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Measurement of the branching fraction of the decay $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ in fully reconstructed events at Belle

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We present an analysis of the exclusive $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ decay, where ℓ represents an electron or a muon, with the assumption of charge-conjugation symmetry and lepton universality. The study exploits the full $\Upsilon(4S)$ data sample collected by the Belle detector, corresponding to 711 fb^{-1} of integrated luminosity. Events are selected by fully reconstructing one B meson in hadronic decay modes, subsequently determining the properties of the other B meson. We extract the signal yields using a binned maximum-likelihood fit to the missing-mass squared distribution in bins of the invariant mass of the two pions or the momentum transfer squared. We measure a total branching fraction of $\mathcal{B}(B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell) = [22.7_{-1.6}^{+1.9}(\text{stat}) \pm 3.5(\text{syst})] \times 10^{-5}$, where the uncertainties are statistical and systematic, respectively. This result is the first reported measurement of this decay.

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I. INTRODUCTION

The reported measurements of exclusive semileptonic $b \rightarrow u\ell\nu_\ell$ decays, with ℓ being either a muon or an electron, do not saturate the inclusive charmless semileptonic $b \rightarrow u\ell\nu_\ell$ decay rate. Summing up all observed exclusive modes, only about 25% of the inclusive rate can be accounted for [1]. The remaining modes pose a sizeable source of systematic uncertainty in inclusive and exclusive semileptonic $b \rightarrow u\ell\nu_\ell$ measurements or in decays in which such processes constitute important backgrounds. The magnitude of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$ [2,3] can be precisely determined by combining measured branching fractions with predictions for the total rate. Three direct methods are considered to be mature at the present time: first, combining the measured branching fraction of $B \rightarrow \pi\ell\bar{\nu}_\ell$ with lattice quantum chromodynamics (QCD) information to determine $|V_{ub}|$ and the nonperturbative form factors in a global fit [1,4]; second, measurement of the inclusive charmless semileptonic branching fraction, which is combined with calculations of the decay rate at next-to-next-leading order in QCD plus nonperturbative parameters, determined in global fits to $b \rightarrow c$ semileptonic decays [1,4]; and last, combining the measured ratio of branching fractions of $\Lambda_b \rightarrow p\ell\bar{\nu}_\ell$ and $\Lambda_b \rightarrow \Lambda_c\ell\bar{\nu}_\ell$ with lattice QCD information to extract the ratio $|V_{ub}|/|V_{cb}|$ [5]. The determinations from these exclusive and inclusive approaches are marginally compatible, yielding a difference more than two standard deviations [6]. A fourth method is the indirect determination of $|V_{ub}|$ with combining angles and other measurements characterizing the unitarity triangle. This indirect method is carried out by

such groups as CKMfitter [7] and UTfit [8]. The values determined in these fits favor the exclusive result.

In this paper, we present the first measurement of the branching fraction of the exclusive channel $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$, where charge-conjugation symmetry and lepton universality are assumed. This channel is of particular interest, as the $\pi^+ \pi^-$ system receives contributions from various resonant and nonresonant states, giving rise to a rich spectroscopy of the system. In this manner, it can serve as a probe to inspect the internal structure of light mesons decaying to a charged-pion pair, given that in semileptonic decays the hadronic and leptonic currents can be treated independently because the latter are not affected by strong interaction [9]. Measurements of branching fractions of this decay will improve the calculation of the $B \rightarrow \pi\pi$ form factors, which are an essential hadronic input for other processes such as the rare flavor-changing-neutral-current decays $B \rightarrow \pi\pi\ell^+\ell^-$ and hadronic decays $B \rightarrow \pi\pi\pi$ [10,11]. The resonant channel $B^+ \rightarrow \rho^0\ell^+\nu_\ell$, which contributes to the $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ final state, has been measured by the CLEO [12], Belle [13,14], and BABAR [15] collaborations. All these results focus on reconstructing the resonant ρ^0 final state and do not measure the full $\pi^+ \pi^-$ invariant-mass spectrum. The exclusive measurement of the $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ decay presented in this paper extends these previous studies. Furthermore, more precise knowledge of the nonresonant $\pi^+ \pi^-$ contributions will help improve future measurements of the ρ^0 final state [16]. With the rapid progress of lattice QCD, we hope that the measured $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ branching fraction and future measurements at Belle II will provide a new avenue to determine $|V_{ub}|$, which is expected to reach a precision at the 2% level [17].

II. DETECTOR, DATA SET, AND MONTE CARLO SIMULATION

The Belle detector is a large-solid-angle magnetic spectrometer consisting of a silicon vertex detector (SVD), a 50-layer central drift chamber (CDC), an array of aerogel threshold Cherenkov counters (ACC), a barrel-like arrangement of time-of-flight scintillation counters

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(TOF), and an electromagnetic calorimeter comprised of CsI(Tl) crystals (ECL). All these components are placed inside a superconducting solenoid coil that provides a 1.5 T magnetic field. An iron flux-return located outside of the coil is instrumented with resistive plate chambers to detect K_L^0 mesons and muons (KLM). More details on the detector can be found in Ref. [18].

We use the entire $\Upsilon(4S)$ data sample of 711 fb^{-1} recorded by Belle at the KEKB asymmetric-energy e^+e^- collider [19]. The sample contains $(772 \pm 11) \times 10^6$ $e^+e^- \rightarrow \Upsilon(4S) \rightarrow B\bar{B}$ events. The Belle detector used two inner subdetector configurations in the course of the experiment. The first arrangement consisted of a 2.0-cm-radius beampipe, and a three-layer SVD used to collect a sample of 152×10^6 $B\bar{B}$ pairs, while the second comprised a 1.5-cm-radius beampipe, a four-layer SVD, and a small-inner-cell CDC employed to record the remaining 620×10^6 $B\bar{B}$ pairs [20].

Monte Carlo (MC) simulated samples are generated with the EvtGen [21] package, and the response of the detector is modeled with GEANT3 [22]. We account for final-state radiation (FSR) effects by using the PHOTOS package [23,24]. A sample of $\Upsilon(4S) \rightarrow B\bar{B}$ events, where both the B mesons decay via the dominant quark-level transition $b \rightarrow cW$ (generic B decays), was generated with a size equivalent to ten times the integrated luminosity of the data sample. Continuum events arising from $e^+e^- \rightarrow q\bar{q}$, where q denotes u, d, s , or c quark, were simulated using PYTHIA6.4 [25] in a sample containing six times the data luminosity. Charmless rare B decays, occurring among others via loop transitions such as $b \rightarrow s$ quark transition or via radiative decays, are generated with a sample size corresponding to 50 times the integrated luminosity of the data sample.

The signal $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ sample is generated with the phase-space (PHSP) model of EvtGen, to make sure that each point in phase space is populated, independent of whether or not it can be produced by an intermediate resonance. Given that branching fraction estimations for the $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ decay in the entire phase space are unavailable from either lattice QCD or QCD sum-rule calculations, we have assumed a branching fraction of 31.7×10^{-5} according to Ref. [26] based on $|V_{ub}/V_{cb}| = 0.083 \pm 0.006$ [5]. We generate 100 million $B\bar{B}$ events, with one B meson decaying generically and the other to the $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ final state. A sample of exclusive semi-leptonic decays proceeding through the CKM-suppressed transition $b \rightarrow u\ell\nu_\ell$ is produced with a sample size equivalent to 20 times the data luminosity. This sample contains the following decays: $B^+ \rightarrow \pi^0\ell^+\nu_\ell$, $B^+ \rightarrow \eta\ell^+\nu_\ell$, $B^+ \rightarrow \eta'\ell^+\nu_\ell$, $B^+ \rightarrow \omega\ell^+\nu_\ell$, $B^+ \rightarrow a_0(980)^0\ell^+\nu_\ell$, $B^+ \rightarrow a_1(1260)^0\ell^+\nu_\ell$, $B^+ \rightarrow a_2(1320)^0\ell^+\nu_\ell$, $B^+ \rightarrow b_1(1235)^0\ell^+\nu_\ell$, $B^+ \rightarrow f_1(1285)\ell^+\nu_\ell$, $B^+ \rightarrow f_2'(1525)\ell^+\nu_\ell$, $B^0 \rightarrow \rho^-\ell^+\nu_\ell$, $B^0 \rightarrow \pi^-\ell^+\nu_\ell$, $B^0 \rightarrow a_0(980)^-\ell^+\nu_\ell$, $B^0 \rightarrow a_1(1260)^-\ell^+\nu_\ell$, $B^0 \rightarrow$

$a_2(1320)^-\ell^+\nu_\ell$, and $B^0 \rightarrow b_1(1235)^-\ell^+\nu_\ell$. These decays are generated using form factor calculations from ISGW2 [27] and light-cone sum rules (LCSR) [28]. We do not consider an inclusive component since the V_{ub} generator [29], used to model this contribution, incorrectly describes nonresonant states in the entire phase space. High-multiplicity mass terms that can contribute to the nonresonant component come from decays such as $B^+ \rightarrow \pi^+\pi^-\pi^0\ell^+\nu_\ell$ and $B^+ \rightarrow \pi^+\pi^-\pi^0\pi^0\ell^+\nu_\ell$. However, after simulating these processes with the PHSP generator and examining their contributions after the full selection, they are found to be negligible and thus are not considered further in this analysis.

We set the branching fractions of the decays $B \rightarrow D\ell\nu_\ell$, $B \rightarrow D^*\ell\nu_\ell$, $B \rightarrow D_1\ell\nu_\ell$, $B \rightarrow D_1'\ell\nu_\ell$, $B \rightarrow D_2^*\ell\nu_\ell$, $B \rightarrow D_0^*\ell\nu_\ell$, and of the known exclusive charmless semi-leptonic B decays to the latest experimental averages [1]. We reweight the Caprini-Lellouch-Neubert (CLN)-based form factors [30] of the decays $B \rightarrow D^{(*)}\ell\nu_\ell$ to the recent world-average values [4], and the form factors of the $B \rightarrow D^{**}\ell\nu_\ell$ decay according to the Leibovich-Ligeti-Stewart-Wise (LLSW) model [31]. We also correct the MC for the efficiency of particle identification of charged tracks, derived from studies using control samples for known processes, as described later in the section about systematic uncertainties associated to the detector simulation. These corrections depend on the kinematics of the particles involved.

III. EVENT SELECTION

This analysis employs a full reconstruction technique [32] based on the NeuroBayes neural-network package [33], in which we reconstruct one B meson (B_{tag}) stemming from the $\Upsilon(4S)$ resonance in 1104 hadronic modes. This tagging technique allows one to determine the properties of the other B meson (B_{sig}) from kinematic constraints. Subsequently, we reconstruct the B_{sig} using the rest of the event, except for the neutrino, which is invisible to the detector.

To distinguish $B\bar{B}$ events from nonhadronic background such as two-photon, radiative Bhabha, and $\tau^+\tau^-$ processes, we implement a selection based on the charged-track multiplicity and total visible energy [34]. Afterward, to reject continuum events, we combine 18 modified Fox-Wolfram [35] moment variables in the neural network used for B_{tag} reconstruction. The output classifier $o_{\text{tag}}^{\text{CS}}$ of the algorithm ranges from zero to unity, with values closer to unity indicating a higher probability of correctly reconstructing a B meson with low contamination of continuum events. We retain candidates with $\ln o_{\text{tag}}^{\text{CS}} > -4.9$ to ensure good quality of the B_{tag} candidate. This requirement is optimized using a figure-of-merit $N_S/\sqrt{N_S + N_B}$, where N_S and N_B are the number of signal and background events, respectively, estimated from MC simulation. With

this selection criterion, we attain a tag-side efficiency of 0.1% and purity of around 23% for charged B mesons reconstructed with the full hadronic tagging algorithm. Data-MC differences in the tagging efficiency have been evaluated in Ref. [14]; they depend on the network output value and the reconstructed B_{tag} channel. We derive an event-by-event correction factor from this study, using a control sample of $B \rightarrow D^{(*)}\ell\nu$ decays on the signal side, to account for these discrepancies.

We require the beam-energy-constrained mass, $M_{\text{bc}} = \sqrt{E_{\text{beam}}^2 - |\vec{p}_{B_{\text{tag}}}|^2}$, to be greater than 5.27 GeV [36]. Here, E_{beam} and $\vec{p}_{B_{\text{tag}}}$ are the beam energy and the three-momentum of the B_{tag} candidate in the $\Upsilon(4S)$ frame, respectively. We select only charged B_{tag} candidates since the signal mode involves charged B mesons.

The charged particles and neutral clusters in the event not associated with the B_{tag} candidate are used in the reconstruction of the B_{sig} candidate. Due to the magnetic field inside the detector, low-momentum charged particles spiral inside the CDC and may lead to multiple track candidates for the same particle. A pair of tracks is regarded as duplicated if they have momenta transverse to the beam direction below 275 MeV, with a small momentum difference (below 100 MeV) and an opening angle either below 15° (same charges) or above 165° (opposite charges). In case such a duplicated pair is identified, the track with the smaller value of the quantity $(5 \times |dr|)^2 + |dz|^2$ is kept, with $|dr|$ and $|dz|$ denoting the distance of closest approach of a given track to the interaction point (IP) in the plane perpendicular, and along the beam direction, respectively. This criterion is optimized using simulated tracks. In addition, we require that all selected tracks satisfy $|dr| < 0.4$ cm and $|dz| < 2.0$ cm.

We identify charged hadrons using the ionization energy loss in the CDC, the time-of-flight in the TOF, and the Cherenkov light yield in the ACC [37]. The identification of charged pions in this analysis has an efficiency of 85% with a kaon misidentification rate of 13%.

We consider events with only a single charged-lepton candidate on the signal side. Electron candidates are identified based on the ratio of the ECL energy to that of the CDC track, the ECL shower shape, the position matching between the CDC track and the ECL cluster, the energy loss in the CDC, and the response of the ACC [38]. Furthermore, we require electrons to have a minimum momentum of 0.3 GeV. Muon candidates are selected using their penetration range and transverse scattering in the KLM [39], and requiring a minimum momentum of 0.6 GeV. In the momentum region relevant to this analysis, the average electron (muon) identification efficiency is about 87% (89%), and the probability of misidentifying a pion as an electron (muon) is 0.15% (1.3%). We veto charged leptons arising from photon conversion in the detector material as well as from J/ψ and $\psi(2S)$ decays if

the lepton candidate, when combined with an oppositely charged particle, gives an invariant mass $M_{\ell\ell}$ satisfying any of the following three conditions: $M_{\ell\ell} < 0.1$ GeV, $M_{\ell\ell} \in [3.00, 3.15]$ GeV, or $M_{\ell\ell} \in [3.60, 3.75]$ GeV.

We reconstruct photons as clusters in the ECL not linked to any track in the CDC. To reject low-energy photons originating from background caused by the beam circulation, we require a minimum energy of 50 MeV, 100 MeV, and 150 MeV in the barrel, forward and backward endcap region, respectively, of the ECL. We reconstruct neutral pions from pairs of photons with an invariant mass in the range 120–150 MeV. The photons forming a neutral pion candidate are rejected from the one to be linked to a charged track. In electron events, we take into account possible bremsstrahlung by searching for low-energy photons ($E_\gamma < 1$ GeV) within a 5° cone around the electron direction. If such a photon is found, it is merged with the electron and the sum of the momenta is taken to be the lepton momentum. If there is more than one photon candidate, only the nearest photon is combined with the electron.

IV. SIGNAL SELECTION AND BACKGROUND SUPPRESSION

After applying the above criteria, we reconstruct the signal decay $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ from the tracks not associated with B_{tag}^- . In this manner, we require exactly three tracks on the signal side, the two charged pions and the lepton. Given that the neutrino is invisible to the detector, we infer its four-momentum from the overall missing momentum of the event, defined as

$$P_{\text{miss}} = P_{\Upsilon(4S)} - P_{B_{\text{tag}}^\pm} - P_{\ell^\mp} - P_{\pi^+} - P_{\pi^-}, \quad (1)$$

where P_i denotes the four-momentum of particle $i = \Upsilon(4S), B_{\text{tag}}^-, \ell, \pi^+, \pi^-$. We determine the missing-mass squared, $M_{\text{miss}}^2 = P_{\text{miss}}^2$, to distinguish semileptonic decays from other processes. For correctly reconstructed semileptonic decays, M_{miss}^2 sharply peaks at 0, whereas other processes have a shoulder typically at positive values.

At this point in the reconstruction, the dominant background is from semileptonic B decays to charmed mesons whose kinematic distributions resemble those of the signal. To suppress this background, we deploy a boosted decision tree (BDT) to recognize $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ decays and identify B -meson decays into other final states. This BDT is also effective against other backgrounds such as continuum, rare and charmless semileptonic B decays. A statistically independent two sets of MC samples for signal and background are prepared. One set is used to train BDT with the stochastic gradient boosting approach from the TMVA software package [40]. Another one set is used for validation of the training. The following input variables are used:

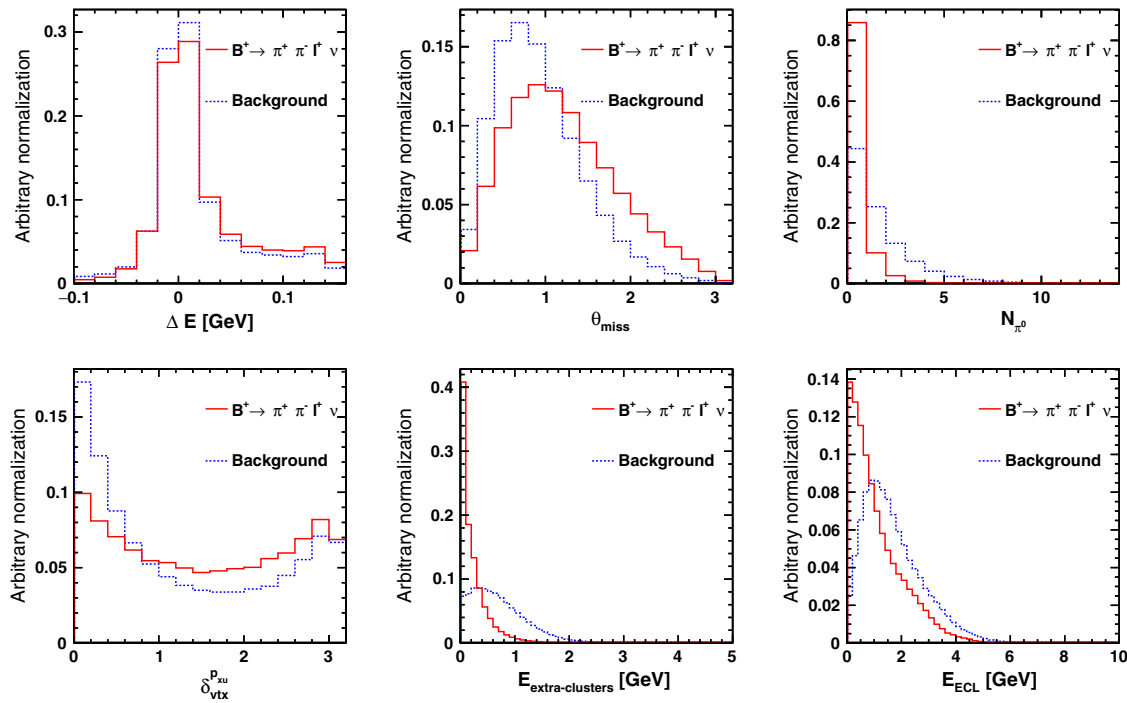


FIG. 1. Shape comparison of the input variables of the BDT before the selection on \mathcal{O}_{BDT} for simulated signal and background events.

- (1) ΔE_{sig} : the difference between the beam and the B_{sig} meson energies in the center-of-mass system (c.m.), which is calculated using the B_{tag} meson, $\Delta E_{\text{sig}} = -\Delta E_{\text{tag}} = -(E_{\text{beam}} - E_{B_{\text{tag}}})$.
- (2) θ_{miss} : the polar angle of the missing momentum in the laboratory frame.
- (3) N_{π^0} : the multiplicity of π^0 candidates on the signal side.
- (4) $\delta_{\text{vtx}}^{\text{p}_{\text{su}}}$: the angle between the signal-side $\pi^+\pi^-$ momentum and the vector connecting the IP and the $\pi^+\pi^-$ decay vertex calculated in the laboratory frame. The distance of the $\pi^+\pi^-$ system to the IP for charmless intermediate states is smaller than that for two-track pairs associated with D^0 and K_S^0 mesons. Thus the angle $\delta_{\text{vtx}}^{\text{p}_{\text{su}}}$ is useful in reducing these background processes.
- (5) $E_{\text{extra-clusters}}$: the total c.m. energy of photons within the barrel region not associated with either the B_{tag} or B_{sig} candidates.
- (6) E_{ECL} : the sum of the clusters in the ECL from the whole event not matching to any tracks but passing the energy thresholds for photons. This calculation also includes ECL clusters made by photons that were incorrectly associated with a track and that satisfy $E_9/E_{25} > 0.94$. The E_9/E_{25} variable quantifies the transverse shower shape in the ECL, defined as the ratio of energy deposited in the 3×3 array of crystals centered on the crystal with the highest energy to that in the corresponding 5×5 array of crystals. This variable is suitable to separate

overlapping hits in the ECL crystals caused by hadronic interaction with charged tracks and photons. For photons E_9/E_{25} peaks near one, whereas for charged tracks it tends to have lower values.

Distributions of the above six variables for signal and background (with arbitrary normalizations) are shown in Fig. 1.

We choose a selection criterion on the BDT output classifier by optimizing the figure-of-merit $N_S/\sqrt{N_S + N_B}$. The distributions of the BDT classifier \mathcal{O}_{BDT} for the signal, B -meson decays to charm mesons and other backgrounds are shown in Fig. 2. We validate the description of the variables used in the BDT using the sideband of the M_{miss}^2

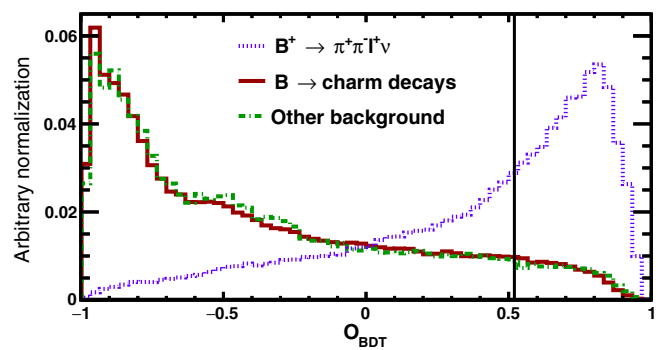


FIG. 2. Shapes of the BDT output for the signal and the major background processes, as predicted by MC. The vertical line shows the minimum requirement applied on this variable.

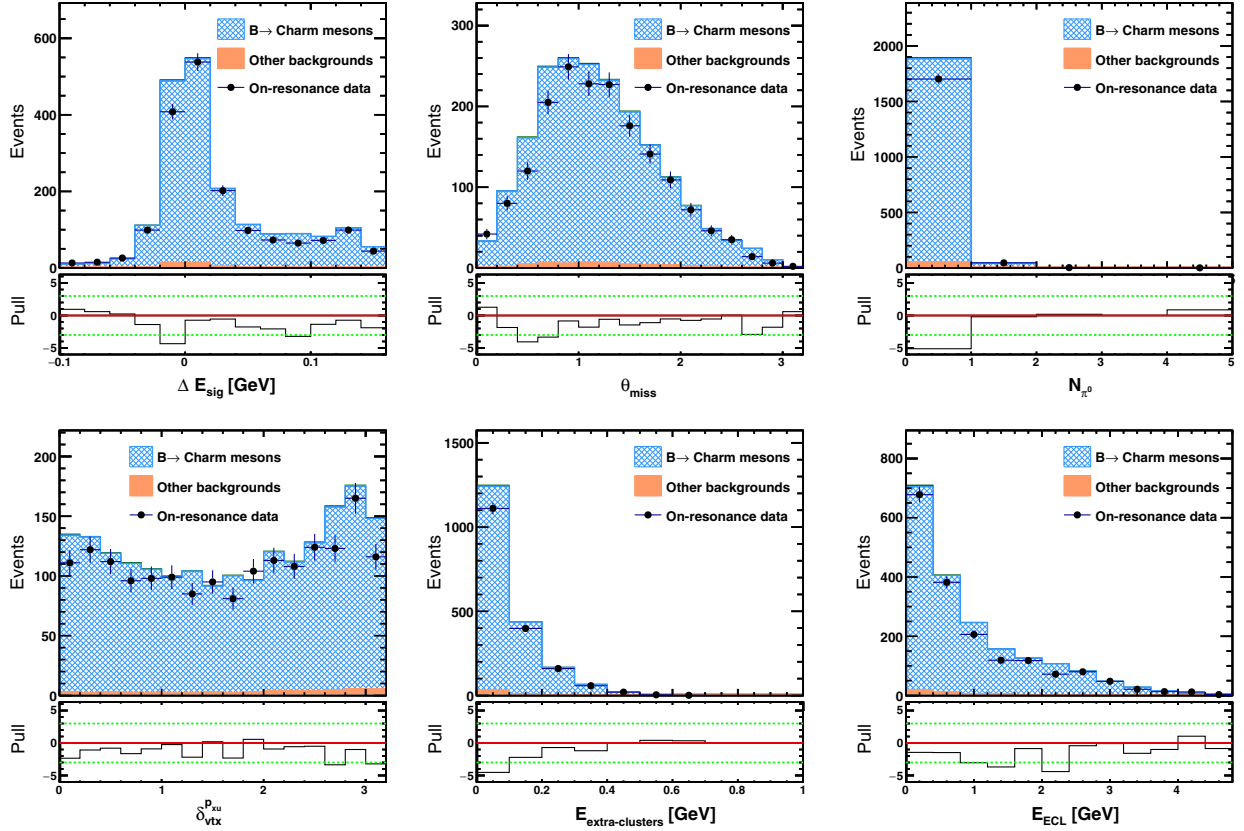


FIG. 3. Distributions of the input variables of the BDT in the sidebands of the missing-mass squared, after the selection on \mathcal{O}_{BDT} . The shaded histogram shows the contribution from B decays to charm mesons, while the solid histogram shows the contributions from other processes. The pull values are presented underneath each plot to show the level of data-MC difference. MC events are normalized to the integrated luminosity.

distribution, defined as $M_{\text{miss}}^2 > 2 \text{ GeV}^2$. These distributions are shown in Fig. 3.

V. SIGNAL EXTRACTION

We perform a binned extended maximum-likelihood fit to the M_{miss}^2 spectrum using histogram templates derived from MC simulation to determine the signal yield. We use a bin width of 0.2 GeV^2 in the range $[-1.0, 6.0] \text{ GeV}^2$. Because of the negligible contribution of the continuum, $b \rightarrow u\ell\nu$, and rare $b \rightarrow s$ mediated decays, we combine these into a single component and fix their yields to the MC expectation (referred to as fixed background in the following). We thus distinguish among three components in our fit:

- (1) the signal $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$,
- (2) $B \rightarrow X_c\ell\nu$ (X_c is a charm meson) and
- (3) the fixed background,

where yields of the first two components are floated in the fit.

To allow for a $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ decay-model-independent interpretation of the result, we analyze the measured yields in bins of $M_{\pi\pi} = \sqrt{(P_{\pi^+} + P_{\pi^-})^2}$ and $q^2 = (P_{\ell^+} + P_{\nu_\ell})^2$ using three fit configurations. The minimum value for $M_{\pi\pi}$

corresponds to twice the mass of a charged pion, that is 0.28 GeV , whereas the maximum value is about the mass of the B^\pm meson, which is approximately 5.28 GeV . On the other hand, q^2 ranges from 0 GeV^2 to approximately 25 GeV^2 . The first configuration employs a fit of the dipion invariant-mass distribution, referred to as $1D(M_{\pi\pi})$ in the following. In the second configuration, abbreviated as $2D$, we carry out a two-dimensional analysis and measure partial branching fractions in bins of $M_{\pi\pi}$ and q^2 . Finally, in the third configuration, we perform the measurement in bins of q^2 , and denote this configuration as $1D(q^2)$. We use 13 bins in the $1D(M_{\pi\pi})$ configuration, consisting of 11 bins with a uniform width in the dipion mass of 80 MeV , and two additional bins corresponding to the low dipion mass ($M_{\pi^+\pi^-} < 0.46 \text{ GeV}$) and the high dipion mass ($M_{\pi^+\pi^-} > 1.34 \text{ GeV}$) regions. In the $1D(q^2)$ configuration, we employ 17 bins with a uniform width of 1 GeV^2 and an additional bin accounting for the region $q^2 > 17 \text{ GeV}^2$. In the $2D$ configuration, we employ five bins of 300 MeV in the dipion mass and, depending on the size of the data sample for these regions, we split the q^2 distribution into either two or three bins. Hence, for $M_{\pi^+\pi^-} < 0.6 \text{ GeV}$ we use $q^2 \leq 8 \text{ GeV}^2$ and $q^2 > 8 \text{ GeV}^2$;

for $M_{\pi^+\pi^-} > 1.5 \text{ GeV}$ we use $q^2 \leq 4 \text{ GeV}^2$ and $q^2 > 4 \text{ GeV}^2$. For the remaining $M_{\pi^+\pi^-}$ bins, we separate q^2 into three regions: $q^2 \leq 4 \text{ GeV}^2$, $4 < q^2 [\text{GeV}^2] \leq 8$, and $q^2 > 8 \text{ GeV}^2$. For the highest bin in the $1D(M_{\pi\pi})$ configuration ($M_{\pi^+\pi^-} > 1.34 \text{ GeV}$), we separate the $B \rightarrow X_c \ell \nu$ background into two components: one containing B meson decays to D^0 mesons as a cross-feed ($B \rightarrow D^0 \ell \nu$), and another involving the remaining charmed mesons (rest of $B \rightarrow X_c \ell \nu$). The decay $B^+ \rightarrow \bar{D}^0 \ell^+ \nu_\ell$ with $D^0 \rightarrow \pi^+ \pi^-$ also peaks at $M_{\text{miss}}^2 \approx 0 \text{ GeV}^2$ in the dipion mass ($M_{\pi^+\pi^-}$) region from 1.85 to 1.88 GeV, with relatively small contamination from other processes. In this mass window, we measure $\mathcal{B}(B^+ \rightarrow \bar{D}^0 \ell^+ \nu_\ell) = (2.83 \pm 0.54)\%$, where the uncertainty is only statistical, and the result is compatible with the world average $\mathcal{B}(B^+ \rightarrow \bar{D}^0 \ell^+ \nu_\ell)_{\text{PDG}} = (2.33 \pm 0.10)\%$ [1]. We fix this component in MC according to the measured event yield in data and add it to the fixed background shape and yield. The detector resolution for the dipion mass and q^2 are about 4 MeV and $5 \times 10^{-2} \text{ GeV}^2$, respectively. These values are significantly smaller than the bin sizes used in our measurement, and hence no additional corrections to account for migrations between the reconstructed bins are applied.

Figure 4 shows the projection of the fit results in the $1D(M_{\pi\pi})$ configuration in three regions of the dipion mass:

a low-mass region ($M_{\pi^+\pi^-} \leq 0.62 \text{ GeV}$), an intermediate-mass region dominated by the ρ^0 meson ($0.62 < M_{\pi^+\pi^-} [\text{GeV}] \leq 0.94$), and a high-mass region ($M_{\pi^+\pi^-} > 0.94 \text{ GeV}$) where we can also observe contributions from the $B^+ \rightarrow \bar{D}^0(\pi^+\pi^-)\ell^+\nu_\ell$ decay. Tables I–III list the fit results for every bin in the three configurations. In these tables, we provide the χ^2 value and number of degrees of freedom to verify the goodness of fit, following the χ^2 calculation of Baker and Cousins [41], which applies to fits derived from a maximum-likelihood method where the data obey Poisson statistics. The fit procedure is validated by generating an ensemble of pseudoexperiments using the fitted number of signal and background events in each of the bins. No bias in the coverage of the reported uncertainties is observed. The recovered central values show a small bias, which we include into the systematic uncertainties (discussed in the next section). To validate our measurement, we used control samples following a selection procedure similar to that implemented for the signal. For that purpose, we study four channels in the $B^+ \rightarrow \bar{D}^0 \ell^+ \nu_\ell$ decay, with the D^0 meson reconstructed as a combination of two charged hadrons and the possibility to include a neutral pion: $K^-\pi^+$, K^-K^+ , $\pi^+\pi^-\pi^0$ and $K^-\pi^+\pi^0$. The measured branching fractions are in agreement with the world averages [1].

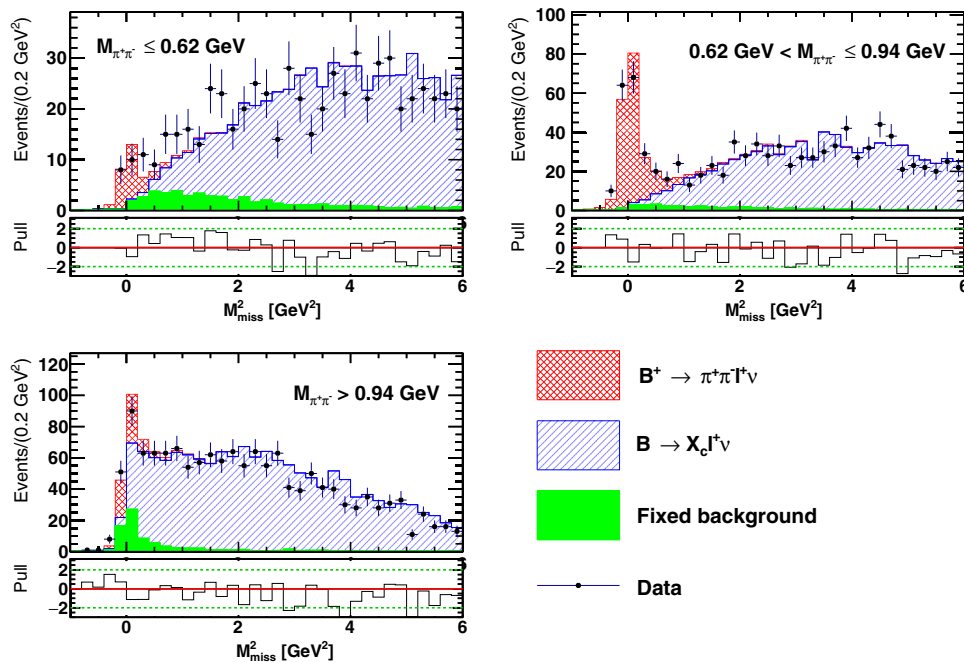


FIG. 4. Projection of the $1D(M_{\pi\pi})$ configuration fit results in the M_{miss}^2 distribution (points with error bars) in three regions of the dipion mass as labeled: (upper left) low-mass region ($M_{\pi^+\pi^-} \leq 0.62 \text{ GeV}$), (upper right) around the ρ^0 meson ($0.62 \text{ GeV} < M_{\pi^+\pi^-} \leq 0.94 \text{ GeV}$) and (lower left) high-mass region ($M_{\pi^+\pi^-} > 0.94 \text{ GeV}$). The fit components are shown as the colored histograms as given in the lower right. The peaking structure in the fixed background around the signal region in the high dipion mass range is due to the $B^+ \rightarrow \bar{D}^0(\pi^+\pi^-)\ell^+\nu_\ell$ decay.

TABLE I. Event yields for the signal and background processes in the $B^+ \rightarrow \pi^+\pi^-\ell^+\nu$ decay obtained from an extended binned maximum-likelihood fit to the M_{miss}^2 distribution in bins of $M_{\pi^+\pi^-}$. The χ^2 per number of degrees of freedom (NDF) and the probability of the fit (Prob.) are provided.

| Bin | $M_{\pi\pi}$ [GeV] | Signal | $B^+ \rightarrow X_c \ell \nu$ | Fixed | Total MC | Data | χ^2/NDF | Prob. [%] | |
|-----|-------------------------------|------------------------|--------------------------------------|--------------------------------|----------|-------|---------------------|-----------|------|
| 1 | $M_{\pi\pi} < 0.46$ | $7.1^{+4.1}_{-3.2}$ | 195.0 ± 14.6 | 20.2 | 222.3 | 225 | 27.5/33 | 73.7 | |
| 2 | $0.46 \leq M_{\pi\pi} < 0.54$ | $10.0^{+4.4}_{-3.5}$ | 146.7 ± 12.7 | 17.1 | 173.8 | 179 | 30.2/33 | 60.7 | |
| 3 | $0.54 \leq M_{\pi\pi} < 0.62$ | $10.6^{+4.3}_{-3.5}$ | 190.1 ± 14.2 | 14.8 | 215.5 | 216 | 38.3/33 | 24.3 | |
| 4 | $0.62 \leq M_{\pi\pi} < 0.70$ | $23.3^{+6.2}_{-5.4}$ | 185.4 ± 14.1 | 9.3 | 218.0 | 220 | 27.9/33 | 71.7 | |
| 5 | $0.70 \leq M_{\pi\pi} < 0.78$ | $90.3^{+10.7}_{-10.0}$ | 234.8 ± 16.0 | 12.4 | 337.5 | 337 | 45.9/33 | 6.8 | |
| 6 | $0.78 \leq M_{\pi\pi} < 0.86$ | $50.5^{+8.1}_{-7.4}$ | 151.6 ± 12.8 | 12.3 | 214.4 | 214 | 30.2/33 | 60.8 | |
| 7 | $0.86 \leq M_{\pi\pi} < 0.94$ | $29.6^{+6.4}_{-5.7}$ | 108.5 ± 10.8 | 7.8 | 145.9 | 146 | 43.6/33 | 10.4 | |
| 8 | $0.94 \leq M_{\pi\pi} < 1.02$ | $10.2^{+4.2}_{-3.4}$ | 102.4 ± 10.3 | 6.1 | 118.7 | 119 | 15.2/33 | 99.7 | |
| 9 | $1.02 \leq M_{\pi\pi} < 1.10$ | $8.9^{+3.7}_{-3.0}$ | 127.6 ± 11.3 | 4.0 | 140.5 | 140 | 26.3/33 | 78.9 | |
| 10 | $1.10 \leq M_{\pi\pi} < 1.18$ | $5.7^{+3.1}_{-2.4}$ | 149.2 ± 12.1 | 2.9 | 157.8 | 158 | 40.0/33 | 18.8 | |
| 11 | $1.18 \leq M_{\pi\pi} < 1.26$ | $15.7^{+5.0}_{-4.2}$ | 186.6 ± 13.8 | 3.0 | 205.3 | 205 | 41.2/33 | 15.5 | |
| 12 | $1.26 \leq M_{\pi\pi} < 1.34$ | $11.8^{+4.2}_{-3.4}$ | 221.4 ± 14.9 | 3.1 | 236.3 | 236 | 30.5/33 | 59.3 | |
| | | | Rest of | | | | | | |
| | | | $B^+ \rightarrow \bar{D}^0 \ell \nu$ | $B^+ \rightarrow X_c \ell \nu$ | | | | | |
| 13 | $M_{\pi\pi} \geq 1.34$ | $23.4^{+14.7}_{-13.6}$ | 175.3 ± 49.4 | 289.3 ± 44.0 | 68.0 | 556.0 | 556 | 28.3/32 | 65.4 |

TABLE II. Event yields for the signal and background processes in the $B^+ \rightarrow \pi^+\pi^-\ell^+\nu$ decay obtained from an extended binned maximum-likelihood fit to the M_{miss}^2 distribution in bins of q^2 . The χ^2 per number of degrees of freedom (NDF) and the probability of the fit (Prob.) are provided.

| Bin | q^2 [GeV 2] | Signal | $B^+ \rightarrow X_c \ell \nu$ | Fixed | Total MC | Data | χ^2/NDF | Prob. [%] |
|-----|--------------------|----------------------|--------------------------------|-------|----------|------|---------------------|-----------|
| 1 | $q^2 < 1$ | $16.5^{+6.8}_{-6.1}$ | 126.4 ± 12.1 | 20.6 | 163.5 | 163 | 32.1/33.0 | 51.2 |
| 2 | $1 \leq q^2 < 2$ | $11.4^{+6.0}_{-5.1}$ | 150.5 ± 12.9 | 15.2 | 177.1 | 176 | 34.9/33.0 | 37.9 |
| 3 | $2 \leq q^2 < 3$ | $13.0^{+5.6}_{-4.7}$ | 166.3 ± 13.4 | 12.8 | 192.1 | 192 | 40.9/33.0 | 16.4 |
| 4 | $3 \leq q^2 < 4$ | $16.0^{+6.6}_{-5.8}$ | 180.2 ± 14.1 | 13.5 | 209.7 | 210 | 32.2/33.0 | 50.9 |
| 5 | $4 \leq q^2 < 5$ | $24.3^{+7.1}_{-6.3}$ | 224.9 ± 15.6 | 13.8 | 263.0 | 263 | 41.9/33.0 | 13.7 |
| 6 | $5 \leq q^2 < 6$ | $12.2^{+5.7}_{-4.9}$ | 212.3 ± 15.0 | 14.0 | 238.5 | 238 | 17.4/33.0 | 98.8 |
| 7 | $6 \leq q^2 < 7$ | $10.8^{+5.1}_{-4.1}$ | 235.6 ± 15.7 | 10.8 | 257.2 | 257 | 54.2/33.0 | 1.1 |
| 8 | $7 \leq q^2 < 8$ | $21.4^{+6.5}_{-5.7}$ | 220.5 ± 15.3 | 10.5 | 252.4 | 253 | 36.0/33.0 | 32.9 |
| 9 | $8 \leq q^2 < 9$ | $9.6^{+4.7}_{-3.9}$ | 220.5 ± 15.1 | 9.5 | 239.6 | 239 | 34.1/33.0 | 41.6 |
| 10 | $9 \leq q^2 < 10$ | $30.8^{+6.7}_{-6.0}$ | 199.0 ± 14.6 | 9.3 | 239.1 | 239 | 36.5/33.0 | 30.9 |
| 11 | $10 \leq q^2 < 11$ | $11.6^{+5.0}_{-4.1}$ | 159.4 ± 13.0 | 9.2 | 180.2 | 181 | 19.2/33.0 | 97.3 |
| 12 | $11 \leq q^2 < 12$ | $16.3^{+4.9}_{-4.1}$ | 122.1 ± 11.4 | 7.2 | 145.6 | 146 | 35.4/33.0 | 35.4 |
| 13 | $12 \leq q^2 < 13$ | $19.4^{+5.3}_{-4.5}$ | 93.7 ± 10.0 | 6.1 | 119.2 | 119 | 16.5/33.0 | 99.3 |
| 14 | $13 \leq q^2 < 14$ | $15.4^{+4.7}_{-4.0}$ | 66.1 ± 8.5 | 5.7 | 87.2 | 87 | 21.5/33.0 | 93.8 |
| 15 | $14 \leq q^2 < 15$ | $15.1^{+4.9}_{-4.1}$ | 37.1 ± 6.6 | 5.9 | 58.1 | 61 | 24.3/28.0 | 66.3 |
| 16 | $15 \leq q^2 < 16$ | $10.8^{+4.0}_{-3.2}$ | 24.1 ± 5.3 | 4.8 | 39.7 | 41 | 17.4/23.0 | 79.1 |
| 17 | $16 \leq q^2 < 17$ | $12.3^{+4.4}_{-3.7}$ | 18.5 ± 5.0 | 4.8 | 35.6 | 36 | 13.0/23.0 | 95.2 |
| 18 | $q^2 \geq 17$ | $32.3^{+6.8}_{-6.1}$ | 7.8 ± 4.1 | 7.4 | 47.5 | 50 | 14.5/18.0 | 69.4 |

VI. SYSTEMATIC UNCERTAINTIES

Various sources of systematic uncertainties considered in this analysis fall into three categories: those related to detector performance, those due to the modeling of the signal and background processes, and those associated with

the fitting procedure. In most cases, we estimate the systematic uncertainties by varying each fixed parameter in the simulation by one standard deviation up and down ($\pm 1\sigma$) and repeating the fit to the M_{miss}^2 distribution. The relative difference between the signal yield from the nominal fit and that with the parameter varied is taken

TABLE III. Event yields for the signal and background processes in the $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu$ decay obtained from an extended binned maximum-likelihood fit to the M_{miss}^2 distribution in bins of $M_{\pi^+ \pi^-}$ and q^2 . The χ^2 per number of degrees of freedom (NDF) and the probability of the fit (Prob.) are provided.

| Bin | $M_{\pi\pi}$ [GeV] | q^2 [GeV ²] | Signal | $B^+ \rightarrow X_c \ell \nu$ | Fixed | Total MC | Data | χ^2/NDF | Prob.[%] | |
|---------|-----------------------------|---------------------------|-------------------------|---|---|----------|-------|---------------------|----------|------|
| 1 | $M_{\pi\pi} \leq 0.6$ | $q^2 \leq 8$ | $9.8^{+4.6}_{-3.7}$ | 218.2 ± 15.3 | 15.6 | 243.5 | 249 | 50.1/33 | 2.8 | |
| 2 | $M_{\pi\pi} \leq 0.6$ | $8 < q^2$ | $15.8^{+3.5}_{-4.6}$ | 275.9 ± 17.4 | 33.1 | 324.8 | 329 | 30.6/33 | 58.5 | |
| 3 | $0.6 < M_{\pi\pi} \leq 0.9$ | $q^2 \leq 4$ | $29.5^{+6.4}_{-5.7}$ | 134.9 ± 12.1 | 9.8 | 174.2 | 175 | 31.4/33 | 54.7 | |
| 4 | $0.6 < M_{\pi\pi} \leq 0.9$ | $4 < q^2 \leq 8$ | $34.8^{+7.0}_{-6.2}$ | 216.9 ± 15.1 | 9.9 | 261.5 | 262 | 24.4/33 | 86.0 | |
| 5 | $0.6 < M_{\pi\pi} \leq 0.9$ | $8 < q^2$ | $116.2^{+12.2}_{-11.5}$ | 318.8 ± 18.7 | 22.5 | 457.6 | 457 | 39.0/33 | 21.8 | |
| 6 | $0.9 < M_{\pi\pi} \leq 1.2$ | $q^2 \leq 4$ | $8.0^{+3.7}_{-2.9}$ | 110.8 ± 10.6 | 5.8 | 124.6 | 124 | 20.4/33 | 95.8 | |
| 7 | $0.9 < M_{\pi\pi} \leq 1.2$ | $4 < q^2 \leq 8$ | $9.2^{+4.0}_{-3.2}$ | 190.2 ± 13.9 | 4.9 | 204.3 | 204 | 32.3/33 | 50.4 | |
| 8 | $0.9 < M_{\pi\pi} \leq 1.2$ | $8 < q^2$ | $27.6^{+6.4}_{-5.6}$ | 169.8 ± 13.4 | 6.6 | 204.1 | 204 | 39.5/33 | 20.3 | |
| 9 | $1.2 < M_{\pi\pi} \leq 1.5$ | $q^2 \leq 4$ | $11.3^{+4.3}_{-3.5}$ | 142.4 ± 12.1 | 4.1 | 157.8 | 158 | 36.7/33 | 30.0 | |
| 10 | $1.2 < M_{\pi\pi} \leq 1.5$ | $4 < q^2 \leq 8$ | $9.7^{+4.3}_{-3.5}$ | 227.1 ± 15.1 | 2.5 | 239.2 | 239 | 25.6/33 | 81.8 | |
| 11 | $1.2 < M_{\pi\pi} \leq 1.5$ | $8 < q^2$ | $13.2^{+4.4}_{-3.7}$ | 132.5 ± 11.6 | 2.4 | 148.1 | 148 | 27.2/33 | 75.0 | |
| Rest of | | | | | | | | | | |
| 12 | $M_{\pi\pi} > 1.5$ | $q^2 \leq 4$ | $8.5^{+10.1}_{-9.0}$ | $B^+ \rightarrow D^0 \ell \nu$ 72.1 ± 17.3 | $B^+ \rightarrow X_c \ell \nu$ 63.1 ± 13.2 | 36.2 | 179.8 | 180 | 25.4/32 | 79.1 |
| 13 | $M_{\pi\pi} > 1.5$ | $4 < q^2$ | $7.6^{+7.2}_{-4.9}$ | 96.4 ± 22.4 | 93.2 ± 20.2 | 27.8 | 225.1 | 222 | 27.8/32 | 68.1 |

as the systematic uncertainty. We calculate these uncertainties separately for each bin in our measurement.

A. Signal and background modeling

The sources of uncertainties related to the modeling of physical processes include the lack of precise knowledge of hadronic form factors that describe a specific decay, and the relative contributions of background processes. To assess the systematic uncertainty arising from the signal modeling, we compare the signal reconstruction efficiency calculated for each bin in $M_{\pi\pi}$, q^2 , or $(M_{\pi\pi}, q^2)$, using the phase space $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ and other B semileptonic channels with an intermediate resonance decaying to a $\pi^+ \pi^-$ pair. As these channels simulate the same final state, the resulting efficiencies should be similar. Nonetheless, resonances do not span as much of the domain in the phase space as an inclusive simulation since they have a finite width; hence their coverage in the dipion mass is essentially limited to the interval $[M_R - 2\Gamma_R, M_R + 2\Gamma_R]$, with M_R the nominal mass of the resonance and Γ_R its decay width. The range of q^2 varies with the resonant state as the maximum value depends on the mass of the resonance through $q_{\text{max}}^2 = (M_B - M_R)^2$, where M_B is the mass of the B meson. We thus simulate semileptonic B decays with four intermediate resonances covering the phase space of the $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ decay, namely $f_0(500)$, ρ^0 , $f_2(1270)$, and $\rho^0(1450)$, and produce these with the phase space and ISGW2 [27] models. Furthermore, we use form factors from LCSR calculations for the $B^+ \rightarrow \rho^0 \ell^+ \nu_\ell$ and the $B^+ \rightarrow f_2(1270) \ell^+ \nu_\ell$ decays according to Refs. [28,42], respectively. We calculate the root mean square error between the nominal efficiency (phase space $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$) and the resonant models

valid for a given bin as the systematic uncertainty due to signal modeling. In addition, we also consider the finite size of the MC sample used to estimate the signal reconstruction efficiency. We include this error in the systematic uncertainty due to reconstruction efficiency. The values of the efficiencies used for this assessment are presented in the Appendix in Tables VII–IX for the 1D($M_{\pi\pi}$), 1D(q^2), and 2D fit binning configurations, respectively.

Given that the continuum background is almost negligible after all selection, we compare continuum MC events with the off-resonance data applying a loose selection to assign the uncertainty due to the description of this process. Consequently, we determine an asymmetric variation in the continuum normalization ($^{+50\%}_{-20\%}$) and repeat the fit with these changes. Contributions from rare decays are also very small. To evaluate their effects on our measurement, we carry out 1000 pseudoexperiments (using the same prescription described in Sec. V) with and without this component. The systematic uncertainty is then derived from the difference in mean values from both ensembles for each bin. To assess the impact of the background shape on the calculation of the branching fraction, we reweight a specific decay in the MC sample with another model. Specifically, we adjust the CLN-based form factors [30] of the $B \rightarrow D^* \ell \nu_\ell$ decays in the MC sample to the new world-average values [4]. Similarly, we reweight the form factors for the $B \rightarrow D^{**} \ell \nu_\ell$ decays from the ISGW2 [27] to the LLSW model [31]. In both cases, we add in quadrature the change in the branching fraction due to variation of each form factor to obtain a total uncertainty associated with these sources. The $B \rightarrow \pi \ell \nu_\ell$ and $B \rightarrow \omega \ell \nu_\ell$ MC events are generated with LCSR form factors taken from Ref. [28]. We reweight the $B \rightarrow \omega \ell \nu_\ell$ form factors to the calculation

of Ref. [43] and use the difference in efficiencies compared to the nominal sample as the uncertainty. The $B \rightarrow \pi \ell \nu_\ell$ form factors are reweighted to the Bourrely-Caprini-Lellouch model [44], which combines information from the measured spectra, light-cone sum rules (valid at low q^2) and lattice QCD (valid at high q^2), and the same procedure to calculate the uncertainty is used. We also reweight the form factors of the $B \rightarrow \eta^{(\prime)} \ell \nu_\ell$ decay from the ISGW2 [27] and LCSR models according to [45]. Other exclusive charmless semileptonic B decays considered in this analysis are generated with the ISGW2 model. As they do not have well-established form factors derived from QCD calculations, we compare their shapes with those produced using the phase space and FLATQ2 generators [21,46].

We correct the branching fractions of the $B \rightarrow (D^{(*,**)}, \pi, \eta^{(\prime)}, \omega) \ell \nu_\ell$ decay modes according to the world-averages [1] and vary these values within their measured uncertainties as presented in Table IV. For the unmeasured exclusive charmless semileptonic B decays, we assign a $\pm 100\%$ uncertainty in the variation of the branching fraction. We modify the contribution of the secondary leptons relative to the total uncertainty in the measurement of the branching fraction of the decay chain $B^+ \rightarrow X_{\bar{c}} \ell^+ \nu_\ell$ with $X_{\bar{c}} \rightarrow \ell^- + \text{anything}$. To consider the effect of the BDT selection on our result, we evaluate the data-MC difference in efficiency and find it to be negligible as compared to the statistical error.

To assess the effect of inclusive charmless semileptonic B decays, we include an additional component to the fixed background simulated with the De Fazio-Neubert model [29]. The differences with respect to the nominal fit are taken as a systematic uncertainty.

B. Detector simulation

Since the analysis relies extensively on MC simulation, the detection of final-state particles affects the reconstruction of signal and background decays and the subsequent extraction of signal yields used in the measurement of the branching fractions. The efficiency for detecting these particles in data usually differs from that in MC simulation, for which we apply a correction derived from independent control samples. We take the total uncertainty associated with this correction as a systematic uncertainty. These uncertainties include those related to charged lepton and pion identification efficiencies. Analogously to secondary leptons, charged tracks misidentified as leptons, i.e., fake leptons, can also originate from the continuum and charmed semileptonic B decays. To inspect their effect, we compare the fake rate between data and MC using a control sample of $D^{*+} \rightarrow D^0(K^-\pi^+)\pi^+$ decay and determine a weight factor for each lepton type. We then correct the contribution of fake leptons in MC events and vary the central value by its error. We assign the relative difference between the fit results as the uncertainties associated with fake leptons.

To assess the size of the uncertainty due to FSR, we prepare histogram templates normalized to the fit results in data using two versions of the signal component: one where the signal is generated with the PHOTOS package (as in the nominal fit) and another without it. We then carry out 1000 pseudoexperiments for each case and take 20% of the mean difference in the signal yields from these two scenarios [15,47,48].

C. Fit procedure

We perform 5000 pseudoexperiments to validate our fit procedure, obtaining pull distributions that take into

TABLE IV. Decay channels that are corrected in the MC, with their branching fractions in the MC, world-average, and their respective correction (weight).

| Decay mode | MC | World-average \mathcal{B} [1,4] | Weight |
|---|-----------------------|--|--------|
| $B^- \rightarrow D^0 \ell^- \bar{\nu}_\ell$ | 2.31×10^{-2} | $(2.33 \pm 0.10) \times 10^{-2}$ | 1.01 |
| $B^- \rightarrow D^{*0} \ell^- \bar{\nu}_\ell$ | 5.79×10^{-2} | $(5.59 \pm 0.19) \times 10^{-2}$ | 0.97 |
| $B^- \rightarrow D_1^0 \ell^- \bar{\nu}_\ell, D_1^0 \rightarrow D^{*+} \pi^-$ | 5.40×10^{-3} | $(2.8 \pm 0.1 \pm 1.5) \times 10^{-3}$ | 0.52 |
| $B^- \rightarrow D_2^{*0} \ell^- \bar{\nu}_\ell, D_2^{*0} \rightarrow D^{*+} \pi^-$ | 8.20×10^{-4} | $(7.7 \pm 0.6 \pm 0.4) \times 10^{-4}$ | 0.94 |
| $B^- \rightarrow D_1^{\prime 0} \ell^- \bar{\nu}_\ell, D_1^{\prime 0} \rightarrow D^{*+} \pi^-$ | 5.40×10^{-3} | $(1.3 \pm 0.3 \pm 0.2) \times 10^{-3}$ | 0.24 |
| $B^- \rightarrow D_0^{*0} \ell^- \bar{\nu}_\ell, D_0^{*0} \rightarrow D^+ \pi^-$ | 6.10×10^{-3} | $(2.8 \pm 0.3 \pm 0.4) \times 10^{-3}$ | 0.46 |
| $\bar{B}^0 \rightarrow D^0 \ell^- \bar{\nu}_\ell$ | 2.13×10^{-3} | $(2.20 \pm 0.10) \times 10^{-2}$ | 1.03 |
| $\bar{B}^0 \rightarrow D^{*+} \ell^- \bar{\nu}_\ell$ | 5.33×10^{-3} | $(4.88 \pm 0.10) \times 10^{-2}$ | 0.92 |
| $B^- \rightarrow \pi^0 \ell^- \bar{\nu}_\ell$ | 7.80×10^{-5} | $(7.80 \pm 0.27) \times 10^{-5}$ | 1.07 |
| $B^- \rightarrow \rho^0 \ell^- \bar{\nu}_\ell$ | 1.49×10^{-4} | $(1.58 \pm 0.11) \times 10^{-4}$ | 1.06 |
| $B^- \rightarrow \omega \ell^- \bar{\nu}_\ell$ | 1.15×10^{-4} | $(1.19 \pm 0.09) \times 10^{-4}$ | 1.04 |
| $B^- \rightarrow \eta \ell^- \bar{\nu}_\ell$ | 8.40×10^{-5} | $(3.8 \pm 0.6) \times 10^{-5}$ | 0.45 |
| $B^- \rightarrow \eta' \ell^- \bar{\nu}_\ell$ | 3.30×10^{-5} | $(2.3 \pm 0.8) \times 10^{-5}$ | 0.70 |
| $\bar{B}^0 \rightarrow \pi^+ \ell^- \bar{\nu}_\ell$ | 1.36×10^{-4} | $(1.45 \pm 0.05) \times 10^{-4}$ | 1.07 |
| $\bar{B}^0 \rightarrow \rho^+ \ell^- \bar{\nu}_\ell$ | 2.77×10^{-4} | $(2.94 \pm 0.21) \times 10^{-4}$ | 1.06 |

account the asymmetric statistical uncertainties. These distributions exhibit Gaussian behavior with a slight deviation from zero in the mean, which in most cases is an effect at the 1% level. We do not correct the signal yields or their uncertainties; instead we assign a systematic uncertainty due to the fit procedure. The size of the systematic uncertainty is estimated by the difference between the mean signal yield of the ensemble of pseudoexperiments and the signal yield used as the central value in the generation of the ensemble. This is evaluated separately for each bin.

Tables X–XII, in the Appendix, list the systematic uncertainties for the 1D($M_{\pi\pi}$), 1D(q^2), and 2D configurations, respectively.

To estimate correlations among the systematic uncertainties of the values of the partial branching fractions for each bin, we consider two scenarios. The first corresponds to uncertainties derived from the variation of one parameter in the MC simulation, such as those involving normalization of a background component or branching fractions of some decay processes. We characterize each component by a Gaussian distribution with a width equal to the systematic uncertainty investigated, draw a random variable for each parameter, and repeat the entire analysis procedure 1000 times. For each systematic uncertainty, we associate a correlation matrix COR_{ij} calculated as:

$$\text{COR}_{ij} = \frac{\langle (\Delta\mathcal{B}^i - \overline{\Delta\mathcal{B}^i})(\Delta\mathcal{B}^j - \overline{\Delta\mathcal{B}^j}) \rangle}{\sigma_i \sigma_j}, \quad (2)$$

where the indices i, j run over the bins in the sample, $\overline{\Delta\mathcal{B}^i}$ is the mean of the randomly generated partial branching fractions for the i -th bin, σ_i is its standard deviation, and $\langle \rangle$ denotes an average over the 1000 iterations. We then compute the associated covariance matrix as

$$\text{COV}_{ij} = \text{COR}_{ij} \sigma_i \sigma_j. \quad (3)$$

The second scenario applies to systematic uncertainties assessed under a different procedure, e.g., signal model dependence or FSR, among others. In this case, we evaluate the effect of a particular systematic uncertainty k in the i -th bin, σ_i^k , on $\Delta\mathcal{B}^i$ and assign it to $\xi_i^k = \sigma_i^k \Delta\mathcal{B}^i$ and from this quantity, determine the corresponding covariant matrix as

$$\text{COV}_{ij}^k = \xi_i^k \xi_j^k. \quad (4)$$

We provide the total systematic correlation matrices for the different fit scenarios in Tables XIII–XV.

D. Normalization uncertainties

The uncertainty in the measurement of the number of B -meson pairs produced is 1.4%, while that in the

branching fraction of $\Upsilon(4S) \rightarrow B^+B^-$ is 1.17%. We assume a 0.35% uncertainty in the track-finding efficiency for each charged particle reconstructed on the signal side and add each contribution linearly. Finally, we take the uncertainty due to the tagging efficiency correction, which originates from incorrect assumptions of the hadronic branching fractions on the tag side, as 4.2% [14]. These uncertainties are assumed to be 100% correlated across all bins and are also included in the correlation matrices of Tables XIII–XV.

VII. RESULTS AND DISCUSSION

The main result of this analysis is the total branching fraction for the $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ decay. Since we carry out this measurement in bins of the kinematic variables $M_{\pi\pi}$ or q^2 , we calculate the total branching fraction as the sum over all bins of partial branching fractions, $\mathcal{B}(B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell) = \sum_i \Delta\mathcal{B}^i$ with

$$\Delta\mathcal{B}^i = \frac{1}{4} \frac{Y_{\text{signal}}^i}{\epsilon_i} \frac{1}{\mathcal{B}(\Upsilon(4S) \rightarrow B^+B^-) N_{B\bar{B}}}. \quad (5)$$

Here, Y_{signal}^i denotes the signal yield measured in the i -th bin, ϵ_i is the corresponding reconstruction efficiency, $\mathcal{B}(\Upsilon(4S) \rightarrow B^+B^-) = (51.4 \pm 0.6)\%$, and $N_{B\bar{B}} = (771.6 \pm 10.6) \times 10^6$ is the number of $B\bar{B}$ events in the complete $\Upsilon(4S)$ dataset. We determine the ϵ_i values from MC simulation, with corrections applied for data-MC differences detector performance. The factor of 4 in the denominator averages the observed branching fraction across the four channels: $B^+ \rightarrow \pi^+\pi^-e^+\nu_e$, $B^- \rightarrow \pi^+\pi^-e^-\bar{\nu}_e$, $B^+ \rightarrow \pi^+\pi^-\mu^+\nu_\mu$, and $B^- \rightarrow \pi^+\pi^-\mu^-\bar{\nu}_\mu$.

The values of the input parameters for Eq. (5), as well as the partial branching fractions for each bin, are presented in Table V for the 1D($M_{\pi\pi}$) and 2D configurations, and in Table VI for the 1D(q^2) configuration. Adding the partial branching fractions, the total branching fraction for each configuration results in

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell)[1D(M_{\pi\pi})] \\ = [22.3_{-1.8}^{+2.0}(\text{stat}) \pm 4.0(\text{syst})] \times 10^{-5}, \end{aligned} \quad (6)$$

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell)[1D(q^2)] \\ = [22.7_{-1.6}^{+1.9}(\text{stat}) \pm 3.5(\text{syst})] \times 10^{-5}, \end{aligned} \quad (7)$$

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell)[2D] \\ = [23.0_{-1.8}^{+2.0}(\text{stat}) \pm 3.0(\text{syst})] \times 10^{-5}, \end{aligned} \quad (8)$$

where the quoted uncertainties are statistical and systematic, respectively. As the $[1D(q^2)]$ result lies between the

TABLE V. Signal yields (Y_{signal}^i), signal reconstruction efficiency (ϵ^i), and partial branching fractions ($\Delta\mathcal{B}^i$) for each bin i in the 1D ($M_{\pi\pi}$) and 2D configurations with the bin number convention as defined in Tables I and III, respectively. The first quoted uncertainty is statistical, and the second is systematic.

| Bin | 1D ($M_{\pi\pi}$) configuration | | | 2D configuration | | |
|-----|-----------------------------------|-----------------------|---------------------------------|-------------------------|-----------------------|---------------------------------|
| | Y_{signal}^i | $\epsilon^i[10^{-4}]$ | $\Delta\mathcal{B}^i[10^{-5}]$ | Y_{signal}^i | $\epsilon^i[10^{-4}]$ | $\Delta\mathcal{B}^i[10^{-5}]$ |
| 1 | $7.1^{+4.1}_{-3.2}$ | 7.92 ± 0.66 | $0.57^{+0.33}_{-0.25} \pm 0.09$ | $9.8^{+4.6}_{-3.7}$ | 7.39 ± 0.57 | $0.84^{+0.39}_{-0.32} \pm 0.18$ |
| 2 | $10.0^{+4.4}_{-3.5}$ | 8.20 ± 0.77 | $0.77^{+0.34}_{-0.27} \pm 0.16$ | $15.8^{+5.5}_{-4.6}$ | 8.45 ± 0.63 | $1.18^{+0.41}_{-0.34} \pm 0.20$ |
| 3 | $10.6^{+4.3}_{-3.5}$ | 7.75 ± 0.68 | $0.86^{+0.35}_{-0.28} \pm 0.24$ | $29.5^{+6.4}_{-5.7}$ | 8.61 ± 0.60 | $2.16^{+0.47}_{-0.42} \pm 0.23$ |
| 4 | $23.3^{+6.2}_{-5.4}$ | 7.82 ± 0.64 | $1.88^{+0.50}_{-0.44} \pm 0.41$ | $34.8^{+7.0}_{-6.2}$ | 8.35 ± 0.63 | $2.63^{+0.53}_{-0.47} \pm 0.32$ |
| 5 | $90.3^{+10.7}_{-10.0}$ | 9.32 ± 0.66 | $6.11^{+0.72}_{-0.68} \pm 1.12$ | $116.2^{+12.2}_{-11.5}$ | 7.98 ± 0.48 | $9.18^{+0.96}_{-0.91} \pm 1.02$ |
| 6 | $50.5^{+8.1}_{-7.4}$ | 7.76 ± 0.58 | $4.10^{+0.70}_{-0.66} \pm 0.74$ | $8.0^{+3.7}_{-2.9}$ | 7.20 ± 0.46 | $0.70^{+0.32}_{-0.25} \pm 0.20$ |
| 7 | $29.6^{+6.4}_{-5.7}$ | 8.18 ± 0.57 | $2.28^{+0.49}_{-0.44} \pm 0.36$ | $9.2^{+4.0}_{-3.2}$ | 9.07 ± 0.56 | $0.64^{+0.28}_{-0.22} \pm 0.11$ |
| 8 | $10.2^{+4.2}_{-3.4}$ | 8.47 ± 0.57 | $0.76^{+0.31}_{-0.25} \pm 0.12$ | $27.6^{+6.4}_{-5.6}$ | 9.78 ± 0.50 | $1.78^{+0.41}_{-0.36} \pm 0.25$ |
| 9 | $8.9^{+3.7}_{-3.0}$ | 8.79 ± 0.56 | $0.64^{+0.27}_{-0.22} \pm 0.11$ | $11.3^{+4.3}_{-3.5}$ | 7.82 ± 0.43 | $0.91^{+0.35}_{-0.28} \pm 0.12$ |
| 10 | $5.7^{+3.1}_{-2.4}$ | 8.98 ± 0.56 | $0.40^{+0.22}_{-0.17} \pm 0.07$ | $9.7^{+4.3}_{-3.5}$ | 8.45 ± 0.49 | $0.72^{+0.32}_{-0.26} \pm 0.08$ |
| 11 | $15.7^{+5.0}_{-4.2}$ | 9.04 ± 0.55 | $1.09^{+0.35}_{-0.29} \pm 0.12$ | $13.2^{+4.4}_{-3.7}$ | 8.97 ± 0.49 | $0.93^{+0.31}_{-0.26} \pm 0.11$ |
| 12 | $11.8^{+4.2}_{-3.4}$ | 8.20 ± 0.52 | $0.91^{+0.32}_{-0.26} \pm 0.14$ | $8.5^{+10.1}_{-9.0}$ | 6.77 ± 0.12 | $0.79^{+0.94}_{-0.84} \pm 0.26$ |
| 13 | $23.4^{+14.7}_{-13.6}$ | 7.45 ± 0.10 | $1.98^{+1.24}_{-1.15} \pm 0.46$ | $7.6^{+7.2}_{-4.9}$ | 8.55 ± 0.20 | $0.56^{+0.53}_{-0.36} \pm 0.08$ |

[1D($M_{\pi\pi}$)] and [2D] results, with the difference in central values being negligible as compared to the quoted systematic uncertainty, we take the [1D(q^2)] measurement as our final result:

TABLE VI. Signal yields (Y_{signal}^i), signal reconstruction efficiency (ϵ^i), and partial branching fractions ($\Delta\mathcal{B}^i$) for each bin i in the 1D(q^2) configuration with the bin number convention defined according to Table II. The first quoted uncertainty is statistical, and the second is systematic.

| Bin | Y_{signal}^i | $\epsilon^i[10^{-4}]$ | $\Delta\mathcal{B}^i[10^{-5}]$ |
|-----|-----------------------|-----------------------|---------------------------------|
| 1 | $16.5^{+6.8}_{-6.1}$ | 6.27 ± 0.19 | $1.66^{+0.68}_{-0.61} \pm 0.53$ |
| 2 | $11.4^{+6.0}_{-5.1}$ | 6.97 ± 0.22 | $1.03^{+0.54}_{-0.46} \pm 0.25$ |
| 3 | $13.0^{+5.6}_{-4.7}$ | 7.45 ± 0.25 | $1.10^{+0.47}_{-0.40} \pm 0.20$ |
| 4 | $16.0^{+6.6}_{-5.8}$ | 7.56 ± 0.27 | $1.33^{+0.55}_{-0.48} \pm 0.26$ |
| 5 | $24.3^{+7.1}_{-6.3}$ | 8.13 ± 0.31 | $1.88^{+0.55}_{-0.49} \pm 0.29$ |
| 6 | $12.2^{+5.7}_{-4.9}$ | 8.60 ± 0.35 | $0.89^{+0.42}_{-0.36} \pm 0.10$ |
| 7 | $10.8^{+5.1}_{-4.1}$ | 8.43 ± 0.38 | $0.81^{+0.38}_{-0.31} \pm 0.13$ |
| 8 | $21.4^{+6.5}_{-5.7}$ | 9.17 ± 0.44 | $1.47^{+0.45}_{-0.39} \pm 0.17$ |
| 9 | $9.6^{+4.7}_{-3.9}$ | 8.03 ± 0.45 | $0.75^{+0.37}_{-0.31} \pm 0.13$ |
| 10 | $30.8^{+6.7}_{-6.0}$ | 8.96 ± 0.53 | $2.17^{+0.47}_{-0.42} \pm 0.23$ |
| 11 | $11.6^{+5.0}_{-4.1}$ | 9.52 ± 0.60 | $0.77^{+0.33}_{-0.27} \pm 0.09$ |
| 12 | $16.3^{+4.9}_{-4.1}$ | 9.14 ± 0.66 | $1.12^{+0.34}_{-0.28} \pm 0.14$ |
| 13 | $19.4^{+5.3}_{-4.5}$ | 8.62 ± 0.72 | $1.42^{+0.39}_{-0.33} \pm 0.17$ |
| 14 | $15.4^{+4.7}_{-4.0}$ | 10.1 ± 0.88 | $0.96^{+0.29}_{-0.25} \pm 0.15$ |
| 15 | $15.1^{+4.9}_{-4.1}$ | 8.65 ± 0.93 | $1.10^{+0.36}_{-0.30} \pm 0.17$ |
| 16 | $10.8^{+4.0}_{-3.2}$ | 9.31 ± 1.12 | $0.73^{+0.27}_{-0.22} \pm 0.12$ |
| 17 | $12.3^{+4.4}_{-3.7}$ | 8.94 ± 1.28 | $0.87^{+0.31}_{-0.26} \pm 0.15$ |
| 18 | $32.3^{+6.8}_{-6.1}$ | 7.85 ± 0.88 | $2.59^{+0.55}_{-0.49} \pm 0.47$ |

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell) \\ = [22.7^{+1.9}_{-1.6}(\text{stat}) \pm 3.5(\text{syst})] \times 10^{-5} \end{aligned} \quad (9)$$

In the three configurations, our measurement is dominated by systematic uncertainties. The most significant source of systematic uncertainty comes from signal modeling. The value given in Eq. (9) is the first reported measurement of the branching fraction for the $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ decay. A correlation matrix between the measurements using the 18 bins in q^2 and the 13 bins in $M_{\pi\pi}$ is provided in Table XVI.

Figure 5 shows the dependence of the partial branching fractions on the $\pi^+\pi^-$ invariant mass and the squared momentum transfer. Though a detailed analysis of the resonant and nonresonant composition of the dipion mass spectrum is beyond the scope of this paper due to the limited statistics, we can observe a dominant peak associated with the ρ^0 meson and a small bump around the mass window for the $f_2(1270)$ meson. In a previous analysis [14], using the same data set, the contribution of the ρ^0 meson corresponded to a branching fraction of $\mathcal{B}(B^+ \rightarrow \rho^0\ell^+\nu_\ell) = [18.3 \pm 1.0(\text{stat}) \pm 1.0(\text{syst})] \times 10^{-5}$. The excess observed in the dipion mass distribution motivates the study of other exclusive charmless semileptonic B decays with masses above 1 GeV in the next-generation B factory experiment Belle II [17]. Our measurement of $B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell$ should help the modeling of B semileptonic decays and thus increase the precision with which the CKM matrix element $|V_{ub}|$ can be measured.

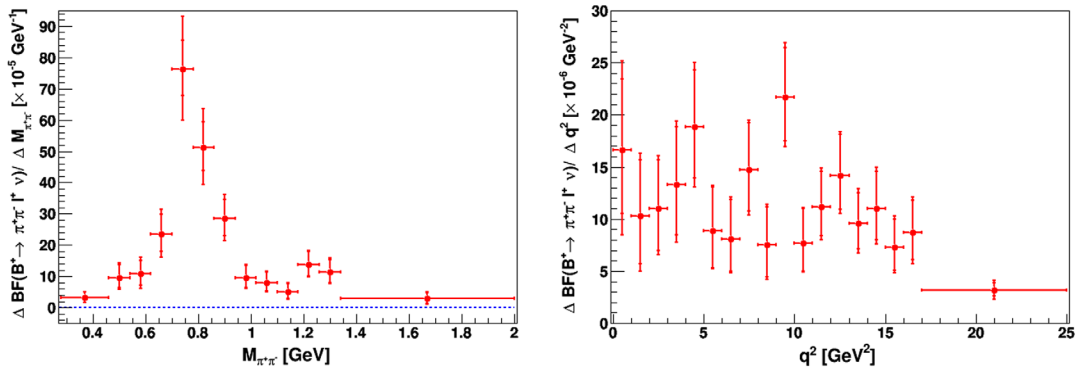


FIG. 5. Partial branching fractions for the decay $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ in bins of: (left) the $\pi^+ \pi^-$ invariant mass according to the results in the $1D(M_{\pi\pi})$ configuration, and (right) the momentum-transfer squared according to the results in the $1D(q^2)$ configuration. As there is no upper limit in the $\pi^+ \pi^-$ invariant mass, we use a cutoff at 2 GeV for the left.

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APPENDIX: SUPPLEMENTARY TABLES

TABLE VII. Reconstruction efficiency ($\times 10^{-4}$) in $M_{\pi\pi}$ bins for the phase space $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ and for $B^+ \rightarrow X \ell^+ \nu_\ell$, with the intermediate resonance X simulated using different MC generators. The bin-number convention is defined in Table I.

| Bin | PHSP $\pi^+ \pi^-$ | ISGW2 $f_0(500)$ | PHSP $f_0(500)$ | ISGW2 ρ^0 | LCSR [28] ρ^0 | PHSP ρ^0 | ISGW2 $f_2(1270)$ | PHSP $f_2(1270)$ | LCSR [42] $f_2(1270)$ | ISGW2 $\rho(1450)^0$ | PHSP $\rho(1450)^0$ |
|-----|--------------------|------------------|-----------------|----------------|--------------------|---------------|-------------------|------------------|-----------------------|----------------------|---------------------|
| 1 | 7.92±0.66 | 8.06±0.21 | 8.17±0.21 | | 9.74±0.69 | 9.27±0.68 | | | | | |
| 2 | 8.20±0.77 | 8.27±0.29 | 8.18±0.29 | 10.05±0.70 | 9.40±0.46 | 8.89±0.45 | | | | | |
| 3 | 7.75±0.68 | 8.39±0.28 | 7.96±0.27 | 9.43±0.46 | 9.08±0.28 | 8.64±0.27 | | | | | |
| 4 | 7.82±0.64 | 8.21±0.30 | 8.28±0.30 | 9.01±0.28 | 8.54±0.17 | 8.20±0.16 | | | | | |
| 5 | 9.32±0.66 | 8.10±0.36 | 8.33±0.37 | 9.08±0.17 | 8.28±0.17 | 8.06±0.18 | | | | | |
| 6 | 7.76±0.58 | 8.61±0.48 | 8.31±0.47 | 8.93±0.17 | 8.91±0.31 | 8.70±0.31 | | | | | |
| 7 | 8.18±0.57 | | | 9.34±0.32 | 8.81±0.47 | 9.19±0.48 | 10.12±0.96 | 9.63±0.94 | 8.32±0.87 | 8.73±0.96 | 9.67±1.01 |
| 8 | 8.47±0.57 | | | 9.57±0.49 | 10.02±0.69 | 10.17±0.70 | 9.14±0.64 | 9.71±0.66 | 7.74±0.59 | 11.70±0.97 | 8.13±0.80 |
| 9 | 8.79±0.56 | | | 9.35±0.67 | | | 8.9±0.41 | 9.45±0.42 | 8.67±0.41 | 11.91±0.83 | 8.93±0.72 |
| 10 | 8.98±0.56 | | | | | | 8.91±0.27 | 8.49±0.26 | 8.23±0.26 | 9.55±0.63 | 8.63±0.60 |
| 11 | 9.04±0.55 | | | | | | 8.40±0.24 | 8.55±0.25 | 8.23±0.24 | 9.99±0.54 | 9.49±0.52 |
| 12 | 8.2±0.52 | | | | | | 8.74±0.23 | 8.56±0.22 | 9.02±0.23 | 9.69±0.19 | 9.05±0.19 |
| 13 | 7.45±0.10 | | | | | | | | | | |

TABLE VIII. Reconstruction efficiency ($\times 10^{-4}$) in q^2 bins for the phase space $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ and for $B^+ \rightarrow X \ell^+ \nu_\ell$, with the intermediate resonance X simulated using different MC generators. The bin-number convention is defined in Table II.

| Bin | PHSP $\pi^+ \pi^-$ | ISGW2 $f_0(500)$ | PHSP $f_0(500)$ | ISGW2 ρ^0 | LCSR [28] ρ^0 | PHSP ρ^0 | ISGW2 $f_2(1270)$ | PHSP $f_2(1270)$ | LCSR [42] $f_2(1270)$ | ISGW2 $\rho(1450)^0$ | PHSP $\rho(1450)^0$ |
|-----|--------------------|------------------|-----------------|----------------|--------------------|---------------|-------------------|------------------|-----------------------|----------------------|---------------------|
| 1 | 6.27±0.19 | 8.10±0.40 | 7.94±0.40 | 7.95±0.83 | 8.76±0.48 | 7.63±0.31 | 8.46±0.43 | 8.13±0.39 | 8.55±0.28 | 7.53±0.63 | 7.91±0.44 |
| 2 | 6.97±0.22 | 7.28±0.38 | 7.55±0.39 | 10.14±0.75 | 8.84±0.46 | 7.81±0.32 | 8.75±0.42 | 7.73±0.38 | 8.61±0.30 | 8.62±0.62 | 8.31±0.46 |
| 3 | 7.45±0.25 | 8.17±0.41 | 7.78±0.4 | 9.15±0.61 | 8.26±0.43 | 7.67±0.32 | 8.82±0.42 | 7.95±0.40 | 8.22±0.32 | 9.55±0.61 | 9.41±0.50 |
| 4 | 7.56±0.27 | 8.39±0.43 | 8.26±0.43 | 9.17±0.55 | 9.06±0.44 | 7.94±0.34 | 8.89±0.42 | 8.15±0.42 | 8.30±0.36 | 9.72±0.59 | 9.87±0.53 |
| 5 | 8.13±0.31 | 8.15±0.43 | 8.25±0.43 | 9.51±0.51 | 8.56±0.42 | 8.06±0.35 | 8.82±0.42 | 9.16±0.45 | 8.26±0.40 | 10.88±0.61 | 9.24±0.53 |
| 6 | 8.60±0.35 | 8.85±0.46 | 8.22±0.44 | 10.13±0.49 | 8.35±0.41 | 9.06±0.38 | 8.62±0.42 | 9.02±0.47 | 8.95±0.47 | 10.21±0.57 | 8.52±0.53 |
| 7 | 8.43±0.38 | 8.20±0.46 | 8.84±0.47 | 9.84±0.46 | 8.37±0.40 | 8.44±0.37 | 9.07±0.45 | 9.44±0.49 | 8.43±0.52 | 11.48±0.60 | 9.12±0.57 |
| 8 | 9.17±0.44 | 8.24±0.47 | 8.82±0.49 | 9.40±0.43 | 9.31±0.42 | 8.60±0.39 | 9.14±0.47 | 9.15±0.51 | 9.06±0.63 | 10.88±0.59 | 10.63±0.64 |
| 9 | 8.03±0.45 | 8.58±0.49 | 9.02±0.51 | 8.63±0.39 | 8.75±0.40 | 8.50±0.40 | 8.37±0.47 | 9.83±0.55 | 10.15±0.80 | | |
| 10 | 8.96±0.53 | 9.75±0.54 | 8.71±0.51 | 9.50±0.40 | 9.44±0.41 | 8.91±0.42 | 8.68±0.52 | 9.62±0.57 | 9.84±0.97 | | |
| 11 | 9.52±0.60 | 9.41±0.55 | 8.84±0.53 | 9.62±0.39 | 9.14±0.40 | 9.06±0.44 | 9.01±0.58 | 8.66±0.57 | 10.07±1.25 | | |
| 12 | 9.14±0.66 | 9.13±0.56 | 8.44±0.54 | 8.96±0.37 | 9.10±0.40 | 9.04±0.46 | 9.44±0.67 | 10.29±0.67 | 7.56±1.46 | | |
| 13 | 8.62±0.72 | 8.44±0.56 | 8.19±0.55 | 9.58±0.38 | 8.50±0.39 | 8.84±0.47 | | | | | |
| 14 | 10.14±0.88 | 8.66±0.59 | 8.68±0.59 | 9.65±0.38 | 9.06±0.41 | 9.52±0.51 | | | | | |
| 15 | 8.65±0.93 | 8.00±0.60 | 8.86±0.63 | 8.92±0.37 | 9.01±0.42 | 8.91±0.52 | | | | | |
| 16 | 9.31±1.12 | 8.33±0.65 | 8.24±0.64 | 8.58±0.37 | 9.10±0.43 | 8.82±0.56 | | | | | |
| 17 | 8.94±1.28 | 8.3±0.69 | 8.64±0.70 | 9.12±0.40 | 8.43±0.43 | 8.54±0.59 | | | | | |
| 18 | 7.85±0.88 | 7.07±0.36 | 7.76±0.38 | 8.56±0.27 | 8.60±0.29 | 9.09±0.43 | | | | | |

TABLE IX. Reconstruction efficiency ($\times 10^{-4}$) in $M_{\pi\pi}$ and q^2 bins for the phase space $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ and for $B^+ \rightarrow X \ell^+ \nu_\ell$, with the intermediate resonance X simulated using different MC generators. The bin-number convention is defined in Table III.

| Bin | PHSP $\pi^+ \pi^-$ | ISGW2 $f_0(500)$ | PHSP $f_0(500)$ | ISGW2 ρ^0 | LCSR [28] ρ^0 | PHSP ρ^0 | ISGW2 $f_2(1270)$ | PHSP $f_2(1270)$ | LCSR [42] $f_2(1270)$ | ISGW2 $\rho(1450)^0$ | PHSP $\rho(1450)^0$ |
|-----|--------------------|------------------|-----------------|------------------|--------------------|------------------|-------------------|------------------|-----------------------|----------------------|---------------------|
| 1 | 7.39 ± 0.57 | 8.02 ± 0.21 | 7.88 ± 0.21 | 9.65 ± 0.72 | 9.19 ± 0.64 | 8.92 ± 0.52 | | | | | |
| 2 | 8.45 ± 0.63 | 8.46 ± 0.22 | 8.35 ± 0.22 | 10.30 ± 0.49 | 9.97 ± 0.50 | 9.80 ± 0.58 | | | | | |
| 3 | 8.61 ± 0.60 | 8.43 ± 0.35 | 8.32 ± 0.34 | 8.78 ± 0.37 | 8.49 ± 0.25 | 7.73 ± 0.18 | | | | | |
| 4 | 8.35 ± 0.63 | 8.14 ± 0.37 | 8.21 ± 0.37 | 9.61 ± 0.27 | 8.32 ± 0.23 | 8.24 ± 0.21 | | | | | |
| 5 | 7.98 ± 0.48 | 8.01 ± 0.28 | 8.56 ± 0.29 | 8.88 ± 0.13 | 8.61 ± 0.14 | 8.68 ± 0.16 | | | | | |
| 6 | 7.20 ± 0.46 | 7.79 ± 0.76 | 8.78 ± 0.81 | 10.68 ± 1.10 | 8.66 ± 0.63 | 7.64 ± 0.46 | 8.79 ± 0.47 | 8.84 ± 0.48 | 8.05 ± 0.34 | 9.65 ± 0.96 | 8.39 ± 0.71 |
| 7 | 9.07 ± 0.56 | 8.30 ± 0.85 | 9.46 ± 0.91 | 9.84 ± 0.65 | 9.47 ± 0.59 | 9.20 ± 0.55 | 9.26 ± 0.51 | 10.12 ± 0.56 | 8.34 ± 0.51 | 11.18 ± 0.94 | 8.72 ± 0.80 |
| 8 | 9.78 ± 0.50 | 10.64 ± 0.81 | 9.28 ± 0.76 | 9.65 ± 0.35 | 10.05 ± 0.40 | 10.04 ± 0.48 | 9.26 ± 0.51 | 9.12 ± 0.47 | 9.30 ± 0.91 | 11.30 ± 0.67 | 9.40 ± 0.72 |
| 9 | 7.82 ± 0.43 | | | | | | 8.54 ± 0.26 | 7.71 ± 0.24 | 8.23 ± 0.19 | 8.79 ± 0.51 | 8.72 ± 0.38 |
| 10 | 8.45 ± 0.49 | | | | | | 8.51 ± 0.27 | 8.82 ± 0.29 | 8.40 ± 0.29 | 9.86 ± 0.45 | 8.52 ± 0.42 |
| 11 | 8.97 ± 0.49 | | | | | | 8.51 ± 0.29 | 9.24 ± 0.29 | 9.60 ± 0.62 | 10.14 ± 0.38 | 10.00 ± 0.46 |
| 12 | 6.77 ± 0.12 | | | | | | 9.08 ± 0.54 | 7.89 ± 0.49 | 9.56 ± 0.43 | 8.46 ± 0.43 | 8.58 ± 0.34 |
| 13 | 8.55 ± 0.20 | | | | | | 9.53 ± 0.51 | 9.23 ± 0.52 | 9.65 ± 0.78 | 10.17 ± 0.30 | 9.55 ± 0.34 |

TABLE X. Relative systematic uncertainties in percent for the fits in $M_{\pi\pi}$ bins. The bin-number convention is defined in Table I.

| Source | Bin 1 | Bin 2 | Bin 3 | Bin 4 | Bin 5 | Bin 6 | Bin 7 | Bin 8 | Bin 9 | Bin 10 | Bin 11 | Bin 12 | Bin 13 |
|--|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|--------|
| FF $B \rightarrow D^{(*)}\ell\nu$ | 0.12 | 0.04 | 0.10 | 0.10 | 0.03 | 0.02 | 0.06 | 0.07 | 0.12 | 0.07 | 0.10 | 0.05 | 0.65 |
| FF $B \rightarrow D^{**}\ell\nu$ | 0.79 | 0.27 | 0.37 | 0.52 | 0.29 | 0.08 | 0.13 | 0.38 | 0.47 | 0.21 | 0.23 | 0.06 | 1.40 |
| Shapes $B \rightarrow X_u\ell\nu$ | 2.11 | 1.39 | 0.49 | 1.10 | 0.17 | 0.37 | 0.39 | 0.56 | 0.13 | 1.52 | 0.06 | 0.06 | 0.11 |
| $\mathcal{B}(B \rightarrow D^{(*)}\ell\nu)$ | 0.37 | 0.01 | 0.34 | 0.19 | 1.70 | 0.16 | 0.05 | 0.14 | 0.23 | 0.14 | 0.31 | 0.12 | 0.41 |
| $\mathcal{B}(B \rightarrow D^{**}\ell\nu)$ | 0.12 | 0.08 | 0.38 | 0.21 | 0.05 | 0.08 | 0.14 | 0.12 | 0.14 | 0.09 | 0.14 | 0.20 | 0.39 |
| $\mathcal{B}(B \rightarrow X_u\ell\nu)$ | 6.54 | 3.02 | 4.14 | 3.44 | 0.92 | 1.74 | 5.13 | 3.48 | 0.55 | 0.41 | 0.43 | 0.31 | 0.94 |
| Continuum | 0.58 | 0.39 | 0.09 | 0.24 | 0.08 | 0.09 | 0.30 | 0.99 | 0.09 | 0.60 | 0.20 | 0.34 | 1.85 |
| Rare | 0.80 | 1.04 | 0.62 | 0.65 | 0.15 | 0.25 | 0.41 | 0.61 | 0.53 | 1.49 | 0.65 | 0.48 | 4.80 |
| Sec. Leptons | 0.05 | 0.13 | 0.00 | 0.19 | 0.02 | 0.01 | 0.14 | 0.14 | 0.16 | 0.01 | 0.00 | 0.17 | 1.14 |
| Fake leptons | 0.76 | 0.07 | 0.11 | 0.24 | 0.04 | 0.06 | 0.03 | 0.90 | 0.11 | 0.23 | 0.41 | 0.06 | 2.00 |
| ℓ ID | 1.85 | 1.90 | 1.90 | 1.87 | 1.93 | 1.90 | 1.93 | 1.83 | 1.89 | 1.85 | 1.89 | 1.89 | 2.02 |
| π ID | 0.98 | 0.98 | 0.95 | 1.00 | 0.98 | 1.00 | 1.01 | 0.99 | 1.00 | 1.01 | 1.04 | 1.02 | 1.18 |
| FSR | 0.00 | 0.15 | 0.20 | 0.28 | 0.37 | 0.11 | 0.27 | 0.37 | 1.04 | 1.46 | 1.38 | 0.58 | 0.53 |
| Signal model | 2.56 | 14.4 | 24.7 | 18.2 | 15.9 | 15.1 | 10.4 | 11.4 | 15.4 | 14.9 | 5.86 | 12.2 | 21.6 |
| Fit procedure | 3.01 | 1.42 | 1.43 | 0.19 | 0.33 | 0.27 | 1.40 | 1.09 | 1.82 | 2.35 | 1.15 | 1.07 | 2.70 |
| Nom. Eff. Stats. | 8.33 | 9.39 | 8.77 | 8.18 | 7.08 | 7.47 | 6.97 | 6.73 | 6.37 | 6.24 | 6.08 | 6.34 | 1.34 |
| BDT selection | 0.84 | 1.16 | 0.00 | 0.32 | 0.77 | 1.64 | 1.42 | 2.30 | 1.22 | 0.68 | 1.73 | 4.01 | 2.18 |
| No. BB pairs | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 |
| $\mathcal{B}(\Upsilon(4S) \rightarrow B^+B^-)$ | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 |
| Tracking efficiency | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 |
| Tagging efficiency | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 |
| Nonresonant $B \rightarrow X_u\ell\nu$ | 9.86 | 9.00 | 3.77 | 5.58 | 1.99 | 3.17 | 5.74 | 3.92 | 3.37 | 3.51 | 2.55 | 1.69 | 2.14 |
| Total | 16.1 | 20.4 | 27.3 | 21.7 | 18.4 | 18.1 | 15.7 | 15.4 | 18.0 | 17.7 | 10.6 | 15.4 | 23.4 |

TABLE XI. Relative systematic uncertainties in percent for the fits in q^2 bins. The bin-number convention is defined in Table II.

| Source | Bin 1 | Bin 2 | Bin 3 | Bin 4 | Bin 5 | Bin 6 | Bin 7 | Bin 8 | Bin 9 | Bin 10 | Bin 11 | Bin 12 | Bin 13 | Bin 14 | Bin 15 | Bin 16 | Bin 17 | Bin 18 |
|--|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| FF $B \rightarrow D^{(*)}\ell\nu$ | 1.19 | 0.85 | 0.40 | 0.40 | 0.20 | 0.45 | 0.56 | 0.71 | 1.08 | 0.20 | 1.07 | 0.10 | 0.04 | 0.07 | 0.05 | 0.02 | 0.15 | 0.03 |
| FF $B \rightarrow D^{**}\ell\nu$ | 1.93 | 2.36 | 1.27 | 1.84 | 1.05 | 1.09 | 1.12 | 0.73 | 0.72 | 0.29 | 0.66 | 0.10 | 0.07 | 0.20 | 0.15 | 0.21 | 0.22 | 0.08 |
| Shapes $B \rightarrow X_u\ell\nu$ | 0.20 | 0.36 | 0.42 | 0.33 | 0.23 | 0.32 | 0.25 | 0.26 | 0.11 | 0.08 | 0.21 | 0.22 | 0.37 | 0.35 | 1.32 | 1.30 | 0.40 | 1.89 |
| $\mathcal{B}(B \rightarrow D^{(*)}\ell\nu)$ | 1.92 | 3.10 | 2.05 | 2.35 | 1.41 | 2.63 | 1.17 | 1.11 | 1.79 | 0.24 | 0.91 | 0.06 | 0.04 | 0.08 | 0.09 | 0.05 | 0.24 | 0.07 |
| $\mathcal{B}(B \rightarrow D^{**}\ell\nu)$ | 0.66 | 0.78 | 0.42 | 0.61 | 0.31 | 0.37 | 0.39 | 0.22 | 0.18 | 0.08 | 0.17 | 0.02 | 0.02 | 0.04 | 0.05 | 0.02 | 0.08 | 0.02 |
| $\mathcal{B}(B \rightarrow X_u\ell\nu)$ | 0.68 | 0.71 | 0.15 | 1.07 | 0.83 | 0.67 | 0.66 | 0.53 | 0.53 | 0.42 | 1.85 | 1.06 | 1.37 | 1.20 | 3.81 | 2.30 | 0.95 | 1.88 |
| Continuum | 1.70 | 0.93 | 0.34 | 0.43 | 0.13 | 0.45 | 0.32 | 0.41 | 0.54 | 0.02 | 0.19 | 0.03 | 0.28 | 0.07 | 0.01 | 0.27 | 0.40 | 0.14 |
| Rare | 7.44 | 1.47 | 0.73 | 0.97 | 0.60 | 0.30 | 0.31 | 0.46 | 1.09 | 0.34 | 0.44 | 0.29 | 0.30 | 0.20 | 0.39 | 0.01 | 0.25 | 0.16 |
| Sec. Leptons | 0.29 | 0.14 | 0.16 | 0.20 | 0.01 | 0.32 | 0.03 | 0.01 | 0.11 | 0.02 | 0.00 | 0.01 | 0.00 | 0.01 | 0.00 | 0.01 | 0.07 | 0.03 |
| Fake leptons | 2.56 | 1.05 | 0.20 | 0.17 | 0.11 | 0.15 | 0.04 | 0.23 | 0.09 | 0.04 | 0.20 | 0.02 | 0.16 | 0.01 | 0.40 | 0.08 | 0.55 | 0.27 |
| ℓ ID | 1.99 | 2.03 | 2.08 | 2.02 | 2.04 | 1.98 | 1.96 | 1.91 | 1.85 | 1.87 | 1.82 | 1.83 | 1.74 | 1.83 | 1.83 | 1.89 | 1.89 | 1.89 |
| π ID | 1.23 | 1.21 | 1.19 | 1.17 | 1.15 | 1.12 | 1.11 | 1.07 | 1.06 | 1.03 | 0.97 | 0.96 | 0.93 | 0.84 | 0.81 | 0.81 | 0.81 | 0.68 |
| FSR | 0.19 | 0.50 | 0.06 | 0.58 | 0.05 | 0.06 | 1.05 | 0.65 | 2.11 | 0.03 | 0.36 | 0.43 | 0.24 | 0.46 | 0.73 | 0.92 | 0.35 | 0.35 |
| Signal model | 29.8 | 23.0 | 16.6 | 18.2 | 13.7 | 8.79 | 13.7 | 8.76 | 14.1 | 6.24 | 5.44 | 8.15 | 5.69 | 10.9 | 4.44 | 8.21 | 4.88 | 10.2 |
| Nom. Eff. Stats. | 3.03 | 3.16 | 3.36 | 3.57 | 3.81 | 4.07 | 4.51 | 4.80 | 5.60 | 5.92 | 6.30 | 7.22 | 8.35 | 8.68 | 10.8 | 12.0 | 14.3 | 11.2 |
| Fit procedure | 0.27 | 1.79 | 1.46 | 2.09 | 0.99 | 1.27 | 2.43 | 1.18 | 2.36 | 1.08 | 3.67 | 0.68 | 1.29 | 1.00 | 2.50 | 2.53 | 1.44 | 1.19 |
| BDT selection | 2.31 | 1.88 | 1.55 | 2.22 | 1.00 | 1.65 | 0.88 | 2.19 | 1.08 | 0.76 | 1.07 | 0.64 | 0.21 | 0.32 | 0.74 | 0.21 | 0.32 | 2.15 |
| No. $B\bar{B}$ pairs | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 |
| $\mathcal{B}(\Upsilon(4S) \rightarrow B^+B^-)$ | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 |
| Tracking efficiency | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 |
| Tagging efficiency | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 |
| Nonresonant $B \rightarrow X_u\ell\nu$ | 0.42 | 1.75 | 1.54 | 1.25 | 0.82 | 1.64 | 0.93 | 1.87 | 5.21 | 2.60 | 2.59 | 3.07 | 4.12 | 5.19 | 6.62 | 4.63 | 4.07 | 5.26 |
| Total | 31.7 | 24.5 | 18.1 | 19.9 | 15.4 | 11.7 | 15.7 | 11.8 | 17.4 | 10.5 | 11.1 | 12.5 | 12.2 | 15.9 | 15.5 | 17.0 | 17.3 | 18.3 |

TABLE XII. Relative systematic uncertainties in percent for the fits in $M_{\pi\pi}$ and q^2 bins. The bin-number convention is defined in Table III.

| Source | Bin 1 | Bin 2 | Bin 3 | Bin 4 | Bin 5 | Bin 6 | Bin 7 | Bin 8 | Bin 9 | Bin 10 | Bin 11 | Bin 12 | Bin 13 |
|--|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|--------|
| FF $B \rightarrow D^{(*)}\ell\nu$ | 0.18 | 0.07 | 0.06 | 0.03 | 0.05 | 0.09 | 0.06 | 0.08 | 0.10 | 0.14 | 0.10 | 2.39 | 1.13 |
| FF $B \rightarrow D^{**}\ell\nu$ | 1.34 | 0.18 | 0.21 | 0.16 | 0.10 | 0.51 | 0.69 | 0.17 | 0.23 | 0.38 | 0.20 | 1.59 | 0.76 |
| Shapes $B \rightarrow X_u\ell\nu$ | 2.24 | 2.49 | 0.56 | 0.63 | 0.57 | 0.62 | 0.26 | 0.27 | 0.01 | 0.12 | 0.05 | 0.48 | 0.09 |
| $\mathcal{B}(B \rightarrow D^{(*)}\ell\nu)$ | 0.37 | 0.01 | 0.34 | 0.19 | 1.70 | 0.16 | 0.05 | 0.14 | 0.23 | 0.14 | 0.31 | 0.12 | 0.41 |
| $\mathcal{B}(B \rightarrow D^{**}\ell\nu)$ | 0.12 | 0.08 | 0.38 | 0.21 | 0.05 | 0.08 | 0.14 | 0.12 | 0.14 | 0.09 | 0.14 | 0.20 | 0.39 |
| $\mathcal{B}(B \rightarrow X_u\ell\nu)$ | 6.54 | 3.02 | 4.14 | 3.44 | 0.92 | 1.74 | 5.13 | 3.48 | 0.55 | 0.41 | 0.43 | 0.31 | 0.94 |
| Continuum | 0.53 | 0.25 | 0.51 | 0.02 | 0.07 | 0.82 | 0.44 | 0.27 | 1.00 | 0.03 | 0.19 | 2.93 | 2.76 |
| Rare | 1.04 | 0.47 | 0.95 | 0.32 | 0.14 | 2.10 | 0.33 | 0.23 | 2.20 | 0.45 | 0.60 | 8.47 | 1.78 |
| Sec. Leptons | 0.17 | 0.00 | 0.05 | 0.04 | 0.02 | 0.04 | 0.58 | 0.07 | 0.18 | 0.03 | 0.02 | 1.24 | 0.15 |
| Fake leptons | 0.25 | 0.68 | 0.15 | 0.01 | 0.07 | 0.72 | 0.01 | 0.29 | 0.35 | 0.15 | 0.13 | 1.32 | 1.02 |
| ℓ ID | 1.89 | 1.85 | 1.86 | 2.00 | 1.90 | 1.83 | 1.97 | 1.83 | 1.89 | 1.97 | 1.83 | 2.08 | 1.94 |
| π ID | 1.12 | 0.81 | 1.14 | 1.04 | 0.85 | 1.10 | 1.05 | 0.90 | 1.08 | 1.04 | 0.97 | 1.23 | 1.15 |
| FSR | 0.14 | 0.00 | 0.06 | 0.22 | 0.09 | 0.24 | 0.13 | 0.16 | 0.03 | 0.99 | 0.48 | 1.49 | 1.29 |
| Signal model | 18.2 | 12.7 | 4.46 | 6.90 | 6.98 | 26.9 | 13.4 | 11.4 | 8.95 | 7.73 | 8.79 | 29.9 | 13.1 |
| Nom. Eff. Stats. | 7.71 | 7.46 | 6.97 | 7.54 | 6.02 | 6.39 | 6.17 | 5.11 | 5.50 | 5.80 | 5.46 | 1.77 | 2.34 |
| Fit procedure | 1.88 | 0.93 | 0.85 | 0.82 | 0.21 | 1.48 | 2.03 | 0.57 | 2.41 | 1.65 | 0.96 | 6.89 | 1.35 |
| BDT selection | 1.49 | 0.21 | 0.99 | 0.65 | 0.86 | 1.66 | 1.33 | 1.11 | 3.60 | 2.52 | 2.61 | 2.51 | 1.54 |
| No. $B\bar{B}$ pairs | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 | 1.40 |
| $\mathcal{B}(\Upsilon(4S) \rightarrow B^+B^-)$ | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 | 1.17 |
| Tracking efficiency | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 | 1.05 |
| Tagging efficiency | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 | 4.20 |
| Nonresonant $B \rightarrow X_u\ell\nu$ | 0.42 | 3.80 | 1.02 | 2.59 | 2.50 | 1.25 | 1.09 | 1.45 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Total | 21.8 | 16.6 | 10.8 | 12.3 | 11.1 | 28.4 | 16.7 | 14.1 | 12.7 | 11.4 | 11.9 | 32.8 | 15.0 |

TABLE XIII. Systematic uncertainty correlation matrix of the $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ measurement in bins of the dipion mass, 1D($M_{\pi\pi}$) configuration. The binning convention is defined in Table I.

| Bin | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 0.787 | 0.529 | 0.635 | 0.523 | 0.595 | 0.777 | 0.722 | 0.562 | 0.573 | 0.735 | 0.547 | 0.327 |
| 2 | | 1 | 0.918 | 0.966 | 0.919 | 0.948 | 0.990 | 0.967 | 0.936 | 0.934 | 0.907 | 0.902 | 0.749 |
| 3 | | | 1 | 0.986 | 0.986 | 0.981 | 0.919 | 0.944 | 0.979 | 0.967 | 0.824 | 0.934 | 0.871 |
| 4 | | | | 1 | 0.981 | 0.989 | 0.964 | 0.971 | 0.980 | 0.971 | 0.874 | 0.938 | 0.837 |
| 5 | | | | | 1 | 0.994 | 0.934 | 0.953 | 0.991 | 0.979 | 0.882 | 0.969 | 0.859 |
| 6 | | | | | | 1 | 0.961 | 0.976 | 0.989 | 0.976 | 0.905 | 0.975 | 0.842 |
| 7 | | | | | | | 1 | 0.981 | 0.951 | 0.945 | 0.946 | 0.933 | 0.761 |
| 8 | | | | | | | | 1 | 0.959 | 0.949 | 0.931 | 0.956 | 0.792 |
| 9 | | | | | | | | | 1 | 0.992 | 0.898 | 0.972 | 0.870 |
| 10 | | | | | | | | | | 1 | 0.896 | 0.958 | 0.860 |
| 11 | | | | | | | | | | | 1 | 0.925 | 0.681 |
| 12 | | | | | | | | | | | | 1 | 0.831 |
| 13 | | | | | | | | | | | | | 1 |

TABLE XIV. Systematic uncertainty correlation matrix of the $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ measurement in bins of the momentum transfer square, 1D(q^2) configuration. The binning convention is defined in Table II.

| Bin | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|-----|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 0.961 | 0.949 | 0.950 | 0.936 | 0.850 | 0.920 | 0.840 | 0.867 | 0.711 | 0.622 | 0.746 | 0.584 | 0.763 | 0.419 | 0.598 | 0.423 | 0.686 |
| 2 | | 1 | 0.991 | 0.992 | 0.978 | 0.918 | 0.969 | 0.901 | 0.929 | 0.779 | 0.709 | 0.806 | 0.656 | 0.823 | 0.499 | 0.668 | 0.488 | 0.748 |
| 3 | | | 1 | 0.995 | 0.993 | 0.948 | 0.985 | 0.937 | 0.949 | 0.835 | 0.761 | 0.854 | 0.716 | 0.864 | 0.558 | 0.721 | 0.553 | 0.795 |
| 4 | | | | 1 | 0.989 | 0.944 | 0.985 | 0.932 | 0.944 | 0.821 | 0.761 | 0.843 | 0.703 | 0.852 | 0.553 | 0.714 | 0.541 | 0.787 |
| 5 | | | | | 1 | 0.961 | 0.991 | 0.954 | 0.948 | 0.870 | 0.797 | 0.888 | 0.760 | 0.888 | 0.605 | 0.763 | 0.607 | 0.826 |
| 6 | | | | | | 1 | 0.962 | 0.984 | 0.951 | 0.930 | 0.891 | 0.926 | 0.843 | 0.914 | 0.722 | 0.832 | 0.714 | 0.880 |
| 7 | | | | | | | 1 | 0.962 | 0.963 | 0.885 | 0.836 | 0.901 | 0.785 | 0.901 | 0.641 | 0.796 | 0.642 | 0.844 |
| 8 | | | | | | | | 1 | 0.960 | 0.957 | 0.911 | 0.957 | 0.881 | 0.944 | 0.765 | 0.872 | 0.762 | 0.920 |
| 9 | | | | | | | | | 1 | 0.912 | 0.865 | 0.929 | 0.846 | 0.948 | 0.739 | 0.847 | 0.703 | 0.894 |
| 10 | | | | | | | | | | 1 | 0.952 | 0.991 | 0.971 | 0.974 | 0.883 | 0.952 | 0.890 | 0.964 |
| 11 | | | | | | | | | | | 1 | 0.933 | 0.946 | 0.914 | 0.911 | 0.943 | 0.878 | 0.924 |
| 12 | | | | | | | | | | | | 1 | 0.969 | 0.991 | 0.882 | 0.962 | 0.888 | 0.979 |
| 13 | | | | | | | | | | | | | 1 | 0.959 | 0.962 | 0.987 | 0.956 | 0.973 |
| 14 | | | | | | | | | | | | | | 1 | 0.878 | 0.956 | 0.868 | 0.979 |
| 15 | | | | | | | | | | | | | | | 1 | 0.959 | 0.945 | 0.925 |
| 16 | | | | | | | | | | | | | | | | 1 | 0.961 | 0.980 |
| 17 | | | | | | | | | | | | | | | | | 1 | 0.920 |
| 18 | | | | | | | | | | | | | | | | | | 1 |

TABLE XV. Systematic uncertainty correlation matrix of the $B^+ \rightarrow \pi^+ \pi^- \ell^+ \nu_\ell$ measurement in bins of the dipion mass and the momentum transfer square, 2D configuration. The binning convention is defined in Table III.

| Bin | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 0.940 | 0.767 | 0.861 | 0.881 | 0.907 | 0.936 | 0.934 | 0.843 | 0.856 | 0.877 | 0.806 | 0.844 |
| 2 | | 1 | 0.867 | 0.950 | 0.965 | 0.906 | 0.958 | 0.968 | 0.874 | 0.899 | 0.914 | 0.769 | 0.832 |
| 3 | | | 1 | 0.969 | 0.950 | 0.693 | 0.846 | 0.856 | 0.876 | 0.925 | 0.902 | 0.545 | 0.683 |
| 4 | | | | 1 | 0.992 | 0.790 | 0.911 | 0.922 | 0.883 | 0.931 | 0.921 | 0.636 | 0.746 |
| 5 | | | | | 1 | 0.829 | 0.932 | 0.951 | 0.897 | 0.938 | 0.938 | 0.677 | 0.789 |
| 6 | | | | | | 1 | 0.962 | 0.956 | 0.867 | 0.860 | 0.900 | 0.902 | 0.909 |
| 7 | | | | | | | 1 | 0.992 | 0.939 | 0.952 | 0.969 | 0.858 | 0.904 |
| 8 | | | | | | | | 1 | 0.931 | 0.946 | 0.969 | 0.837 | 0.904 |
| 9 | | | | | | | | | 1 | 0.969 | 0.972 | 0.781 | 0.856 |
| 10 | | | | | | | | | | 1 | 0.991 | 0.758 | 0.852 |
| 11 | | | | | | | | | | | 1 | 0.788 | 0.874 |
| 12 | | | | | | | | | | | | 1 | 0.911 |
| 13 | | | | | | | | | | | | | 1 |

TABLE XVI. Approximated correlation matrix between the 18 bins in q^2 and 13 bins in $M_{\pi\pi}$ derived from data by requiring $M_{\text{miss}}^2 < 0.5 \text{ GeV}^2$.

| Bin | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0.000 | 0.000 | 0.091 | 0.000 | 0.067 | 0.085 | 0.028 | 0.048 | 0.000 | 0.057 | 0.036 | 0.182 | 0.281 |
| 2 | 0.000 | 0.000 | 0.050 | 0.038 | 0.091 | 0.046 | 0.000 | 0.052 | 0.052 | 0.000 | 0.039 | 0.050 | 0.283 |
| 3 | 0.000 | 0.000 | 0.058 | 0.044 | 0.021 | 0.027 | 0.035 | 0.000 | 0.000 | 0.000 | 0.091 | 0.000 | 0.276 |
| 4 | 0.050 | 0.000 | 0.050 | 0.038 | 0.054 | 0.046 | 0.030 | 0.052 | 0.052 | 0.000 | 0.000 | 0.000 | 0.307 |
| 5 | 0.045 | 0.091 | 0.045 | 0.069 | 0.033 | 0.107 | 0.083 | 0.048 | 0.095 | 0.000 | 0.000 | 0.000 | 0.271 |
| 6 | 0.171 | 0.057 | 0.000 | 0.000 | 0.083 | 0.053 | 0.000 | 0.000 | 0.060 | 0.000 | 0.045 | 0.000 | 0.217 |
| 7 | 0.000 | 0.000 | 0.000 | 0.000 | 0.022 | 0.055 | 0.107 | 0.000 | 0.062 | 0.000 | 0.185 | 0.000 | 0.211 |
| 8 | 0.000 | 0.051 | 0.051 | 0.039 | 0.112 | 0.048 | 0.093 | 0.000 | 0.000 | 0.000 | 0.120 | 0.000 | 0.218 |
| 9 | 0.000 | 0.000 | 0.000 | 0.111 | 0.080 | 0.069 | 0.044 | 0.000 | 0.000 | 0.092 | 0.000 | 0.000 | 0.139 |
| 10 | 0.000 | 0.000 | 0.000 | 0.000 | 0.096 | 0.074 | 0.095 | 0.110 | 0.000 | 0.197 | 0.164 | 0.210 | 0.112 |
| 11 | 0.000 | 0.000 | 0.000 | 0.150 | 0.145 | 0.062 | 0.000 | 0.000 | 0.000 | 0.165 | 0.103 | 0.000 | 0.094 |
| 12 | 0.000 | 0.078 | 0.078 | 0.000 | 0.171 | 0.110 | 0.000 | 0.082 | 0.082 | 0.000 | 0.000 | 0.078 | 0.019 |
| 13 | 0.000 | 0.208 | 0.000 | 0.053 | 0.152 | 0.162 | 0.168 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 14 | 0.000 | 0.161 | 0.000 | 0.061 | 0.030 | 0.189 | 0.098 | 0.085 | 0.085 | 0.000 | 0.000 | 0.081 | 0.000 |
| 15 | 0.000 | 0.000 | 0.000 | 0.000 | 0.177 | 0.076 | 0.195 | 0.085 | 0.085 | 0.000 | 0.000 | 0.000 | 0.000 |
| 16 | 0.000 | 0.000 | 0.091 | 0.069 | 0.100 | 0.085 | 0.165 | 0.000 | 0.095 | 0.000 | 0.000 | 0.000 | 0.000 |
| 17 | 0.167 | 0.084 | 0.000 | 0.000 | 0.214 | 0.078 | 0.000 | 0.088 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 18 | 0.224 | 0.000 | 0.112 | 0.213 | 0.267 | 0.105 | 0.034 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

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