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Optimal periodic dividend strategies for spectrally negative Lévy processes with fixed transaction costs

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Abstract

Maximising dividends is one classical stability criterion in actuarial risk theory. Motivated by the fact that dividends are paid periodically in real life, *periodic* dividend strategies were recently introduced (Albrecher, Gerber and Shiu, 2011b). In this paper, we incorporate fixed transaction costs into the model and study the optimal periodic dividend strategy with fixed transaction costs for spectrally negative Lévy processes.

The value function of a periodic (b_u, b_l) strategy is calculated by means of exiting identities and Itô's excursion when the surplus process is of unbounded variation. We show that a sufficient condition for optimality is that the Lévy measure admits a density which is completely monotonic. Under such assumptions, a periodic (b_u, b_l) strategy is confirmed to be optimal.

Results are illustrated.

Keywords: Optimal periodic dividends, SNLP, Fixed transaction costs

JEL codes: C44, C61, G24, G32, G35

1. Introduction

The first to study the now so-called “stability problem” were Lundberg (1909); Cramér (1930), with the traditional stability criterion being the probability of ruin (Bühlmann, 1970; Gerber, 1972). A major criticism of this set-up is that companies do not let their surplus grow to infinity (as the probability of ruin criterion suggests they should), and they do distribute profits to their beneficiaries eventually. This means that the ruin probability calculated does not actually represent the probability of ruin of the company (even with infinite horizon) - the calculations are flawed (as first argued by de Finetti, 1957). In addition, making decisions based on probability of ruin does not capture the risk and reward trade-off which companies typically face. Because of this, Bruno de Finetti (1957) first introduced an alternative formulation where distribution of surplus, or ‘dividends’, is allowed and the stability criterion is the maximised expected present value of dividends paid until ruin. This formulation is arguably more balanced, as neither paying too much nor too little will maximise the dividends. A strategy that maximises dividends is called an optimal strategy and the form of an optimal strategy is of particular interest. Since then, the optimal dividend problem for an insurance company has been studied intensively (see, for instance, Avanzi, 2009; Albrecher and Thonhauser, 2009).

While this literature does not belong to corporate finance, and does not really mean to (directly) inform companies how one should pay dividends in real life, the qualitative results we obtain from the modelling

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can only be improved by making the dividend strategies more realistic (see Avanzi, Tu and Wong, 2016a, for a formal discussion of what ‘realistic’ means in this context).

In this spirit, periodic dividend strategies were introduced by Albrecher, Cheung and Thonhauser (2011a) and have caught some recent attention as they capture the periodicity of dividend payments in real life. A periodic dividend strategy refers to the scenario when dividends can only be paid at some “specified” times. One motivation of this setting is that companies typically distribute (part of) their surplus to shareholders (as dividends or share buy-backs, for instance) at specific times in a year. Unfortunately, paying dividends at deterministic times introduces technical difficulties as one needs to keep track of the time until the next payment time. However, an *Erlang- n* random variable can be used as an approximation to a deterministic constant. This technique was first used in ruin theory by Asmussen, Avram and Usabel (2002) to approximate the probability of ruin in finite time. The same technique was subsequently introduced in the dividend setting by Albrecher, Cheung and Thonhauser (2011a). They considered the case when the dividend payment times (also called “decision times” as the dividends are “decided” and paid instantly at those times) are random variables and the solvency of the company is also considered at that same period. This means that a negative surplus is possible as long as it reverts to a non-negative value at the next observation time. This is related to the concept of Parisian (soft) ruin, where it is argued that companies do not go bankrupt instantaneously and may be able to recover before bankruptcy. Avanzi, Cheung, Wong and Woo (2013) studied periodic barrier strategies with continuous monitoring of solvency, that is, when ruin happens as soon as the surplus hits 0 (the assumption in this paper).

In this paper, we determine the optimal periodic dividend strategy under spectrally negative Lévy process, in presence of *fixed* transaction costs (see Remark 1.1 below). Here, transaction costs refer to the costs of transferring the surplus of the company to the shareholders. This includes both explicit components (e.g., tax and administrative costs), but also potentially implicit components (e.g., opportunity costs, penalty if it is undesirable to pay too often). The studies of optimal dividend strategies under fixed transaction costs have been done in ‘continuous’ decision making models (‘continuous’ here is mentioned as opposed to ‘periodic’; see also Avanzi, Tu and Wong, 2016b, for a discussion of the interaction between periodic and continuous dividend decisions); see for example Jeanblanc-Picqué and Shiryaev (1995, in the Brownian model), Loeffen (2008a, for spectrally negative Lévy processes), Bayraktar, Kyprianou and Yamazaki (2014, in the dual model) and Chen, Yuen and Wang (2020, with taxes). Although inspired by the Erlang- n technique, we only consider the case when $n = 1$ in this paper to enable neat expressions and formula, see Albrecher, Ivanovs and Zhou (2016). This means that a dividend decision time is activated when a Poisson process jumps.

Remark 1.1. *Transaction costs are typically comprised of two components, proportional costs and fixed costs. For example, if a dividend amount of ξ is paid, the cost is $\rho\xi + \kappa$, where $\kappa \geq 0$ and $1 > \rho \geq 0$. Note that the complexity of the problem does not increase with the presence of proportional cost ρ (e.g. tax) because it can be removed by scaling the risk metric (or currency) with a ratio of $(1 - \rho)^{-1}$ and considering another fixed cost κ' . Hence, we can assume without loss of generality that there are no proportional costs.*

Lévy processes (Bertoin, 1998) encompass a wide class of models present in the literature, including the Cramér-Lundberg (Gerber, 1969), Brownian (Gerber, 1970), and dual models (Avanzi, Gerber and Shiu, 2007). When there are no positive jumps—models are then referred to as ‘spectrally negative’—fluctuation theory takes the nicest form and various quantities can be expressed explicitly in terms of scale functions W_q (see Kyprianou, 2014, p. 239, for a remark regarding their historical development).

Thanks to the recent development in the theories regarding spectrally negative Lévy processes, the Cramér-Lundberg model is often extended and studied as a spectrally negative Lévy process. Classical examples of spectrally negative Lévy processes include Brownian motion with drift, Cramér-Lundberg risk processes and α -stable processes with stability parameter $\alpha \in (1, 2)$. More recently, a new family of examples known as Gaussian Tempered Stable Convolution class has been derived in Hubalek and Kyprianou (2011) (See also Loeffen, 2008a, 2009a,b; Avram, Palmowski and Pistorius, 2007; Wang and Zhou, 2018; Wang and Xu, 2020; Xu, Wang and Garrido, 2020).

For spectrally negative Lévy processes (“SNLP”), it is known that in general barrier types of strategy are not necessarily optimal (e.g., Gerber, 1969; Azcue and Muler, 2005). It is also observed in Avram, Palmowski and Pistorius

(2007) that the shape of the scale function W_q plays an important role in the optimal dividend problem for spectrally negative Lévy process. In particular, one sufficient condition for the barrier type of strategy to be optimal is that the Lévy measure has a completely monotonic density (e.g. Cramér-Lundberg under certain conditions **such as with hyper exponentially distributed jumps**, or (one-sided) tempered stable processes as discussed above); see Loeffen (2008b). Under such assumption, Noba, Pérez, Yamazaki and Yano (2018) recently proved that a periodic barrier strategy is optimal when the surplus is a spectrally negative Lévy process. Extending Loeffen (2008b) and Noba, Pérez, Yamazaki and Yano (2018), we show that a periodic (b_u, b_l) strategy is also optimal under SNLP with the same assumption on the Lévy measure, when fixed transaction costs on dividends are present. Sometimes, forced capital injection is used to replace the sufficient condition on the Lévy process, see e.g. Kulenko and Schmidli (2008).

The paper is organised as follows. In Section 2, the Mathematical model is introduced. Following that, Sections 3 and 4 briefly review some results in fluctuation theory for Lévy processes, and the well-known sufficient optimality result in the literature, respectively. A verification lemma is then presented in Section 6. In Sections 7-9, a candidate strategy is constructed and proved to be optimal. Convergence results for $\kappa \downarrow 0$ is shown in Section 10. Section 11 illustrates and Section 12 concludes.

2. The model

In this paper we use the standard set-up for stochastic processes (e.g. Bertoin, 1998, Chapter O). A spectrally negative Lévy process on the real line $Y = \{Y(t); t \geq 0\}$ is defined through its characteristic exponent, i.e.

$$\mathbb{E}[e^{\theta Y(t)}] = e^{t\psi_Y(\theta)} \quad (2.1)$$

and

$$\psi_Y(\theta) = c\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty, 0)} (e^{\theta s} - 1 - \theta s 1_{\{s > -1\}}) \Upsilon(ds), \quad (2.2)$$

with

$$\int_{(-\infty, 0)} (1 \wedge z^2) \Upsilon(dz) < \infty, \quad (2.3)$$

where (c, σ, Υ) are the Lévy triplet of Y . In order to avoid trivial cases, we also require that Y does not have a monotonic path. In this paper, we will use \mathbb{P}_x and \mathbb{E}_x to denote the probability measure and expectation for quantities for $X := x + Y$ instead of Y , for $x \in \mathbb{R}$. For example $\mathbb{P}_x(X \in B) := \mathbb{P}(x + Y \in B)$ for a “measurable” set B . Note in particular we have (\mathbb{P} -a.s.) $X(0) = x$.

Periodic dividend decision (payment) times, or in short decision times, are the times when the Poisson process (independent of X) with rate γ , $N_\gamma(t)$, jumps from $i - 1$ to i , i.e. the set $\mathbb{T} = \{T_i, i \in \mathbb{N}\}$ with

$$T_i = \inf\{t \geq 0 : N_\gamma(t) = i | N_\gamma(0) = 0\}. \quad (2.4)$$

Let $\mathbb{F} := \{\mathcal{F}(t); t \geq 0\}$ be the filtration generated by the process (X, N_γ) . Then, a periodic dividend strategy $\pi := \{D^\pi(t); t \geq 0\}$ is a non-decreasing, right-continuous and \mathbb{F} -adapted process where the cumulative amount of dividends $D^\pi = \{D^\pi(t); t \geq 0\}$ admits the form

$$D^\pi(t) = \int_{[0, t]} \nu^\pi(s) dN_\gamma(s), \quad t \geq 0, \quad D^\pi(0) = 0.$$

Hence, the dividend amount paid at T_i is $\xi_i^\pi := \nu^\pi(T_i)$ (the increment of D^π at T_i) and the strategy π can also be specified in terms of $\{\xi_i^\pi; i \in \mathbb{N}\}$. The modified surplus $X^\pi = \{X^\pi(t); t \geq 0\}$ is defined as

$$X^\pi(t) = X(t) - D^\pi(t) \quad (2.5)$$

and the ruin time τ^π is defined as

$$\tau^\pi = \inf\{t \geq 0 : X^\pi(t) < 0\}, \quad (2.6)$$

with the convention

$$\inf \emptyset = \infty. \quad (2.7)$$

We now introduce some constraints for a periodic dividend strategy to be admissible. Since we are not allowed to inject capital to the company and a dividend payment cannot exceed the current surplus, a periodic strategy π is admissible if it satisfies the following restriction:

$$X^\pi(T_i) \geq 0, \quad \forall T_i < \tau^\pi, \quad i \in \mathbb{N}. \quad (2.8)$$

Intuitively, given that a fixed transaction cost $\kappa > 0$ is incurred on each dividend payment, the amount of dividend should be large enough to pay the transaction cost, i.e.

$$\xi_i^\pi \geq \kappa \text{ if } \xi_i^\pi \neq 0. \quad (2.9)$$

This holds naturally (see property 4 in Remark 2.1 below).

We can see from the above definitions that not paying any dividend is also allowed. In this case, no transaction cost is incurred. We denote Π the set of all admissible strategies and Π_κ the set of all admissible strategy such that (2.9) holds. Note when X is of unbounded variation (i.e. with diffusion), it is possible that a dividend payment can cause ruin, which refers to liquidation of the company, i.e. the company chooses to close its business by distributing all the available surplus.

Lastly, we introduce the time preference parameter $\delta > 0$. The value function of a strategy π , $\pi \in \Pi$ with initial surplus x is denoted as $V_\kappa(x; \pi)$ with the following definition:

$$V_\kappa(x; \pi) = \mathbb{E}_x \left[\sum_{i=1}^{\infty} e^{-\delta T_i} (\xi_i^\pi - \kappa) 1_{\{\xi_i^\pi > 0\}} 1_{\{T_i \leq \tau^\pi\}} \right]. \quad (2.10)$$

Our goal is to find an optimal strategy π_κ^* (if exists) such that

$$V_\kappa(x; \pi_\kappa^*) = v_\kappa(x) := \sup_{\pi \in \Pi} V_\kappa(x; \pi) (\geq 0). \quad (2.11)$$

Remark 2.1. *From the definitions of V_κ and Π_κ , we have for any $0 \leq \kappa_1 \leq \kappa_2$*

1. $\pi \in \Pi_{\kappa_2} \implies \pi \in \Pi_{\kappa_1}$ and
2. $\pi \in \Pi_{\kappa_2} \implies V_{\kappa_1}(x; \pi) \geq V_{\kappa_2}(x; \pi)$ for all $x \geq 0$, and
3. $v_{\kappa_1}(x) \geq v_{\kappa_2}(x)$ for all $x \geq 0$, and
4. $V_\kappa(x; \pi) \geq 0$ for all $x \geq 0$ and $\pi \in \Pi_\kappa$,
5. $v_\kappa(x) = \sup_{\pi \in \Pi_\kappa} V_\kappa(x; \pi)$.

Further justification for item 5. is provided in Avanzi, Lau and Wong (2020).

Thanks to the fifth property in Remark 2.1, it is sufficient to only consider the strategies in Π_κ . Therefore, in the remaining of this paper, we restrict ourselves to strategies in Π_κ .

Note that ruin is immediate when $X(0) = x < 0$, which implies for any strategy π .

$$V_\kappa(x; \pi) = 0, \quad x < 0. \quad (2.12)$$

Definition 2.1 (Periodic (b_u, b_l) strategy). *A periodic (b_u, b_l) strategy with $0 \leq b_l \leq b_u$ is the strategy that pays $x - b_l$ whenever the surplus x is above or equal to b_u , at decision times. This reduces the surplus level to b_l .*

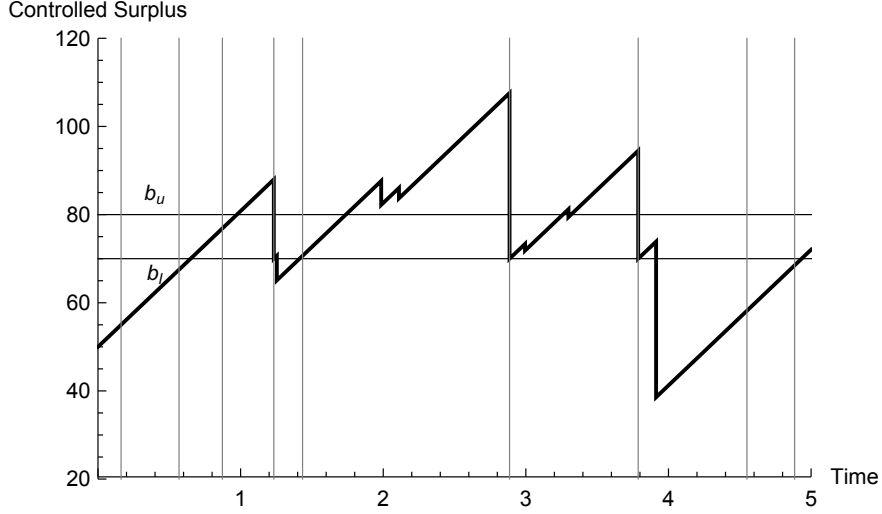


Figure 1: An illustration of a periodic (b_u, b_l) strategy. The vertical lines represent the (Poissonian) dividend decision times.

By denoting the strategy as π_{b_u, b_l} , we have

$$\xi_i^{\pi_{b_u, b_l}} = [X^{\pi_{b_u, b_l}}(T_i-) - b_l] 1_{\{X^{\pi_{b_u, b_l}}(T_i-) \geq b_u\}}. \quad (2.13)$$

Clearly, we have $\pi_{b_u, b_l} \in \Pi_\kappa \iff b_u - b_l \geq \kappa$.

Definition 2.2. Similarly, a periodic barrier strategy at barrier level $b > 0$, denoted as π_b , is defined as

$$\xi_i^{\pi_b} = [X^{\pi_b}(T_i-) - b] 1_{\{X^{\pi_b}(T_i-) \geq b\}}. \quad (2.14)$$

3. Scale functions

This section very briefly review knowledge of (fluctuation theory of spectrally negative Lévy processes and) scale functions for our purpose, i.e. to calculate the value function of a periodic (b_u, b_l) strategy. Interested reader can refer to standard textbook such as Bertoin (1998) and Kyprianou (2006). The tool we are going to use is fluctuation theory for Lévy processes which is quite standard within the community. Specifically, we will borrow the recent results from the Section 6 of Pérez and Yamazaki (2018). To fully understand the results, we recommend the work of Bertoin (1998), Kyprianou (2014), Loeffen, Renaud and Zhou (2014), Pantí (2017), Chaumont and Doney (2005), Pardo, Pérez and Rivero (2015), Avram, Pérez and Yamazaki (2018) and Pérez and Yamazaki (2018) (in the order), together with the references therein.

The q -scale function, W_q , for $x \geq 0$, $q \geq 0$ is defined through the inverse Laplace transform of $\frac{1}{\psi(\theta) - q}$, i.e.

$$\int_0^\infty e^{-\theta x} W_q(x) dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \phi_q,$$

where

$$\phi_q = \sup\{s \geq 0 : \psi(s) = q\}.$$

Next, for $x \geq 0$, the “tilted” q -scale function is define as

$$Z_q(x, \theta) = e^{\theta x} \left(1 - (\psi(\theta) - q) \int_0^x e^{-\theta y} W_q(y) dy \right). \quad (3.1)$$

In particular, when $\theta = 0$, we write

$$Z_q(x) = Z_q(x, 0) = 1 + q \int_0^x W_q(y) dy. \quad (3.2)$$

When $\theta = \phi_{q+r}$, we define

$$Z_{r,q}(x) := Z_q(x, \phi_{r+q}) = r \int_0^\infty e^{-\phi_{r+q}u} W_q(x+u) du, \quad (3.3)$$

where the last equality comes from $\int_0^\infty e^{-\phi_{r+q}u} W_q(u) du = 1/r$. Lastly, we define for $b \geq 0$

$$W_{r,q,b}(x) := W_q(x) + r \int_b^x W_{q+r}(x-y) W_q(y) dy, \quad (3.4)$$

where the integral vanishes when $x \leq b$.

Notice that when $x < 0$, all functions defined above are extended with

$$W_q(x) = 0,$$

which implies

$$Z_q(x) = 1, \quad (3.5)$$

$$\overline{Z}_q(x) = x. \quad (3.6)$$

We also define the integral of functions by adding an overhead line to it, e.g.

$$\overline{W}_q(x) = \int_0^x W_q(y) dy,$$

$$\overline{\overline{W}}_q(x) = \int_0^x \overline{W}_q(y) dy,$$

$$\overline{Z}_q(x) = \int_0^x Z_q(y) dy,$$

for $x \geq 0$.

We end this section with a remark that except in a few cases explicit calculation of W_q is difficult, if not impossible. We refer to Remark 1.1 in Loeffen, Renaud and Zhou (2014) for a review in the numerical aspect of scale functions.

4. Additional assumption for optimality

It is well known that for a spectrally negative Lévy process, barrier type of strategy is in general not optimal, e.g. see Azcue and Muler (2005). However, with the additional assumption that the Lévy measure has completely monotonic density, the shape of the scale function is “nice” and barrier type of strategy is optimal, see e.g. Loeffen (2008b), Loeffen (2008a), Noba, Pérez, Yamazaki and Yano (2018).

In this section, we follow the lines of Loeffen (2008b) and assume that the Lévy measure of X has completely monotonic density, i.e. the Lévy measure of the dual process $-X$, $\overline{\Upsilon}$, admits a density η , whose n^{th} derivative, $\eta^{(n)}$, exists for all $n \in \mathbb{N}$ with

$$(-1)^n \eta^{(n)}(x) \geq 0, \quad x > 0. \quad (4.1)$$

Note that this is a known sufficient condition for a (continuously decided) barrier strategy to be optimal. In general, the optimal strategy for a Cramér-Lundberg model is a band strategy as discovered by Azcue and Muler (2005), where many different “bands” are possible. Although being artificial, we note that if the optimal strategy is a barrier strategy, the optimal strategy is then characterised by the barrier level(s), which offers much simplicity to obtain qualitative insights.

Lastly, note that while a “log-convex” condition on the Lévy measure (see Loeffen and Renaud, 2010) **is weaker, and** would imply a completely monotonic Lévy density (which is the assumption we will make throughout the paper), we chose to directly assume the latter because it offers an explicit formula for the calculation of the first derivative of the value function in the upper branch (see Lemma 8.3).

5. Preliminary results

From Noba, Pérez, Yamazaki and Yano (2018), we know that the value function of a periodic barrier strategy at barrier level $b \geq 0$, π_b is given by

$$V_0(x; \pi_b) = \frac{\gamma}{\phi_{\gamma+\delta} Z'_{\gamma,\delta}(b)} \left(W_{\gamma,\delta,b}(x) - \gamma W_\delta(b) \overline{W}_{\gamma+\delta}(x-b) \right) - \gamma \overline{W}_{\gamma+\delta}(x-b), \quad x \in \mathbb{R}, \quad (5.1)$$

$$V_0(x; \pi_b) = \frac{\gamma}{\phi_{\gamma+\delta} Z'_{\gamma,\delta}(b)} W_\delta(x), \quad x \leq b \quad (5.2)$$

when there are no fixed transaction costs, i.e. $\kappa = 0$, where we recall the function $W_{\gamma,\delta,b}$ is defined in (3.4). In addition, when the Lévy measure of X has completely monotonic density,

1. We have $W_q \in \mathcal{C}^\infty$ and

$$W_q(x) = \frac{\partial}{\partial y} \phi(y) \Big|_{y=q} e^{\phi_q x} - \int_0^\infty e^{-xt} \mu_q(dt), \quad (5.3)$$

for some finite measure μ_q .

2. We have $W_\delta''' > 0 \implies$ there exists $\bar{b} \geq 0$ such that $W_\delta''(b) < 0$, $b \in (0, \bar{b})$, $W_\delta''(\bar{b}) = 0$ and $W_\delta''(b) > 0$, $b \in (\bar{b}, \infty)$.
3. For $x \geq b$, $V_0'(x; \pi_b)$ is given by

$$V_0'(x; \pi_b) = K + \gamma \frac{\int_0^\infty e^{-tx} g(t, b) \mu_{q+r}(dt)}{\phi_{q+r} Z'_{\gamma,\delta}(b)}, \quad (5.4)$$

where $K \in (0, 1)$ and

$$g(t, b) = t + \gamma W_\delta(0) + \gamma \int_0^b e^{ut} (W_\delta'(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma,\delta}(b)) du - \frac{\phi_{q+r}}{t} Z'_{\gamma,\delta}(b). \quad (5.5)$$

4. By defining the function h as

$$h(x) := e^{-\phi_{\gamma+\delta} x} Z''_{\gamma,\delta}(x) = \gamma \int_x^\infty e^{-\phi_{\gamma+\delta} y} W_\delta''(y) dy, \quad x > 0, \quad (5.6)$$

where the second equality is equation (4.11) in Noba, Pérez, Yamazaki and Yano (2018), we have that

$$\text{either } h(x) \geq 0 \text{ for } x \geq 0 \text{ or there exists a } b^* > 0 \text{ such that } h(x) \leq 0 \iff x \in (0, b^*]. \quad (5.7)$$

5. We have $b^* > 0 \iff h(0) \geq 0$. Moreover, it holds that

$$v_0(x) = V_0(x; \pi_{b^*}), \quad x \geq 0 \quad (5.8)$$

and

$$V_0'(x; \pi_{b^*}) < 1, \quad x \geq b^*. \quad (5.9)$$

6. Verification lemma

A function F is said to be smooth if $F \in \mathcal{C}^2$ (resp. $F \in \mathcal{C}^1$) if X is of unbounded variation (resp. bounded variation). The extended generator for X , \mathcal{L} , applied on a function F is given by

$$\mathcal{L}F(x) := cF'(x) + \frac{\sigma^2}{2} F''(x) + \int_{(-\infty, 0)} [F(x+s) - F(x) - F'(x)s 1_{\{|s|<1\}}] \Upsilon(ds) \quad (6.1)$$

if it is well defined, where the term $\frac{\sigma^2}{2} F''(x)$ is understood to be vanished if X is of bounded variation (no Gaussian component). The following lemma characterises sufficient conditions for a candidate strategy to satisfy in order to be optimal.

Lemma 6.1. Suppose $\pi \in \Pi_\kappa$ and its value function $H(x) := V_\kappa(x; \pi)$ satisfies

1. H is smooth,
2. $H \geq 0$ on \mathbb{R} ,
3. $(\mathcal{L} - \delta)H(x) + \gamma \max_{l \in [0, x]} \left((l - \kappa)1_{\{l > 0\}} + H(x - l) - H(x) \right) \leq 0, x \geq 0.$

Then the strategy π is optimal, i.e. $H(x) = v_\kappa(x)$ for all $x \in \mathbb{R}$.

The proof follows from Appendix B in Avanzi, Lau and Wong (2020).

7. Value function and the choice of (b_u, b_l)

7.1. On the value function and its smoothness

The value function of a periodic (b_u, b_l) strategy is given by the following lemma.

Lemma 7.1. The value function of a periodic (b_u, b_l) strategy is given by

$$V_\kappa(x; \pi_{b_u, b_l}) = \frac{\gamma \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right)}{\phi_{\gamma+\delta} Z_{\gamma, \delta}(b_u) - \gamma W_\delta(b_l)} \left(W_{\gamma, \delta, b}(x) - \gamma W_\delta(b_l) \overline{W}_{\gamma+\delta}(x - b_u) \right) - \gamma \left(\overline{\overline{W}}_{\gamma+\delta}(x - b_u) + (g - \kappa) \overline{W}_{\gamma+\delta}(x - b_u) \right), \quad (7.1)$$

where

$$g := b_u - b_l \quad (7.2)$$

and we recall that $W_{\gamma, \delta, b}$ is defined in (3.4).

If furthermore the smoothness condition (defined as)

$$V_\kappa(b_u; \pi_{b_u, b_l}) = V_\kappa(b_l; \pi_{b_u, b_l}) + b_u - b_l - \kappa \quad (7.3)$$

holds, the value function of a periodic (b_u, b_l) strategy reduces to the value function of a periodic barrier strategy at barrier level b_u without transaction costs, i.e.

$$V_\kappa(x; \pi_{b_u, b_l}) = V_0(x; \pi_{b_u}) = \frac{\gamma}{\phi_{\gamma+\delta} Z'_{\gamma, \delta}(b_u)} \left(W_{\gamma, \delta, b}(x) - \gamma W_\delta(b_u) \overline{W}_{\gamma+\delta}(x - b_u) \right) - \gamma \overline{\overline{W}}_{\gamma+\delta}(x - b_u). \quad (7.4)$$

Proof. Results from Pérez and Yamazaki (2018) can be adapted to our context. See Appendix A for details. Note that our proof does not use the additional assumption on the Lévy measure, thus is true for general spectrally negative Lévy processes. \square

When $x \leq b_u$, the value function is given by

$$V(x; \pi_{b_u, b_l}) = \frac{\gamma \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right)}{\phi_{\gamma+\delta} Z_{\gamma, \delta}(b_u) - \gamma W_\delta(b_l)} W_\delta(x), \quad x \leq b_u. \quad (7.5)$$

Hence, the smoothness condition $V_\kappa(b_u; \pi_{b_u, b_l}) = V_\kappa(b_l; \pi_{b_u, b_l}) + g - \kappa$ is equivalent to

$$\begin{aligned} & \frac{\gamma \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right)}{\phi_{\gamma+\delta} Z_{\gamma, \delta}(b_u) - \gamma W_\delta(b_l)} (W_\delta(b_u) - W_\delta(b_l)) = g - \kappa \\ \iff & \gamma \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right) (W_\delta(b_u) - W_\delta(b_l)) - (g - \kappa) (\phi_{\gamma+\delta} Z_{\gamma, \delta}(b_u) - \gamma W_\delta(b_l)) = 0 \\ \iff & \frac{\gamma}{\phi_{\gamma+\delta}} (W_\delta(b_u) - W_\delta(b_l)) + (g - \kappa) (\gamma W_\delta(b_u) - \phi_{\gamma+\delta} Z_{\gamma, \delta}(b_u)) = 0 \\ \iff & (g - \kappa) Z'_{\gamma, \delta}(b_u) - \frac{\gamma}{\phi_{\gamma+\delta}} (W_\delta(b_u) - W_\delta(b_l)) = 0, \end{aligned}$$

or

$$\Gamma_{b_l}(g) := (g - \kappa)Z'_{\gamma,\delta}(b_l + g) - \frac{\gamma}{\phi_{\gamma+\delta}}(W_\delta(b_l + g) - W_\delta(b_l)) = 0, \quad (7.6)$$

with $g = b_u - b_l$.

We want now to show that for all $b_l \geq 0$, there is a unique $g \geq \kappa$ such that $\Gamma_{b_l}(g) = 0$, which is precisely the following lemma.

Lemma 7.2. *For every $b_l \geq 0$, there exist a unique $b_u > b_l + \kappa$ such that the smoothness condition holds, i.e. $\Gamma_{b_l}(b_u - b_l) = 0$.*

Proof. First, we have

$$\begin{aligned} \Gamma_{b_l}(\kappa) &= (\kappa - \kappa)Z'_{\gamma,\delta}(b_l + \kappa) - \frac{\gamma}{\phi_{\gamma+\delta}}(W_\delta(b_l + \kappa) - W_\delta(b_l)) \\ &= -\frac{\gamma}{\phi_{\gamma+\delta}}(W_\delta(b_l + \kappa) - W_\delta(b_l)) \\ &< 0 \end{aligned}$$

as W_δ is an increasing function.

We proceed to show $\lim_{g \rightarrow \infty} \Gamma_{b_l}(g) = +\infty$. By differentiating (7.6) with respect to g and denote $b_u = b_l + g$, we get

$$\begin{aligned} \frac{\partial}{\partial g} \Gamma_{b_l}(g) &= (g - \kappa)Z''_{\gamma,\delta}(b_l + g) + Z'_{\gamma,\delta}(b_l + g) - \frac{\gamma}{\phi_{\gamma+\delta}}W'_\delta(b_l + g) \\ &= (g - \kappa)Z''_{\gamma,\delta}(b_l + g) + \frac{1}{\phi_{\gamma+\delta}} \left(\phi_{\gamma+\delta}Z'_{\gamma,\delta}(b_l + g) - \gamma W'_\delta(b_l + g) \right) \\ &= \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right) e^{\phi_{\gamma+\delta}b_u} h(b_u). \end{aligned} \quad (7.7)$$

Hence, by letting $g \rightarrow \infty$ and using $W''_\delta > 0$ (the second item in Section 5), we have

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{\partial}{\partial g} \Gamma_{b_l}(g) &= \lim_{g \rightarrow \infty} \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right) \gamma e^{\phi_{\gamma+\delta}b_u} \int_{b_u}^{\infty} e^{-\phi_{\gamma+\delta}y} W''_\delta(y) dy \\ &> \lim_{g \rightarrow \infty} \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right) \gamma e^{\phi_{\gamma+\delta}b_u} \int_{b_u}^{\infty} e^{-\phi_{\gamma+\delta}y} dy W''_\delta(y_0), \text{ large enough } g \text{ s.t. } W''_\delta(y_0) > 0 \\ &= +\infty \end{aligned} \quad (7.8)$$

and hence $\lim_{g \rightarrow \infty} \Gamma_{b_l}(g) = +\infty$ as desired. By the continuity of Γ_{b_l} and equation (7.6), we know that there exists a root for $\Gamma_{b_l}(g) = 0$. We now show that such root is unique.

From equation (7.7), we know that

$$\frac{\partial}{\partial g} \Gamma_{b_l}(g) = \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right) Z''_{\gamma,\delta}(b_u) = \left(\frac{1}{\phi_{\gamma+\delta}} + g - \kappa \right) e^{\phi_{\gamma+\delta}b_u} h(b_u).$$

Hence, if $\Gamma_{b_l}(g) = 0$ and $\frac{\partial}{\partial g} \Gamma_{b_l}(g) \leq 0$, we have

$$\frac{\partial}{\partial g} \Gamma_{b_l}(g) \leq 0 \implies \begin{cases} Z''_{\gamma,\delta}(b_u) \leq 0 \iff Z'_{\gamma,\delta}(b_u) \leq \frac{\gamma}{\phi_{\gamma+\delta}} W'_\delta(b_u) \\ b^* > 0 \text{ exists and } \kappa \leq b_u \leq b^* < \bar{b} \implies W'_\delta(b_u) \leq W'_\delta(y), y \leq b_u \end{cases}, \quad (7.9)$$

where we recall \bar{b} is defined in the second item in Section 5 and the second implication follows from (5.7).

Consequently, we have

$$\begin{aligned}
\Gamma_{b_l}(g) &= (g - \kappa)Z'_{\gamma,\delta}(b_u) - \frac{\gamma}{\phi_{\gamma+\delta}}(W_\delta(b_u) - W_\delta(b_l)) \\
&\leq (g - \kappa)\frac{\gamma}{\phi_{\gamma+\delta}}W'_\delta(b_u) - \frac{\gamma}{\phi_{\gamma+\delta}}(W_\delta(b_u) - W_\delta(b_l)) \\
&= \frac{\gamma}{\phi_{\gamma+\delta}}\left((g - \kappa)W'_\delta(b_u) - gW'_\delta(\alpha)\right), \exists \alpha \in [b_l, b_u] \text{ by Mean Value Theorem} \\
&= \frac{\gamma}{\phi_{\gamma+\delta}}\left((g(W'_\delta(b_u) - W'_\delta(\alpha)) - \kappa W'_\delta(b_u))\right) \\
&< 0,
\end{aligned} \tag{7.10}$$

which contradicts with the assumption that $\Gamma_{b_l}(g) = 0$. Hence, we have that $\Gamma_{b_l}(g) = 0$ implies $\frac{\partial}{\partial g}\Gamma_{b_l}(g) > 0$, which show that the root for $\Gamma_{b_l}(g) = 0$ is unique, by the continuity of Γ_{b_l} . Moreover, the root is also continuous in b_l because Γ_{b_l} defined in (7.6), as a 2 parameter function, is continuous in (b_u, b_l) . \square

Remark 7.1. *In view of the proof of Lemma 7.2, we have*

$$\Gamma_{b_l}(b_u - b_l) = 0 \implies \frac{\partial}{\partial g}\Gamma_{b_l}(g)|_{g=b_u-b_l} > 0 \implies h(b_u) > 0 \implies b_u > b^*, \tag{7.11}$$

which also implies that when the smoothness condition is met, we have

$$Z''_{\gamma,\delta}(b_u) > 0. \tag{7.12}$$

For a (b_u, b_l) strategy such that the smoothness condition (7.6) holds, we also call the strategy ‘‘smooth (b_u, b_l) strategy’’ and denote it as $\pi_{b_u, b_l}^{\kappa, s}$. In particular, Lemma 7.2 assures its existence. If $V'_\kappa(0; \pi_{b_u, 0}^{\kappa, s}) \leq 1$, the smooth (b_u, b_l) strategy is also called ‘‘optimal (b_u, b_l) strategy’’. Otherwise, if $V'_\kappa(0; \pi_{b_u, 0}^{\kappa, s}) > 1$ and $V'_\kappa(b_l; \pi_{b_u, b_l}^{\kappa, s}) = 1$, we call the smooth (b_u, b_l) strategy ‘‘optimal (b_u, b_l) strategy’’. The notation for an optimal (b_u, b_l) strategy is $\pi_{b_u, b_l}^{\kappa, *}$.

In the remaining of this paper, unless otherwise specified, when considering the properties of a smooth (resp. optimal) (b_u, b_l) strategy, we assume that the barriers and the fixed transaction costs κ are given. In this spirit, we denote its value function V_s (resp. V_*). If the dependence on the barriers or the transaction costs need to be stressed, we write the value function explicitly as $V_\kappa(\cdot; \pi_{b_u, b_l}^{\kappa, s})$ (resp. $V_\kappa(\cdot; \pi_{b_u, b_l}^{\kappa, *})$).

7.2. Existence of the lower barrier b_l and liquidation at first opportunity strategies

As explained earlier, if X is of unbounded variation (e.g. if a diffusion component exists) then $b_l = 0$ corresponds to a liquidation at first opportunity. This is because the surplus is ruined as soon as it reaches 0. On the other hand, if X is of bounded variation then ruin does not occur when the surplus is 0 because of the spectrally negative nature of the surplus dynamics. These cases occur when $V'_\kappa(0; \pi_{b_u, 0}^{\kappa, s}) \leq 1$, as we have $\pi_{b_u, 0}^{\kappa, *}$ by definition (which is $\pi_{b_u, 0}^{\kappa, s}$). This is illustrated in Section 11.3.

The existence of an ‘‘optimal (b_u, b_l) strategy’’ requires more care when $V'_\kappa(0; \pi_{b_u, 0}^{\kappa, s}) > 1$. The following lemma asserts the existence of an optimal (b_u, b_l) strategy.

Lemma 7.3. *If $V'_\kappa(0; \pi_{b_u, 0}^{\kappa, s}) > 1$, then there exist a $b_l \in (0, b^*)$ such that $V'_\kappa(b_l; \pi_{b_u, b_l}^{\kappa, s}) = 1$.*

Proof. First, notice that $b_l \mapsto V'_\kappa(b_l; \pi_{b_u, b_l}^{\kappa, s})$ is continuous in b_l due to Lemma 7.2 and (7.5). Hence, by continuity it suffices to show that

$$V'_{b_l}(b_l) < 1 \text{ for all } b_l \geq b^*, \tag{7.13}$$

which is equivalent to

$$\begin{aligned}
0 &< \phi_{\gamma+\delta}Z'_{\gamma,\delta}(b_u) - \gamma W'_\delta(b_l) \\
&= \phi_{\gamma+\delta}Z'_{\gamma,\delta}(b_l) - \gamma W'_\delta(b_l) + \phi_{\gamma+\delta}(Z'_{\gamma,\delta}(b_u) - Z'_{\gamma,\delta}(b_l)) \\
&= Z''_{\gamma,\delta}(b_l) + \phi_{\gamma+\delta}(Z'_{\gamma,\delta}(b_u) - Z'_{\gamma,\delta}(b_l))
\end{aligned}$$

for all $b_l \geq b^*$. This is true because $b_l \geq b^* \implies h(x) \geq 0$, $x \geq b_l \implies Z''_{\gamma,\delta}(b_l) \geq 0$ and $Z'_{\gamma,\delta}(b_u) > Z'_{\gamma,\delta}(b_l)$, implying that the sum is positive. \square

We have now proved the existence of an optimal (b_u, b_l) strategy. Combining Lemma 7.3 with Remark 7.1, for a $\pi_{b_u, b_l}^{\kappa, *}$, we have

$$b_l < b^* < b_u. \quad (7.14)$$

8. The derivative of the value function, V_*

The existence of a $\pi_{b_u, b_l}^{\kappa, *}$ strategy was shown in the previous sections. In this section, we investigate the properties of the value function of a given $\pi_{b_u, b_l}^{\kappa, *}$ to prepare for the proof of its optimality. Our goal is to show (8.5).

We start by showing that

$$V_*'(x) > 1, \quad x < b_l \text{ if } b_l > 0, \quad (8.1)$$

which is a direct consequence of (7.14). To be more specific, W_δ is decreasing on $[0, \bar{b}]$ and $b^* < \bar{b}$, implying that W_δ is decreasing on $[0, b_l]$. For a periodic (b_u, b_l) strategy, the lower branch of the value function given by (7.5) is proportional to W_δ , hence is decreasing to $V_*'(b_l) = 1$ on $[0, b_l]$.

Next, We show that the following lemma holds.

Lemma 8.1. *The inequality $V_*'(b_u) < 1$ holds.*

Proof. Using (7.4), we have

$$\begin{aligned} &\iff V_*'(b_u) < 1 \\ &\iff \phi_{\gamma+\delta} Z'_{\gamma,\delta}(b_u) > \gamma W'_\delta(b_u) \\ &\iff Z''_{\gamma,\delta}(b_u) > 0, \end{aligned}$$

where the last line is true from (7.12). \square

Due to the shape of W'_δ , i.e. the second item in Section 5, we have the following corollary.

Corollary 8.2. *For $x \in (b_l, b_u]$,*

$$V_*'(x) < 1. \quad (8.2)$$

Proof. Here we have 2 cases. If $b_u \leq \bar{b}$, then the derivative of W_δ is decreasing on $[0, b_u]$. Hence, by (7.5) V_* is also decreasing on $[0, b_u]$. In particular, we have $V_*'(x) < V_*'(b_l) \leq 1$ for $x \in (b_l, b_u]$.

On the other hand if $b_u > \bar{b}$, then the derivative of W_δ is decreasing on $(0, \bar{b})$ and increasing on $(\bar{b}, b_u]$. Similar to the previous case, we have $V_*'(x) < \max\{V_*'(b_l), V_*'(b_u)\} \leq 1$ for $x \in (b_l, b_u]$. \square

Next, we want to show the derivative of the value function is less than one beyond b_u , which is the consequence of the following lemma because of (5.9).

Lemma 8.3. *Recall that $V_0(x; \pi_b)$ is the value function of a periodic barrier strategy at barrier level b without transaction costs. For $x \geq b_u (\geq b^*)$, we have*

$$V_*'(x) \leq V'_0(x; \pi_{b^*}). \quad (8.3)$$

Proof. Recall that for $x \geq b$, we have

$$V'_0(x; \pi_b) = K + \gamma \frac{\int_0^\infty e^{-tx} g(t, b) \mu_{q+r}(dt)}{\phi_{q+r} Z'_{\gamma,\delta}(b)},$$

where $K \in (0, 1)$ and

$$g(t, b) = t + \gamma W'_\delta(0) + \gamma \int_0^b e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma,\delta}(b)) du - \frac{\phi_{q+r}}{t} Z'_{\gamma,\delta}(b).$$

Since $\pi_{b_u, b_l}^{\kappa, *}$ is also a smooth (b_u, b_l) strategy, we have from (7.4) $V_*(x) = V_0(x; \pi_{b_u})$. Hence, in order to show (8.3), it suffices to show that

$$\frac{g(t, b_u)}{Z'_{\gamma, \delta}(b_u)} \leq \frac{g(t, b^*)}{Z'_{\gamma, \delta}(b^*)}$$

holds. Note that we have $Z''_{\gamma, \delta}(b_u) > 0$ from (7.12), which implies $Z'_{\gamma, \delta}(b^*) < Z'_{\gamma, \delta}(b_u)$. Hence, it suffices to show

$$\int_0^{b_u} e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma, \delta}(b_u)) du \leq \int_0^{b^*} e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma, \delta}(b^*)) du. \quad (8.4)$$

Recall that we have $V'_\kappa(b_l; \pi_{b_u, b_l}^{\kappa, *}) \leq 1$ by the definition of $\pi_{b_u, b_l}^{\kappa, *}$. By direct computing using (5.2), we have $-\phi_{q+r} Z'_{\gamma, \delta}(b_u) \leq -W'_\delta(b_l)$. In addition, we have $W'_\delta(b_l) > W'_\delta(b^*)$ and $W'_\delta(u) \leq W'_\delta(b_l)$, $u \in [b_l, b_u]$, due to the shape of the scale function W_δ , see the second item in Section 5. If $b^* = 0$, we have $b_l = 0$ and hence

$$\begin{aligned} \int_0^{b_u} e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma, \delta}(b_u)) du &= \int_{b_l}^{b_u} e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma, \delta}(b_u)) du \\ &\leq \int_{b_l}^{b_u} e^{ut} (W'_\delta(u) - W'_\delta(b_l)) du \\ &< 0 \\ &= \int_0^{b^*} e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{r} Z'_{\gamma, \delta}(b^*)) du. \end{aligned}$$

On the other hand, if $b^* > 0$, we have

$$\begin{aligned} \int_0^{b_u} e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma, \delta}(b_u)) du &\leq \int_0^{b_u} e^{ut} (W'_\delta(u) - W'_\delta(b_l)) du \\ &= \int_0^{b_l} e^{ut} (W'_\delta(u) - W'_\delta(b_l)) du + \int_{b_l}^{b_u} e^{ut} (W'_\delta(u) - W'_\delta(b_l)) du \\ &< \int_0^{b_l} e^{ut} (W'_\delta(u) - W'_\delta(b)) du + \int_{b_l}^{b_u} e^{ut} (W'_\delta(u) - W'_\delta(b_l)) du \\ &< \int_0^b e^{ut} (W'_\delta(u) - W'_\delta(b)) du + \int_{b_l}^{b_u} e^{ut} (W'_\delta(u) - W'_\delta(b_l)) du \\ &< \int_0^{b^*} e^{ut} (W'_\delta(u) - W'_\delta(b^*)) du = \int_0^{b^*} e^{ut} (W'_\delta(u) - \frac{\phi_{q+r}}{\gamma} Z'_{\gamma, \delta}(b^*)) du. \end{aligned}$$

□

To sum up, we have

$$\begin{cases} V'_{b_l^*}(x) \leq 1, & x \geq 0, & \text{if } b_l = 0, \\ V'_{b_l^*}(x) \begin{cases} > 1, & x < b_l \\ = 1, & x = b_l \\ < 1, & x > b_l \end{cases}, & \text{if } b_l > 0. \end{cases} \quad (8.5)$$

The first case ($b_l = 0$) corresponds to a liquidation at first opportunity when X is of unbounded variation (see Section 7.2).

9. Optimality

In this section, we verify that the strategy $\pi_{b_u, b_l}^{\kappa, *}$ is optimal by arguing that all conditions in Lemma 6.1 are satisfied by its value function.

First, recall from (7.4) that when the smoothness condition is met, the value function at barrier level b_u is the same as the value function of a periodic barrier strategy without transaction costs. It was shown in Noba, Pérez, Yamazaki and Yano (2018) that the value function of a periodic barrier strategy belongs to the class of $\mathcal{C}^2(0, \infty)$ (resp. $\mathcal{C}^1(0, \infty)$) if X is of unbounded (resp. bounded) variation. Therefore, we conclude that the value function of the strategy $\pi_{b_u, b_l}^{\kappa, *}$ is smooth (see Section 6 for the definition of smoothness). The second condition is satisfied directly from the definition of value function. Conditions 3 can be shown to be met by proceeding in a similar fashion to Lemma 9.5 in Avanzi, Lau and Wong (2020), given the range of the derivative of the value function specified in (8.5).

To conclude, we present the following theorem and corollary.

Theorem 9.1. *The optimal strategy $\pi_{b_u, b_l}^{\kappa, *}$ is optimal, i.e. $V_\kappa(s; \pi_{b_u, b_l}^{\kappa, *}) = v_\kappa(x)$, $x \geq 0$.*

Corollary 9.2. *There is only 1 pair of (b_u, b_l) which qualifies to be a $\pi_{b_u, b_l}^{\kappa, *}$. We write the pair (b_u^*, b_l^*) and the strategy $\pi_{b_u^*, b_l^*}$ is optimal.*

The proof follows from Proposition 11.3 in Avanzi, Lau and Wong (2020). Therefore, we can conclude that there are (unique) (b_u^*, b_l^*) such that $\pi_{b_u^*, b_l^*}$ is optimal.

10. Convergence result when $\kappa \downarrow 0$

In this section, we show in Lemmas 10.2 and 10.3 that when $\kappa \downarrow 0$, $b_u^* - b_l^* \rightarrow 0$ so that combining with (7.14), it can be concluded that $b_u^* \rightarrow b^*$ and $b_l^* \rightarrow b^*$. In particular, convergence in barriers will imply convergence in both strategy π (convergence of D^π with probability one) and value function V . Note it can be shown that the convergence is monotonic (i.e. $b_u^* - b_l^* \downarrow 0$) with some additional steps. Since the latter implications can be obtained easily once the convergence of barriers is shown, we will only show the convergence of the barriers b_u^* and b_l^* . For a numerical illustration, we refer to Figure 4a.

In order to show our main result, we need to first show the following lemma, which should hold intuitively.

Lemma 10.1. *We have $\lim_{\kappa \downarrow 0} v_\kappa(x) = v_0(x)$ for all $x \geq 0$.*

Proof. By the definition in equation (2.10), we see that

$$V_0(x; \pi) = V_\kappa(x; \pi) + \kappa \mathbb{E} \sum_{i \in \mathbb{N}} e^{-\delta T_i} \mathbf{1}_{\{T_i \leq \tau^\pi\}}$$

for $\pi \in \Pi$. Therefore, by taking supremum, we have

$$v_0(x) \leq \sup_{\pi \in \Pi} V_\kappa(x; \pi) + \kappa \mathbb{E} \sum_{i \in \mathbb{N}} e^{-\delta T_i} = v_\kappa(x) + \kappa \mathbb{E} \sum_{i \in \mathbb{N}} e^{-\delta T_i} \leq v_0(x) + \kappa \mathbb{E} \sum_{i \in \mathbb{N}} e^{-\delta T_i},$$

where the term with κ is positive and finite. Therefore, by taking $\kappa \downarrow 0$, it is readily deduced that $\lim_{\kappa \downarrow 0} v_\kappa(x) = v_0(x)$ for all $x \geq 0$. \square

We are now ready to show our main result when $b^* > 0$ in the following lemma.

Lemma 10.2. *Suppose $b^* > 0$, we have that when $\kappa \downarrow 0$, $b_u^* - b_l^* \rightarrow 0$.*

Proof. By the hypothesis that $b^* > 0$ and Lemma 10.1, we have that

1. W'_δ is decreasing on $(0, \bar{b})$ which contains $(0, b^*)$;
2. $d_{[0, b^*]}(v_\kappa, v_0) \rightarrow 0$ when $\kappa \downarrow 0$, where $d_K(\cdot, \cdot)$ is the distance induced by the uniform norm for continuous functions on a compact set K .

Now, suppose the statement in the lemma is false, then there is a $\varepsilon > 0$ and a decreasing sequence $\{\kappa_n; n \in \mathbb{N}\}$ such that $\kappa_n \downarrow 0$ and $b_u^*(\kappa_n) - b_l^*(\kappa_n) > \varepsilon$ for all $n \in \mathbb{N}$. For convenience, we denote the barriers (b_u^n, b_l^n) .

Let's first consider the case there are infinite many n such that $b_l^n \geq \bar{\delta} > 0$ so by taking a subsequence if necessary, we can work with $b_l^n \geq \bar{\delta}$. Since

$$v_0(x) = V_0(x; \pi_{b^*}) = \frac{\gamma}{\phi_{\gamma+\delta} Z'_{\gamma,\delta}(b^*)} W_\delta(x), \quad v_{\kappa_n}(x) = V_0(x; \pi_{b_u^n}) = \frac{\gamma}{\phi_{\gamma+\delta} Z'_{\gamma,\delta}(b_u^n)} W_\delta(x)$$

for $x \leq \bar{\delta}$, we see that $b_u^n \rightarrow b^*$ by the second item at the beginning of the proof. In particular, we can assume that $b_u^n = b^* + \delta_n$, where $\delta_n \downarrow 0$ (by passing a subsequence if necessary). This means we can assume that $b^* - b_l^n > \varepsilon_1$ (with $\varepsilon_1 \in (0, \varepsilon)$) without loss of generality. By the second point above, for large enough n , we have

$$v_{\kappa_n}(b^*) - v_0(b^*) = \varepsilon_2, \quad v_{\kappa_n}(b^* - \varepsilon_1) - v_0(b^* - \varepsilon_1) = \varepsilon_3$$

for some $\varepsilon_2, \varepsilon_3 > 0$ arbitrarily small absolute values. This implies

$$v_{\kappa_n}(b^*) - v_{\kappa_n}(b^* - \varepsilon_1) = v_0(b^*) - v_0(b^* - \varepsilon_1) + \varepsilon_2 - \varepsilon_3,$$

or

$$\int_{b^* - \varepsilon_1}^{b^*} v'_{\kappa_n}(y) dy = \int_{b^* - \varepsilon_1}^{b^*} v'_0(y) dy + \varepsilon_2 - \varepsilon_3.$$

Note from Corollary 8.2, the integral on the left is strictly less than ε_1 , while the integral on the right is strictly larger than ε_1 (see Section 5). Since $\varepsilon_2 - \varepsilon_3$ can be made arbitrarily small (in absolute value) by increasing n , we arrive at a contradiction.

For the case where $b_l^n \downarrow 0$, the above argument applies if we choose $\varepsilon_1 = (b^*/2) \vee (b^* - \varepsilon)$. □

The next lemma deals with the case $b^* = 0$.

Lemma 10.3. *Suppose $b^* = 0$, we have that when $\kappa \downarrow 0$, $b_u^* - b_l^* \rightarrow 0$.*

Proof. Note when $b^* = 0$, we have that $b_l^* = 0$ so essentially we need to show $b_u^* \rightarrow 0$. On the other hand, from $b^* = 0$, we know that $W'_\delta(0+) < \infty$. Similar to Lemma 10.2, we work with a decreasing sequence of κ , say $\{\kappa_n, n \in \mathbb{N}\}$, with $\kappa_n \downarrow 0$.

Define the function

$$\tilde{\Gamma}_\kappa(\varepsilon) := \varepsilon Z'_{\gamma,\delta}(\kappa + \varepsilon) - \frac{\gamma}{\phi_{\gamma+\delta}} (W_\delta(\kappa + \varepsilon) - W_\delta(0)), \quad (10.1)$$

which is the smoothness condition when $b_l = 0$, see (7.6), when b_u is decomposed to $b_l + \kappa + \varepsilon$. From this and that $Z''_{\gamma,\delta} > 0$ on $(0, \infty)$ (because $b^* = 0$), we have

$$\begin{aligned} \tilde{\Gamma}_\kappa(\varepsilon) &= \int_0^\varepsilon Z'_{\gamma,\delta}(\kappa + y) dy - \frac{\gamma}{\phi_{\gamma+\delta}} \int_0^\varepsilon W'_\delta(\kappa + y) dy - \frac{\gamma}{\phi_{\gamma+\delta}} \int_0^\kappa W'_\delta(y) dy \\ &\geq \frac{1}{\phi_{\gamma+\delta}} \int_0^\varepsilon Z''_{\gamma,\delta}(\kappa + y) dy - \frac{\gamma}{\phi_{\gamma+\delta}} \int_0^\kappa W'_\delta(y) dy. \end{aligned}$$

Since the second term goes to zero when $\kappa_n \downarrow 0$ and the term inside the first integral is strictly positive, we have $\varepsilon \rightarrow 0$ for $\tilde{\Gamma}_\kappa(\varepsilon) = 0$, when $\kappa_n \downarrow 0$.

This completes the proof. □

11. Numerical Illustrations

Our base model is a diffusion model with mixed exponential downward jumps, i.e.

$$X(t) = ct + \sigma W(t) - \sum_{k=1}^{N(t)} G_k,$$

where $W = \{W(t); t \geq 0\}$ is a standard Brownian motion, $N = \{N(t); t \geq 0\}$ is a Poisson process with rate λ , and where G_k are sampled i.i.d. from a mixed exponential distribution, i.e.

$$\mathbb{P}(G_k \leq x) = p_1 e^{-\beta_1 x} + p_2 e^{-\beta_2 x}, \quad x \geq 0, \quad \forall k$$

with $p_1 + p_2 = 1$, $p_1, p_2, \beta_1, \beta_2 > 0$.

In this case, the Laplace exponent (minus q) is given by

$$\psi(\theta) - q = c\theta + \frac{\sigma^2}{2}\theta^2 + \lambda \left(p_1 \frac{\beta_1}{\beta_1 + \theta} + p_2 \frac{\beta_2}{\beta_2 + \theta} \right) - \lambda - q.$$

It is very easy to see that there are 4 distinct real roots (1 positive and 3 negative) for $\psi(\theta) - q = 0$. Denote them r_j , $j = 1, 2, 3, 4$ with order $r_i > r_j$ for $i < j$. Since the function $1/(\psi(\theta) - q)$ is a rational function, we can further express it (using partial fraction) as

$$\frac{1}{\psi(\theta) - q} = \sum_j \frac{A_j}{\theta - r_j}$$

with

$$A_j = \frac{1}{\psi'(r_j)}.$$

Note

$$\int_0^\infty e^{-\theta x} e^{r_j x} dx = \int_0^\infty e^{-(\theta - r_j)x} dx = \frac{1}{\theta - r_j}$$

and from the uniqueness of Laplace transform, we can deduce that

$$W_q(x) = \sum_j A_j e^{r_j x}.$$

Therefore, all other scale functions can be computed explicitly easily.

To find the optimal barriers (b_u^*, b_l^*) , we make use of Lemmas 7.2 and 7.3. To be more specific, we perform the following:

1. Find b^* using (5.7). Specifically, if $Z''_{\gamma, \delta}(0) \geq 0$, then set $b^* = 0$, otherwise, solve b^* such that $Z''_{\gamma, \delta}(b^*) = 0$. This can be done by (1) trying a large enough b such that $Z''_{\gamma, \delta}(b) > 0$ following by (2) a bisection method on the range $[0, b]$.
2. Write a function on $b_l \in [0, b^*]$ to output b_u from Lemma 7.2 with a similar method as the previous step (using range $[\max(\kappa, b^*), b]$ for large enough b), then calculate the derivative of the value function at b_l and return this number. Say we call this function G .
3. Find b_l^* using Lemma 7.3. Specifically, if $G(0) \leq 1$, then we set $b_l^* = 0$, otherwise we can obtain b_l^* by solving $G(b_l^*) = 1$ via a bisection method on the range $[0, b^*]$. Use Lemma 7.2 to calculate b_u^* from b_l^* .

Remark 11.1. *We remark that gradient descent type of methods typically do not work well here because a relatively large increment of the parameters (barriers) only results in a small change of the objective function (i.e. plateau). Therefore, analytic methods (those used in this paper) are needed. Perhaps more importantly, this shows that in practice one typically has more flexibility to deviate from the optimal strategy to incorporate other considerations.*

Our choice of parameters is $\lambda = 10$, $p_1 = 0.9$, $\beta_1 = 1.9$, $\beta_2 = 0.19$, $c = 11$, $\sigma = 1$, $\gamma = 1$, $\delta = 0.2$ and $\kappa = 0.2$. The Brownian motion term are used to model uncertainty (e.g. of the expenses), while the jump terms are used to model small and (ten times) larger claims, which occur at an average rate of 10% of total claims. Here, the value of c is chosen such that the profit loading is 10% and the expected profit per unit of time (net drift) of the process is 1. In the following, we will use μ and ζ^2 to denote the expected value and the variance of the expected profit (increment over one unit of time of the surplus before dividends), respectively. That is,

$$\mu := c - \lambda \left(\frac{p_1}{\beta_1} + \frac{p_2}{\beta_2} \right) = 1, \quad (11.1)$$

$$\zeta^2 := \sigma^2 + \frac{p_1}{\beta_1^2} + \frac{p_2}{\beta_2^2}. \quad (11.2)$$

In addition, we also use M to denote the ratio between the expected value of the large claims and that of the small claims, i.e.

$$M := \frac{\beta_1}{\beta_2} = 10. \quad (11.3)$$

11.1. The first derivative of the optimal value function beyond the upper barrier

Generally speaking, we expect the optimal value function v (see Equation (2.11)) to be concave because we expect the law of diminishing return holds. However, with the presence of fixed transaction costs, this is not necessarily the case. To see why it is possible to violate the concavity property, we shall consider the scenario when the surplus is high and the fixed transaction cost is also high. In such scenario, an incremental increase in surplus would actually decrease the ratio of the transaction costs to the first dividend payment (provided the company has not ruined yet), achieving a (relatively) higher return. Hence, the first order derivative v' is increasing, as shown in Figure 2b. Remarkably, such a case also seems to hold even when the fixed transaction cost is low, see Figure 2a. We therefore conjecture that this would be the general case. We also plot the second order derivative for reference. The horizontal line in Figures 2a and 2b are the asymptotes $\gamma/(\gamma + \delta)$.

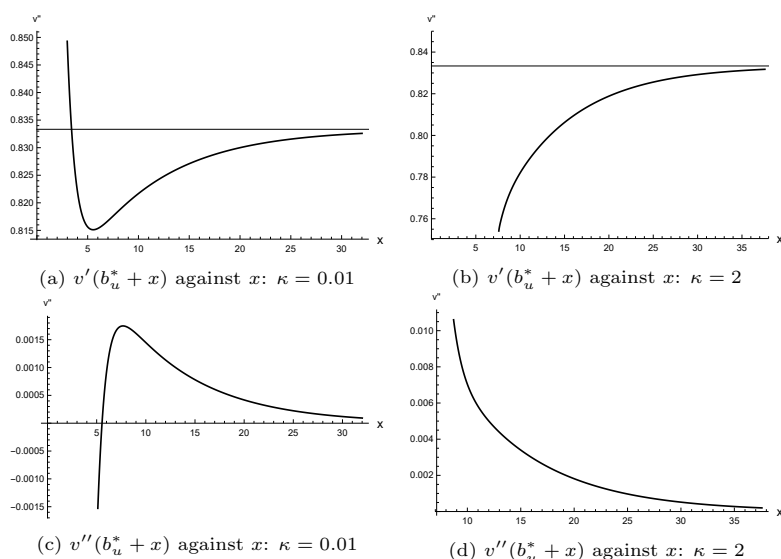


Figure 2: The derivatives $v'(b_u^* + x)$ (top row) and $v''(b_u^* + x)$ (bottom row) against x

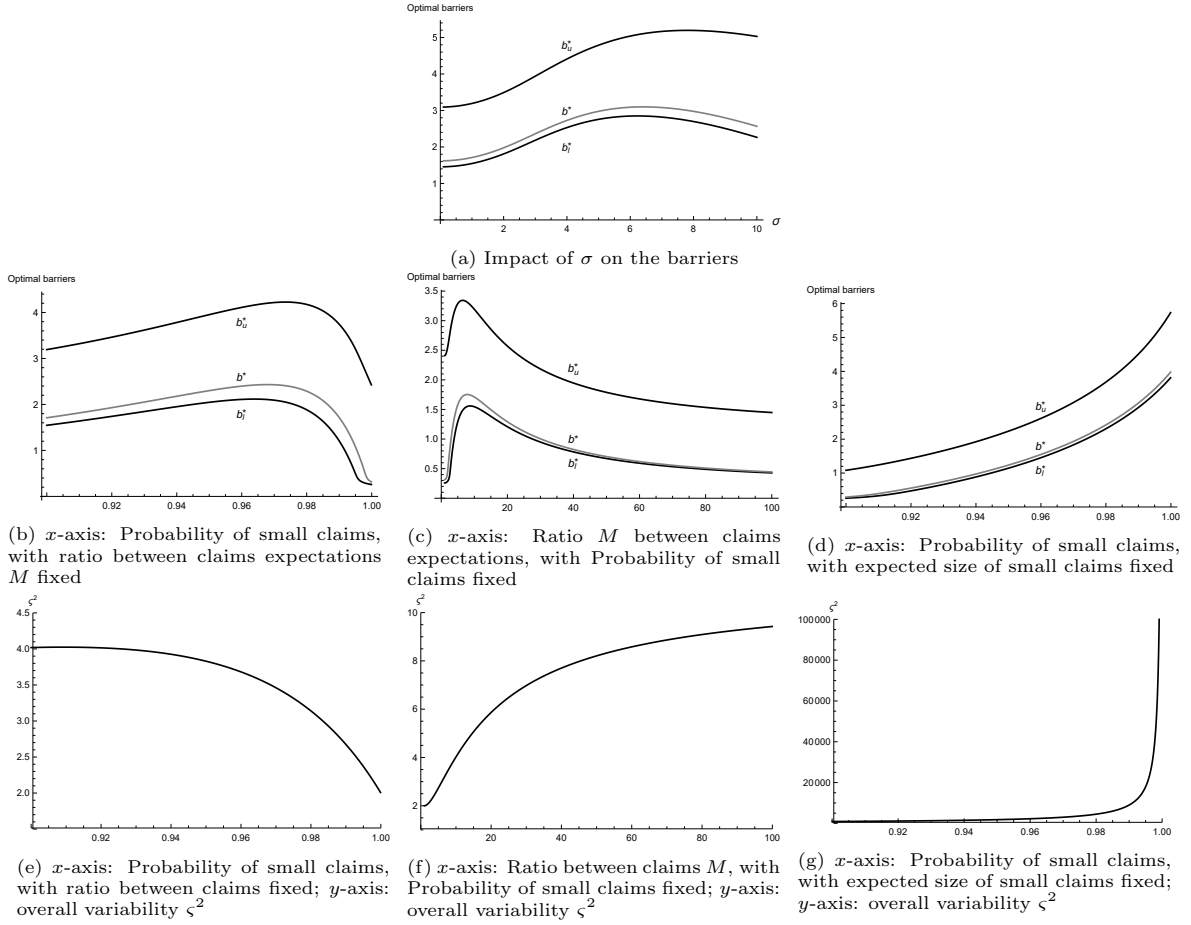


Figure 3: Impact of surplus variability on the optimal barriers

11.2. The impact of risk on the optimal barriers

Figure 3 shows the impact of the changes of parameters (σ , p_1 and M) on the optimal barrier levels (b_l^* , b^* and b_u^*), where the variability measure ζ^2 corresponding to the changes are also plotted.

Figure 3a exhibits an increasing then decreasing behaviour. This is because the increased variability would induce cautiousness at first and then deem to be too high for the business to be sustainable, leading eventually to an optimal liquidation at first opportunity.

For the adjustments of the jumps, we hold the overall expected claim amount per unit time μ fixed and adjust the parameters accordingly. The left column (Figures 3b,3e) adjusts p_1 , the probabilities of having small jumps (where β_1 and β_2 , the expected sizes of small and large claims are adjusted accordingly so that their ratio M , as well as μ , are fixed). As we see in Figure 3e, when the probability of the occurrence of large jumps decreases, the variability of the process decreases. However, barriers in (Figure 3b) are increasing then decreasing, as a trade-off between occurrence of large jumps (which is decreasing) and their size (which is increasing) operates.

The middle column (Figures 3c,3f) adjusts M , the ratio between the expected sizes of the large and small jumps (where β_1 and β_2 , the expected sizes of small and large claims are adjusted accordingly so that μ is fixed). In Figure 3f, when the ratio between the large and small claims increases, the variability (ζ^2) of the process increases to a limit. The barriers in Figure 3c (and beyond) seem to display a convergent behaviour which agrees with Figure 3f.

For another comparison, we increase the magnitude of the extreme events while decreases its probability

of occurrence in Figures 3d and 3g. To achieve this, in the right column (Figures 3d,3g) we fix the expected value of the small jumps ($1/\beta_1$), decrease the probability of large claims (p_2) but at the same time increase its expected value ($1/\beta_2$). We keep the overall expected claims (μ) constant for a fair comparison. It is remarkable that the barriers don't seem to decline to a liquidation-at-first opportunity, even though ζ^2 becomes very large. Here scarcity of large events seem to overpower the size of the events, even though optimal barriers are still increasing.

11.3. The impact of the surplus process parameters on the optimal barriers

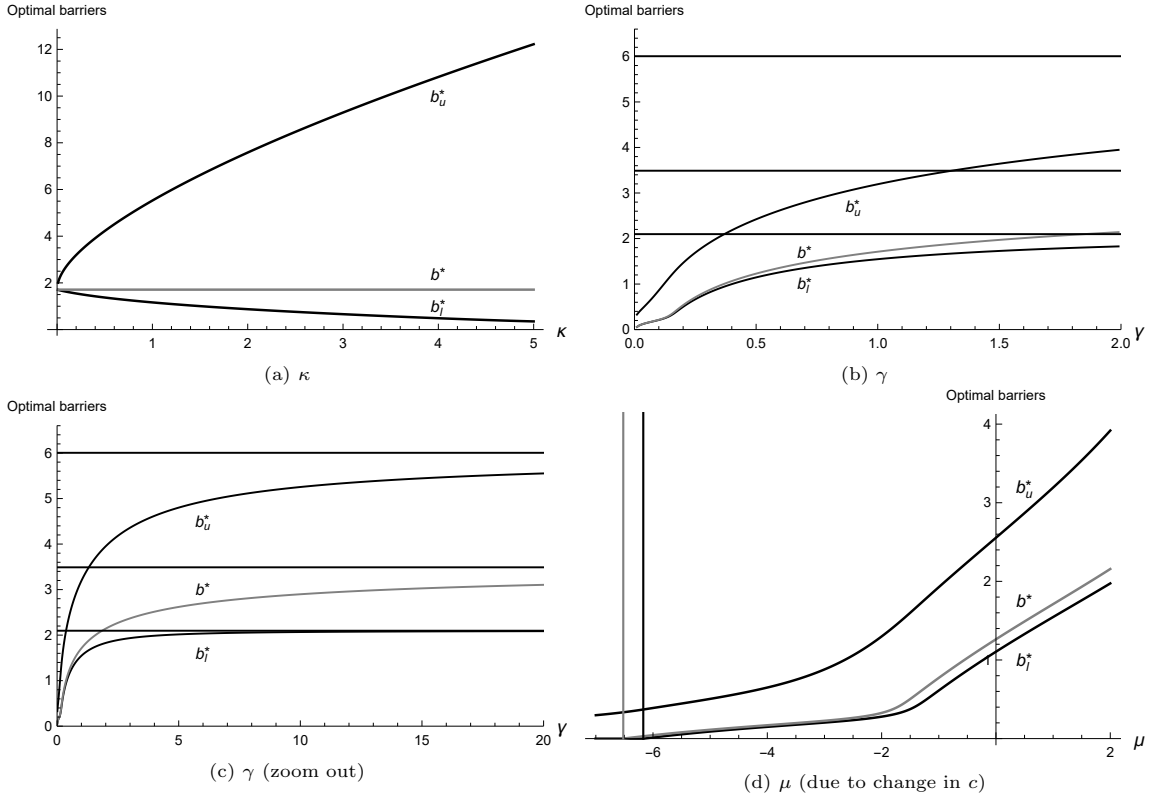


Figure 4: Sensitivities to parameters (II): Others

Figure 4 describes the sensitivities of the barriers to changes in the level of fixed transaction costs, the frequencies of dividend payment opportunities and the premium rate c . Figure 4a shows that when the fixed transaction costs increase, the optimal periodic barriers b_u^* and b_l^* are moving further away from the periodic barrier b^* . This is consistent with the results in the literature (e.g. Bayraktar, Kyprianou and Yamazaki, 2014).

Figures 4b, 4c show the monotonic increase in the optimal barriers with increasing dividend payment frequency γ . Ultimately they will converge to the barriers when dividends can be paid at any time. It is quite interesting that when γ is small, the lower optimal barrier b_l^* behaves similarly to the optimal barrier without the fixed transaction costs. This suggests that the fixed transaction costs are mainly compensated by the upper optimal barrier b_u^* , as b_l cannot go below 0, lest a liquidation at first opportunity occurs, which is not optimal in those cases unless γ is extremely small (< 0.002).

Finally, Figure 4d plots the change of the optimal barriers corresponding to the changes of the expected gain per unit time μ , solely due to the change of the premium rate c . Interestingly, the lower barriers b_l^* and b^* (when there is no transaction costs) are hardly zero unless μ is too negative. This displays a very

different behaviour compared to the case when the surplus process is spectrally positive (i.e. deterministic costs and random gain) where the optimal lower barrier is zero as long as the business is non-profitable indicated by a non-positive μ . This is because in our case, as long as the premium rate c is positive, there is a benefit in having a small but positive surplus as the surplus “pushes up” in-between downwards jumps there are chances that no claims arrive before the next jump. On the other hand, once c becomes too small it makes sense to liquidate at first opportunity. This happens on the solid black vertical line. The threshold in absence of fixed transaction costs is different and is illustrated with a solid grey vertical line, which is lower due to absence of transaction costs.

12. Conclusion

In this paper, we determined the form of the optimal periodic dividend strategy when there are fixed transaction costs, when the dividend decisions are Poissonian, and where the underlying model is a spectrally negative Lévy process. Extending papers such as Pérez and Yamazaki (2018), we were able to compute the value function of a periodic (b_u, b_l) strategy concisely in terms of scale functions.

Using an additional assumption that the Lévy measure has completely monotonic density and imposed the same 2 conditions as Avanzi, Lau and Wong (2020) on the parameters b_u and b_l , it was then confirmed that the periodic (b_u^*, b_l^*) (exists and) is optimal.

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A. Value function of a (b_u, b_l) strategy in SNLP

We adapt the approach in Pérez and Yamazaki (2018) where the scale of the process is shifted b_u units down, i.e. there are numbers $a < d < -\kappa < 0 < b$ such that whenever the process X is above or at 0 at exponential times ($t = e_\gamma$), it jumps to d . We denote the difference (the dividend) is denoted as $dL_\gamma^d(t)$ (with the aggregate version being $L_\gamma^d(t)$) and the process is killed upon exiting the interval $[a, b]$.

For $a < 0$, we define for any measurable function f

$$\mathcal{M}_a^{(\gamma, \delta)} f(x) := f(x - a) + \gamma \int_0^x W_\delta(x - y) f(y - a) dy \quad (\text{A.1})$$

and

$$W_{\gamma, \delta, a}(x) := \mathcal{M}_a^{(\gamma, \delta)} W_\delta(x). \quad (\text{A.2})$$

In particular, we have $W_{\gamma,\delta,a}(0) = W_\delta(-a)$. For $a \leq x \leq b$, we have the standard 2 sided exiting identities

$$\mathbb{E}_x(e^{-\delta\tau_b^+}; \tau_b^+ < \tau_a^-) = \frac{W_\delta(x-a)}{W_\delta(b-a)}, \quad (\text{A.3})$$

$$\mathbb{E}_x(e^{-\delta\tau_a^-}; \tau_b^+ > \tau_a^-) = Z_\delta(x-a, \theta) - Z_\delta(b-a, \theta) \frac{W_\delta(x-a)}{W_\delta(b-a)}. \quad (\text{A.4})$$

In addition, from equation (2.21), (5.1) and Lemma 5.2 in Pérez and Yamazaki (2018), we have

$$\mathbb{E}_x(e^{-(\gamma+\delta)\tau_0^-} W_\delta(X(\tau_0^-) - a); \tau_0^- < \tau_b^+) = W_{\gamma,\delta,a}(x) - \frac{W_{\gamma+\delta}(x)}{W_{\gamma+\delta}(b)} W_{\gamma,\delta,a}(b), \quad (\text{A.5})$$

$$\mathbb{E}_x(e^{-\delta e_\gamma}; e_\gamma < \tau_b^+ \wedge \tau_0^-) = \gamma \left(\frac{W_{\gamma+\delta}(x)}{W_{\gamma+\delta}(b)} \overline{W}_{\gamma+\delta}(b) - \overline{W}_{\gamma+\delta}(x) \right), \quad (\text{A.6})$$

and

$$\mathbb{E}_x(e^{-\delta e_\gamma} X(e_\gamma); e_\gamma < \tau_b^+ \wedge \tau_0^-) = \gamma \left(\frac{W_{\gamma+\delta}(x)}{W_{\gamma+\delta}(b)} \overline{\overline{W}}_{\gamma+\delta}(b) - \overline{\overline{W}}_{\gamma+\delta}(x) \right), \quad (\text{A.7})$$

where e_γ is an independent exponential random variable with mean $1/\gamma$.

By denoting our quantity of interest

$$f_{a,b,d}(x) := \mathbb{E}_x \left(\int_0^{\tau_b^+ \wedge \tau_a^-} e^{-\delta t} dL_\gamma^d(t) \right) \quad (\text{A.8})$$

and using the strong Markov property, we have for $x \leq 0$,

$$f_{a,b,d}(x) = \mathbb{E}_x(e^{-\delta\tau_0^+}; \tau_0^+ < \tau_a^-) f_{a,b,d}(0) = \frac{W_\delta(x-a)}{W_\delta(-a)} f_{a,b,d}(0).$$

Hence, for $x \geq 0$, we have by the strong Markov property

$$\begin{aligned} f_{a,b,d}(x) &= \mathbb{E}_x(e^{-\delta e_\gamma} X(e_\gamma); e_\gamma < \tau_0^- \wedge \tau_b^+) + \mathbb{E}_x(e^{-\delta e_\gamma}; e_\gamma < \tau_0^- \wedge \tau_b^+) (f_{a,b,d}(d) - d - \kappa) \\ &\quad + \mathbb{E}_x(e^{-(\gamma+\delta)\tau_0^-} W_\delta(X(\tau_0^-) - a); \tau_0^- < \tau_b^+) \frac{f_{a,b,d}(0)}{W_\delta(-a)} \\ &= \mathbb{E}_x(e^{-\delta e_\gamma} X(e_\gamma); e_\gamma < \tau_0^- \wedge \tau_b^+) \\ &\quad + \mathbb{E}_x(e^{-\delta e_\gamma}; e_\gamma < \tau_0^- \wedge \tau_b^+) \left(\frac{W_\delta(d-a)}{W_\delta(-a)} f_{a,b,d}(0) - d - \kappa \right) \\ &\quad + \mathbb{E}_x(e^{-(\gamma+\delta)\tau_0^-} W_\delta(X(\tau_0^-) - a); \tau_0^- < \tau_b^+) \frac{f_{a,b,d}(0)}{W_\delta(-a)}. \end{aligned}$$

Hence, via equations (A.7), (A.6) and (A.5), we have

$$\begin{aligned} f_{a,b,d}(x) &= \gamma \left(\frac{W_{\gamma+\delta}(x)}{W_{\gamma+\delta}(b)} \overline{\overline{W}}_{\gamma+\delta}(b) - \overline{\overline{W}}_{\gamma+\delta}(x) \right) \\ &\quad + \gamma \left(\frac{W_{\gamma+\delta}(x)}{W_{\gamma+\delta}(b)} \overline{W}_{\gamma+\delta}(b) - \overline{W}_{\gamma+\delta}(x) \right) \times \left(\frac{W_\delta(d-a)}{W_\delta(-a)} f_{a,b,d}(0) - d - \kappa \right) \\ &\quad + \left(W_{\gamma,\delta,a}(x) - \frac{W_{\gamma+\delta}(x)}{W_{\gamma+\delta}(b)} W_{\gamma,\delta,a}(b) \right) \frac{f_{a,b,d}(0)}{W_\delta(-a)}. \end{aligned}$$

Since we are only interested in the case when $b \rightarrow \infty$, we should take the limit before calculating $f_{a,b,d}(0)$. By using (A.15), (A.16) and (A.17), we get

$$\begin{aligned}
f_{a,d}(x) &:= \lim_{b \rightarrow \infty} f_{a,b,d}(x) \\
&= \gamma \left(\frac{1}{\phi_{\gamma+\delta}^2} W_{\gamma+\delta}(x) - \overline{\overline{W}}_{\gamma+\delta}(x) \right) \\
&\quad + \gamma \left(\frac{1}{\phi_{\gamma+\delta}} W_{\gamma+\delta}(x) - \overline{W}_{\gamma+\delta}(x) \right) \times \left(\frac{W_\delta(d-a)}{W_\delta(-a)} f_{a,d}(0) - d - \kappa \right) \\
&\quad + (W_{\gamma,\delta,a}(x) - W_{\gamma+\delta}(x) Z_{\gamma,\delta}(-a)) \frac{f_{a,d}(0)}{W_\delta(-a)} \\
&= \gamma \left(\frac{1}{\phi_{\gamma+\delta}^2} W_{\gamma+\delta}(x) - \overline{\overline{W}}_{\gamma+\delta}(x) \right) + \gamma(-d - \kappa) \left(\frac{1}{\phi_{\gamma+\delta}} W_{\gamma+\delta}(x) - \overline{W}_{\gamma+\delta}(x) \right) \\
&\quad + \frac{f_{a,d}(0)}{W_\delta(-a)} \left\{ r W_\delta(d-a) \left(\frac{1}{\phi_{\gamma+\delta}} W_{\gamma+\delta}(x) - \overline{W}_{\gamma+\delta}(x) \right) + W_{\gamma,\delta,a}(x) - W_{\gamma+\delta}(x) Z_{\gamma,\delta}(-a) \right\}. \quad (\text{A.9})
\end{aligned}$$

When X is of bounded variation, $W_{\gamma+\delta}(0) > 0$, and hence we put $x = 0$ in equation (A.9) to obtain

$$\begin{aligned}
f_{a,d}(0) &= \gamma \frac{1}{\phi_{\gamma+\delta}^2} W_{\gamma+\delta}(0) + \gamma(-d - \kappa) \frac{1}{\phi_{\gamma+\delta}} W_{\gamma+\delta}(0) \\
&\quad + \frac{f_{a,d}(0)}{W_\delta(-a)} \left\{ \gamma W_\delta(d-a) \frac{1}{\phi_{\gamma+\delta}} W_{\gamma+\delta}(0) + W_\delta(-a) - W_{\gamma+\delta}(0) Z_{\gamma,\delta}(-a) \right\} \\
&= \frac{W_{\gamma+\delta}(0)}{\phi_{\gamma+\delta}} \left(\gamma \left(\frac{1}{\phi_{\gamma+\delta}} - d - \kappa \right) \right) + \frac{W_{\gamma+\delta}(0)}{\phi_{\gamma+\delta}} \frac{f_{a,d}(0)}{W_\delta(-a)} \left\{ \gamma W_\delta(d-a) - \phi Z_{\gamma,\delta}(-a) \right\} + f_{a,d}(0),
\end{aligned}$$

or

$$\frac{f_{a,d}(0)}{W_\delta(-a)} = \frac{\gamma \left(\frac{1}{\phi_{\gamma+\delta}} - d - \kappa \right)}{\phi_{\gamma+\delta} Z_{\gamma,\delta}(-a) - \gamma W_\delta(d-a)}. \quad (\text{A.10})$$

When X is of unbounded variation, by denoting the event

$$E_B := \{e_\gamma < \tau_0^-\} \cup \{\tau_b^+ < \zeta\} \cup \{\tau_a^- < \zeta\}, \quad (\text{A.11})$$

where ζ is the lifetime of an excursion away from 0, $a \leq 0 \leq b$. In addition, we denote T_{E_B} the first time an excursion in the event E_B occurs and $l_{T_{E_B}}$ the starting point of the excursion, i.e.

$$l_{T_{E_B}} := \sup\{t < T_{E_B} : X(t) = 0\}. \quad (\text{A.12})$$

From equation (6.6) in Pérez and Yamazaki (2018), by denoting $\overline{T}_0^- = l_{T_{E_B}} + \tau_0^- \circ \vartheta_{l_{T_{E_B}}}$, where ϑ is the shifting operator, we have

$$\mathbb{E}(e^{-\delta(l_{T_{E_B}} + e_\gamma)} X(l_{T_{E_B}} + e_\gamma); l_{T_{E_B}} + e_\gamma < \overline{T}_0^- \wedge \tau_b^+) = \gamma \frac{W_\delta(-a)}{W_{\gamma,\delta,a}(b)} \overline{\overline{W}}_{\gamma+\delta}(b), \quad (\text{A.13})$$

$$\mathbb{E}(e^{-\delta(l_{T_{E_B}} + e_\gamma)}; l_{T_{E_B}} + e_\gamma < \overline{T}_0^- \wedge \tau_b^+) = \gamma \frac{W_\delta(-a)}{W_{\gamma,\delta,a}(b)} \overline{W}_{\gamma+\delta}(b). \quad (\text{A.14})$$

Regarding the limiting behaviour when $b \rightarrow \infty$, we have

$$\lim_{b \rightarrow \infty} \frac{\overline{\overline{W}}_{\gamma+\delta}(b)}{W_{\gamma+\delta}(b)} = \frac{1}{\phi_{\gamma+\delta}^2}, \quad (\text{A.15})$$

$$\lim_{b \rightarrow \infty} \frac{\overline{W}_{\gamma+\delta}(b)}{W_{\gamma+\delta}(b)} = \frac{1}{\phi_{\gamma+\delta}}, \quad (\text{A.16})$$

and

$$\lim_{b \rightarrow \infty} \frac{W_{\gamma, \delta, a}(b)}{W_{\gamma + \delta}(b)} = Z_{\gamma, \delta}(-a). \quad (\text{A.17})$$

On the other hand, when X is of unbounded variation, we proceed as in section 6 in Pérez and Yamazaki (2018) to yield

$$\begin{aligned} f_{a, b, d}(0) &= \mathbb{E}(e^{-\delta(l_{T_{EB}} + e_\gamma)} X(l_{T_{EB}} + e_\gamma); l_{T_{EB}} + e_\gamma < \bar{T}_0^- \wedge \tau_b^+) \\ &\quad + \mathbb{E}(e^{-\delta(l_{T_{EB}} + e_\gamma)}; l_{T_{EB}} + e_\gamma < \bar{T}_0^- \wedge \tau_b^+) \times (f_{a, b, d}(d) - d - \kappa) \\ &= \gamma \frac{W_\delta(-a)}{W_{\gamma, \delta, a}(b)} \overline{\overline{W}}_{\gamma + \delta}(b) + \gamma \frac{W_\delta(-a)}{W_{\gamma, \delta, a}(b)} \overline{W}_{\gamma + \delta}(b) (f_{a, b, d}(d) - d - \kappa). \end{aligned}$$

By passing the limit $b \rightarrow \infty$, we get

$$\begin{aligned} f_{a, d}(0) &= \frac{\gamma W_\delta(-a)}{\phi^2 Z_{\gamma, \delta}(-a)} + \frac{\gamma W_\delta(-a)}{\phi Z_{\gamma, \delta}(-a)} \left(\frac{W_\delta(d-a)}{W_\delta(-a)} f_{a, b, d}(0) - d - \kappa \right) \\ &= \frac{\gamma W_\delta(-a)}{\phi_{\gamma + \delta} Z_{\gamma, \delta}(-a)} \left(\frac{1}{\phi_{\gamma + \delta}} - d - \kappa \right) + \frac{\gamma W_\delta(d-a)}{\phi_{\gamma + \delta} Z_{\gamma, \delta}(-a)} f_{a, b, d}(0), \end{aligned}$$

or

$$\phi_{\gamma + \delta} Z_{\gamma, \delta}(-a) f_{a, d}(0) = \gamma W_\delta(-a) \left(\frac{1}{\phi_{\gamma + \delta}} - d - \kappa \right) + \gamma W_\delta(d-a) f_{a, d}(0),$$

which yields

$$\frac{f_{a, d}(0)}{W_\delta(-a)} = \frac{\gamma \left(\frac{1}{\phi_{\gamma + \delta}} - d - \kappa \right)}{\phi_{\gamma + \delta} Z_{\gamma, \delta}(-a) - \gamma W_\delta(d-a)},$$

the same as (A.10).

Thus, plugging (A.10) back to (A.9), we obtain

$$\begin{aligned} f_{a, d}(x) &= \gamma \left(\frac{1}{\phi_{\gamma + \delta}^2} W_{\gamma + \delta}(x) - \overline{\overline{W}}_{\gamma + \delta}(x) \right) + \gamma(-d - \kappa) \left(\frac{1}{\phi_{\gamma + \delta}} W_{\gamma + \delta}(x) - \overline{W}_{\gamma + \delta}(x) \right) \\ &\quad + \frac{f_{a, d}(0)}{W_\delta(-a)} \left\{ \gamma W_\delta(d-a) \left(\frac{1}{\phi_{\gamma + \delta}} W_{\gamma + \delta}(x) - \overline{W}_{\gamma + \delta}(x) \right) + W_{\gamma, \delta, a}(x) - W_{\gamma + \delta}(x) Z_{\gamma, \delta}(-a) \right\} \\ &= \frac{W_{\gamma + \delta}(x)}{\phi_{\gamma + \delta}} \gamma \left(\frac{1}{\phi_{\gamma + \delta}} - d \right) - \gamma \left(\overline{\overline{W}}_{\gamma + \delta}(x) - d \overline{W}_{\gamma + \delta}(x) - \kappa \overline{W}_{\gamma + \delta}(x) \right) \\ &\quad + \frac{f_{a, d}(0)}{W_\delta(-a)} \left\{ W_{\gamma, \delta, a}(x) - \gamma W_\delta(d-a) \overline{W}_{\gamma + \delta}(x) \right\} \\ &\quad + \frac{f_{a, d}(0)}{W_\delta(-a)} \left\{ \gamma W_\delta(d-a) - \phi_{\gamma + \delta} Z_{\gamma, \delta}(-a) \right\} \frac{1}{\phi_{\gamma + \delta}} W_{\gamma + \delta}(x) \\ &= \frac{W_{\gamma + \delta}(x)}{\phi_{\gamma + \delta}} \gamma \left(\frac{1}{\phi_{\gamma + \delta}} - d - \kappa \right) - \gamma \left(\overline{\overline{W}}_{\gamma + \delta}(x) - d \overline{W}_{\gamma + \delta}(x) - \kappa \overline{W}_{\gamma + \delta}(x) \right) \\ &\quad + \frac{f_{a, d}(0)}{W_\delta(-a)} \left\{ W_{\gamma, \delta, a}(x) - \gamma W_\delta(d-a) \overline{W}_{\gamma + \delta}(x) \right\} \\ &\quad - \gamma \left(\frac{1}{\phi_{\gamma + \delta}} - d - \kappa \right) \frac{W_{\gamma + \delta}(x)}{\phi_{\gamma + \delta}}, \end{aligned} \quad (\text{A.18})$$

or

$$\begin{aligned} f_{a, d}(x) &= \frac{\gamma \left(\frac{1}{\phi_{\gamma + \delta}} - d - \kappa \right)}{\phi_{\gamma + \delta} Z_{\gamma, \delta}(-a) - \gamma W_\delta(d-a)} \left(W_{\gamma, \delta, a}(x) - \gamma W_\delta(d-a) \overline{W}_{\gamma + \delta}(x) \right) \\ &\quad - \gamma \left(\overline{\overline{W}}_{\gamma + \delta}(x) - d \overline{W}_{\gamma + \delta}(x) - \kappa \overline{W}_{\gamma + \delta}(x) \right). \end{aligned} \quad (\text{A.19})$$

Next, the smoothness condition ((7.3) shifted b_u units downward)

$$f_{a,d}(0) = f_{a,d}(d) - d - \kappa \quad (\text{A.20})$$

can be rewritten as

$$-d - \kappa = \frac{f_{a,d}(0)}{W_\delta(-a)} (W_\delta(-a) - W_\delta(d-a)). \quad (\text{A.21})$$

Hence, in view of (A.18), we have

$$\begin{aligned} & f_{a,d}(x) \\ &= \frac{f_{a,d}(0)}{W_\delta(-a)} \left(W_{\gamma,\delta,a}(x) - \gamma W_\delta(d-a) \overline{W}_{\gamma+\delta}(x) \right) - \gamma \left(\overline{\overline{W}}_{\gamma+\delta}(x) - d \overline{W}_{\gamma+\delta}(x) - \kappa \overline{W}_{\gamma+\delta}(x) \right) \\ &= \frac{f_{a,d}(0)}{W_\delta(-a)} \left(W_{\gamma,\delta,a}(x) - \gamma W_\delta(d-a) \overline{W}_{\gamma+\delta}(x) \right) - \gamma(-d - \kappa) \overline{W}_{\gamma+\delta}(x) - \gamma \overline{\overline{W}}_{\gamma+\delta}(x) \\ &= \frac{f_{a,d}(0)}{W_\delta(-a)} \left(W_{\gamma,\delta,a}(x) - \gamma W_\delta(d-a) \overline{W}_{\gamma+\delta}(x) \right) - \gamma \frac{f_{a,d}(0)}{W_\delta(-a)} (W_\delta(a) - W_\delta(d-a)) \overline{W}_{\gamma+\delta}(x) - \gamma \overline{\overline{W}}_{\gamma+\delta}(x) \\ &= \frac{f_{a,d}(0)}{W_\delta(-a)} \left(W_{\gamma,\delta,a}(x) - \gamma W_\delta(d-a) \overline{W}_{\gamma+\delta}(x) + \gamma W_\delta(d-a) \overline{W}_{\gamma+\delta}(x) - \gamma W_\delta(-a) \overline{W}_{\gamma+\delta}(x) \right) - \gamma \overline{\overline{W}}_{\gamma+\delta}(x) \\ &= \frac{f_{a,d}(0)}{W_\delta(-a)} \left(W_{\gamma,\delta,a}(x) - \gamma W_\delta(-a) \overline{W}_{\gamma+\delta}(x) \right) - \gamma \overline{\overline{W}}_{\gamma+\delta}(x). \end{aligned} \quad (\text{A.22})$$

Now, plugging in $x = 0$ in (A.19), we get

$$f_{a,d}(0) = \frac{\gamma(\frac{1}{\phi} - d - \kappa)}{\phi_{\gamma+\delta} Z_{\gamma,\delta}(-a) - \gamma W_\delta(d-a)} W_\delta(-a),$$

or

$$\frac{f_{a,d}(0)}{W_\delta(-a)} = \frac{\frac{\gamma}{\phi_{\gamma+\delta}} + \gamma \frac{f_{a,d}(0)}{W_\delta(-a)} (W_\delta(-a) - W_\delta(d-a))}{\phi_{\gamma+\delta} Z_{\gamma,\delta}(-a) - \gamma W_\delta(d-a)},$$

which results in

$$\frac{f_{a,d}(0)}{W_\delta(-a)} = \frac{\gamma}{\phi_{\gamma+\delta}} \frac{1}{\phi_{\gamma+\delta} Z_{\gamma,\delta}(-a) - \gamma W_\delta(-a)}. \quad (\text{A.23})$$

Plugging this back to (A.22), we retrieve the value function of a periodic barrier strategy (at barrier level a), which appears in Pérez and Yamazaki (2018).

Finally, we shall perform a horizontal transformation of the axis such that we have $0 < \kappa < g = b_u - b_l$, $0 < b_u, b_l$, the process is ruin when downcrossing 0 and whenever at Poissonian times the process is above b_u it jumps to b_l . This gives the value function of a periodic (b_u, b_l) strategy as desired.