State estimation of non-linear systems over random access wireless networks*

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Abstract— **We study emulation-based state estimation for non-linear plants that communicate with a remote observer over a shared wireless network subject to packet losses. To reduce bandwidth usage, a stochastic communication protocol is employed to determine which node should be given access to the network. We describe the overall wireless system as a hybrid model, which allows us to capture the behaviour both between and at transmission instants, whilst covering network features such as random transmission instants, packet losses, and stochastic scheduling. Under this setting, we provide sufficient conditions on the transmission rate that guarantee an input-to-state stability property for the corresponding estimation error system. We illustrate our results with an example of Lipschitz non-linear plants.**

I. Introduction

Advances in wireless technology are revolutionising how control systems exchange information with physical processes [1]. In wireless networked control systems (WNCSs), the sensor and actuator information is transmitted to a remote controller over a shared wireless network. WNCSs provide several advantages over control systems based on wired counterparts, e.g. improved flexibility, reduced maintenance costs, and simple deployment of additional measurement points. Particularly, wireless technology has proven to be successful in replacing wired control systems with wireless ones even in complex industrial environments, see e.g. [2]. However, wireless networks also introduce communication constraints that include, but are not limited to, packet losses, data collisions, time-varying transmission instants, and delays. Therefore, in order to have efficient design solutions for these systems, the wireless network and its corresponding constraints have to be considered explicitly in the analysis.

We restrict our attention to wireless networks subject to random packet losses, random transmission instants, and that implement stochastic scheduling protocols to avoid data collisions. Communication protocols determine which node should obtain access to the shared wireless network at a particular transmission instant. In the literature, we can find static protocols like round-robin [3], in which nodes are assigned to a particular timeslot in a predetermined and cyclic manner, and dynamic protocols like try-once-discard [4], where the node with the greatest weighted error will be granted access to the network. Moreover, stochastic (also called random access) protocols can also be used to avoid collisions [5], and these have received a lot of attention recently in the literature since they model carrier-sense multiple access with collision avoidance (CSMA/CA) protocols that arise in IEEE 802.15.4-based industrial wireless networks such as WirelessHART for instance [6], [7].

Stochastic protocols have been widely studied in the context of control design of linear WNCS in [8], [9], [10], [11], [12], in [13] for a class of non-linear WNCS, and our previous works [5], [7] for general non-linear WNCS. Recently, stochastic protocols have also been studied for the remote state estimation problem, see e.g. [14], [15] for results on linear systems, and [16], [17], [18], [19] for results on special classes of non-linear systems. Particularly, all these works focus on discrete-time models (either linear or special class of non-linear) for the underlying WNCS, and consider equidistant and deterministic transmissions for the network. Additionally, none of these works considered packet losses in their analysis. To the best of our knowledge, there are no available results in the literature that provide a fully non-linear framework for the remote state estimation of WNCS under stochastic scheduling protocols, random packet losses, and random transmission instants.

In response to the above discussion, this paper provides an emulation-based framework for state estimation of general non-linear plants that communicate with the observer via a wireless network that adopts a stochastic protocol and is subject to random packet losses and transmissions. As our first contribution, we propose a hybrid model for the WNCS that captures the continuous dynamics of both the plant and observer, and the discrete dynamics of the network in terms of transmissions and dropouts. This class of models encompasses both the linear and classes of non-linear models found in the aforementioned literature, and it allows us to capture in a higher-fidelity fashion the effects of the underlying wireless network. Our modelling tools are inspired by the $\mathcal{L}_p$ stability work in [5], where a similar stochastic model is presented for the controller design problem. We emphasise that there are some important technical differences in the analysis since the observer design problem considered in this paper requires convergence of the estimation error (in absence of inputs) as opposed to only $\mathcal{L}_p$ stability, and this has to be handled carefully given the random transmission instants and dropouts. Using the obtained WNCS model, we provide a sufficient condition on the rate of transmission that ensures an input-to-state stability (ISS) property on the corresponding estimation error system. This sufficient condition translates into a bound on the transmission rate under which, in absence of inputs, the mean of the estimation error converges to zero.

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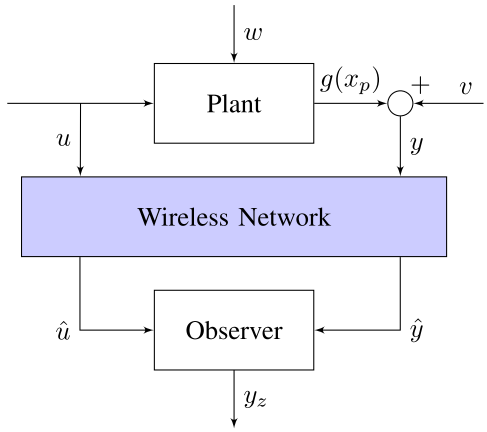

Fig. 1. Block diagram of our estimation architecture.

Notation: Let $\mathbb{R}_{>0} := (0, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{N} := \{0, 1, 2, \dots\}$, and $\mathbb{N}_{>0} := \mathbb{N} \backslash \{0\}$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\overline{x} = (|x_1|, \dots, |x_n|)$. For any matrix M, the entries of $\overline{M}$ are the absolute values of the corresponding entries of M. Given $f : \mathbb{R} \to \mathbb{R}^n$, $\|f\|_{\mathcal{L}_\infty} := \operatorname{ess\,sup}_{t \in \mathbb{R}} |f(t)|$, and $\|f\|_{\mathcal{L}_\infty[a,b]} := \operatorname{ess\,sup}_{t \in [a,b]} |f(t)|$. Also, $f(t^+) := \lim_{s \to t, s > t} f(s)$. Given $a \in (0, \infty]$, $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ with $a = \infty$, and unbounded. For $a, b \in (0, \infty]$, $\gamma : [0, a) \times [0, b) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KK}$ if, for any $(s_1, s_2) \in [0, a) \times [0, b)$, $\gamma(s_1, \cdot), \gamma(\cdot, s_2) \in \mathcal{K}$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$, $\forall t \geq 0$, and if $\beta(s, \cdot)$ is non-increasing and satisfies $\lim_{t \to \infty} \beta(s, t) = 0$, $\forall s \geq 0$.

II. Problem Setting

We study the scenario depicted in Fig. 1, where a non-linear plant communicates with an observer via a random access wireless network subject to packet losses. The plant model is given by

$$\dot{x}_p = f(x_p, u, w), \quad y = g(x_p) + v, \tag{1}$$

where $x_p \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the external disturbance, $y \in \mathbb{R}^{n_y}$ is the plant output affected by the noise $v \in \mathbb{R}^{n_y}$, and $n_x, n_u, n_w, n_y \in \mathbb{N}_{>0}$. The functions $u : \mathbb{R} \to \mathbb{R}^{n_u}$ and $v : \mathbb{R} \to \mathbb{R}^{n_v}$ are Lebesgue measurable and differentiable almost everywhere. Moreover, these functions and their time-derivatives are assumed to have a finite $\mathcal{L}_\infty$ norm.

To design the observer we adopt an emulation-based approach. That is, we first design an observer that estimates the state of the plant (1)—in some appropriate sense—without taking into account the wireless network. Any of the available techniques for non-linear observer design can be used at this stage. For the sake of generality, we assume the designed observer has the form

$$\dot{z} = f_z(z, u, y - y_z), \ \tilde{x}_p = g_z(z), \ y_z = g(\tilde{x}_p), \tag{2}$$

where $z \in \mathbb{R}^{n_z}$ is the observer state, $n_z \in \mathbb{N}_{>0}$, $\tilde{x}_p \in \mathbb{R}^{n_x}$ is the estimate of the state x_p, and $y_z \in \mathbb{R}^{n_y}$ is the output estimate. Note that we allow the dimension of the observer to be different than the system dimension, hence covering immersion-based and reduced-order observers for instance. It is implicit in (2) that we measure u and y, whereas w and v are unmeasured. The stability property we will prove is natural for this setting.

In the next stage of emulation, the observer is implemented over the network as per Fig. 1, and thus the goal of this paper is to provide conditions on the observer and the network under which the state estimate $\tilde{x}_p$ (approximately) converges to the state of the plant x_p. We emphasise that in the wireless scenario in Fig. 1, the observer has no longer access to (y, u) as in the classical state estimation case, but to the wireless versions $(\hat{y}, \hat{u})$ that are discussed next.

A. Wireless network dynamics

Before introducing the dynamics of the wireless signals $(\hat{y}, \hat{u})$, we define the transmission times and the packet loss process. In wireless networks, because of synchronisation routines, acknowledgement packets, waiting times, etc., transmissions instants are naturally stochastic.

Assumption 1: Consider a Poisson point process $r(t)$ with rate $\boldsymbol{\lambda}$ that satisfies $r(t) = 0$ for $t \in [0, t_0)$ and $r(t) = k$ for $t \in [t_{k-1}, t_k)$. The sequence of transmission instants $\{t_k\}_{k \in \mathbb{N}}$ is defined inductively by: $t_0 = \tau_0$ with $\tau_0 \sim \text{Exp}(\boldsymbol{\lambda})$, and for each $k \in \mathbb{N}_{>0}$, $t_k = t_{k-1} + \tau_k$, with $\tau_k \sim \text{Exp}(\boldsymbol{\lambda})$, where the sequence $\{\tau_k\}_{k \in \mathbb{N}}$ is i.i.d. $\square$

The times $\{t_k\}_{k \in \mathbb{N}}$ are also called *arrival times* in the literature, $\{\tau_k\}_{k \in \mathbb{N}}$ are called *intertransmission times* (or *interarrival times*), $\bar{\tau} := 1/\boldsymbol{\lambda}$ represents the *average inter-transmission time*, and $\boldsymbol{\lambda}$ is the *arrival rate*. Throughout this paper, we will use the terms arrival rate and transmission rate interchangeably. The exponential distribution that governs each τ_k describes the time between transmissions. Poisson processes are a natural modelling tool in communication network engineering and applied probability problems [20].

To model packet loss, we introduce the Bernoulli process $\{\theta_k\}_{k \in \mathbb{N}}$, which satisfies $\theta_k = 1$ for a successful transmission with probability of success equal to p, and $\theta_k = 0$ if the packet is lost with probability $1 - p$. We assume that θ_k is independent of τ_k. Future work will focus on generalising this model to Markovian packet losses.

We next present the dynamics of $\hat{y}$ and $\hat{u}$. We consider a scenario where the sensors and actuators of the plant are grouped into N *nodes* which are connected to the network. At each transmission instant t_k, $k \in \mathbb{N}$, only one node is granted access to the network according to a stochastic scheduling protocol, which will be described further below. Let $(y, u) = (y_1, \dots, y_{n_y}, u_1, \dots, u_{n_u})$ and $(\hat{y}, \hat{u}) = (\hat{y}_1, \dots, \hat{y}_{n_y}, \hat{u}_1, \dots, \hat{u}_{n_u})$. Suppose that the node $j \in \{1, \dots, N\}$, $N \in \{1, \dots, n_y + n_u\}$, is granted access to the network at time t_k, and say that the components y_{j_y} and u_{j_u} of y and u, respectively, are associated to node j, with $j_y \in \{1, \dots, n_y\}$ and $j_u \in \{1, \dots, n_u\}$. Then,

$$(\hat{y}_{j_y}(t_k^+), \hat{u}_{j_u}(t_k^+)) = \begin{cases} (y_{j_y}(t_k), u_{j_u}(t_k)), & \text{if } \theta_k = 1, \\ (\hat{y}_{j_y}(t_k), \hat{u}_{j_u}(t_k)), & \text{if } \theta_k = 0. \end{cases}$$

That is, if a node is granted access to the network, the corresponding components of the received signal are updated with the sent signal provided the transmission was successful ($\theta_k = 1$), and the components are kept unchanged in case of a packet loss ($\theta_k = 0$). On the other hand, every other node that was not granted access to the network satisfies, for any $\theta_k \in \{0,1\}$, $(\hat{y}_{i_y}(t_k^+), \hat{u}_{i_u}(t_k^+)) = (\hat{y}_{i_y}(t_k), \hat{u}_{i_u}(t_k))$, for $i_y \neq j_y$ and $i_u \neq j_u$.

Between transmission instants, $\hat{y}$ and $\hat{u}$ are generated according to the in-network processing implementation. For simplicity, we use zero-order hold devices which translates into $\dot{\hat{y}} = 0$ and $\dot{\hat{u}} = 0$ for $t \in [t_k, t_{k+1}]$ and $k \in \mathbb{N}$. Note that this can be easily relaxed as in e.g. [21].

B. *Observer emulation*

Since the designed observer (2) is implemented over the wireless network, it no longer receives (y, u) but $(\hat{y}, \hat{u})$ according to the dynamics in Section II-A. This implies that

$$\dot{z} = f_z(z, \hat{u}, \hat{y} - \hat{y}_z). \tag{3}$$

Furthermore, we note that (3) does not depend on its output y_z, as in (2), but on $\hat{y}_z$, which is an artificially introduced networked version of y_z. The idea of using $\hat{y}_z$ instead of y_z was suggested in [22, Section VIII] and it allows stronger stability properties for the estimation error system to be established. A similar idea is proposed in [23] for the design of high-gain observers. The variable $\hat{y}_z$ is constructed to evolve along the same vector field as $\hat{y}$ between two successive transmission instants, i.e. $\dot{\hat{y}}_z = 0$ for $t \in [t_k, t_{k+1}]$. Let $y_z = (y_{z,1}, \ldots, y_{z,n_y})$ and $\hat{y}_z = (\hat{y}_{z,1}, \ldots, \hat{y}_{z,n_y})$. At each successful transmission of a component of $\hat{y}$, say $\hat{y}_{j_y}$ with $j_y \in \{1, \ldots, n_y\}$, the corresponding component of $\hat{y}_z$, that is $\hat{y}_{z,j_y}$, is reset to y_{z,j_y}, that is

$$\hat{y}_{z,j_y}(t_k^+) = \begin{cases} y_{z,j_y}(t_k), & \text{if } \hat{y}_{j_y}(t_k^+) = y_{j_y}(t_k), \\ \hat{y}_{z,j_y}(t_k), & \text{otherwise.} \end{cases} \tag{4}$$

Remark 1: The stochastic setting we consider in this paper is the estimation counterpart to the controller design setup in [5]. It also extends our previous deterministic work [24] on observer design to the case with random packet losses, random transmission instants, and stochastic protocols. □

III. Random access scheduling

We now introduce the stochastic scheduling mechanism. To facilitate the analysis, we define the notion of *network-induced error*. Particularly, the network-induced error on the plant output is defined as $e^y := \hat{y} - y$, and the error on the input is $e^u := \hat{u} - u$. We also define a network-induced error on the observer output $e^{y_z} := \hat{y}_z - y_z$. Using these definitions, we rewrite (3) as

$$\dot{z} = f_z(z, u + e^u, y - y_z + e^y - e^{y_z}). \tag{5}$$

We can see that the dynamics of the observer are affected by $e^y - e^{y_z}$ and e^u, thus we define the overall network-induced error as $e := (e^y - e^{y_z}, e^u)$, which is partitioned into N nodes as $e = (e_1, \ldots, e_N)$. The introduction of the network-induced error allows us to model transmissions in a simpler way. Specifically, if at transmission instant t_k node $j \in \{1, \ldots, N\}$ is granted network access, and if the transmission is successful, then the dynamics in Section II-A imply the corresponding error component satisfies $e_j(t_k^+) = 0$, while the remaining components are kept unchanged.

As already mentioned, we consider a scenario where transmissions in the wireless network are governed by a random access protocol, also known as stochastic protocol [5]. We define it as follows: At each transmission instant, and randomly, only one of the contending nodes $\{1, \ldots, N\}$ will get network access. Given the network dynamics in Section II-A, we can formally write an equation that describes this mechanism. That is,

$$e(t_k^+) = \theta_k \mathcal{H}_k e(t_k) + (1 - \theta_k) e(t_k), \tag{7}$$

where $\{\mathcal{H}_k\}_{k \in \mathbb{N}}$ are i.i.d. random matrices taking values in the finite set $\mathbb{M} := \{M_1, \ldots, M_N\}$, where each M_j, $j \in \{1, \ldots, N\}$, is such that $M_j e = (e_1, \ldots, e_{j-1}, 0, e_{j+1}, \ldots, e_N)$. In other words, M_j represents node j having access to the network. We emphasise that (7) captures all the wireless network dynamics previously described in Section II-A.

We assume that each node is equally likely to be granted access by the network, i.e. $\Pr\{\mathcal{H}_k = M_j\} = 1/N$. Our results do not rely on this assumption, and a different probability distribution could be assumed. However, choosing different access probabilities for each node may lead to some nodes being favoured over others during contention, which is not the case in some wireless networks like e.g. WirelessHART. Let $Q_k := \theta_k \mathcal{H}_k + (1 - \theta_k) I$. Then, $\Pr\{Q_k = M_j\} = p/N$ for $j \in \{1, \ldots, N\}$, and $\Pr\{Q_k = I\} = 1 - p$. That is, the probability that, at some t_k, a node j is granted access and transmits successfully is equal to p/N. The packet is dropped with probability $1 - p$ otherwise.

In both controller design [25] and observer design [24] without packet losses, the concept of persistently exciting (PE) protocols was adopted in the analysis. PE essentially is a property which ensures every node in the network has transmitted at least once within a finite number of transmissions. Since our setting deals with stochastic packet losses, stochastic transmissions, and a stochastic protocol, we will adopt the stochastic counterpart to PE, which is the so-called *almost surely covering* property introduced in [5].

Definition 1 (Cover time): For any $k \in \mathbb{N}$, the k-th *cover time*, denoted by T_k, is defined as $T_k := \min\big\{j \geq 1 : \{M_1, \ldots, M_N\} \subset \{Q(T_{k-1}), \ldots, Q(T_{k-1} + j - 1)\}\big\}$, and $T_{-1} = 0$. □

Definition 2 (Almost surely covering protocol): A stochastic protocol is *a.s. covering* if $\Pr\{T_k < \infty\} = 1$, $\forall k \geq 0$, with T_k as per Definition 1. □

That is, an a.s. covering protocol will grant network access to every node, at least once, within a finite number of transmissions with probability one, which is the natural extension of the deterministic PE property. We note that (7) satisfies Definition 2 provided $p > 0$.

$$f_\chi(\chi,z,e,u,v,w) \coloneqq f(\chi+g_z(z),u,w) - \frac{\partial g_z}{\partial z} f_z\left(z, u+e^u, g(\chi+g_z(z)) + v - g(g_z(z)) + e^y - e^{y_z}\right), \tag{6a}$$

$$f_z(\chi,z,e,u,v) \coloneqq f_z\left(z, u+e^u, g(\chi+g_z(z)) + v - g(g_z(z)) + e^y - e^{y_z}\right), \tag{6b}$$

$$g_e(\chi,z,e,u,v,w,d) \coloneqq \Big(-\frac{\partial g}{\partial x_p} f(\chi+g_z(z),u,w) - d_v + \frac{\partial g}{\partial \tilde{x}_p}\frac{\partial g_z}{\partial z} f_z(z, u+e^u, g(\chi+g_z(z)) + v - g(g_z(z)) + e^y - e^{y_z}), -d_u \Big). \tag{6c}$$

IV. INPUT-TO-STATE STABILITY

A. Hybrid model

Given the continuous dynamics from the plant/controller, and the discrete dynamics introduced by transmissions and packet losses, we model the WNCS in Fig. 1 as a hybrid system. We introduce the estimation error $\chi \coloneqq x_p - \tilde{x}_p$, and $d \coloneqq (d_u, d_v)$, where $d_u \coloneqq \dot{u}$ and $d_v \coloneqq \dot{v}$. Then, by using the system description in Section II, we can write

$$\dot{\chi} = f_\chi(\chi,z,e,u,v,w), \qquad \forall t \in [t_k, t_{k+1}], \tag{8a}$$
$$\dot{z} = f_z(\chi,z,e,u,v), \qquad \forall t \in [t_k, t_{k+1}], \tag{8b}$$
$$\dot{e} = g_e(\chi,z,e,u,v,w,d), \qquad \forall t \in [t_k, t_{k+1}], \tag{8c}$$
$$\chi(t_k^+) = \chi(t_k), \tag{8d}$$
$$z(t_k^+) = z(t_k), \tag{8e}$$
$$e(t_k^+) = Q_k e(t_k), \tag{8f}$$

where f_χ, f_z and g_e are defined in (6). The model (8) captures all dynamics of the elements in the WNCS from Fig. 1 both between– and at–transmission instants.

We are interested in different properties for the χ–system and the z–system. In particular, we want to prove a convergence property for the estimation error χ, but only some well defined or bounded behaviour for all time is desired for the observer state z. We emphasise that it is only the network-induced constraints (i.e. stochastic protocol, random transmissions, and random dropouts) that introduce randomness in our models, and the exogenous disturbances are $\mathcal{L}_\infty$ signals as stated in Section II.

B. Stability analysis

We now use the WNCS model (8) to provide sufficient conditions that guarantee an ISS stability property in expectation for the estimation error system. First, we state the underlying assumptions.

Assumption 2: There exists an $n_e \times n_e$ matrix A with nonnegative entries and a continuous function $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_u+n_v} \to \mathbb{R}^{n_e}_{\geq 0}$ such that the error dynamics (8c) satisfy

$$\overline{g}_e(\chi,z,e,u,v,w,d) \preceq A\overline{e} + \tilde{y}(\chi,z,u,v,w,d), \tag{9}$$

for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, e \in \mathbb{R}^{n_e}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}$, and $d \in \mathbb{R}^{n_u+n_v}$. □

Assumption 2 is the vector analogue of the standard dissipation-type inequalities often adopted for the $\dot{e}$ dynamics in controller and observer design, see e.g. [25], [26].

Assumption 3: There exists $\gamma_2^\chi \in \mathbb{R}_{\geq 0}$ and $\sigma \in \mathcal{K}_\infty$ such that $\tilde{y}$ in (9) satisfies $|\tilde{y}(\chi,z,u,v,w,d)| \leq \gamma_2^\chi|\chi| + \sigma(|(u,v,w,d)|)$, for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}$, and $d \in \mathbb{R}^{n_u+n_v}$. □

Next, we assume the observer was designed appropriately so that the estimation error converges in absence of external disturbances, noise, and network.

Assumption 4: There exists $\beta_1 \in \mathcal{KL}, \gamma_1^e \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathcal{K}_\infty$ such that, for any $\chi(0) \in \mathbb{R}^{n_x}$ and $(e,v,w) \in \mathcal{L}_\infty$, solutions to (8a) satisfy, for all $t \geq 0$,

$$\begin{aligned}\mathbf{E}\{|\chi(t)|\} &\leq \beta_1(|\chi(0)|,t) + \gamma_1^e \mathbf{E}\left\{\|e\|_{\mathcal{L}_\infty[0,t]}\right\} + \mu\left(\|(v,w)\|_{\mathcal{L}_\infty[0,t]}\right),\\ \mathbf{E}\left\{\|\chi\|_{\mathcal{L}_\infty[0,t]}\right\} &\leq \beta_1(|\chi(0)|,0) + \gamma_1^e \mathbf{E}\left\{\|e\|_{\mathcal{L}_\infty[0,t]}\right\} + \mu\left(\|(v,w)\|_{\mathcal{L}_\infty[0,t]}\right).\end{aligned}$$

□

These ISS conditions are the stochastic counterpart of the ones often adopted when using emulation-based design, see e.g. [21], [24]. These conditions are satisfied in different scenarios, and we present an example in Section V.

Given Assumptions 2–4, the stability of χ– and e–dynamics can be studied separately from the system interconnection (8). Since the z–dynamics of the observer do not necessarily have to converge, we ensure they possess an appropriate behaviour in the sense of no finite escapes for bounded inputs as per the below assumption.

Assumption 5: System (8b) is forward complete in mean with input $(\chi,e,u,v,w) \in \mathcal{L}_\infty$ [27]. That is, there exist $\nu_1,\nu_2,\nu_3 \in \mathcal{K}$ and $c \in \mathbb{R}_{\geq 0}$ such that, for any $z(0) \in \mathbb{R}^{n_z}$ and $(\chi,e,u,v,w) \in \mathcal{L}_\infty$, $\mathbf{E}\{|z(t)|\} \leq \nu_1(t) + \nu_2(|z(0)|) + \nu_3(\mathbf{E}\left\{\|(\chi,e,u,v,w)\|_{\mathcal{L}_\infty[0,t]}\right\}) + c$, for all $t \geq 0$. □

We are now ready to state the main stability results (proofs are omitted due to space constraints). Essentially, we consider system (8) as the interconnection of three subsystems in χ, z and e, and apply small-gain arguments to conclude an ISS stability property for the overall system. We first show that the e–subsystem is IOS with respect to $\tilde{y}$.

Proposition 1: Suppose Assumptions 1–2 hold. Under the stochastic protocol in Section III, there exists a rate of transmission $\boldsymbol{\lambda}$ that satisfies $\rho(\boldsymbol{\lambda}) < 1$, $\boldsymbol{\lambda} > N|A|/p$, where

$$\rho(\boldsymbol{\lambda}) \coloneqq \prod_{j=1}^N \frac{(N-(j-1))p}{N(1-|A|/\boldsymbol{\lambda}-(1-p))-(j-1)p} - 1,$$

such that the e-subsystem (8c), (8f) is input-to-output stable (IOS) in mean from $\tilde{y}$ to e with linear gain

$$\tilde{\gamma}(\boldsymbol{\lambda}) = \frac{\mathbf{E}\{T_k\}(1+\rho(\boldsymbol{\lambda}))}{(\boldsymbol{\lambda}-|A|)(1-\rho(\boldsymbol{\lambda}))}. \tag{10}$$

Formally, $\exists\beta_2 \in \mathcal{KL}$ such that, for all $t \geq 0$,

$$\mathbf{E}\{|e(t)|\} \leq \beta_2(|e(0)|, t)\tilde{\gamma}(\boldsymbol{\lambda})\mathbf{E}\left\{\|\tilde{y}\|_{\mathcal{L}_\infty[0,t]}\right\}. \tag{11}$$

$\square$

We note that $\mathbf{E}\{T_k\}$ is the average of the cover time from Definition 1, and it can be computed as $\mathbf{E}\{T_k\} = NH_N/p$,where $H_N := \sum_{j=1}^{N} 1/j$ is the N-th harmonic number; the proof is not included for space constraints.

Proposition 1 provides sufficient conditions on the arrival rate $\boldsymbol{\lambda}$ so that the e–subsystem is IOS w.r.t. $\tilde{y}$. Since the emulation design ensures ISS properties on the χ–subsystem by means of Assumption 4, and the z–dynamics behave nicely via Assumption 5, we can now state the ISS property for the overall system (8) via a small-gain theorem.

Theorem 1: Suppose Assumptions 1–5 hold. Then, there exists $\boldsymbol{\lambda}^* \in (0, \infty)$ that solves $\tilde{\gamma}(\boldsymbol{\lambda}^*)\gamma_2^{\chi}\gamma_1^{e} = 1$, and for any choice of arrival rate satisfying $\boldsymbol{\lambda} > \boldsymbol{\lambda}^*$, the following holds.

(i) There exist $\beta \in \mathcal{KL}$, $\eta_1 \in \mathcal{K}$, and $\eta_2 \in \mathcal{KK}$ such that

$$\begin{aligned}&\mathbf{E}\{|(\chi(t), e(t))|\} \leq \beta(|(\chi(0), e(0))|, t)\\ &+ \eta_1(\|(v, w)\|_{\mathcal{L}_\infty}) + \eta_2(1/\boldsymbol{\lambda}, \|(u, v, w, d)\|_{\mathcal{L}_\infty}),\end{aligned} \tag{12}$$

for all $(\chi(0), e(0)) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e}$, $(u, v, w, d) \in \mathcal{L}_\infty$, and $t \geq 0$.

(ii) System (8) is forward complete in mean with input $(u, v, w, d) \in \mathcal{L}_\infty$. $\square$

Theorem 1 shows that the mean of both the estimation error χ and the network-induced error e converge to a ball centred at the origin, and whose radius depends on the $\mathcal{L}_\infty$ norm of the input (u, v, w, d). When disturbances and noises are absent, i.e. $w = v = 0$, the mean of the estimation error—*a priori*—does not converges to the origin, since in this case, $\mathbf{E}\{|(\chi(t), e(t))|\} \leq \beta(|(\chi(0), e(0))|, t) + \eta_2(1/\boldsymbol{\lambda}, \|(u, d_u)\|_{\mathcal{L}_\infty})$. However, we can always increase the arrival rate $\boldsymbol{\lambda}$ so that η_2 is small, and thus the effect of (u, d_u) is reduced. Moreover, if the observer had access to the exact u (instead of $\hat{u}$ as before) then the η_2–term would not appear in (12). Lastly, we note that in absence of inputs, i.e. $w = v = u = 0$, the mean of the estimation error indeed asymptotically converges to the origin in (12).

V. Example

A. A class of plants and observers

Consider the following class of Lipschitz non-linear plants

$$\dot{x}_p = A_p x_p + \pi(x_p) + w, \quad y = Cx_p + v, \tag{13}$$

where A_p, C are constant matrices of appropriate dimensions, and $\pi : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is globally Lipschitz with constant $\Pi \geq 0$. We design a high-gain observer of the form $\dot{z} = A_p z + \pi(z) + K(y - y_z)$, $y_z = Cz$, where $z \in \mathbb{R}^{n_x}$ is the estimate of state x_p, and $K \in \mathbb{R}^{n_x \times n_y}$ is designed such that the estimation error converges to zero in absence of disturbances and network. That is, K should be such that the below assumption holds.

Assumption 6: There exists $K \in \mathbb{R}^{n_x \times n_y}$ and a real symmetric positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ such that for $V : \chi \mapsto \chi^T P\chi$, $\langle\nabla V(\chi), (A_p - KC)\chi + \pi(\chi + z) - \pi(z)\rangle \leq -cV(\chi)$ for all $\chi \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_x}$ and some $c \in \mathbb{R}_{>0}$. $\square$

Assumption 6 holds if there exist matrices $P = P^T > 0$ and R of adequate dimensions so that the following LMI condition is satisfied (see e.g. [28], [29]),

$$\begin{bmatrix} A_p^T P + PA_p - R^T C - C^T R + I_{n_x} & P \\ P & -(1/\Pi^2)I_{n_x} \end{bmatrix} < 0.$$

Then, the stabilising gain K is given by $K = P^{-1}R^T$.

We next show that all conditions enabling the use of our main theorem, i.e. Theorem 1, hold for the considered class of systems. Since system (13) has no control input u, the network-induced error is given by $e = e^y - e^{y_z}$. Therefore, the continuous dynamics for the estimation error χ, observer state z, and network-induced error e, are

$$\begin{aligned}\dot{\chi} &= (A_p - KC)\chi + \pi(\chi + z) - \pi(z)\\ &\quad - K(e + v) + w, &\text{(14a)}\\ \dot{z} &= A_p z + KC\chi + \pi(z) + K(e + v), &\text{(14b)}\\ \dot{e} &= -C(A_p - KC)\chi - C(\pi(\chi + z) - \pi(z))\\ &\quad + CK(e + v) - Cw - d_v. &\text{(14c)}\end{aligned}$$

We note from (14) that Assumption 2 holds with $A = \overline{CK}$ and $\tilde{y} = \overline{\tilde{y}_1}$, where $\tilde{y}_1 := -C(A_p - KC)\chi - C(\pi(\chi + z) - \pi(z)) + CKv - Cw - d_v$. With $\tilde{y}$ as above, we can see that Assumption 3 holds with $\gamma_2^{\chi} = |C(A_p - KC)| + \Pi|C|$ and $\sigma(s) = (|CK| + |C| + 1)s$. With the Lyapunov function $V(\chi) = \chi^T P\chi$, computing $\dot{V}$ along the solutions of (14a) together with Assumption 6 allows us to conclude that Assumption 4 holds with $\gamma_1^{e} = \frac{4|PK|}{c\lambda_{\min}(P)}$, $\beta_1(s, t) = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\exp(-ct/8)s$, and $\mu(s) = \frac{4}{c\lambda_{\min}(P)}\max\{|PK|, |P|\}2s$ (see the proof of Proposition 3 in [24] for similar steps). Lastly, Assumption 5 follows directly from the fact that $\pi(\cdot)$ is globally Lipschitz [30]. We have thus verified that the class of Lipschitz non-linear plants and high-gain observers satisfies all the conditions of Theorem 1, which we will use below to compute $\boldsymbol{\lambda}^*$ that ensures the ISS property (12).

B. Single-link rigid robot manipulator

Consider a single-link rigid robot manipulator, whose model belongs to the class (13) with $A_p = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $\pi(x_p) = \begin{bmatrix} 0 \\ -0.1\cos(x_{p,1}) \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, see e.g. [31]. Note that the Lipschitz constant is equal to $\Pi = 0.1$. We use YALMIP to solve the LMI above and obtain the gain $K = \begin{bmatrix} 1.26 & 0.25 \\ 0.25 & 0.51 \end{bmatrix}$, which leads to $c = 1.98$ in Assumption 6. Additionally, consider a wireless network with $N = 2$ nodes (i.e. one for each sensor) and probability of successful transmission $p = 0.7$ (i.e. an average of 30% packet loss). For this WNCS, $\gamma_2^{\chi} = 2.88$ and $\gamma_1^{e} = 2.69$ using the formulas obtained above. To compute the rate of transmission in Theorem 1, we numerically solve $\tilde{\gamma}(\boldsymbol{\lambda}^*)\gamma_2^{\chi}\gamma_1^{e} = 1$ to find that $\boldsymbol{\lambda}^* = 52$ ensures the ISS property (12) for a given probability of success $p = 0.7$. In other words, we need

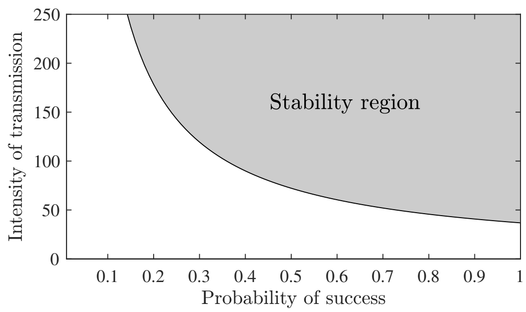

Fig. 2. Region in which the ISS property (12) on the estimation error system holds.

to transmit every $\bar{\tau} = 1/\boldsymbol{\lambda}^* = 19[ms]$, on average, to ensure our state estimation approximates the state of the plant (13) on average, over a random access wireless network with packet loss.

More generally, we compute the arrival rate bound $\boldsymbol{\lambda}^*$ as above for any $p \in (0, 1]$, which leads to Fig. 2. It can be seen that a lower probability of success requires faster transmissions to have effective estimation. On the other hand, a higher quality channel ensures an effective estimation with less frequent transmissions.

VI. Conclusion

We proposed an emulation-based framework for state estimation of general non-linear plants subject to external disturbances and measurement noise, and that communicate with the observer over a wireless network. The network adopts a stochastic protocol for the nodes, and it is subject to both random packet losses and random transmission instants. We provided sufficient conditions on the rate of transmission that ensure convergence of the mean of the estimation error under the network-induced constraints.

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