in networks of oscillators using linearization techniques. This framework offers useful tools for the analysis of local synchronization in various applications, such as synchronization of networks of neural mass models.

In the context of computational neuroscience, there are three different levels of modeling depending on the application and the complexity needed: microscopic, mesoscopic, and macroscopic. Microscopic levels aim to model a single neuron, such as Hodgkin-Huxley model [6] or FitzHugh model [7]. Mesoscopic models concentrate on modeling the behavior of populations of neurons, such as cortical columns. Popular types of mesoscopic descriptions of cortical dynamics include the Wilson and Cowan model [8] and the Jansen and Rit model [9]. Macroscopic models aim to describe the whole brain and they are sometimes represented as networks of microscopic or mesoscopic models. The Wilson-Cowan model is of great interest since it is parsimonious, as it considers both excitatory and inhibitory populations of neurons, and reproduces self-sustained oscillations observed in EEG signals. In particular, local synchronization of this model has been investigated in the literature using the centre manifold theorem [10] and the notion of phase response curves [11]. These approaches only deal with weak coupling. However, synchronization is observed in Wilson-Cowan networks with intermediate or strong coupling.

In this manuscript, we demonstrate that the framework by Shafi et al. [5] can be adapted to study local synchronization of Wilson-Cowan networks with arbitrary coupling strengths. As far as we are aware, this is a new result. A novelty of our work comes from the fact that Wilson-Cowan networks do not exhibit synchronization for weak and strong coupling and, hence, we had to use an instability result for the linearized model based on the Chetaev theorem to prove this fact. This is different from results [5], where stability of the linearized system was shown for weak and strong coupling. This current paper demonstrates that studying local synchronization via the approach of [5] can be extended to other classes of networks, such as networks of Wilson-Cowan oscillators.

The paper is organized as follow. In Sections II and III, we briefly introduce the Wilson-Cowan model of a single population, as well as the network of such models, and then we state the problem. In Section IV, the synchronization conditions are stabilized for three cases: a network with weak, intermediate, and strong coupling. Simulation results and the conclusion are presented in Sections V and VI, respectively.

II. Wilson-Cowan Model

The neural mass model of Wilson and Cowan [8] characterizes the behavior of spatially localized neural populations via a lumped parameter description. Throughout the paper, the Wilson and Cowan oscillator is the agent in our network. We will use the terms Wilson and Cowan model/oscillator and agent interchangeably. The model contains an excitatory

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and an inhibitory neural population that are coupled together and that interact with other populations in a larger network. In this model, the neural populations are described by a linear first-order system coupled with a sigmoid non-linearity that converts the average membrane potential of a neural population into an average pulse density of action potentials. Considering one excitatory and one inhibitory population as a single "node", the Wilson-Cowan model is described by

$$\dot{x}_i = -\Lambda_i x_i + F_i\left(\Upsilon_i + \Xi_i x_i + I_i\right), \tag{1}$$

where $x_i = [x_{E_i}, x_{I_i}]^T \in \mathbb{R}^2$ is a stack vector of the average membrane potentials of the excitatory and inhibitory populations, x_{E_i} and x_{I_i}, respectively. The variables $I_i = [I_{E_i}, I_{I_i}]^T \in \mathbb{R}^2$ are the inputs from neighboring populations and,

$$\begin{aligned}\Lambda_i &= \begin{bmatrix}\alpha_{E_i} & 0\\ 0 & \alpha_{I_i}\end{bmatrix} \quad , \quad F_i = vec(f_i, f_i),\\ \Upsilon_i &= \begin{bmatrix}\rho_{x_{E_i}}\\ \rho_{x_{I_i}}\end{bmatrix} \quad , \quad \Xi_i = \begin{bmatrix}a_i & -b_i\\ c_i & -d_i\end{bmatrix},\end{aligned} \tag{2}$$

where $f_i : \mathbb{R} \to \mathbb{R}$ is sigmoid function given by

$$f_i(s_i) = \frac{1}{1+\exp(-r_i s_i)} \quad , \quad r_i > 0. \tag{3}$$

In order to interconnect the Wilson-Cowan oscillators, it is assumed that the excitatory neural population in one node is coupled to the excitatory neural population in another node. The same coupling configuration is assumed for coupling between inhibitory populations in two distinct nodes. For clarity of explanation, it is also assumed that the excitatory and inhibitory neuronal populations within a node are coupled together with the same gain. In the other words, if a node i is coupled to a node j with coupling gain l_{ij}, then the excitatory neural population and inhibitory neural population in node i are coupled to the excitatory and inhibitory neural populations in node j with the same coupling gain l_{ij}, respectively. We note that this assumption is somewhat restrictive as these two interconnections can have different coupling gains in general [10], [12]. Although the interconnection between nodes has been originally considered as a direct coupling, it was proposed that diffusive coupling can be utilized to control oscillatory behaviors and, in particular, synchrony behavior of populations [12].

In this case, the model of N interconnected Wilson-Cowan nodes is represented by the following state space model:

$$\dot{\mathbf{x}} = -\mathbf{\Lambda x} + \mathbf{F}\big(\mathbf{\Upsilon} + \mathbf{\Xi x} - (L \otimes D)\mathbf{x}\big), \tag{4}$$

where $\mathbf{x} = \mathrm{vec}(x_1, \dots, x_N)$, $\mathbf{\Lambda} = \mathrm{diag}(\Lambda_1, \Lambda_2, \dots, \Lambda_N)$, $\mathbf{\Upsilon} = \mathrm{vec}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N)$, $\mathbf{F} = \mathrm{vec}(F_1, F_2, \dots, F_N)$, $\mathbf{\Xi} = \mathrm{diag}(\Xi_1, \Xi_2, \dots, \Xi_N)$ and $D = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. The matrix L describes interconnections between different nodes.

III. Problem Statement

Motivated by the network of Wilson-Cowan models, we consider a network of N homogeneous interconnected agents in which every agent is described by the dynamical system,

$$\dot{x}_i = -\Lambda x + F(\Upsilon + \Xi x_i + D\sum_{j=1}^{N} l_{ij}(x_j - x_i)), \tag{5}$$

where $x_i \in \mathbb{R}^n$ is the state vector, $\Upsilon \in \mathbb{R}^n$ represents an external input applied to each agent, and $\Xi \in \mathbb{R}^{n\times n}$ shows the internal coupling among states of the agent. $F = (f_1, f_2, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function that is sufficiently smooth to guarantee the existence of a solution. $D \in \mathbb{R}^{n\times n}$ shows the inner coupling between states of all agents, while the interconnection between agents is described by a graph Laplacian matrix $L \in \mathbb{R}^{N\times N}$ (specified by following assumption).

Assumption 1: The Laplacian graph is symmetric zero-sum row defined as

$$L = EE^T, \tag{6}$$

where E is the incidence matrix. We also assume that the network graph is connected, which implies $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$.
In particular, we assume that in the absence of interconnections each agent admits an oscillatory behaviour that results from an asymptotically stable limit cycle.

Assumption 2: Each agent (5) has a periodic solution $\bar{x}(t) = \bar{x}(t+T)$ produced by an asymptotically stable limit cycle that satisfies

$$\dot{\bar{x}}_i = -\Lambda\bar{x} + F(\Upsilon + \Xi\bar{x}). \tag{7}$$

The analysis that follows is similar to Shafi et al. [5], where we linearized (5) around the oscillatory trajectory $\bar{x}(t)$. The linearized trajectory is given by

$$\begin{aligned}\dot{x}_i =& -\Lambda x_i + F(\Upsilon + \Xi\bar{x})\\ &+ A(t)\big(\Xi(x_i - \bar{x}) + D\sum_{j=1}^{N} l_{ij}(x_j - x_i)\big)\ ,\end{aligned} \tag{8}$$

where $A(t) = \frac{\partial f}{\partial x}|_{\Upsilon+\Xi\bar{x}}$. Now, we define $\tilde{x}_i = x_i - \bar{x}$, the difference between trajectories of each agent and the oscillatory trajectory. Considering (7) and (8), the dynamics of $\tilde{x}_i$ can be expressed as

$$\dot{\tilde{x}}_i = -\Lambda\tilde{x}_i + A(t)\big(\Xi\tilde{x}_i + D\sum_{j=1}^{N} l_{ij}(\tilde{x}_j - \tilde{x}_i)\big)\ . \tag{9}$$

Using the interconnection strategy L, the aggregated linearized system can be written as

$$\dot{\tilde{x}} = (I \otimes (-\Lambda I + A(t)\Xi) - L \otimes A(t)D)\tilde{x}\ . \tag{10}$$

Let U be a unitary similarity transformation, such that it diagonalizes the Laplacian matrix, i.e $L = U\Sigma U^T$. Now changing to a new coordinate $\tilde{y} = (U^{-1} \otimes I)\tilde{x}$ leads to the following error dynamics in $\tilde{y}$,

$$\begin{aligned}\dot{\tilde{y}} &= (U^{-1} \otimes I)\Big(I \otimes (-\Lambda I + A(t)\Xi) - L \otimes A(t)D\Big)\\ &\qquad \times (U \otimes I)\tilde{y}\\ &= \Big(I \otimes (-\Lambda I + A(t)\Xi) - \Sigma \otimes A(t)D\Big)\tilde{y}.\end{aligned} \tag{11}$$

Since Σ is a diagonal matrix, (11) is decoupled into N independent systems described by

$$\dot{\tilde{y}}_k = \big((-\Lambda I + A(t)\Xi) - \lambda_k A(t)D\big)\tilde{y}_k \ , \quad k = 1, \dots, N, \tag{12}$$

where $\tilde{y}_k \in \mathbb{R}^n$ and λ_k are eigenvalues of the Laplacian matrix. If (12) is asymptotically stable for $k = 2, \dots, N$, then the underlying network becomes completely synchronized, i.e $x_i(t) - x_j(t) \to 0, i, j = 1, \dots, N$ [5]. Hence, in the rest of this section, we drop subscript k and consider λ as coupling strength to investigate the asymptotically stability of the system,

$$\dot{\tilde{y}} = \big((-\Lambda I + A(t)\Xi) - \lambda A(t)D\big)\tilde{y} \ . \tag{13}$$

IV. Stability Analysis of the Linearized Model

In this section, we present sufficient conditions for stability of the linearized model (13). Similar to [5], we split the interval $\lambda \in (0, \infty)$ into three subintervals and find a corresponding λ that leads to stability or instability of (13). For small values of λ, we present two propositions that provide sufficient conditions for stability and instability of the linearized model (13). We also show that (13) always becomes unstable for sufficiently large value of λ. For intermediate values of λ, we adopt the numerical approach proposed in [5] to find values that make (13) stable.

A. Synchronization with Weak Coupling

For weak coupling, we utilized the two-time scale averaging method [5], [13]. Due to assumption 2, (13) has a T-periodic solution for $\lambda = 0$ that is associated with a stable limit cycle in the original system (7). We denote as $\phi(t, t_0)$ the principle state transition matrix of periodic dynamics and

$$\dot{\tilde{y}} = \big(-\Lambda I + A(t)\Xi\big)\tilde{y} \ . \tag{14}$$

The Floquet theory [14] indicates that $\phi(t, t_0)$ is a T-periodic matrix that can be written as

$$\phi(t, t_0) = W(t)\exp(H(t - t_0))R(t_0) \ , \tag{15}$$

where $W(t)$ is also a T-periodic matrix and $R(t) = W^{-1}(t)$. The columns of W denoted by w_i and rows of $R(t)$ denoted by r_j^T are orthonormal, $r_j^T w_i = \delta_{ij}$. The matrix H is known as the *monodromy matrix*, which satisfies

$$\phi(t_0 + T, t_0) = J\exp(HT)J^{-1},$$

where J is a matrix that contains eigenvectors of $\phi(t_0 + T, t_0)$. For a stable limit cycle, the monodromy matrix H has the structure

$$H = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & H_2 \end{bmatrix} \ , \tag{16}$$

where H_2 is an $(n-1)\times(n-1)$ Hurwitz matrix that contains non-zero Floqouet exponents. Even though computing the analytical form of $\phi(t, t_0)$, H_2, $W(t)$, and $R(t)$ is probably impossible, there are effective approaches that can be used to compute the matrices numerically; for example, an approach was proposed by Demir et al. [15]. Floquet theory implies that the change of coordinate $[w, r]^T = \tilde{y}R(t)$ transforms (13) to the representation

$$\begin{bmatrix} \dot{w} \\ \dot{r} \end{bmatrix} = \left(\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & H_2 \end{bmatrix} - \lambda R(t)A(t)DW(t) \right) \begin{bmatrix} w \\ r \end{bmatrix}. \tag{17}$$

Under weak coupling, i.e. $D = \epsilon D_0$ with small ϵ, analysis of stability of (13) can be conducted using a two-time scale averaging method. Proposition 1 states an instability condition of (13), while Proposition 2 provides sufficient conditions for asymptotic stability of (13).

Proposition 1: (**Instability with Weak Coupling**) Let r_1^T and w_1 be the first row and column of $R(t)$ and $W(t)$ in (15), respectively. For every matrix D, if

$$\int_0^T r_1^T(\tau)A(\tau)Dw_1(\tau)d\tau < 0 \tag{18}$$

then there exists ϵ^* such that for every $\epsilon \in (0, \epsilon^*)$ the system

$$\dot{\tilde{y}} = \big((-\Lambda I + A(t)\Xi) - \epsilon A(t)D\big)\tilde{y} \tag{19}$$

is unstable. The value of ϵ^* can be obtained from Lemma 1.

Proof: Proposition 1 is proved by applying Lemma 1 (see Appendix VI) to the system (17).

Proposition 2: (**Stability with Weak Coupling**) Let r_1^T and w_1 be the first row and column of $R(t)$ and $W(t)$ in (15), respectively. For every matrix D, if

$$\int_0^T r_1^T(\tau)A(\tau)Dw_1(\tau)d\tau > 0 \tag{20}$$

then there exists ϵ^* such that for every $\epsilon \in (0, \epsilon^*)$ the system

$$\dot{\tilde{y}} = \big((-\Lambda I + A(t)\Xi) - \epsilon A(t)D\big)\tilde{y} \tag{21}$$

is asymptotically stable. Furthermore, decompose matrix $A(t)D$ with dimensions consistent with the monodromy matrix so that

$$A(t)D = \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{bmatrix}.$$

Assume that there exists $\iota_{ij} > 0$ such that $\|\Psi_{ii}(t)\| \leq \iota_{ij}$ for all $t \geq 0$. Then

$$\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3\},$$

where

$$\begin{aligned} \epsilon_1 &= \frac{1}{2T\iota_{11}} \\ \epsilon_2 &= \frac{1}{\iota_{22}} \end{aligned} \tag{22}$$

and ϵ_3 satisfies the following inequality

$$-(\epsilon_3\gamma - 2\epsilon_3^2\gamma T\iota_{11}) + \frac{1}{4}\big(\epsilon_3\gamma\iota_{12} + \epsilon_3\iota_{21} + \epsilon_3^2(2T\iota_{21}\iota_{11})\big)^2 \times (1 - \epsilon_3\iota_{22}) < 0$$

where $\gamma = \frac{1}{1-2\epsilon_1 T\iota_{11}}$.

Proof: Proposition 2 can be proved by applying Lemma A.1 in [5] to the system (17).

Remark 1: Lemma A.1 in [5] states a result for stability analysis using two-time scale averaging. Lemma A.1 in appendix VI is a counterpart of Lemma A.1 in [5] that presents an instability condition using two-time scale averaging.

Remark 2: For the Wilson-Cowan model, the matrix $A(t) \in \mathbb{R}^{2\times 2}$ has the structure

$$A(t) = \begin{bmatrix} \frac{\partial f}{\partial s}|_{s=[\Upsilon+\Xi\bar{x}]_1} & 0 \\ 0 & \frac{\partial f}{\partial s}|_{s=[\Upsilon+\Xi\bar{x}]_2} \end{bmatrix}. \tag{23}$$

Considering the structure of matrix D, condition (18) in Proposition 1 can be written as

$$\int_0^T \big(r_{11}(\tau)A_{11}(\tau)w_{11}(\tau) - r_{12}(\tau)A_{22}(\tau)w_{21}(\tau)\big)d\tau < 0, \tag{24}$$

and the condition (20) of Proposition 2 can be written as

$$\int_0^T \big(r_{11}(\tau)A_{11}(\tau)w_{11}(\tau) - r_{12}(\tau)A_{22}(\tau)w_{21}(\tau)\big)d\tau > 0. \tag{25}$$

B. Synchronization under intermediate coupling

In this section, we briefly present the numerical approach proposed by Shafi et al. [5] to study stability of linear periodic time-varying systems. We represent the matrix λD as

$$\lambda D = M + B\Delta C, \tag{26}$$

where $M \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$, and $\Delta \in \mathbb{R}^{m\times m}$ is a diagonal matrix whose diagonal entries vary in $[-1, 1]$. Then, the system (13) can be written as a linear periodic time-varying system with block uncertainty Δ:

$$\begin{aligned} \dot{\tilde{y}} &= \bar{A}(t)\tilde{y} + \bar{B}(t)q \\ z &= \bar{C}\tilde{y} \\ q &= \Delta z, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \bar{A}(t) &= -\Lambda I + A(t)\Xi - A(t)M \\ \bar{B}(t) &= -A(t)B \\ \bar{C} &= C \end{aligned} \tag{28}$$

and $z \in \mathbb{R}^m$ and $q \in \mathbb{R}^m$ are the outputs of the linear periodic time-varying system and uncertainty block. In order to check stability of (27), the structured singular value (SSV) analysis is performed on a truncated harmonic state space model of (27). Under the assumption that $\bar{A}(t), \bar{B}(t)$ are continuous functions of t with the Fourier series with the fundamental frequency w_f, the state space model (27) can be represented in the form of a harmonic state space model

$$\begin{aligned} sY &= (\bar{\mathcal{A}} - \mathcal{N})Y + \bar{\mathcal{B}}Q \\ Z &= \mathcal{C}Y \\ Q &= \tilde{\Delta}Z, \end{aligned} \tag{29}$$

where $\bar{\mathcal{A}}, \mathcal{N}, \bar{\mathcal{B}}$ are infinite-dimensional matrices with complex entries. We refer to Shafi et al. [5] and Zhou and Hagiwara [16] for further details on such a harmonic state space representation. Although the harmonic state space model (29) is an infinite dimensional model, it can be approximated with a truncated model that considers the dominant harmonics in the Fourier series of $\tilde{y}(t)$ [17]. Once the truncated model of (29) is computed, the structured singular value μ can be calculated using a numerical approach such as *mussv* in the Robust Control Toolbox. If $\mu \leq 1$, then (13) is stable for all matrices D described by (26). If $\mu > 1$, we can compute the matrix Δ with the smallest norm that leads to instability of (13). It is worth mentioning that the matrices in (26) cover all possible coupling strengths for the intermediate coupling.

C. Synchronization with Strong Coupling

In this section, we analyze the stability of (13) for the Wilson-Cowan model (1) when the coupling is strong. Note that we drop subscript i due to considering identical nodes in our problem setup. Although the result of this section should be extended to apply to a more general model (13), it gives conditions to check stability of (13) for the Wilson-Cowan model. Before presenting the main result of this section, we state a fact on $f(\cdot)$ that is exploited to prove Proposition 3.

Fact 1: The sigmoid function $f(s)$ in (3) is a continuous function and its derivative is positive and bounded, i.e. $0 \leq f_{min} \leq f'(s) \leq f_{max}$ for all $s \in \mathbb{R}$. In addition, $f'(s) \to 0$ iff $s \to \pm\infty$.

Proposition 3: (**Instability with Strong Coupling for the Wilson-Cowan Model**) Consider the dynamical system of described by the state space model

$$\dot{\tilde{y}} = \big((-\Lambda I + A(t)\Xi) - \epsilon A(t)D\big)\tilde{y}\ , \tag{30}$$

where Λ, Ξ, D are given by (2) and $A(t)$ is computed from linearization of (1) around the limit cycle. Then, there exists a positive scalar ϵ^* such that (30) is unstable for every $\epsilon \geq \epsilon^*$. Furthermore, $\epsilon^* = \max\{a, d, \epsilon_1\}$ and ϵ_1 can be obtained from the following optimization problem,

$$\begin{gathered} \epsilon_1 = \min_{\epsilon>0} \epsilon \\ s.t. \\ \begin{bmatrix} \alpha_E + (\epsilon - a)\underline{\Theta}_{11} & -\frac{1}{2}\big(|b|\overline{\Theta}_{11} + |c|\overline{\Theta}_{22}\big) \\ -\frac{1}{2}\big(|b|\overline{\Theta}_{11} + |c|\overline{\Theta}_{22}\big) & -\alpha_I + (\epsilon - d)\underline{\Theta}_{22} \end{bmatrix} > 0 \end{gathered} \tag{31}$$

where a and d are diagonal elements of Ξ in (2) , and $0 < \underline{\Theta}_{ii} \leq A_{ii}(t) \leq \overline{\Theta}_{ii},\ i = 1, 2$ for all $t \geq 0$.

Proof: For the Wilson-Cowan model, the matrix $A(t) \in \mathbb{R}^{2\times 2}$ is a diagonal matrix that is given by (23). Due to Fact 1, there exists a non-zero positive scalar $\underline{\Theta}_{ii}, \overline{\Theta}_{ii}$ such that $0 < \underline{\Theta}_{ii} \leq A_{ii}(t) \leq \overline{\Theta}_{ii},\ i = 1, 2$ for all $t \geq 0$. Now, consider a candidate Lyapunov function $V(\tilde{y}_1, \tilde{y}_2) = \frac{1}{2}\tilde{y}_2^2 - \frac{1}{2}\tilde{y}_1^2$. Taking the derivative along the dynamics of the system leads to

$$\begin{aligned} \dot{V} =& \big(-\alpha_I + (\epsilon - d)A_{22}(t)\big)\tilde{y}_2^2 + \big(cA_{22}(t) + bA_{11}(t)\big)\tilde{y}_1\tilde{y}_2 \\ &- \big(-\alpha_E + (a - \epsilon)A_{11}(t)\big)\tilde{y}_1^2. \end{aligned} \tag{32}$$

If $\epsilon \geq \max\{a, d\}$, then

$$\begin{aligned} \dot{V} \geq& \big(-\alpha_I + (\epsilon - d)\underline{\Theta}_{22}\big)|\tilde{y}_2|^2 + \big(\alpha_E + (\epsilon - a)\underline{\Theta}_{11}\big)|\tilde{y}_1|^2 \\ &- \big(|c|\overline{\Theta}_{22} + |b|\overline{\Theta}_{11}\big)|\tilde{y}_1||\tilde{y}_2|\ . \end{aligned} \tag{33}$$

If (31) holds, then $\dot{V} > 0$. For large values of ϵ, the first and second terms of (33) are positive; hence (31) has a unique solution. In the set $\Omega \triangleq \{(\hat{y}_1, \hat{y}_2) \in \mathbb{R}^2 \mid |\hat{y}_2|^2 > |\hat{y}_1|^2\}$, the Lyapunov function is positive and, hence, according to the *Chetaev's* theorem [18], the origin of (30) is unstable. This completes the proof.

V. Simulation Results

In this section, we present simulation results to validate the proposed approach to the study of synchronization in a network of Wilson-Cowan models. It is known that for some set of variables, the Wilson-Cowan model exhibits an oscillatory behavior resulting from the existence of a stable limit cycle that appears from the Andronov-Hopf-bifurcation of equilibrium point [10]. In our simulations, we choose the following set of variables: $a = 10, b = 10, c = 10, d = -2, \rho_x = 2, \rho_y = -6$ that guarantees oscillatory behavior. First, we compute the matrix $A(t)$ in (13) using the oscillatory behavior of each node. In the next step, we study the effect of weak coupling on stability of (13). We then check the condition (18) in Proposition 1 and (20) in Proposition 2 using the numerical approach.

For a Wilson-Cowan model, the instability condition holds, which means that there exists an ϵ_w such that, for every interconnection gain smaller than this value, (13) is unstable. The value of ϵ_w is computed from the proof of Lemma 1 in Appendix VI. In our case, $\epsilon_w = 0.034$. In the next step, we apply Proposition 3 to find the minimum value of ϵ_s such that, for every $\epsilon \geq \epsilon_s$, (13) is unstable. Following the proof of the proposition, we obtain $\epsilon_s = 15$. Then, the computational algorithm in the Section IV-B is exploited for coupling strength between between ϵ_w and ϵ_s . Note that the matrix D can be written in the form of (26) as

$$M = \begin{bmatrix} \frac{\epsilon_s+\epsilon_w}{2} & 0 \\ 0 & -\frac{\epsilon_s+\epsilon_w}{2} \end{bmatrix}, B = \begin{bmatrix} \frac{\epsilon_s-\epsilon_w}{2} & 0 \\ 0 & \frac{\epsilon_s-\epsilon_w}{2} \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The first fifteen harmonics of a Fourier series are sufficient to approximate the oscillatory trajectories in order to find the truncated harmonic state space model.

To reduce numerical computation errors, we split the interval between $[0.034, 15]$ into five subintervals. The computational results show that (29) is stable for all $\epsilon \in [3.2, 14]$. The results of the analysis for low, intermediate, and strong coupling conclude that, for a given Laplacian graph L, the Wilson-Cowan network is synchronized if non-zero eigenvalues of the Laplacian matrix are in the interval $[3.2, 14]$. In order to verify this observation, we consider a network of Wilson-Cowan models with three nodes and the Laplacian matrix as

$$L = \begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix},$$

where k is the coupling strength. It is straightforward to see that the Laplacian matrix has three eigenvalues: $\lambda_1 = 0, \lambda_2 = \lambda_3 = 3k$. This implies that the underlying network becomes synchronized if $3k \in [3.2, 14]$. Figure 1 shows the trajectories of each node in the underlying network when the coupling strength is $k = 4$. It is observed that the network becomes synchronized for this coupling strength.

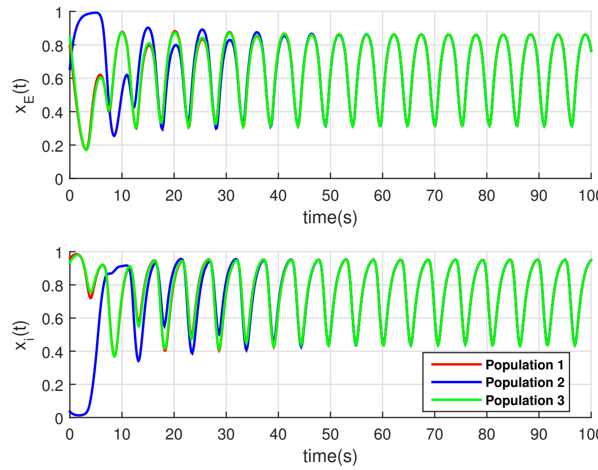

Fig. 1. The trajectories of populations in the Wilson-Cowan network with three populations with coupling gain $k = 4$.

VI. Conclusion

A new procedure was introduced to analyze synchronization for networks of homogeneous Wilson-Cowan models coupled via diffusive-type interconnections. Using the linearized model around a limit cycle, analytic results have been developed that can be utilized to check the existence or absence of synchronous behavior. We have proved that the network of Wilson-Cowan models always becomes desynchronized for large values of coupling strength. A computational approach has been presented for intermediate coupling strengths. As an example, we applied the proposed result to a network of diffusive Wilson-Cowan oscillators. Simulation results showed that the network becomes synchronized for a coupling strength at an intermediate value.

APPENDIX

Instability Using two-time averaging

Lemma 1: Consider the linear time-varying system described by the state space model

$$\begin{bmatrix} \dot{\hat{w}} \\ \dot{\hat{r}} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & G \end{bmatrix} - \epsilon \begin{bmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{bmatrix} \right) \begin{bmatrix} \hat{w} \\ \hat{r} \end{bmatrix}, \quad (34)$$

where $\hat{w} \in \mathbb{R}^p$ and $\hat{r} \in \mathbb{R}^q$. Assume that $\Psi_{ij}(t),\ i, j = 1, 2$ are T-periodic piecewise continuous functions bounded by the scaler $\iota_{ij} > 0$, i.e $|\Psi_{ij}(t)| \leq \iota_{ij}$ for all $t \geq 0$. Now consider the average system described by

$$\dot{\bar{w}} = -\epsilon\bar{\Psi}_{11}\bar{w} \quad , \quad \bar{\Psi}_{11} = \frac{1}{T}\int_0^T \Psi_{11}(\tau)d\tau. \quad (35)$$

If $\bar{\Psi}_{11}$ and G are Hurwitz, then there exists ϵ^* such that origin of the original system (34) is unstable for every $\epsilon \in (0, \epsilon^*)$.

Similar to [5], [13], we change the coordinate of system by defining the new state $\breve{w}$ as

$$\hat{w} \triangleq O\breve{w} = \left[I - \epsilon \int_0^T (\Psi_{11}(\tau) - \bar{\Psi}_{11}) d\tau \right] \breve{w}. \tag{36}$$

Note that since $|\int_0^T (\Psi_{11}(\tau) - \bar{\Psi}_{11}) d\tau| \leq 2T\iota_{11}$, if $\epsilon < \epsilon_1 \triangleq \frac{1}{2T\iota_{11}}$, then the inverse of O exists and, therefore, $|O^{-1}| \leq \gamma = \frac{1}{1-2\epsilon_1 T\iota_{11}}$. In this case, the original system (34) is represented in the new coordinates as

$$\begin{bmatrix} \dot{\breve{w}} \\ \dot{\hat{r}} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & G \end{bmatrix} - \epsilon \begin{bmatrix} \mathfrak{M}_{11}(t) & \mathfrak{M}_{12}(t) \\ \mathfrak{M}_{21}(t) & \mathfrak{M}_{22}(t) \end{bmatrix} \right) \begin{bmatrix} \breve{w} \\ \hat{r} \end{bmatrix}, \tag{37}$$

where

$$\begin{aligned} \mathfrak{M}_{11} &= O^{-1}\Psi_{11}(t)O + O^{-1}(\bar{\Psi}_{11} - \Psi_{11}(t)) \\ \mathfrak{M}_{12} &= O^{-1}\Psi_{12}(t) \\ \mathfrak{M}_{21} &= \Psi_{21}(t)O \\ \mathfrak{M}_{22} &= \Psi_{22}(t). \end{aligned} \tag{38}$$

Since the matrices $\bar{\Psi}_{11}$ and G are Hurwitz, there exists positive definite matrices P_w and P_r such that

$$\begin{aligned} P_w\bar{\Psi}_{11} + \bar{\Psi}_{11}^T P_w &= -I \\ P_r G + G^T P_r &= -I. \end{aligned} \tag{39}$$

Now, define the Lyapunov function as $V(\breve{w}, \hat{r}) = \breve{w}^T P_w \breve{w} - \hat{r}^T P_r \hat{r}$. Taking the derivative of the Lyapunov function leads to

$$\begin{aligned} \dot{V} \geq & (\epsilon\gamma - 4T\epsilon^2\gamma\iota_{11}^2|P_w|)|\breve{w}|^2 - 4\epsilon\gamma\iota_{12}|P_w||\breve{w}||\hat{r}| \\ & + (1 - 2\epsilon\iota_{22}|P_r|)|\hat{r}|^2 \\ & - 2\epsilon\iota_{21}(1 + 2\epsilon T\iota_{11})|P_r||\hat{r}||\breve{w}|. \end{aligned} \tag{40}$$

If $\epsilon \leq \frac{\epsilon_1}{2|P_w|}$ and $\epsilon \leq \epsilon_2 \triangleq \frac{1}{\iota_{22}|P_r|}$, the first and third terms of (40) are positive and hence there exists ϵ such that the derivative of the Lyapunov function is positive. Indeed, using the Schur complement, the positiveness of the right hand side of (40) is equivalent to existence of positive value ϵ_3 such that the following matrix inequality holds,

$$\begin{bmatrix} \epsilon\gamma - 4T\epsilon^2\gamma\iota_{11}^2|P_w| & \digamma \\ \digamma & 1 - 2\epsilon\iota_{22}|P_r| \end{bmatrix} > 0, \tag{41}$$

where $\digamma = -\big(\epsilon\gamma\iota_{12}|P_w| + \epsilon\iota_{21}(1 + 2\epsilon T\iota_{11})|P_r|\big)$. Now, let us define $\epsilon^* \triangleq \min(\epsilon_1, \frac{\epsilon_1}{2|P_w|}\epsilon_2, \epsilon_3)$. Then, for every $\epsilon \in (0, \epsilon^*)$, the inverse of the coordinate change exists and the derivative of the Lyapunov function is positive. In the set $\Omega \triangleq \{(\breve{w}, \hat{r}) \in \mathbb{R}^{p+q} \mid \lambda_{min}(P_w)\breve{w}^T\breve{w} > \lambda_{max}(P_z)\hat{r}^T\hat{r}\}$, the Lyapunov function is positive and hence according to Chetaev's theorem [18], the origin of (34) is unstable. This completes the proof.

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