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Interacting Quarter-Plane Lattice Walk Problems: Solutions and Proofs

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Interacting Quarter-Plane Lattice Walk Problems: Solutions and Proofs

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February 23, 2021

A thesis submitted for the degree of Doctor of Philosophy

Declaration

This thesis is about the research done between March 2017 and May 2020 at the School of Mathematics and Statistics, the University of Melbourne, Melbourne, Australia.

Except for the work that has been explicitly cited, I declare that the content in this thesis is my original work. The original work is mostly in Chapters 7 to 10 and 12. In Chapters 7, 9 and 12 we do proofs and discussions. We prove several theorems and provide related discussions. In Chapters 8 to 10, we do calculations. We study all 23 quarter-plane lattice walks associated with finite groups[1] and exactly solve 21 of them.

Ruijie Xu
February 23, 2021

[1]See Section 3.5 for the definition of the symmetry group of a quarter-plane lattice walk model

Abstract

Lattice walk problems in the quarter-plane have been widely studied in recent years. The main objective is to calculate the number of configurations, that is the number of n-step walks ending at certain points or alternatively, the generating function of the walks. In combinatorics, physics and probability theory, other properties such as asymptotic behavior and the algebra of the generating functions are also of interest.

In this thesis we focus on solving quarter-plane lattice walks with interactions via the kernel method. We assign interaction a to the x-axis, b to the y-axis and c to the origin. We denote $q_{n,k,l,h,u,v}$ as the number of n-step paths that start at $(0,0)$ and end at point (k,l), and which visit vertices on the horizontal boundary (except the origin) h times, vertices on the vertical boundary (except the origin) v times and the origin u times. Then $Q(x,y,t) = \sum_{n,k,l,h,u,v} q_{n,k,l,h,u,v} x^k y^l t^n a^h b^v c^u$ is the generating function of the walk. Our aim is to find whether the interactions will affect the solubility of quarter-plane lattice walk models and the properties of the generation functions $Q(x,y,t)$. In particular, we give solutions to all 23 quarter-plane lattice walks associated with finite groups. We solve 21 of them explicitly, writing an expression for the generating functions $Q(x,y,t)$. The other 2 are only partly solved. We discuss when the generating function $Q(x,y,t)$ become D-algebraic, D-finite and algebraic by the variations of a, b, c.

We also compare the solutions obtained by the kernel method and the solutions obtained by an analytic method based on homomorphisms on a Riemann surface. We combine the solutions in another form with the solution of the kernel method, and prove the properties of the generating function $Q(x,y,t)$. We observe that the solution obtained using the kernel method and other analytic methods are consistent with one another.

Acknowledgements

First I'd like to thank my supervisors Aleks Owczarek, Richard Brak and Nicholas Beaton for their guidance for my research and my thesis. I'd like to thank Aleks for accepting me a PhD student, giving me a topic that suits me and guiding my PhD research. I'd like to thank Nick and Richard for the helping with my proofs, calculations and writing. And many thanks to Nick for going through the thesis together with me and helping me revise and typeset it.

Second I'd like to thank all staff in the School of Mathematics and Statistics at the University of Melbourne for helping me with different affairs. I'd like to thank all my fellows that worked together with me. And special thanks to Micheal Swaddle for helping me with the English of my thesis.

Thirdly, I'd like to thank my family and all the friends that supported me for my PhD career.

Finally, I would say, it's a good experience to be a PhD student at the University of Melbourne. I learned a lot here.

Ruijie Xu
February 23, 2021

Preface

The work in this thesis is done under the supervision of Aleks Owczarek and Nicholas Beaton. In Chapters 4 to 6 we introduce the kernel method and past works [13, 7] in quarter-plane lattice walk problems solved using the kernel method. In Chapter 11, we introduce the solution of quarter-plane lattice walks by analytic methods [33].

The main original work is in Chapters 7 to 10 and 12. In Chapters 7, 9 and 12 we provide proofs and discussions. We prove some theorems about the solubility of the models and the properties of the generating functions. In Chapters 8 to 10, we do calculations. We study all 23 quarter-plane lattice walk models associated with finite groups[2] and exactly solve 21 of them. The work in Chapter 8 on reverse Kreweras walks is published in [5]. The work on Kreweras walks is currently on the arXiv [6] and will appear in the proceedings of the 2019 conference *Transient Transcendence in Transylvania* (Springer Proceedings in Mathematics and Statistics).

The calculations in Chapters 8 and 10 can be found in the *Mathematica* files attached.

[2]See Section 3.5 for the definition of the symmetry group of a quarter-plane lattice walk model

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Chapter 1

Introduction

1.1 Background of lattice walk problems

A *lattice walk* is a walk with a fixed step set (allowed lengths and directions), ending on the grid (for our purposes, the grid will always be $\mathbb{Z}^2$). A famous example of a class of lattice walks is Dyck paths [28]. These are lattice walks in the first quadrant $\mathbb{N}^2$ which begin at the origin and end at $(2n, 0)$ with step set $\{\nearrow, \searrow\}$. Enumeration of lattice walks is a classical combinatorial problem, which involves counting those walks ending at each point on the lattice. We may have one single walk or several non-intersecting walks with different kinds of boundaries. Weights or interactions can also be assigned to the steps to vary the configurations. If we interpret the weight and interaction as a probability or surface energy, the lattice walk problem can also be viewed as a probability or a physics problem. The importance of lattice walks is that they not only have straightforward applications in probability theory and statistical physics, but also lead to the development of new enumeration tools. A direct example is that continued fractions can be interpreted as weighted Dyck paths [35]. Since continued fractions can be used to represent sequence like the Stirling numbers, Bell numbers or Motzkin numbers, we can also use lattice walks to describe these numbers. Many combinatorial objects, such as trees, tableaux or permutations have direct bijections with lattice walk models [1, 15, 31, 30]. Some physical models such as ASEP (asymmetric simple exclusion processes) [9] can also be solved using a corresponding lattice walk model [25, 23].

1.2 Quarter-plane lattice walks

In this thesis, we will study lattice walk models in the quarter-plane of $\mathbb{Z}^2$. The step set is $\mathbb{S} \subseteq \{\nearrow, \uparrow, \nwarrow, \leftarrow, \swarrow, \downarrow, \searrow, \rightarrow\}$, that is, $\{(1,1), (0,1), (0,-1), (-1,0), (-1,-1), (0,-1), (1,-1), (1,0)\}$. We wish to calculate $Q_{i,j,n}$, the number of paths ending at (i, j) with length n. A useful way to study this problem is to study the generating function $Q(x, y, t)$, defined as

$$Q(x, y, t) = \sum_{i,j,n} q_{i,j,n} x^i y^j t^n. \tag{1.2.1}$$

The study of quarter-plane lattice walks has a long history because of the connection to random walks in probability theory. Probabilists omit the variable t and instead assign a probability to each step. Then a lattice walk is a typical Markov process and the $Q_{i,j}$ is treated as the stationary distribution [42]. In [33], Fayolle et al solved all the models using analytic methods.

Quarter-plane lattice walk models are also of interest in combinatorics. The properties of the generating functions can depend on the region of the lattice containing the walks. For lattice walks in the plane without any restrictions, $Q(x, y, t)$ is a rational function of x, y, t. For walks in the upper half plane, $Q(x, y, t)$ is an algebraic function [20, 2]. Walks with algebraic generating functions are of special interest since these walks can be factored into smaller objects of the same type [16]. In the quarter-plane, there are both algebraic and non-algebraic cases. The generating functions may be D-finite, D-algebraic or hyper-transcendental[1]. These properties can be further applied to the discussion of other problems such as finding the asymptotic behavior of $Q_{i,j,n}$. In combinatorics, not only the explicit solutions are of interest, but also these properties.

Quarter-plane lattice walk models without interactions were systematically studied in [20, 61, 14]. There are four special models: Kreweras, reverse Kreweras, double Kreweras, and Gessel's walks which all have algebraic generating functions. The method of solving quarter-plane lattice walk models varies greatly, especially for these algebraic ones. For example, solutions for Kreweras walks range from combinatorics, probability theory, algebraic geometry, differential equations, linear algebra and computer algebra [8, 33, 16, 44, 54, 61]. For other models, there are some attempts in combinatorics [40] or probability theory [38]. But mostly, all the solutions are through three main ideas: the kernel method [20, 54], Riemann boundary value problems [61, 48] and homomorphisms on Riemann surfaces [33, 29, 34]. These three main ideas will be introduced in this work. We will solve most models by the kernel method, and prove properties of the walk by the homomorphism on a Riemann surface. Some recent works have extended these methods to walks avoiding a quarter-plane [18] and walks in three dimensions [11], but we will not discuss these models here.

1.3 Application of quarter-plane lattice walks with interactions to physics

The idea of adding interactions to quarter-plane models comes from physics. Some recent publications [66, 57, 24] modelled the growth of a polymer using an interacting lattice walk model with one direction. For example, in [66], the authors studied a two-dimensional model of interacting directed polymers above an impenetrable surface. They model the polymers as two walks starting at the origin which can step north-east or south-east with unit step length. Weights (Boltzmann weights) are assigned when the two walks touch (weight a) or when either touches the surface (weight b). In physics, the focus is on the singularity structure of the generating function, which determines the phase diagram of the model.

This directed polymer model from statistical physics is in bijection with a certain quarter-plane walk problem (see Figure 1.1). If we use the following mapping:

$$\begin{pmatrix}\nearrow\\\nearrow\end{pmatrix} \mapsto (\rightarrow) \qquad \begin{pmatrix}\nearrow\\\searrow\end{pmatrix} \mapsto (\nwarrow) \qquad \begin{pmatrix}\searrow\\\searrow\end{pmatrix} \mapsto (\leftarrow) \qquad \begin{pmatrix}\searrow\\\nearrow\end{pmatrix} \mapsto (\searrow)$$

then a pair of directed walks corresponds to a quarter-plane walk whose allowed steps are $\{\leftarrow, \rightarrow, \nwarrow, \searrow\}$. This quarter-plane walk model has interactions when the walk touches the x-axis and the y-axis. The weights are the same as in the corresponding directed polymer model, namely weight a on the x-axis and weight b on the y-axis.

This idea can then be applied to general quarter-plane models. In recent work [7], the authors introduced the interactions to all quarter-plane walk models with a finite group[2] and tried to solve them by the kernel method. In particular, they assigned weights a and b to the two axes and ab to the origin. The interactions change the property of the generating

[1]See definitions in Chapter 2.

[2]See Section 4.3 for the definition of the symmetry group of a quarter-plane lattice walk model.

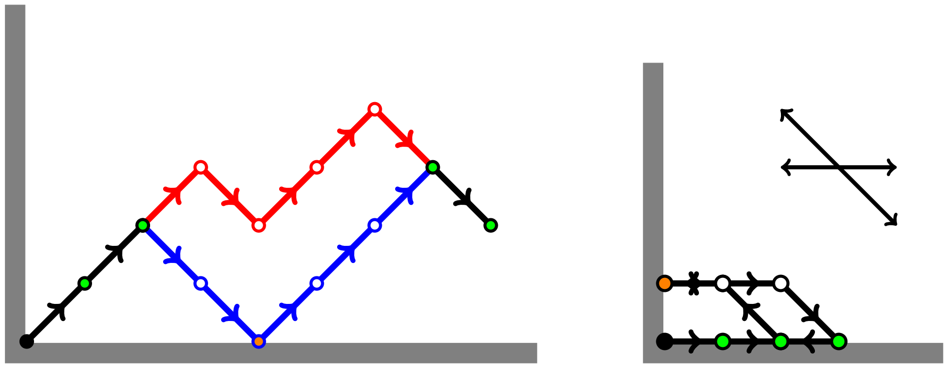


Figure 1.1: Left: A pair of interacting directed paths with weights associated with shared vertices and visits to the bottom boundary. Right: The corresponding quarter-plane lattice path with weights associated with visits to the two boundaries.

functions. For some of the models, the generating function remains the same as non-interacting cases. For some models, the generating functions change from algebraic to D-finite or from D-finite to D-algebraic. Many of the models were not solved by the kernel method although the non-interacting models were solved explicitly in [20, 17].

1.4 Outline of this thesis

We want to generally study quarter-plane walks with interactions via the kernel method. In this work, we further assign an independent interaction c to the origin. Thus, the interactions at the origin are not just treated as interactions with both axes, but also as special point boundary interactions. We want to know whether the point boundary interactions have the same effect as the line boundary interactions. There are several models that have not been solved in full generality in [7], such as Kreweras walks and reverse Kreweras walks. They were only solved with conditions $a = b$. We will address these problems here.

The structure of this work will be as follows:

In Chapter 2, we introduce some basic concepts and knowledge. In Chapter 3, we will go through basic lattice walk problems. These include lattice walks in the bulk, binomial paths, and quarter-plane lattice walks without interactions. Then, the kernel method is introduced in Chapter 4. In Chapter 5, we will go through all the models without interactions that were solved by the kernel method.

In Chapter 6, we introduce interactions into the quarter-plane lattice walk problems. We construct the models, and solve them via the kernel method. We also discuss the properties of those generating functions we obtain. Chapter 7 covers some lemmas and theorems required for further proofs. Most solutions are illustrated in Chapter 8, including Kreweras walks and reverse Kreweras walks. These two walks are special algebraic walks without interactions which were not fully solved in [7]. In Chapter 9, we will conclude the general process of solving quarter-plane lattice walks via the kernel method and make some improvements. Later in Chapter 10, we apply these improvements and partly solve Gessel's walks and double Kreweras walks (which are the other two special models which are algebraic without interactions).

After solving all the 23 models, we move onto some other ideas. In Chapter 11, we introduce Riemann boundary value problems (RBVP) and the homomorphism method on a Riemann surface discovered by Fayolle et al [33]. These two ideas provide solutions to those models that we failed to solve using the kernel method. The disadvantage is that these solutions do not directly show the properties of generating functions. So in Chapter 12, we try to prove the properties of the generating functions via the solution form of these two analytic methods. These discussions are not only valid for the models that we have solved, but also some models that cannot be solved via the kernel method. Chapter 13 is the conclusion and perspective.

Chapter 2

Notation and basic knowledge

The discussion of lattice walk models may include calculation of generating functions and some concepts in field extensions and group operations. We first introduce some basic concepts and define some notation used in this thesis.

2.1 Formal power series and operations

Suppose $f(x)$ can be expanded as a Laurent series in some domain. We write $[x^i]f(x)$ for the coefficient of $[x^i]$ in the Laurent series of $f(x)$. Respectively, $[x^>]f(x)$, $[x^<]f(x)$ and $[x^\geq]f(x)$ are those series of positive, negative and non-negative powers of x.

We may discuss a series in some field or ring. For a ring or field $\mathbb{K}$, we denote

1. $\mathbb{K}[t]$ as the set of polynomials in t with coefficients in $\mathbb{K}$;
2. $\mathbb{K}[t, \frac{1}{t}]$ as the set of polynomials in t and $\frac{1}{t}$ with coefficients in $\mathbb{K}$.
3. $\mathbb{K}[[t]]$ as the set of formal power series in t with coefficients in $\mathbb{K}$;
4. $\mathbb{K}((t))$ as the set of Laurent series in t with coefficients in $\mathbb{K}$;
5. $\mathbb{K}(t)$ as the set of rational functions in t with coefficients in $\mathbb{K}$.

The definition extends to multiple variables. For example $\mathbb{R}(x)((t))$ refers to the set of Laurent series in t with coefficients in the set of rational functions of x with real coefficients. Notice that the definitions here are not equivalent to fields [50]. Generally speaking, a series $x(t)$ in $\mathbb{K}[[t]]$ and a series $y(t)$ in $\mathbb{K}((t))$ can both be elements in the field $\mathbb{K}(t)$, which is a field generated by a transcendental element t. We write them separately since they have different convergent regions of t. We may also use the notation $\mathbb{K}[[x]][1/x]$, which refers to the set of polynomials in $1/x$ with coefficients in the set of formal series in x. This gives a Laurent series with finite $[x^<]$ terms. These definitions are proposed for distinguishing series that we meet later.

For a series in $\mathbb{K}((x))$, when it has finitely many $[x^<]$ terms, we say it has *valuation* n if the minimal degree of x is n. When $n > 0$, we may also say it has degree n in x.

2.2 Generating functions

To study a counting problem, a useful method is to consider the generating functions. For a two-dimensional lattice walk, we denote $q_{n,k,l,h,v,u}$ as the number of paths of length n that start at $(0, 0)$ and end at (k, l), and which visit vertices on the horizontal boundary (except the origin) h times, vertices on the vertical boundary (except the origin) v times and the origin u times[1]. The generating function of these numbers is

$$Q(t;x,y;a,b,c) \equiv Q(x,y) = \sum_n t^n \sum_{k,l,h,v,u} q_{n,k,l,h,v,u} x^k y^l a^h b^v c^u \equiv \sum_n t^n Q_n(x,y)$$
$$\equiv \sum_{k,l} q_{k,l}(t) x^k y^l \qquad (2.2.1)$$

We calculate the total weight of paths ending at (k,l) of length n by taking the coefficient of $x^k y^l t^n$ of $Q(x,y)$. Some boundary terms of this function are also of interest. We define line-boundary terms:

$$[y^i]Q(t;x,y;a,b,c) \equiv Q_{-,i}(x) = \sum_n t^n \sum_{k,h,v,u} q_{n,k,i,h,v,u} x^k a^h b^v c^u \qquad (2.2.2)$$

This is the generating function of walks ending on the line $y = i$. $Q_{i,-}(y)$ is defined similarly. For walks ending on the line $y = 0$ ($x = 0$), we prefer a more convenient expression $Q(x,0)$ or $Q(0,y)$, which are equivalent to $Q_{-,0}(x)$ or $Q_{0,-}(y)$.

Furthermore, we define the generating functions of walks ending on a line of diagonal direction:

$$Q_j^{\mathrm{d}}(x) = \sum_n t^n \sum_{i,h,v,u} q_{n,i,i+j,h,v,u} x^i a^h b^v c^u \qquad (2.2.3)$$

Note that the variable x in $Q_j^{\mathrm{d}}(x)$ marks the x-coordinate of the endpoint of the walk.

Finally we define point boundary terms:

$$[x^i y^j]Q(t;x,y;a,b,c) \equiv Q_{i,j} = \sum_n t^n \sum_{h,v,u} q_{n,i,j,h,v,u} a^h b^v c^u. \qquad (2.2.4)$$

This is the generating function of walks ending at point (i,j).

2.3 Transformations and groups

A symmetric transformation of variables (x,y) is a biholomorphic mapping:

$$\delta : (x,y) \to (\delta_x(x,y), \delta_y(x,y)) \qquad (2.3.1)$$

For shorthand we will write $\delta_x(x,y) \equiv \delta_x$ and $\delta_y(x,y) \equiv \delta_y$.

We sometimes wish to apply a transformation δ to an entire expression (like an equation or a matrix) termwise. In this case we will use the operator $\delta_{(\delta_x,\delta_y)}$. For example, to apply the transformation $(x,y) \to (\frac{1}{x}, y)$ to an equation E we will write $\delta_{(\frac{1}{x},y)}(E)$.

We say an expression has δ *symmetry* if it is invariant under δ, ie. $\delta_{(\delta_x,\delta_y)}(f(x,y)) = f(\delta_x,\delta_y) = f(x,y)$.

For shorthand we will write the composition of two transformations ϕ, ψ as $\psi\phi$. Since all the biholomorphic mappings used in this thesis have inverses, the symmetric transformations form a group. For the groups mentioned in this work, we write D_n as the dihedral group of order $2n$.

[1] An alternative is to count walks by the number of *edges* on the boundary that they visit. It is another weighting scheme which gives different equations

2.4 Field extensions and Galois theory

We will meet some concepts in field extensions in our proofs of functional properties. We take our definitions from [49]. All fields considered in this thesis are of characteristic 0.

Let $\mathbb{F}$ be a field. If $\mathbb{F}$ is a subfield of a field $\mathbb{E}$, then we also say $\mathbb{E}$ is an extension field of $\mathbb{F}$.

Let $\mathbb{F}$ be a subfield of a field $\mathbb{E}$. An element α of $\mathbb{E}$ is algebraic over $\mathbb{F}$ if there exist $a_0, a_1 \ldots a_n \in \mathbb{F}$ such that

$$a_0 + a_1\alpha + a_2\alpha^2 + \ldots a_n\alpha^n = 0 \tag{2.4.1}$$

A field extension $\mathbb{E}$ of $\mathbb{F}$ is algebraic if every element in $\mathbb{E}$ is algebraic over $\mathbb{F}$.

An embedding of a field $\mathbb{F}$ into a field $\mathbb{E}$ is defined as a homomorphism,

$$\sigma : \mathbb{F} \to \mathbb{E}. \tag{2.4.2}$$

Then σ induces an isomorphism of $\mathbb{E}$ to its image $\sigma(\mathbb{E})$. An embedding can also be treated as a field extension. We say an embedding $\tau : \mathbb{E} \to \mathbb{L}$ is over σ if $\tau\mathbb{F} = \sigma\mathbb{F}$.

The algebraic extension that appears in our work is the embedding

$$\mathbb{F} \to \mathbb{F}(x_1, x_2, \ldots x_n) \tag{2.4.3}$$

where x_i are the roots of an irreducible polynomial function $f(x)$.

Such an extension is called a *Galois extension*. It has two properties:

1. **Normal**: Every polynomial in $\mathbb{F}$ can be factored into linear factors in $\mathbb{F}(x_1, x_2, \ldots x_n)$.

2. **Separable**: The number of distinct roots of every minimal polynomial in $\mathbb{F}$ is equal to the degree of the polynomial. A minimal polynomial of α is the polynomial f satisfying $f(\alpha) = 0$ with minimal degree in α.

We may have embeddings $\sigma : \mathbb{F}(x_1, x_2, \ldots, x_n) \to \mathbb{F}(x_1, x_2, \ldots, x_n)$ over $\mathbb{F}$. They form an automorphism group of the Galois extension $\mathbb{F} \to \mathbb{F}(x_1, x_2, \ldots, x_n)$, called the *Galois group*. We have the Galois theorem,

Theorem 1. *Let $\mathbb{E}$ be a Galois extension of $\mathbb{F}$. Let G be the Galois group. Then $\mathbb{F}$ is the fixed field of G.*

The fixed field means every element σ in the group is the identity automorphism for all elements in the field. Galois theory tells us, if for any $\sigma \in G$, $\sigma f = f$, then $f \in \mathbb{F}$. If we write $f(x)$ as $a_0(x - x_1)(x - x_2) \ldots (x - x_n)$, then

$$f(x) = \sigma f(x) = a_0(x - \sigma x_1)(x - \sigma x_2) \ldots (x - \sigma x_n). \tag{2.4.4}$$

That is, σ permutes the roots. If $f(x)$ is irreducible, σ traverses all the roots. Notice that $f(x)$ is a symmetric function of all x_i. So a function $g(x_1, x_2 \ldots x_n) \in \mathbb{F}$ if and only if $g(x_1, x_2 \ldots x_n)$ modulo every $f(x_i)$ is a symmetric function of $x_1, x_2 \ldots x_n$.

2.5 Functional properties

Here we discuss some properties of the generating functions. A function $f(x)$ can be included in the following four cases:

1. **Algebraic:** $f(x)$ is algebraic over x if there exists a polynomial equation $P(f, x) = 0$ with coefficients in $\mathbb{C}(x)$.

2. **D-finite:** $f(x)$ is D-finite (short for differentiably finite, sometimes also called holonomic) over x if there exists a linear equation $L(f, f', \dots f^{(n)}) = 0$ with coefficients in $\mathbb{C}(x)$.

3. **D-algebraic:** $f(x)$ is D-algebraic (differentiably algebraic, sometimes called hyper-algebraic) over x if there exists a polynomial equation $P(f, f', \dots f^{(n)}) = 0$ with coefficients in $\mathbb{C}(x)$.

4. **Hyper-transcendental** $f(x)$ does not satisfy any of the previous three conditions.

Here we have

$$\textbf{Algebraic} \subset \textbf{D-finite} \subset \textbf{D-algebraic} \tag{2.5.1}$$

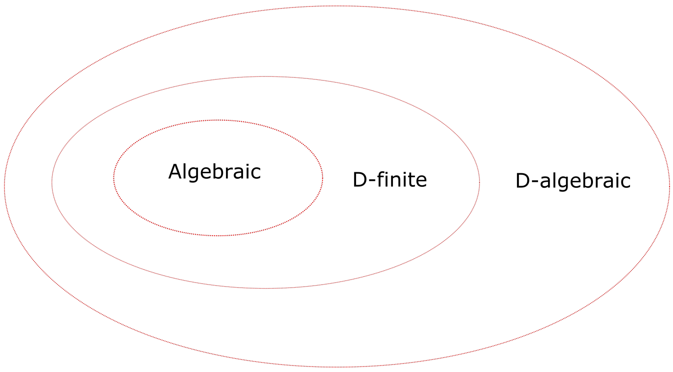


Figure 2.1: The relation between algebraic, D-finite and D-algebraic

In multivariate cases, $f(x_1, x_2 \dots x_k)$ is said to be D-finite if there exist k linear equations $L_i(f, \frac{\partial f}{\partial x_i}, \dots \frac{\partial^n f}{\partial x_i^n}) = 0$, $(i = 1, 2, \dots k)$ with coefficients in $\mathbb{C}(x_1, x_2 \dots, x_n)$. This is the normal definition of D-finite functions [52]. We do not use the regular definition for multi-variable cases, since we are going to prove some function is D-finite by proving that it satisfies a set of linear differential equations. We extend the definition of D-finite to D-finite over certain variables.

1. **D-finite:** $f(x_1, x_2, \dots x_n)$ is D-finite over x_i if there exists a linear equation $L(f, \frac{\partial f}{\partial x_i}, \dots \frac{\partial^n f}{\partial x_i^n}) = 0$ with coefficients in $\mathbb{C}(x_i)$. All other $x_j (j \neq i)$ are treated as parameters here.

In this case, we do not have any restrictions on the other x_j in $L(f, \frac{\partial f}{\partial x_i}, \dots \frac{\partial^n f}{\partial x_i^n}) = 0$. The coefficients in P need not be polynomials of x_j for $i \neq j$. Thus, a function $f(x,t)$ can be D-finite over x but D-algebraic over t. Later we will see many functions we study have this property. For these kinds of function, we can also directly say they are D-algebraic.

2.6 Riemann surface

Our work does not require much knowledge of algebraic topology, but some basic concepts of Riemann surfaces is needed. Here we give a brief introduction to the basic concepts.

A manifold $\mathbb{M}$ is a topological space such that each point has a neighborhood homeomorphic to Euclidean space (ie. an open disc on complex plane) [37]. A Riemann surface is a one-dimensional complex manifold. For example, the complex plane is a Riemann surface. A sphere is another Riemann surface. It is homeomorphic to the complex plane $\cup\ \infty$. An important concept in Riemann surfaces is genus. Every compact Riemann surface is homeomorphic to a sphere with g handles. The number g is called the genus [10]. Generally speaking, it describes how many 'holes' a surface has. For example, the crucial Riemann surface in our work is the torus. The torus has genus 1.

An important property of a genus 1 surface is that it contains two different kind of circles which are not homotopic to a point. We consider a point as a degenerate circle. Two circles $a(t), b(t)$ are homotopic to each other if they can be continuously deformed to each other [63]. Mathematically speaking, we have a two variable function $f(x,t)$ for $x \in [0,1]$, $t \in [0,1]$ such that $f(0,t) = a(t)$, $f(1,t) = b(t)$ and this function is continuous on $[0,1] \times [0,1]$.

In Figure 2.2 we have two circles a and b that cannot be deformed to a point without breaking. Any circle that can be deformed to a is treated as a. We may denote all these circles as a homotopy class a.

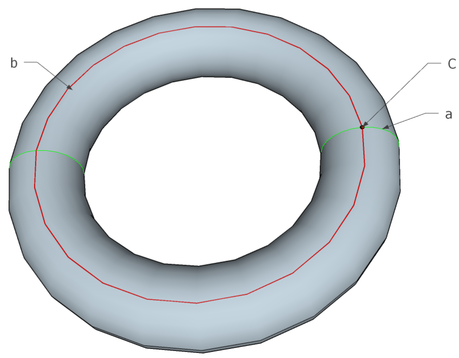


Figure 2.2: A sketch of the torus. Circles a and b are not homotopic to a point and not homotopic to each other. They intersect at point **C**.

Suppose we have a single valued function on the torus. On each point **C**, f has its value $f(\mathbf{C})$. But, as a manifold, we prefer to parameterize the torus by a complex number. So we try to parameterize the torus by z and study $f(z)$. However z is not a single value function of **C**. If we start at some point **C** on the surface and go around a (see Figure 2.2), we return to the original value $f(z)$ but the variable z is changed to $z + z_1$ where z_1 is some value related to a. A similar phenomenon appears if we go around b. So f is a periodic function with two periods z_1, z_2 such that

$$f(z) = f(z + mz_1 + nz_2), \qquad m, n \in \mathbb{Z} \tag{2.6.1}$$

$f(z)$ is called a *doubly periodic* function of z. Actually, by this parameterization, we are taking the universal cover of the surface. The points on the universal cover are single values of z. To define the universal covering, we first define the covering manifold.

Definition 2. *Let X and Y be two Riemann surfaces and $h : Y \to X$ a holomorphic mapping of Y onto X. The pair (Y, h) is called a cover and Y is a covering manifold of X.*

If, for a neighborhood V of y on Y, $h(V)$ is not a one-one mapping, y is called a branch point. For example if the mapping h is defined as $x = y^n$, then in the neighborhood U of 0 in X, each point is covered by n points of y in V. If $n = 1$, h is a one-one mapping for some $V \to U$, then y is called a regular point. Now suppose we have a covering manifold without a branch point. The surface Y may still cover X multiple times. For example, if the Riemann surface is an infinitely long cylinder with circumference 1, we can stick strips $[k, k+1) \times [-\infty, +\infty)$ $(k \in \mathbb{Z})$ together and form a complex plane. Each point on the cylinder corresponds to an infinite number of points on the complex plane. If every point $x \in X$ is covered by s points in Y, we call Y an s sheet cover of x. The previous example is a typical infinite sheet cover.

Definition 3. *Among all the covering manifolds of X without branch points, there is one that can cover any other covering manifold of X. We call it the universal covering.*

For a torus, the universal covering is the complex plane. We can parameterize it by ω. Let $\omega_1 = h^{-1}(a)$, $\omega_2 = h^{-1}(b)$. Each parallelogram $[k\omega_2, (k+1)\omega_2) \times [n\omega_1, (n+1)\omega_1)$ on the complex plane can map to one sheet of the entire torus. Thus the complex plane is an infinite sheeted cover of the torus. The complex plane can be parameterized by a coordinate (x, y) $(\omega = x + yi)$, The function $F(x, y) = h^{-1}f(\mathbf{C})$ is now a single valued function of each pair x, y.

2.7 Elliptic functions

Following the previous section, we are studying a doubly periodic function F on the universal covering of a torus. A meromorphic doubly periodic function $F(\omega)$ is also called an elliptic function. To study elliptic functions we introduce their standard form, the Weierstrass $\wp$-functions [4], defined as

$$\wp(\omega; \omega_1, \omega_2) = \wp(\omega) = \frac{1}{\omega^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(\omega - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}\right] \tag{2.7.1}$$

where m, n are two integers and ω_1, ω_2 are two complex numbers where $\Im(\frac{\omega_1}{\omega_2}) > 0$[2]. If not mentioned, we will always omit the periods and write $\wp(\omega; \omega_1, \omega_2)$ as $\wp(\omega)$. $\wp(\omega)$ satisfies the equation

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3 \tag{2.7.2}$$

where

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^4} \tag{2.7.3}$$

$$g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^6} \tag{2.7.4}$$

and $\wp'(\omega)$ is the derivative of $\wp(\omega)$.

When $\omega = 0$, $\omega = \omega_1$ or $\omega = \omega_2$, $\wp(\omega)$ and $\wp'(\omega)$ both diverge. There are three points where $\wp'(\omega) = 0$ in each period, usually denoted as e_1, e_2, e_3:

$$e_1 = \wp\left(\frac{\omega_2}{2}\right), \qquad e_2 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right), \qquad e_3 = \wp\left(\frac{\omega_1}{2}\right). \tag{2.7.5}$$

Every elliptic function can be written as a rational function of $\wp(\omega)$ and $\wp'(\omega)$ [4]. The idea behind this is the fact that two elliptic function with the same singularities differ by a constant factor. We may use the Weierstrass function to discuss different elliptic functions in a unified way.

Further, we define the integral of $\wp(\omega)$ as

$$\zeta(\omega; \omega_1, \omega_2) = \zeta(\omega) = \frac{1}{\omega} + \sum_{m,n\neq(0,0)} \left[\frac{1}{\omega - m\omega_1 - n\omega_2} + \frac{1}{m\omega_1 + n\omega_2} + \frac{\omega}{(m\omega_1 + n\omega_2)^2}\right] \tag{2.7.6}$$

$\zeta(\omega)$ is not an elliptic function, but it will also be used in our work.

[2] $\Im(x)$ refer to the imaginary part of x

Chapter 3

Foundations in lattice walk models

We have defined lattice walks in Chapter 1. In this chapter, we go through the basic ideas, concepts and methods of solving lattice walk models. We will see how the complexity grows when boundaries are added to the walks.

3.1 Basic lattice walk models

A lattice walk is a walk, with fixed step sets (allowed lengths and directions), ending on the grid. It is a basic combinatorial object and has many applications, for example in the study of trees, words and tableaux.

Let us first consider a typical walk on one-dimensional grid (see Figure 3.1). The grid can be treated as points on the x-axis. The allowed steps are $\{\rightarrow, \leftarrow\}$. That is each step, one can walk from the point k to $k+1$ or from k to $k-1$. Assuming the starting point is the origin 0, we denote $q_{n,k}$ as the number of walks ending at point k with step number n. By direct interpretation, we have the following equation,

$$\begin{aligned} &q_{k,n} = q_{k-1,n-1} + q_{k+1,n-1}, && (n > 0) \\ &q_{k,0} = 0, && (k \neq 0) \\ &q_{0,0} = 1 \end{aligned} \tag{3.1.1}$$

which means a walk ending at point k is formed by a walk ending at point $k-1$ with another $\rightarrow$ step or a walk ending at point $k+1$ with another $\leftarrow$ step. The case of $n = 0$ is the initial condition.

The generating function of this lattice walk model is defined as

$$Q(x,t) = \sum_{k,n} q_{k,n} x^k t^n, \qquad (n \geq 0) \tag{3.1.2}$$

To calculate $Q(x,t)$, we multiply (3.1.1) by $x^k t^n$ and sum all the equations together. We do this summation by two steps. First, sum all the equations with same degree of x together. We define $q_k(t)$ as the generating function of walks ending at point k,

$$q_k(t) = \sum_{n} q_{k,n} t^n \tag{3.1.3}$$

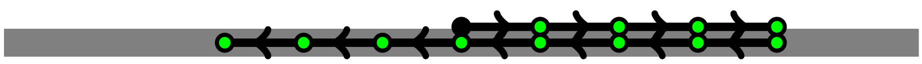


Figure 3.1: A typical one-dimensional lattice walk beginning at $(0,0)$ and ending at $(-3,0)$.

Then we have

$$\begin{aligned} q_k(t)x^k &= t(q_{k-1}(t) + q_{k+1}(t))x^k, \quad (k > 0) \\ q_0(t) &= 1 + t(q_{-1}(t) + q_{+1}(t)) \end{aligned} \tag{3.1.4}$$

Then summing all the equations with different powers of x, we have

$$Q(x,t) = 1 + t\left(Q(x,t)x + \frac{Q(x,t)}{x}\right). \tag{3.1.5}$$

We then easily find that the generating function of one-dimensional walks with allowed steps $\{\rightarrow, \leftarrow\}$ is

$$Q(x,t) = \frac{1}{1 - t(x + \frac{1}{x})} \tag{3.1.6}$$

The expression (3.1.5) can also be explained directly. In (3.1.5),

- 1 refers the generating function of a zero-step walk.
- $Q(x,t)x$ refers the generating function of a walk ending with a $\rightarrow$.
- $\frac{Q(x,t)}{x}$ refers the generating function of a walk ending with a $\leftarrow$.

A walk with allowed steps $\{\leftarrow, \rightarrow\}$ can be decomposed into these three parts. Sum these together and this gives a direct derivation of (3.1.5).

Walks in a higher dimensional space are studied similarly. For a two dimensional walk with allowed steps $\{\leftarrow, \rightarrow, \uparrow, \downarrow\}$ (see Figure 3.2), let $q_{i,j,n}$ be the number of walks starting from the origin, ending at point (i,j) with n steps. We have

$$\begin{aligned} q_{i,j,n} &= q_{i-1,j,n-1} + q_{i,j-1,n-1} + q_{i+1,j,n-1} + q_{i,j+1,n-1}, && (n > 0) \\ q_{i,j,0} &= 0, && (i,j) \neq (0,0) \\ q_{0,0,0} &= 1. \end{aligned} \tag{3.1.7}$$

Multiply (3.1.7) by $x^i y^j t^n$ on both sides and sum for all i, j, n. By a calculation similar to the previous discussion, we find the generating function $Q(x,y,t)$ is

$$Q(x,y,t) = \frac{1}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)} \tag{3.1.8}$$

Comparing (3.1.1) and (3.1.7) we may notice these two equations are in the same form. To discuss them uniformly, we define the allowed steps set $\mathbf{S}$. For example, $\mathbf{S} = \{\leftarrow, \rightarrow, \uparrow, \downarrow\}$ in the previous case. We define $S(\vec{x})$ as the step generator of $\mathbf{S}$,

$$S(\vec{x}) = \sum_{(i_1 i_2 \ldots i_k) \in \mathbf{S}} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}, \tag{3.1.9}$$

where $\vec{x}$ refers to $\{x_1, x_2 \ldots x_k\}$. Here $x_1, x_2 \ldots x_k$ are the generating variables and k is the number of dimensions. Then the generating function satisfies the following equation

$$Q(\vec{x}, t) = 1 + tS(\vec{x})Q(\vec{x}, t) \tag{3.1.10}$$

Thus, we assert that the generating function of a walk without any boundaries is written as

$$Q(\vec{x}, t) = \frac{1}{1 - tS(\vec{x})} \tag{3.1.11}$$

It is a rational function of $\vec{x}$ and t.

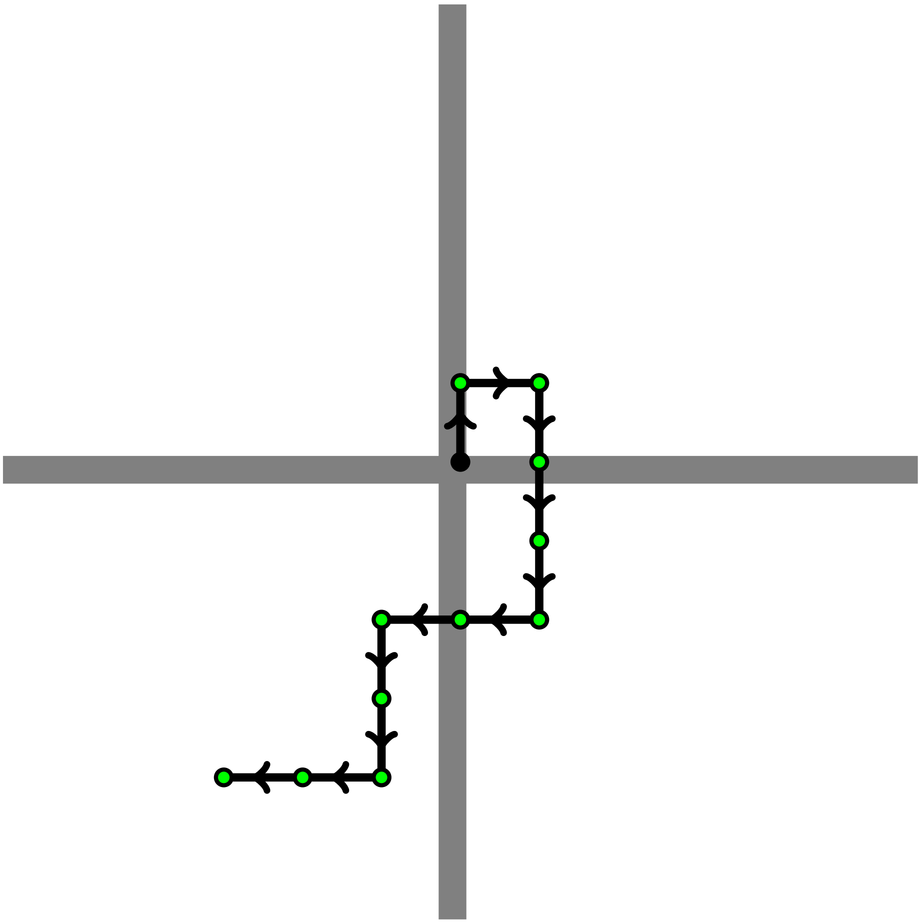


Figure 3.2: A typical two dimensional lattice walk beginning from (0,0) and ending at $(-3,-4)$.

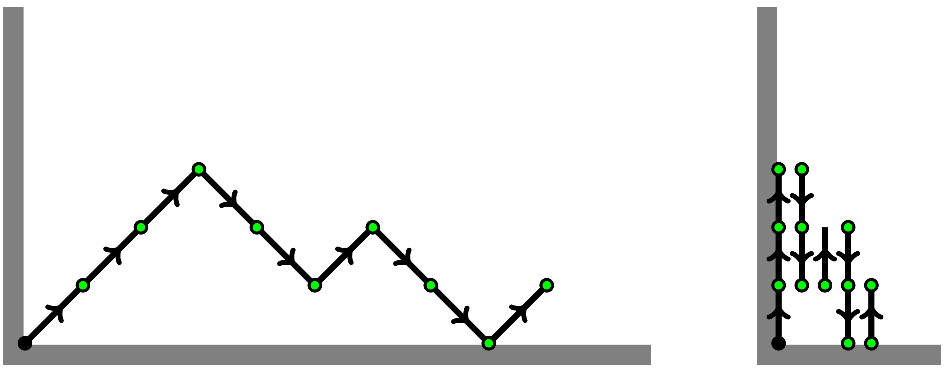


Figure 3.3: Walks with allowed steps $\{\nearrow, \searrow\}$ can be treated as one dimensional walks with allowed steps $\{\uparrow, \downarrow\}$.

3.2 Binomial paths and how boundary conditions affect the walks

We have seen that the generating function of a lattice walk in the whole plane has a rational solution. Now we introduce a boundary into the walk. We begin with a classical model.

The allowed steps are $\{\nearrow, \searrow\}$. The starting point is the origin $(0, 0)$. Each step of the walk is only allowed to end on or above the x-axis. Such a walk is called Ballot path [62]. If the walk finally ends on the x-axis, we call it a *Dyck path* [28]. Let $q_{k,n}$ be the number of walks ending at (k, n). Since each step the walk goes one unit length to the positive direction of x, n also counts the step number. So although it is a walk in two dimensional space, it can be represented by a one dimensional model (see Figure 3.3). That is a walk on the x-axis, starting at 0 with allowed steps $\{\leftarrow, \rightarrow\}$ and can not end on the negative half axis. The general recurrence can be written as

$$\begin{aligned} q_{k,n} &= q_{k-1,n-1} + q_{k+1,n-1}, && (n > 0 \cap k > 0) \\ q_{0,n} &= q_{1,n-1}, && (n > 0) \\ q_{k,n} &= 0, && (k < 0) \\ q_{0,0} &= 1 \end{aligned} \tag{3.2.1}$$

Comparing (3.2.1) and (3.1.1), the recurrence relations are still the same away from the boundary when $k > 0$ since they have the same step generator. When $k = 0$, models with a boundary obey different recurrence relations.

Now we can multiply both sides of (3.2.1) by $y^k x^n$ and sum them together. Similarly, we first take the sum for the x index and define $q_k(x) = \sum_n q_{k,n} x^n$ as the number of walks ending on the line $y = k$. The recurrence of $q_k(x)$ can be written as

$$\begin{aligned} q_k(x) &= x(q_{k-1}(x) + q_{k+1}(x)), \quad (k > 0) \\ q_0(x) &= 1 + x q_1(x) \end{aligned} \tag{3.2.2}$$

Sum all these together, we have

$$Q(x, y) = 1 + x\left(y + \frac{1}{y}\right) Q(x, y) - \frac{x}{y} Q(x, 0) \tag{3.2.3}$$

Thus,

$$Q(x, y) = \frac{1 - \frac{x}{y} Q(x, 0)}{1 - x\left(y + \frac{1}{y}\right)} \tag{3.2.4}$$

The expression (3.2.3) can also be derived directly using another interpretation. A nonempty walk consists of a walk followed by a $\nearrow$ step or a walk that does not end on the x-axis followed by a $\searrow$ step. The generating function of the second case is $Q(x,y) - Q(x,0)$. By (3.2.3), we can see that with one boundary, the generating function still has the same denominator. The numerator contains boundary terms.

We can see with a boundary, the generating function cannot be solved by rearranging (3.2.3). The functional equation (3.2.4) has another unknown function $Q(x,0)$. There are several ways to deal with this situation. The method we focus on is called the kernel method. It will be discussed in Chapter 4. Here, we demonstrate how it works. We first treat the denominator as a function of y:

$$xy^2 - y + x = 0. \tag{3.2.5}$$

It is a quadratic equation with two roots:

$$Y_0 = \frac{1-\sqrt{1-4x^2}}{2x}, \qquad Y_1 = \frac{1+\sqrt{1-4x^2}}{2x} \tag{3.2.6}$$

Notice that if $y \to Y_0, Y_1$, the denominator of $Q(x,y)$ tends to 0. If we want $Q(x,y)$ to be a finite value, the numerator should also tend to 0. The problem is which root we should choose. Let $x \to 0$, then $Y_1 \to \infty$ and $Y_0 \to 0$. Since $Q(x,y)$ is a formal series of x, y, the convergent area is some domain near $(0,0)$. Thus (x, Y_0) must be inside the convergent domain while (x, Y_1) stay outside the convergent area when $x \to 0$. Now we substitute Y_0 into the numerator and let it become 0. This gives the solution of $Q(x,0)$,

$$Q(x,0) = \frac{1-\sqrt{1-4x^2}}{2x^2} \tag{3.2.7}$$

Substituting $Q(x,0)$ back into (3.2.4), we solve $Q(x,y)$.

$$Q(x,y) = \frac{1 - \frac{x}{y}\frac{1-\sqrt{1-4x^2}}{2x^2}}{1 - x\left(y + \frac{1}{y}\right)} \tag{3.2.8}$$

In combinatorics, we have another way to calculate the generating function for Dyck paths. The generating function $Q(x,0)$ can be calculated directly by the following equation.

$$Q(x,0) = 1 + x^2 Q(x,0)Q(x,0) \tag{3.2.9}$$

This equation reflects the unique factorization of Dyck path: every path (except the zero-step path) can be factorized into two Dyck paths (the two factors $Q(x,0)Q(x,0)$) with the second path raised up a step (the x^2 factor, see Figure 3.4).

Combinatorialists have many different ways to deal with lattice walk problems. But most of these methods do not have a unified process and rely on some special tactics. We do not discuss too many of them here.

Now we go to the two-dimensional cases. The two-dimensional models are derived similarly. For example, the generating function of walks with allowed steps $\{\leftarrow, \rightarrow, \uparrow, \downarrow\}$ starting from the origin and restricted to the upper half plane is written as

$$Q(x,y,t) = \frac{1 - \frac{t}{y}Q(x,0)}{1 - t(x + \frac{1}{x} + y + \frac{1}{y})} \tag{3.2.10}$$

The equation (3.2.10) can be derived by both methods discussed above. Notice that $Q(x,y,t)$ still follows a similar expression. It has a denominator $1 - tS(\vec{x})$ and a numerator with linear

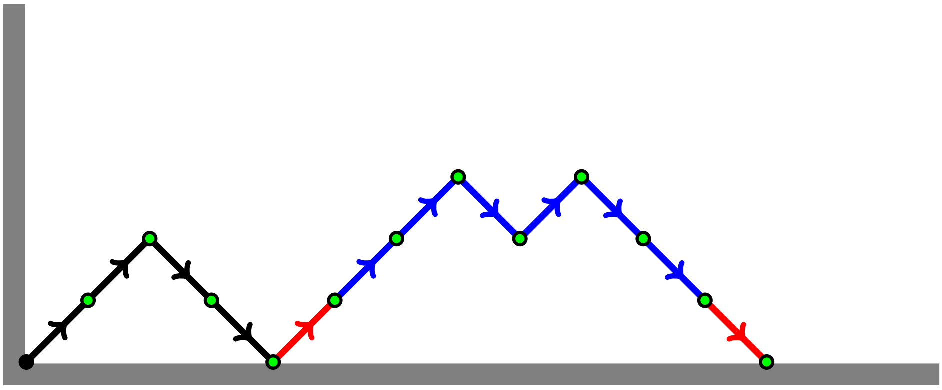


Figure 3.4: A Dyck path can be constructed by a Dyck path (black) followed by a lifted Dyck path(blue), connected by the red steps.

terms of the boundary function $Q(x,0)$. The functional equation will also have this property when we have two boundaries, since the process of derivation is similar. We may conclude the general form of the generating function:

$$Q(\vec{x},0) = \frac{1 - \sum_i tf_i(\vec{x})Q_i(\vec{x},0)}{1 - tS(\vec{x})} \tag{3.2.11}$$

where $Q_i(\vec{x},0)$ refers to the boundary terms and $f_i(\vec{x})$ are the coefficients of the boundary terms. The denominator $1 - tS(\vec{x})$ is normally called the *kernel* of the model.

3.3 Binomial paths with interactions

In the generating function $Q(x,0)$, we assign t to track the length of a path. We may also assign other variables to track other values. For example, some combinatorialists not only want to know the number of walks that end on the x-axis (ie. Dyck paths), but also the number of times that a walk touches the x-axis (starting point excluded). To calculate this, we include another variable a in the generating function. The generating function $Q(x,0,a)$ is calculated as follows,

$$Q(x,0,a) = 1 + ax^2Q(x,0,a)Q(x,0), \tag{3.3.1}$$

where $Q(x,0,a) = \sum_{i,l,n} q_{i,l,n}x^i a^l t^n$ and $q_{i,l,n}$ is the number of paths ending at $(i,0)$ with step length n and have visited the x-axis l times. The equation (3.3.1) can be understood by observing that a Dyck path ending on the x-axis l times consists of a Dyck path ending on the x-axis $l-1$ times followed by a lifted Dyck path (see Figure 3.4).

The coefficient of $a^l x^n$ of $Q(x,0,a)$ is the number of Dyck paths of length n that touch the x-axis l times. With the help of a, we can get more information about this model. This example is actually a combinatorial interpretation of boundary interactions. We add an interaction a_i to steps with some certain condition i. The new generating function $Q(\vec{x},t,a_1,a_2\dots a_i)$ gives extra information.

3.4 Two-boundary cases: quarter-plane walks

Now we move on to the two-boundary quarter-plane models. We still begin with the simplest case, walks with allowed steps $\{\rightarrow,\uparrow,\leftarrow,\downarrow\}$. We have both the x- and y-axes as boundaries.

The walk is restricted to the quarter-plane. Let $q_{i,j,n}$ be the number of walks ending at (i,j) with step number n. The recurrence is

$$\begin{aligned}
q_{i,j,n} &= q_{i-1,j,n-1} + q_{i,j-1,n-1} + q_{i+1,j,n-1} + q_{i,j+1,n-1}, && (i>0 \cap j>0 \cap n>0);\\
q_{0,j,n} &= q_{0,j-1,n} + q_{0,j+1,n}, && (j>0 \cap n>0);\\
q_{i,0,n} &= q_{i-1,0,n} + q_{i+1,0,n}, && (i>0 \cap n>0);\\
q_{0,0,n} &= q_{1,0,n} + q_{0,1,n}, && (n>0);\\
q_{0,0,0} &= 1.
\end{aligned} \tag{3.4.1}$$

We may solve $Q(x,y,t)$ by first multiplying (3.4.1) by $x^i y^j t^n$ and summing, but it is more straightforward to work directly with the generating functions.

The set of walks counted by the generating function $Q(x,y,t)$ consists of

- Walks ending away from an axis followed by an extra step belonging to $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$. The generating function of these walks is written as $t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)(Q(x,y,t)-Q(x,0,t)-Q(0,y,t)+Q(0,0))$. Notice that $Q(0,0)$ has been subtracted twice by $Q(x,0)$ and $Q(0,y)$ so we add one back.
- Walks ending on the positive x-axis followed by a $\{\leftarrow, \rightarrow, \uparrow\}$; The generating function is written as $t\left(x+\frac{1}{x}+y\right)(Q(x,0,t)-Q(0,0))$.
- Walks ending on the positive y-axis followed by a $\{\rightarrow, \uparrow, \downarrow\}$; The generating function is written as $t\left(x+y+\frac{1}{y}\right)(Q(0,y,t)-Q(0,0))$.
- Walks ending at $(0,0)$ followed by $\{\uparrow, \rightarrow\}$; The generating function is written as $t(x+y)Q(0,0)$.
- Walks with 0 steps. This generating function is just $t^0 = 1$.

Sum all these terms together, we get the functional equation

$$\begin{aligned}
Q(x,y,t) = 1 + t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)(Q(x,y,t)-Q(x,0,t)-Q(0,y,t)+Q(0,0))\\
+ t\left(x+\frac{1}{x}+y\right)(Q(x,0)-Q(0,0)) + t\left(x+y+\frac{1}{y}\right)(Q(0,y)-Q(0,0))\\
+ t(x+y)Q(0,0).
\end{aligned} \tag{3.4.2}$$

Thus $Q(x,y,t)$ is written as

$$Q(x,y,t) = \frac{1-\frac{1}{y}Q(x,0,t)-\frac{1}{x}Q(0,y,t)}{1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)} \tag{3.4.3}$$

The expression (3.4.3) still follows the general form of (3.2.11). The kernel method cannot be applied to (3.2.11) directly since we have two boundary terms $Q(x,0)$ and $Q(0,y)$. To solve this equation, we first notice that if we let $x \to \frac{1}{x}$, (3.4.3) becomes

$$Q\left(\frac{1}{x},y,t\right) = \frac{1-\frac{1}{y}Q\left(\frac{1}{x},0,t\right)-xQ(0,y,t)}{1-t(x+\frac{1}{x}+y+\frac{1}{y})} \tag{3.4.4}$$

Notice that the denominator of both (3.4.3) and (3.4.4) is $1 - t(x + \frac{1}{x} + y + \frac{1}{y})$, and its roots are,

$$Y_1 = \frac{-(x+1-\frac{1}{t}) + \sqrt{(x+1-\frac{1}{t})^2 - 4}}{2} \tag{3.4.5}$$

$$Y_2 = \frac{-(x+1-\frac{1}{t}) - \sqrt{(x+1-\frac{1}{t})^2 - 4}}{2} \tag{3.4.6}$$

Note that Y_1 is the root that converges as $t \to 0$. Now we substitute Y_1 into the numerator of both (3.4.3) and (3.4.4), so that

$$1 - \frac{-(x+1-\frac{1}{t}) - \sqrt{(x+1-\frac{1}{t})^2 - 4}}{2} Q(x,0,t) - \frac{1}{x} Q(0,Y_1,t) = 0 \tag{3.4.7}$$

$$1 - \frac{-(x+1-\frac{1}{t}) - \sqrt{(x+1-\frac{1}{t})^2 - 4}}{2} Q\left(\frac{1}{x},0,t\right) - xQ(0,Y_1,t) = 0 \tag{3.4.8}$$

We eliminate $Q(0,Y_1,t)$ and find a relation between $Q(x,0,t)$ and $Q\left(\frac{1}{x},0,t\right)$. Simplifying this, we have

$$xQ(x,0,t) - \frac{1}{x}Q\left(\frac{1}{x},0,t\right) = \left(x - \frac{1}{x}\right) \frac{-(x+1-\frac{1}{t}) + \sqrt{(x+1-\frac{1}{t})^2 - 4}}{2} \tag{3.4.9}$$

Now, take the $[x^>]$ terms of both sides of (3.4.9). Notice that $[x^>](\frac{1}{x}Q\left(\frac{1}{x},0\right)) = 0$. Thus,

$$Q(x,0,t) = \frac{1}{x}[x^>]\left[\left(x - \frac{1}{x}\right) \frac{-(x+1-\frac{1}{t}) + \sqrt{(x+1-\frac{1}{t})^2 - 4}}{2}\right] \tag{3.4.10}$$

This method is called the *obstinate kernel method*, which will be discussed in detail in Chapter 4. For quarter-plane walks, there are other combinatorial methods solving models with two boundaries. See [40] for detailed information. We do not consider them here.

3.5 General constructions and solutions of quarter-plane walks without interactions

The previous construction of a functional equation for walks with steps $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$ can be generalized to all quarter-plane lattice walk models without interactions. From now on, we omit the variable t in $Q(x,y,t)$, $Q(x,0,t)$ and $Q(0,y,t)$. We just write them as $Q(x,y)$, $Q(x,0)$ and $Q(0,y)$. We should keep in mind that they are all defined as formal series of t. Suppose the lattice walk has allowed steps $\mathbf{S}$. Generally, if we write

$$S(x,y) = \frac{A_{-1}(x)}{y} + A_0(x) + A_1(x)y = \frac{B_{-1}(y)}{x} + B_0(y) + B_1(y)x \tag{3.5.1}$$

then, $Q(x,y)$ can be constructed as follows.

- Walks ending inside quarter-plane followed by an extra step belonging to $\mathbf{S}$:

  the generating function is written as $tS(x,y)(Q(x,y) - Q(x,0) - Q(0,y) + Q(0,0))$.

- Walks ending on the positive x-axis followed by a step in $\mathbf{S}$ not going downwards: the generating function is written as $t\left(S(x,y) - A_{-1}(x)\frac{1}{y}\right)(Q(x,0) - Q(0,0))$.
- Walks ending on the positive y-axis follows by a step in $\mathbf{S}$ not going leftwards: the generating function is written as $t\left(S(x,y) - B_{-1}(y)\frac{1}{x}\right)(Q(0,y) - Q(0,0))$.
- Walks ending at $(0,0)$ followed by a step in $\mathbf{S}$ going rightwards or upwards: the generating function is written as $t[x^{\geq}y^{\geq}]S(x,y)Q(0,0)$.
- Walks with 0 steps:

  The generating function is 1.

Summing all these together, we have

$$\begin{aligned}(1 - tS(x,y))Q(x,y) = 1 - tA_{-1}(x)\frac{1}{y}Q(x,0) - tB_{-1}(y)\frac{1}{x}Q(0,y) \\ + t\left(A_{-1}(x)\frac{1}{y} + B_{-1}(y)\frac{1}{x} - S(x,y) + [x^{\geq}y^{\geq}]S(x,y)\right)Q(0,0)\end{aligned} \tag{3.5.2}$$

Since $A_{-1}(x)\frac{1}{y} = [y^{<}]S(x,y)$, $B_{-1}(y)\frac{1}{x} = [x^{<}]S(x,y)$, the $Q(0,0)$ term is further written as $t[y^{<}x^{<}]S(x,y)Q(0,0) = tC_{-1,-1}\frac{1}{xy}Q(0,0)$. Then (3.5.2) becomes,

$$(1 - tS(x,y))Q(x,y) = 1 - tA_{-1}(x)\frac{1}{y}Q(x,0) - tB_{-1}(y)\frac{1}{x}Q(0,y) + \frac{tC_{-1,-1}}{xy}Q(0,0) \tag{3.5.3}$$

The functional equation (3.5.3) suits all quarter-plane lattice walks. Now we try to determine if the idea that we used when solving walks with steps $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$ works for other models as well. Notice that in (3.4.4), we let $x \to \frac{1}{x}$ and got a transformed equation. This transformation keeps the kernel unchanged. Such a transformation always exists if the kernel is a quadratic polynomial of x. This is because a quadratic polynomial always has two roots and there is an involution that permutes the roots X_1, X_2. Similarly, if we treat the kernel as a quadratic polynomial of y, we will have an involution that permutes Y_1, Y_2.

These two involutions can be written as

$$\phi : (x,y) \to \left(\frac{B_{-1}(y)}{xB_1(y)}, y\right) \quad \text{and} \quad \psi : (x,y) \to \left(x, \frac{A_{-1}(x)}{yA_1(x)}\right). \tag{3.5.4}$$

Taking these two involutions as generators, we can generate a group.

Definition 4. *For each quarter-plane lattice walk model with kernel $K(x,y)$, the transformations δ such that*

$$\delta K(x,y) = K(x,y) \tag{3.5.5}$$

form a group. We call it the symmetry group of the model or the group of the model.

The symmetry group was first proposed in [53] and systematically studied by Bousquet-Mélou and Mishna in [20].

There are in all 2^8 walk models with unit step length in the quarter-plane. Many of them are one-boundary models. For example, for the model with allowed steps $\{\leftarrow, \rightarrow, \uparrow\}$, only the y-axis is an actual boundary. This implies for a quarter-plane walk to be a two boundary model, we must have $[x^{>}], [x^{<}], [y^{>}], [y^{<}]$ terms in $K(x,y)$. Only 161 out of 2^8 satisfy this condition.

In this case $K(x, y)$ is quadratic in both x and y. However, some of these models are not valid in the quarter-plane, because a walk will never be able to leave the origin. For example, the model with allowed steps $\{\searrow, \nwarrow, \swarrow\}$. We discard these models, then we have 153 left. For the remaining models, we find that if we have $\{\nearrow, \swarrow\}$ in the step set and the other steps are either a subset of $\{\leftarrow, \nwarrow, \uparrow\}$ or a subset of $\{\downarrow, \searrow, \rightarrow\}$, then the walks will never be able to touch one of the boundaries and the quarter-plane restriction is redundant. After discarding these one-boundary cases, we have 138 left. In these non-trivial two-boundary models, some are reflections of others in the line $y = x$. They should be regarded as the same models. We finally find there are 79 different two-boundary models.

Of these 79 models, there are 23 associated with finite symmetry groups. In [20], Bousquet-Mélou and Mishna give a further classification: 16 models are associated with D_2 groups, five models with D_3 groups and two models with D_4 groups. All the walks associated with finite groups are well studied (see Tables 3.1 to 3.3, where we abbreviate D-finite as DF, algebraic as Alg). All of the walks associated with finite groups have D-finite generating functions. That is, the function $Q(x, y)$ is D-finite over x, y, t. There are four special models: Kreweras walks (19th model), reverse Kreweras walks (20th model), double Kreweras walks (21st model) and Gessel's walks (23rd model). Their generating functions are algebraic. In [20], Bousquet-Mélou and Mishna give some general approaches to most models except the 23rd model, Gessel's walks. The solution for Gessel's walks was given later in [17]. They use the kernel method to solve these models (the method we used in solving walks with steps $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$ is one version of the kernel method). We will first discuss the kernel method and then continue with the discussion of quarter-plane lattice walks.

	Model	$Q(x,y)$	Order	Generator
1	$\uparrow\leftarrow\downarrow\rightarrow$	DF	4	$(\frac{1}{x},y),(x,\frac{1}{y})$
2	$\nwarrow\swarrow\searrow\nearrow$	DF	4	$(\frac{1}{x},y),(x,\frac{1}{y})$
3	$\uparrow\nwarrow\swarrow\downarrow\searrow\nearrow$	DF	4	$(\frac{1}{x},y),(x,\frac{1}{y})$
4	$\uparrow\nwarrow\leftarrow\swarrow\downarrow\searrow\nearrow$	DF	4	$(\frac{1}{x},y),(x,\frac{1}{y})$
5	$\nwarrow\downarrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{1}{(x+\frac{1}{x})y}\right)$
6	$\nwarrow\leftarrow\downarrow\rightarrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{1}{(x+\frac{1}{x})y}\right)$
7	$\uparrow\nwarrow\downarrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{1}{(x+1+\frac{1}{x})y}\right)$
8	$\uparrow\nwarrow\leftarrow\downarrow\rightarrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{1}{(x+1+\frac{1}{x})y}\right)$
9	$\uparrow\nwarrow\swarrow\searrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{x+\frac{1}{x}}{(x+1+\frac{1}{x})y}\right)$
10	$\nwarrow\leftarrow\swarrow\uparrow\searrow\rightarrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{x+\frac{1}{x}}{(x+1+\frac{1}{x})y}\right)$
11	$\uparrow\swarrow\downarrow\searrow$	DF	4	$(\frac{1}{x},y),\left(x,(x+1+\frac{1}{x})\frac{1}{y}\right)$
12	$\uparrow\leftarrow\swarrow\downarrow\searrow\rightarrow$	DF	4	$(\frac{1}{x},y),\left(x,(x+1+\frac{1}{x})\frac{1}{y}\right)$
13	$\nwarrow\swarrow\downarrow\searrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{x+1+\frac{1}{x}}{(x+\frac{1}{x})y}\right)$
14	$\nwarrow\leftarrow\swarrow\downarrow\searrow\rightarrow\nearrow$	DF	4	$(\frac{1}{x},y),\left(x,\frac{x+1+\frac{1}{x}}{(x+\frac{1}{x})y}\right)$
15	$\uparrow\swarrow\searrow$	DF	4	$(\frac{1}{x},y),\left(x,(x+\frac{1}{x})\frac{1}{y}\right)$
16	$\uparrow\leftarrow\swarrow\searrow\rightarrow$	DF	4	$(\frac{1}{x},y),\left(x,(x+\frac{1}{x})\frac{1}{y}\right)$

Table 3.1: The step sets, property of generating functions and group generators of walk associated with D_2 groups.

	Model	$Q(x,y)$	Order	Gnerator
17	$\uparrow\leftarrow\searrow$	DF	6	$(\frac{y}{x},y),(x,\frac{x}{y})$
18	$\uparrow\nwarrow\leftarrow\downarrow\searrow\rightarrow$	DF	6	$(\frac{y}{x},y),(x,\frac{x}{y})$
19	$\leftarrow\downarrow\nearrow$	Alg	6	$(\frac{1}{xy},y),(x,\frac{1}{xy})$
20	$\uparrow\swarrow\rightarrow$	Alg	6	$(\frac{1}{xy},y),(x,\frac{1}{xy})$
21	$\uparrow\leftarrow\swarrow\downarrow\rightarrow\nearrow$	Alg	6	$(\frac{1}{xy},y),(x,\frac{1}{xy})$

Table 3.2: Step sets associated with the group D_3.

	Model	$Q(x,y)$	Order	Generator
22	$\nwarrow\leftarrow\searrow\rightarrow$	DF	8	$(\frac{y}{x},y),(x,\frac{x^2}{y})$
23	$\leftarrow\swarrow\rightarrow\nearrow$	Alg	8	$(\frac{1}{xy},y),(x,\frac{1}{x^2y})$

Table 3.3: Step sets associated with the group isomorphic to D_4.

Chapter 4

The kernel method

The main objective of studies in lattice walks is solving the functional equation (3.5.3) satisfied by the generating function. This equation is a trivariate functional equation. Our main tool for solving it is the kernel method. It is an effective approach to handle such situations. The kernel method appears in different aspects of combinatorics [1, 31, 30]. In lattice walk models, many previous works [14, 7, 26] are based on the kernel method. Later we will extend this method and solve the models that have not been solved previously. In this section, we first give a brief introduction to the kernel method.

4.1 The origin of the kernel method

The kernel method has a long history. It is most commonly claimed (for example, in [60]) that the kernel method was first found as an exercise in Knuth's book [45]. It may well have been independently discovered before or after. It has been developed into a general method in [1, 21, 59]. It is a general method of solving multivariate functional equations.

The general idea of the kernel method is making use of the singularities of the functional equation. We have already seen this in Section 3.2. Now we explain the idea of each step of the kernel method. Suppose we have a functional equation

$$F(x,t) = \frac{tF(0,t) - x}{tx^2 - x + t} \tag{4.1.1}$$

Our aim is to solve $F(x,t)$, which is supposed to be in $\mathbb{R}[x][[t]]$ with constant term 1. Now let us look at the denominator. It can be factored as $t(x - x_0)(x - x_1)$, where

$$x_{0,1} = \frac{1 \pm \sqrt{1 - 4t^2}}{2t} \tag{4.1.2}$$

We may try expand $x_{0,1}$ as series in t near $t = 0$,

$$\begin{aligned}
x_0 &= \frac{1}{t} - t - t^3 - 2t^5 + \mathrm{O}(t^6) \\
x_1 &= t + t^3 + 2t^5 + \mathrm{O}(t^6) \\
\frac{1}{x_0} &= t + t^3 + 2t^5 + \mathrm{O}(t^6) \\
\frac{1}{x_1} &= \frac{1}{t} - t - t^3 - 2t^5 + \mathrm{O}(t^6)
\end{aligned} \tag{4.1.3}$$

Notice that $x_1 \sim t$ as $t \to 0$. Suppose we substitute x_1 into (4.1.1) and take the limit $t \to 0$. Since $F(x,t)$ is defined as a power series of t with polynomial coefficients in x, it must converge in this limit. The RHS has $(x-x_1)$ in the denominator. When $x \to x_1$, the denominator tends to 0. So the numerator must also tend to 0, otherwise the RHS is divergent. This gives

$$tF(0,t) - x_1 = 0. \tag{4.1.4}$$

We can thus solve for $F(0,t)$, from which $F(x,t)$ follows directly. The coefficient $[t^{2n}]F(0,t) = \frac{1}{n+1}\binom{2n}{n}$ is the famous Catalan number [65]. Then $F(0,t)$ is the generating function of Dyck paths [28] (recall Section 3.2) of length $2n$ and $F(x,t)$ is the generating function of meanders (a meander is a path that never goes below the horizontal axis, see [2] for a detailed definition).

Another interpretation of the kernel method uses factorization. We still use (4.1.1) as an example. We can write (4.1.1) as

$$F(x,t) = \frac{tF(0,t)-x}{tx^2-x+t} \tag{4.1.5}$$

$$= \frac{tF(0,t)-x}{t(x_0-x_1)}\left(\frac{1}{x-x_0} - \frac{1}{x-x_1}\right) = \frac{tF(0,t)-x}{t(x_0-x_1)}\left(\sum_{i=0}\left(\frac{x}{x_0}\right)^i + \sum_{i=1}\left(\frac{x_1}{x}\right)^i\right) \tag{4.1.6}$$

Since $F(x,t)$ is defined as an element of $\mathbb{R}[x][[t]]$, the RHS is a series in t with coefficients in $\mathbb{R}[x]$ as well. We first factor the denominator and try to expand $\frac{tF(0,t)-x}{tx^2-x+t}$ as a series in t. Notice that $\frac{1}{x_0}$ and x_1 are formal series in t. So $\frac{1}{x-x_1}$ should be expanded as $\sum(\frac{x_1}{x})^i$ and $\frac{1}{x-x_0}$ should be expanded as $\sum(\frac{x}{x_0})^i$. In this expansion, $\sum(\frac{x_1}{x})^i$ is a negative power series of x. While in the LHS, $F(x,t)$ only has positive powers of x. Thus the coefficients of all negative power terms of x should equal 0. This implies

$$(tF(0,t)-x)\sum_{i=1}\left(\frac{x_1}{x}\right)^i = \sum_{i=1}(tF(0,t)-x_1)\left(\frac{x_1}{x}\right)^i - x_1 = \text{terms independent of } x \tag{4.1.7}$$

Thus we have $tF(0,t) - x_1 = 0$. We can see the second interpretation gives the same result as (4.1.4).

We call the denominator of the RHS of (4.1.1) the *kernel*. That is why we denote $(1-tS(\vec{x}))$ in (3.2.11) in lattice walk problems as the kernel.

4.2 Different insights about the kernel for quarter-plane lattice walks

The following discussion requires a simple lemma.

Lemma 5. *1. If $f(y)$ is a formal series in y, we have $[y^0]\left(f(y)\sum(\frac{Y_0}{y})^i\right) = f(Y_0)$.*

2. If $f(\frac{1}{y})$ is a formal series of $\frac{1}{y}$, we have $[y^0]\left(f(\frac{1}{y})\sum(\frac{y}{Y_1})^i\right) = f\left(\frac{1}{Y_1}\right)$.

Proof. Write $f(y)$ as $\sum_i f_i y^i$ and by direct calculation, we have the result. □

Recall from (3.5.1) that the kernel in a quarter-plane walk problem reads

$$\begin{aligned} xyK(x,y) = xy(1-tS(x,y)) &= A_{-1}(x)x + A_0(x)xy + A_1(x)xy^2 \\ &= B_{-1}(y)y + B_0(y)yx + B_1(y)yx^2 \end{aligned} \tag{4.2.1}$$

For two-boundary cases, it is an irreducible quadratic polynomial in x, y and is linear in t. By discussion in [33, Section 2.2], we have four different insights for the equation $xyK(x,y) = 0$.

1. **(Algebraic functions)** We can treat the trivariate polynomial as an algebraic function $K(x, Y(x))$ (or $K(X(y), y)$) and by the first interpretation of the original kernel method, we have

$$xY(x)K(x, Y(x)) = 0 \tag{4.2.2}$$

Then, substituting the roots of (4.2.2) into functional equation (3.5.3), we have

$$1 - \frac{tA_{-1}(x)}{Y_i(x)}Q(x,0) - \frac{tB_{-1}(Y_i(x))}{x}Q(0, Y_i(x)) + \frac{tC_{-1,-1}}{xY_i}Q(0,0) = 0. \tag{4.2.3}$$

This substitution is valid only in the convergent domain of all three functions, $Q(x, y)$, $Q(x, 0)$ and $Q(0, y)$. The convergent domain may be written as

$$\{|x| \leq \rho_x(t)\} \cap \{|Y_i(x)| \leq \rho_y(t)\} \tag{4.2.4}$$

This gives a constraint on which root we can substitute. It will be proved that only one root of the quadratic polynomial is valid for the substitution (Theorem 6).

2. **(Algebraic curve)** Let $\mathbb{B}$ be an algebraic curve in $\mathbb{C}(t)$ defined by the equation

$$xyK(x, y) = 0. \tag{4.2.5}$$

Now $Q(x, y)$ in (3.5.3) converges in $\{|x| \leq \rho_x(t)\} \cap \{y \leq \rho_y(t)\}$. Then we have for all $x, y \in \mathbb{B} \cap \{|x| \leq \rho_x(t)\} \cap \{y \leq \rho_y(t)\}$,

$$1 - tA_{-1}(x)yQ(x, 0) - tB_{-1}(y)xQ(0, y) + \frac{tC_{-1,-1}}{xy}Q(0, 0) = 0. \tag{4.2.6}$$

If $\mathbb{B} \cap \{|x| \leq \rho_x(t)\} \cap \{y \leq \rho_y(t)\}$ is projected on to the $\mathbb{C}_x$ plane by $x \to x, y \to Y_i(x)$, we get (4.2.3) again.

3. **(Factorization)** Since $xyK(x, y)$ is a quadratic polynomial, we can factor it as

$$K(x, y) = \frac{tA_1(x)Y_1(x)}{y}\left(1 - \frac{Y_0(x)}{y}\right)\left(1 - \frac{y}{Y_1(x)}\right). \tag{4.2.7}$$

Then, factor the LHS of (3.5.2), and we have

$$\begin{aligned}
&\frac{txA_1(x)}{Y_1(x)}\left(1 - \frac{y}{Y_1(x)}\right)Q(x, y) \\
&\quad = \frac{xy}{1 - \frac{1}{y}Y_0}\left(1 - \frac{tA_{-1}(x)}{y}Q(x, 0) - \frac{tB_{-1}(y)}{x}Q(0, y) + \frac{tC_{-1,-1}}{xy}Q(0, 0)\right)
\end{aligned} \tag{4.2.8}$$

Thus take $[y^0]$ of (4.2.8). With the help of Lemma 5, we get exactly the same equation as (4.2.3) with $Y(x) = Y_0(x)$.

4. **(Riemann surfaces)** The irreducible equation $xYK(x, Y) = 0$ defines a multi-valued function $Y(x)$. Each $(x, Y_i(x))$ is treated as a part of a Riemann surface. Since

$$Y_{0,1}(x) = \frac{-b(x) \mp \sqrt{\Delta}}{2a(x)}, \tag{4.2.9}$$

the branch points of $Y(x)$ are the roots of Δ. By the property of the roots of Δ discussed in Section 6.2, Δ has three or four roots. We can cut the complex plane along the branch cut $\overrightarrow{x_1x_2}$ and $\overrightarrow{x_3x_4}$ (or $\overrightarrow{x_3\infty}$) and glue the sheets of $Y_1(x)$ and $Y_0(x)$ together. We get a Riemann surface of the algebraic function $xYK(x, Y) = 0$. It is a torus and the genus is 1.

We do similar things to $XyK(X, y) = 0$. Thus, two algebraic equations,

$$xYK(x, Y) = 0 \quad \text{and} \quad XyK(X, y) = 0 \tag{4.2.10}$$

define two Riemann surfaces (see for example [63]). These two surfaces are conformally equivalent. We define a third Riemann surface $\mathbb{S}$, which satisfies

$$x(s)y(s)K(x(s), y(s)) = 0 \tag{4.2.11}$$

with two homomorphisms

$$\begin{aligned} h_x &: \mathbb{S} \to \mathbb{C}_x, \qquad h_x(s) = x(s) \\ h_y &: \mathbb{S} \to \mathbb{C}_y, \qquad h_y(s) = y(s) \end{aligned} \tag{4.2.12}$$

which project S onto $\mathbb{C}_x$ and $\mathbb{C}_y$.

We can write $Q(x, 0) = Q(x(s), 0)$ and $Q(0, y) = Q(0, y(s))$ and we have

$$1 - \frac{tA_{-1}(x(s))}{y}Q(x(s), 0) - \frac{tB_{-1}(y(s))}{x(s)}Q(0, y(s)) + \frac{tC_{-1,-1}}{(x(s), y(s))}Q(0, 0) = 0 \tag{4.2.13}$$

on this surface. By definition, $Q(x, 0)$ and $Q(0, y)$ are formal series of t with polynomial coefficients in x, y. If we fix t to be some small value, then $Q(x, 0)$ and $Q(0, y)$ converge in some finite domains with radii $\rho_x(t), \rho_y(t)$. But when we consider them on the Riemann surface, if the convergent domain does not have dense singularities on its boundary, we can analytically continue $Q(x(s), 0)$ and $Q(0, y(s))$ to the area outside the domain or even to the whole surface.

4.3 Symmetry group

We have seen that the kernel is invariant under transformation by two involutions,

$$\phi : (x, y) \mapsto \left(\frac{B_{-1}(y)}{xB_1(y)}, y\right) \quad \text{and} \quad \psi : (x, y) \mapsto \left(x, \frac{A_{-1}(x)}{yA_1(x)}\right) \tag{4.3.1}$$

They generate a group G which is isomorphic to a dihedral group D_n of order $2n$, with $n \in \mathbb{N} \cup \{\infty\}$. Applying each group element to the functional equation gives us a new equation. It may provide new information, as well as introducing new unknowns. In [20], Bousquet et al introduce two different kinds of kernel method based on these two interpretations: the algebraic kernel method and the obstinate kernel method. We will introduce these two methods in the following sections and generalize them to the problem of solving the quarter-plane lattice walk with boundary interactions.

4.4 Obstinate kernel method

The idea of the obstinate kernel method follows the first insight. It is much like the original kernel method that we discussed in Section 4.1. We may apply the symmetric transformation δ and then substitute the roots of the kernel into the functional equation.

$$1 - \frac{tA_{-1}(\delta_x(x))}{\delta_y(Y_i)}Q(\delta_x(x), 0) - \frac{tB_{-1}(\delta_y(Y_i))}{\delta_x(x)}Q(0, \delta_y(Y_i)) + \frac{tC_{-1,-1}}{\delta_x(x)\delta_y(Y_i)}Q(0, 0) = 0 \tag{4.4.1}$$

Here δ may be ϕ, ψ or any element in the symmetry group. We get an equation with boundary terms $Q(\phi_x(x), 0)$ or $Q(0, \psi_y(Y_i))$ etc. Unfortunately, not all equations we get from this substitution satisfy the convergent condition. In particular, (4.2.3) holds if all three functions $Q(x, y), Q(x, 0), Q(0, y)$ converge. After the symmetrical transformation, (4.4.1) may break this condition. For example, if $\psi_y(Y_0) = Y_1 = \frac{1}{Y_0}$ and $\rho_{Y_i(t)} = 1$, obviously $Q\left(0, \frac{1}{Y_i}\right)$ converges if and only if $Q(0, Y_i)$ diverges. It has been proved that only half of the symmetry group is valid in [33, Section 2.4]. The two involutions, ϕ and ψ can be treated as two Galois automorphisms of $\mathbb{C}[x, Y]/K(x, Y)$ and $\mathbb{C}[X, x]/K(X, y)$. The Galois automorphism permutes the roots. In our situation, the kernel is a quadratic polynomial satisfying the conditions in Theorem 6. By Theorem 6, if one root is a formal series of t, then the other root has to be a series containing $\frac{1}{t}$[1]. Thus we can only apply a Galois automorphism that permutes X_0, X_1 but not the one that permutes Y_0, Y_1. So only half of the symmetry group is valid.

If the symmetry group contains $2n$ elements, it provides n valid equations. These n equations contain different boundary terms. Our aim is to eliminate all or all except one line boundary terms by linear combinations or some other tactics. If it is possible to do this, we are able to solve the problem. Often, in most quarter-plane walk cases, we have $Q(x, 0)$ and $Q\left(\frac{1}{x}, 0\right)$ left after linear combination. We may then separate the $[x^>]$ and $[x^<]$ parts of this equation. This operation may also separate $Q(x, 0)$ and $Q(\frac{1}{x}, 0)$ into two different equations and each equation is in the form of (4.1.1). We solve them directly by the original kernel method.

Notice that separating the $[x^>]$, $[x^<]$ parts may not always separate $Q(x, 0)$ and $Q\left(\frac{1}{x}, 0\right)$ without any side effect. Sometimes, the separation will lead to an infinite number of unknowns. For example, $\frac{1}{(x-X_0)^{\frac{1}{2}}} Q(x, 0)$ cannot be separated. Since $\frac{1}{(x-X_0)^{\frac{1}{2}}} = \sum -\binom{\frac{1}{2}}{i}\left(-\frac{X_0}{x}\right)^i$ is a formal series of $\frac{1}{x}$, $[x^k]\frac{1}{(x-X_0)^{\frac{1}{2}}} Q(x, 0)$ contains an infinite number of $q_{i,0}(t)$ as new unknown functions. We cannot separate such equations. Normally, if the term contains unknown functions such as $Q(x, 0)$, separating the $[x^>]$, $[x^0]$ and $[x^<]$ terms requires the coefficient of $Q(x, 0)$ to be a formal series of x or a function in $\mathbb{R}[x, 1/x]$.

The technique in the obstinate kernel method involves eliminating the $Q(0, \psi_y(Y_i))$ terms. In the past works [16, 20, 17, 7], we have already seen many examples and we will demonstrate it by more examples later.

4.5 Algebraic kernel method

Here we do not attempt to cancel the kernel by substitution, and instead proceed by factorization (using the third insight). Divide the RHS of the functional equation (3.5.3) by the kernel. Now we introduce an important theorem which appears through all the following sections.

Theorem 6. *[20] Consider the irreducible quadratic polynomial function of x, y with the form*

$$P(x, y) = ta(x)y^2 + (tb_1(x) + b_2(x))y + c(x)t. \tag{4.5.1}$$

$1/P(x, y)$ is a formal power series in t with polynomial coefficients in $x, \frac{1}{x}, y, \frac{1}{y}$, and

$$\frac{1}{P(x, y)} = \frac{1}{ta(x)y(Y_0 - Y_1)} \left(\sum_{i \geq 0} \left(\frac{Y_0}{y}\right)^i + \sum_{i \geq 1} \left(\frac{y}{Y_1}\right)^i \right) \tag{4.5.2}$$

[1] That is it is a Laurent series of t and the minimal degree of t is -1.

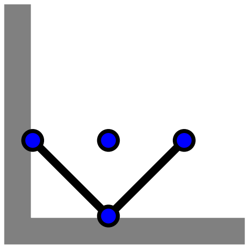


Figure 4.1: The Newton polygon of $ta(x)y^2 + (tb_1(x) + b_2(x))y + c(x)t$, in y, t. A point (i, j) in the picture refers to a $y^i t^j$ term in the polynomial. When we calculate a series root of a polynomial equation, we suppose $y = t^k(f_0 + f(t))$, where $f(t)$ does not have constant terms. Substitute this into the polynomial equation and consider the equality of the least degree of t. We may find the valuations in t of the roots by considering the slopes of the lines on the lower side of the convex hull. In this case one is 1 and the other is -1.

where Y_0 and Y_1 are two roots of $P(x, y)$. Y_0 has valuation[2] *1 in t and Y_1 has valuation -1 in t. Moreover,*

$$[y^j]\frac{y}{P(x,y)} = \begin{cases} \frac{Y_0(x)^{-j}}{ta(x)(Y_0-Y_1)} & \text{if } j \leq 0 \\ \frac{Y_1(x)^{-j}}{ta(x)(Y_0-Y_1)} & \text{if } j \geq 0. \end{cases} \tag{4.5.3}$$

which further implies $[y^j]\frac{1}{P(x,y)}$ is algebraic.

Proof. Factoring $K(x, y)$, we have

$$1/P(x,y) = \frac{1}{ta(x)(y - Y_0)(y - Y_1)} = \frac{1}{ta(x)(Y_0 - Y_1)}\left(\frac{1}{y - Y_0} - \frac{1}{y - Y_1}\right) \tag{4.5.4}$$

The valuation of two roots can be easily derived by the Newton polygon method [3], see Figure 4.1. We can expand $\frac{1}{y-Y_0}$ as $\frac{1}{y}\sum_{i\geq 0}\left(\frac{Y_0}{y}\right)^i$ and $\frac{1}{y-Y_1}$ as $-\frac{1}{Y_1}\sum_{i\geq 0}\left(\frac{y}{Y_1}\right)^i$. Since $Y_0(x)$ has valuation 1 in t and $Y_1(x)$ has valuation -1 in t, these two sums are both formal series of t. By some simple calculations, we get (4.5.2).

The expression (4.5.3) follows immediately. ☐

Remark 1. *Theorem 6 is an example of the application of the Newton polygon method. In lattice walk problems, we try to find solutions of $Q(x, 0), Q(0, 0)$ which are formal series of t. We always expand terms as series of t but not $1/t$. We may extend the result of Theorem 6 to functions of general case $a(x, t)y^2 + b(x, t)y + c(x, t)$. The corresponding result depends on the degree of t in a, b, c.*

Now let us return to the functional equation (3.5.3). Notice that $xyK(x, y)$ is in the form (4.5.1). We can write $\frac{1}{K(x,y)}$ as

$$\frac{1}{K(x,y)} = \frac{1}{t(Y_0 - Y_1)}\left(\sum_{i\geq 0}\left(\frac{Y_0(x)}{y}\right)^i + \sum_{i\geq 0}\left(\frac{y}{Y_1(x)}\right)^i - 1\right), \tag{4.5.5}$$

a sum of a power series of y, t and a series of $\frac{1}{y}, t$, both with coefficients in $\mathbb{R}((x))$. Dividing the RHS of (3.5.3) by the kernel and taking the $[y^0]$ part of the equation, we easily find that

[2] A formal series has valuation n in x means it is a Laurent series in x with smallest exponent n

the LHS is $[y^0]Q(x, y) = Q(x, 0)$. Notice that we have $Q(0, y)$ on the RHS, so that again by Lemma 5, we get an equation with unknown functions $Q(x, 0)$ and $Q(0, Y_0)$ in the RHS. The idea of the algebraic kernel method is, if we can find a linear combination of all $2n$ transformed functional equations, such that the RHS does not contain boundary terms $Q(0, g(y))$[3], we can take the $[y^0]$ part of this linear combination without introducing a new unknown function of y. The term in the LHS is a sum of polynomials multiplying $g(Q(x, y))$, and taking the $[y^0]$ part will give boundary functions of x. Combining the LHS and RHS, we get an equation of x, t. Then if we can separate the $[x^>]$ and $[x^<]$ parts of it, the problem again returns to the two variable original kernel method form.

We will demonstrate how to solve lattice walk problems by these two kernel methods by examples in Chapters 5 and 8.

[3] g refers to a transformation in the symmetry group.

Chapter 5

Quarter-plane walks without interactions

Now we come back to the general process of solving quarter-plane lattice walks without interactions. Recall that among all 79 different quarter-plane lattice walk models, 23 of them are associated with finite groups. We demonstrate how all 23 models were solved by the kernel method. The solutions in this chapter are from Bousquet-Mélou's work [20, 16, 17].

5.1 General approach of the obstinate kernel method for models associated with group D_2

The simplest cases are the models associated with D_2 groups. For all 16 D_2 models, we have $x \to \frac{1}{x}$ symmetry. The functional equation and kernel are defined as in Section 3.5. Suppose the kernel has two roots Y_0, Y_1,

$$\begin{aligned} Y_0 &= \frac{1 - tA_0(x) - \sqrt{\Delta(x)}}{2tA_1(x)} \\ Y_1 &= \frac{1 - tA_0(x) + \sqrt{\Delta(x)}}{2tA_1(x)} \end{aligned} \tag{5.1.1}$$

where

$$\Delta = (1 - tA_0(x))^2 - 4t^2 A_{-1}(x) A_1(x) \tag{5.1.2}$$

By calculation, substituting Y_0 into the equation of $\delta_{(x,y)}$ and $\delta_{(\frac{1}{x},y)}$ [1], we have

$$0 = 1 - \frac{tA_{-1}(x)}{Y_0} Q(x,0) - \frac{tB_{-1}(Y_0)}{x} Q(0,Y_0) + \frac{tC_{-1,-1}}{xY_0} Q(0,0) \tag{5.1.3}$$

$$0 = 1 - \frac{tA_{-1}\left(\frac{1}{x}\right)}{Y_0} Q\left(\frac{1}{x},0\right) - tB_{-1}(Y_0) x Q(0,Y_0) + \frac{tC_{-1,-1}x}{Y_0} Q(0,0) \tag{5.1.4}$$

Eliminating $Q(0,Y_0)$, we obtain an equation with only $Q(x,0)$ and $Q\left(\frac{1}{x},0\right)$.

$$Y_0\left(x - \frac{1}{x}\right) - \left(tA_{-1}(x) x Q(x,0) - \frac{tA_{-1}\left(\frac{1}{x}\right)}{x} Q\left(\frac{1}{x},0\right)\right) = 0 \tag{5.1.5}$$

[1] See Section 2.3 for the definition

Notice that the coefficients of $Q(x,0)$ and $Q\left(\frac{1}{x},0\right)$ are both polynomials of x. This implies that we can separate the $[x^>]$, $[x^0]$, $[x^<]$ terms of (5.1.5) without introducing an infinite number of unknowns, since $tA_{-1}(x)xQ(x,0)$ only has $[x^\geq]$ terms and $\frac{tA_{-1}(\frac{1}{x})}{x}Q\left(\frac{1}{x},0\right)$ only has $[x^\leq]$ terms. With both $[x^0]$ terms $tC_{-1,-1}Q(0,0)$, the final solution is written as

$$Q(x,0) = \frac{1}{tA_{-1}(x)x}\left([x^>]\left(\left(x-\frac{1}{x}\right)Y_0\right) + tC_{-1,-1}Q(0,0)\right) \tag{5.1.6}$$

When $C_{-1,-1}$ is zero, $A_{-1}(x)$ does not contain $\frac{1}{x}$ terms. Notice that the numerator is an $[x^>]$ series, so the right hand side of $Q(x,0)$ is analytic at $x=0$. We can solve $Q(0,0)$ by directly substituting $x=0$ into $Q(x,0)$. If $C_{-1,-1}$ is not zero, we may substitute the roots of $tA_{-1}(x)x$ into the equation and use the original kernel method to solve $Q(0,0)$.

For $Q(0,y)$, we have

$$0 = 1 - \frac{tA_{-1}(X_0)}{y}Q(X_0,0) - \frac{tB_{-1}(y)}{X_0}Q(0,y) + \frac{tC_{-1,-1}}{X_0 y}Q(0,0) \tag{5.1.7}$$

And $Q(x,y)$ can be calculated from $Q(x,0)$ and $Q(0,y)$ directly. Thus all models in Table 3.1 can be solved by the obstinate kernel method.

5.2 Approach of the obstinate kernel method for models associated with group D_3

Models associated with D_3 group are a bit complex. As far as we are aware, the obstinate kernel has not been used to solve reverse Kreweras walks or double Kreweras walks. Among all five D_3 models, only three of the D_3 models have been solved by the obstinate kernel method.

5.2.1 Model 17 and Model 18

For the two D-finite cases, **Models 17** and **18**, the idea is exactly the same as that of the D_2 cases. The symmetry group of these two models is

$$\left\{(x,y), \left(\frac{y}{x},y\right), \left(\frac{1}{y},\frac{x}{y}\right), \left(\frac{1}{y},\frac{1}{x}\right), \left(\frac{y}{x},\frac{1}{x}\right), \left(x,\frac{x}{y}\right)\right\} \tag{5.2.1}$$

The convergence condition shows (x,Y_0), $\left(\frac{Y_0}{x},Y_0\right)$ and $\left(\frac{Y_0}{x},\frac{1}{x}\right)$ are three valid pairs. That is $Q(x,Y_0)$, $Q\left(\frac{Y_0}{x},Y_0\right)$ and $Q\left(\frac{Y_0}{x},\frac{1}{x}\right)$ are all convergent when $t\to 0$. Thus we have three equations

$$0 = 1 - \frac{tA_{-1}(x)}{Y_0}Q(x,0) - \frac{tB_{-1}(Y_0)}{x}Q(0,Y_0) \tag{5.2.2}$$

$$0 = 1 - \frac{tA_{-1}\left(\frac{Y_0}{x}\right)}{Y_0}Q\left(\frac{Y_0}{x},0\right) - \frac{tB_{-1}(Y_0)x}{Y_0}Q(0,Y_0) \tag{5.2.3}$$

$$0 = 1 - tA_{-1}\left(\frac{Y_0}{x}\right)xQ\left(\frac{Y_0}{x},0\right) - \frac{tB_{-1}\left(\frac{1}{x}\right)x}{Y_0}Q\left(0,\frac{1}{x}\right) \tag{5.2.4}$$

Then (5.2.2) $\times\, xY_0$ $-$ (5.2.3) $\times\, \frac{Y_0^2}{x}$ $+$ (5.2.4) $\times\, \frac{Y_0}{x^2}$ gives

$$0 = xY_0 - \frac{Y_0^2}{x} + \frac{Y_0}{x^2} - tA_{-1}(x)xQ(x,0) - \frac{tB_{-1}\left(\frac{1}{x}\right)}{x}Q\left(0,\frac{1}{x}\right) \tag{5.2.5}$$

The equation (5.2.5) has only $Q(x,0)$ and $Q\left(0,\frac{1}{x}\right)$ as unknown functions of x. The coefficients are polynomials. We can separate the $[x^>]$, $[x^<]$ and $[x^0]$ parts of (5.2.5) as per the D_2 cases and solve the problem. Notice that in (5.2.5), $tA_{-1}(x)xQ(x,0)$ only has $[x^\geq]$ terms and $tB_{-1}(\frac{1}{x})\frac{1}{x}Q\left(0,\frac{1}{x}\right)$ only has $[x^\leq]$ terms and their $[x^0]$ terms are both 0. The only unknown function of t in the $[x^>]$ and $[x^<]$ equations is $Q(0,0)$. We can then use the original kernel method to solve everything as per the D_2 models.

5.2.2 Kreweras walks

The previous models still follow the general process of the obstinate kernel method, while Kreweras walks need some further tactics. The functional equation for Kreweras walks is written as

$$K(x,y)Q(x,y) = 1 - \frac{t}{y}Q(x,0) - \frac{t}{x}Q(0,y) \tag{5.2.6}$$

where the kernel is

$$K(x,y) = 1 - t\left(\frac{1}{x} + \frac{1}{y} + xy\right) \tag{5.2.7}$$

The roots are

$$\begin{aligned}
Y_0 &= \frac{-\sqrt{(-4t^2x + t^2/x^2 - 2t/x + 1} - t + x)}{2tx} \\
Y_1 &= \frac{\sqrt{(-4t^2x + t^2/x^2 - 2t/x + 1} - t + x)}{2tx} \\
Y_0 &= t + \frac{t^2}{x} + t^3\left(\frac{1}{x^2} + x\right) + t^4\left(\frac{1}{x^3} + 3\right) + t^5\left(\frac{1}{x^4} + 2x^2 + \frac{6}{x}\right) + \mathrm{O}(t^6) \\
Y_1 &= \frac{1}{tx} - \frac{1}{x^2} - t - \frac{t^2}{x} + t^3\left(-\frac{1}{x^2} - x\right) + t^4\left(-\frac{1}{x^3} - 3\right) + t^5\left(-\frac{1}{x^4} - 2x^2 - \frac{6}{x}\right) + \mathrm{O}(t^6)
\end{aligned} \tag{5.2.8}$$

and we denote $\Delta = (-4t^2x + \frac{t^2}{x^2} - \frac{2t}{x} + 1)$. We further have

$$\begin{aligned}
Y_0 + Y_1 &= -\frac{1}{x^2} + \frac{1}{tx} \\
Y_0Y_1 &= \frac{1}{x}
\end{aligned} \tag{5.2.9}$$

The symmetry group for Kreweras walks is

$$\left\{(x,y), \left(\frac{1}{xy}, y\right), \left(y, \frac{1}{xy}\right), (y,x), \left(\frac{1}{xy}, x\right), \left(x, \frac{1}{xy}\right)\right\} \tag{5.2.10}$$

Now (x, Y_0) and (Y_0, x) are clearly two valid pairs. But we cannot eliminate anything if we just substitute these two pairs. On the other hand $\left(x, \frac{1}{xY_0}\right)$ and $\left(\frac{1}{xY_0}, x\right)$ are clearly two invalid pairs. Now we turn to $\left(Y_0, \frac{1}{xY_0}\right)$ and $\left(\frac{1}{xY_0}, Y_0\right)$. Substituting $\left(\frac{1}{xY_0}, Y_0\right)$ into $Q(x,y)$ gives

$$\begin{aligned}
Q\left(\frac{1}{xY_0}, Y_0\right) &= \sum_{i,j,n} q_{i,j,n}\left(\frac{1}{xY_0}\right)^i Y_0^j t^n = \sum_{i,j,n} q_{i,j,n}\left(\frac{1}{x}\right)^i (txY_1)^{i-j}t^{n-(i-j)} \\
&= \sum_{i,j,n} q_{i,j,n}\left(\frac{1}{x}\right)^i (tY_0)^{j-i}t^{n-(j-i)} \\
&= \sum_{i>j,n} q_{i,j,n}\left(\frac{1}{x}\right)^i (txY_1)^{i-j}t^{n-(i-j)} + \sum_{i\leq j,n} q_{i,j,n}\left(\frac{1}{x}\right)^i (tY_0)^{j-i}t^{n-(j-i)}
\end{aligned} \tag{5.2.11}$$

The second equality holds by (5.2.9).

We have separated the sum into two parts. For $j \geq i$ in the sum, we take the expression $\sum_{i,j,n}(\frac{1}{x})^i(tY_0)^{j-i}t^{n-(j-i)}$. It is convergent when $t \to 0$ clearly. For $j < i$, we take $\sum_{i,j,n}(\frac{1}{x})^i(txY_1)^{i-j}t^{n-(i-j)}$. Notice that the series expansion of Y_1 is $\frac{1}{tx} + \dots$ in (5.2.8), thus $(txY_1)^{i-j}$ is a formal series of t. Further notice $n > (i-j)$, so that $\left(\frac{1}{x}\right)^i (txY_1)^{i-j}t^{n-(i-j)}$ is a formal series of t without constant terms of t. Then the summation is convergent when $t \to 0$. Thus, $\left(Y_0, \frac{1}{xY_0}\right)$ and $\left(\frac{1}{xY_0}, Y_0\right)$ are two valid pairs. They can also be written as (Y_0, Y_1) and (Y_1, Y_0). Actually, we only need to use the pairs (x, Y_0) and (Y_1, Y_0). Substituting these two pairs into the functional equation gives,

$$xtQ(x,0) + Y_0tQ(Y_0,0) = xY_0 \tag{5.2.12}$$

$$Y_0tQ(Y_0,0) + Y_1tQ(Y_1,0) = Y_0Y_1 = \frac{1}{x} \tag{5.2.13}$$

Then (5.2.13) $- 2 \times$ (5.2.12) gives

$$Y_1tQ(Y_1,0) - Y_0tQ(Y_0,0) = 2xtQ(x,0) + \frac{1}{x} - 2xY_0 = 2xtQ(x,0) + \frac{2}{x} + x(Y_1 - Y_0) - \frac{1}{t} \tag{5.2.14}$$

The second equality holds because of (5.2.9). Now divide both sides by $Y_1 - Y_0$ and notice that $Y_1 - Y_0 = \frac{\sqrt{\Delta}}{tx}$. Thus (5.2.14) becomes

$$\frac{Y_0tQ(Y_0,0) - Y_1tQ(Y_1,0)}{Y_0 - Y_1} - x = \frac{2x^2t^2Q(x,0) + 2t - x}{\sqrt{\Delta}} \tag{5.2.15}$$

The LHS of (5.2.15) is a symmetric function of Y_0 and Y_1, so it can be expressed in terms of polynomials of $Y_0 + Y_1$ and Y_0Y_1. By (5.2.9), these are both functions of $\frac{1}{x}$. Thus, the LHS is a series of $\frac{1}{x}$. For the RHS, we use the *canonical factorization* to factor $\sqrt{\Delta}$.

Lemma 7 (The canonical factorization of Δ [20])**.** *We have*

$$\Delta(x) = (1 - tA_0(x))^2 - 4t^2A_{-1}(x)A_1(x). \tag{5.2.16}$$

If $\Delta(x)$ has valuation[2] *$-\delta$, and degree d in x, then $\Delta(x) = 0$ has $\delta + d$ roots. Exactly δ of them (say $X_1, \dots, X_\delta$) are finite (actually vanish) when $t = 0$ and the remaining d roots $(X_{\delta+1}, \dots, X_{\delta+d})$ have negative valuation in t and thus diverge when $t = 0$.*

Then Δ can be factored as:

$$\Delta(x) = \Delta_0\Delta_-\left(\frac{1}{x}\right)\Delta_+(x) \tag{5.2.17}$$

with

$$\Delta_- \equiv \Delta_-\left(\frac{1}{x}, t\right) = \prod_{i=1}^{\delta}(1 - X_i/x) \tag{5.2.18}$$

$$\Delta_+ \equiv \Delta_+\left(\frac{1}{x}, t\right) = \prod_{i=1+\delta}^{\delta+d}(1 - x/X_i) \tag{5.2.19}$$

$$\Delta_0 \equiv \Delta_0(t) = (-1)^\delta \frac{[(\frac{1}{x})^\delta]\Delta(x)}{\prod_{i=1}^{\delta} X_i} = (-1)^d[x^d]\Delta(x) \prod_{i=\delta+1}^{\delta+d} X_i. \tag{5.2.20}$$

[2] A series has valuation $-\delta$ means the minimal degree of x in the series is $-\delta$.

Proof. We factor Δ by its roots. For roots that are finite at $t=0$, we factor them as $x(1-X_i/x)$. For roots that diverge around $t=0$, we factor them as $X_i(1-x/X_i)$. This leads to (5.2.17). □

We can check that $\Delta_0, \Delta_+, \Delta_-$ are formal power series in t with constant terms 1. The canonical factorization provides us a way to separate the $[x^>]$ and $[x^<]$ parts.

Then (5.2.15) becomes

$$\sqrt{\Delta_-}\left(\frac{Y_0tQ(Y_0,0)-Y_1tQ(Y_1,0)}{Y_0-Y_1}-x\right)=\frac{2x^2t^2Q(x,0)+2t-x}{\sqrt{\Delta_0\Delta_+}} \tag{5.2.21}$$

The LHS is a formal series of $\frac{1}{x}$ with polynomial $[x^>]$ part and the RHS is a formal series of x with polynomial $[x^<]$ part. We have successfully separated the $[x^>]$ series and $[x^<]$ series of (5.2.14). Though the $[x^<]$ of LHS contains some unknown series, the $[x^>]$ of (5.2.21) reads

$$-x=\frac{2x^2t^2Q(x,0)+2t-x}{\sqrt{\Delta_0\Delta_+}}-\frac{2t}{\sqrt{\Delta_0}}. \tag{5.2.22}$$

Then we solve $Q(x,0)=\frac{\sqrt{\Delta_0\Delta_+}\left(-x+\frac{2t}{\sqrt{\Delta_0}}\right)-2t+x}{2x^2t^2}$.

Reverse Kreweras and double Kreweras walks have the same symmetry group as Kreweras walks. The general process does not work because of the symmetry group. One may follow the solution of Kreweras walks to solve reverse Kreweras walks or double Kreweras walks. However, this idea is not always effective. For example, for reverse Kreweras walks, the kernel reads

$$K(x,y)=1-t\left(\frac{1}{xy}+x+y\right) \tag{5.2.23}$$

and the roots satisfy the relations

$$\begin{aligned}Y_0+Y_1&=-\left(x-\frac{1}{t}\right)\\ Y_0Y_1&=\frac{1}{x}.\end{aligned} \tag{5.2.24}$$

A symmetric function of Y_0 and Y_1 cannot be claimed to be a function of x or $\frac{1}{x}$, and so the obstinate kernel method stops here.

5.3 Approach of the obstinate kernel method for models associated with group D_4

There are two models associated with group D_4: **Models 22** and **23**.

5.3.1 Model 22

The allowed steps of **Model 22** are $\{\leftarrow, \nwarrow, \rightarrow, \searrow\}$. The functional equation reads

$$K(x,y)Q(x,y)=1-t\left(\frac{y}{x}+\frac{1}{x}\right)Q(0,y)-\frac{txQ(x,0)}{y} \tag{5.3.1}$$

where the kernel is

$$K(x,y)=1-t\left(\frac{x}{y}+\frac{y}{x}+x+\frac{1}{x}\right) \tag{5.3.2}$$

The roots are

$$
\begin{aligned}
Y_0 &= -\frac{\sqrt{\left(-tx^2-t+x\right)^2-4t^2x^2}+tx^2+t-x}{2t} \\
Y_1 &= \frac{\sqrt{\left(-tx^2-t+x\right)^2-4t^2x^2}-tx^2-t+x}{2t}
\end{aligned} \tag{5.3.3}
$$

The symmetry group is

$$
\left\{(x,y),\left(\frac{y}{x},y\right),\left(\frac{x}{y},\frac{x^2}{y}\right),\left(\frac{1}{x},\frac{y}{x^2}\right),\left(\frac{1}{x},\frac{1}{y}\right),\left(\frac{x}{y},\frac{1}{y}\right),\left(\frac{y}{x},\frac{y}{x^2}\right),\left(x,\frac{x^2}{y}\right)\right\} \tag{5.3.4}
$$

generated by

$$
\phi:(x,y)\to\left(\frac{y}{x},y\right) \quad \text{and} \quad \psi:(x,y)\to\left(x,\frac{x^2}{y}\right). \tag{5.3.5}
$$

The valid pairs are $(x,Y_0),(\frac{Y_0}{x},Y_0),(\frac{1}{x},\frac{Y_0}{x^2}),(\frac{Y_0}{x},\frac{Y_0}{x^2})$. Substituting these pairs in, we get four equations with five unknowns:

$$
0 = -\frac{txQ(x,0)}{Y_0}-\frac{t\left(Y_0+1\right)Q\left(0,Y_0\right)}{x}+1 \tag{5.3.6}
$$

$$
0 = -\frac{tQ\left(\frac{Y_0}{x},0\right)}{x}-\frac{Q\left(0,Y_0\right)tx\left(Y_0+1\right)}{Y_0}+1 \tag{5.3.7}
$$

$$
0 = -\frac{t\left(x^2+Y_0\right)Q\left(0,\frac{Y_0}{x^2}\right)}{xY_0}-txQ\left(\frac{Y_0}{x},0\right)+1 \tag{5.3.8}
$$

$$
0 = -\frac{\left(tx^2+tY_0\right)Q\left(0,\frac{Y_0}{x^2}\right)}{x}-\frac{txQ\left(\frac{1}{x},0\right)}{Y_0}+1 \tag{5.3.9}
$$

We can eliminate all unknowns except $Q(x,0)$ and $Q\left(0,\frac{1}{x}\right)$. In particular, $xY_0\times$(5.3.6) $-\frac{Y_0^2}{x}\times$ (5.3.7) $+\frac{Y_0^2}{x^3}\times$ (5.3.8) $-\frac{Y_0}{x^3}\times$ (5.3.9) gives

$$
-tx^2Q(x,0)+\frac{tQ\left(\frac{1}{x},0\right)}{x^2}+\frac{Y_0^2}{x^3}-\frac{Y_0}{x^3}+xY_0-\frac{Y_0^2}{x}=0 \tag{5.3.10}
$$

Notice that $tx^2Q(x,0)$ only contains $[x^>]$ terms and $\frac{tQ(\frac{1}{x},0)}{x^2}$ only contains $[x^<]$ terms. Separating the $[x^>]$, $[x^0]$ and $[x^<]$ terms will directly give the solution to $Q(x,0)$. Then $Q(0,0)$ can be calculated by taking the limit $x\to 0$ and $Q(0,y)$ can be calculated similarly as in (5.1.7). Finally $Q(x,y)$ follows by the functional equation.

5.3.2 Gessel's walks

Model 23 is another algebraic model called Gessel's walks. The approach for Gessel's walks by the kernel method was proposed in [17] in 2016, much later that the solutions of other models. And this is the only model that was only solved by the obstinate kernel method but not the algebraic kernel method. The allowed steps of Gessel's walks are $\{\leftarrow,\swarrow,\rightarrow,\nearrow\}$. The functional equation reads

$$
K(x,y)Q(x,y)=1-t\left(\frac{1}{xy}+\frac{1}{x}\right)Q(0,y)-\frac{t}{xy}Q(x,0)-\frac{t}{xy}Q(0,0) \tag{5.3.11}
$$

where the kernel is

$$K(x,y) = 1 - t\left(x + xy + \frac{1}{x} + \frac{1}{xy}\right) \tag{5.3.12}$$

and its roots are

$$\begin{aligned}
Y_0 &= \frac{1 - t(x+\frac{1}{x}) - \sqrt{(1-t(x+\frac{1}{x}))^2 - 4t^2}}{2tx} \\
Y_1 &= \frac{1 - t(x+\frac{1}{x}) + \sqrt{(1-t(x+\frac{1}{x}))^2 - 4t^2}}{2tx} \\
Y_0 &= \frac{t}{x} + t^2\left(\frac{1}{x^2}+1\right) + t^3\left(\frac{1}{x^3} + x + \frac{3}{x}\right) + t^4\left(\frac{1}{x^4} + x^2 + \frac{6}{x^2} + 6\right) + \mathrm{O}(t^5) \\
Y_1 &= \frac{1}{tx} - \left(\frac{1}{x^2}+1\right) - \frac{t}{x} - t^2\left(\frac{1}{x^2}+1\right) - t^3\left(\frac{1}{x^3} + x + \frac{3}{x}\right) - t^4\left(\frac{1}{x^4} + x^2 + \frac{6}{x^2} + 6\right) + \mathrm{O}(t^5)
\end{aligned} \tag{5.3.13}$$

The roots satisfy

$$\begin{aligned}
Y_0 + Y_1 &= -1 + \frac{1}{xt} - \frac{1}{x^2} \\
Y_0 Y_1 &= \frac{1}{x^2}
\end{aligned} \tag{5.3.14}$$

The symmetry group is

$$\left\{(x,y), \left(\frac{1}{xy}, y\right), \left(\frac{1}{xy}, x^2 y\right), \left(\frac{1}{x}, x^2 y\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(xy, \frac{1}{y}\right), \left(xy, \frac{1}{x^2 y}\right), \left(x, \frac{1}{x^2 y}\right)\right\} \tag{5.3.15}$$

Obviously, (x, Y_0) and $(\frac{1}{x}, x^2 Y_0)$ are valid pairs. For the others, we may write,

$$Q(xY_1, Y_0) = \textstyle\sum_{i,j,n} (xY_1)^i Y_0^j t^n = \sum_{i,j,n} x^i (Y_1 Y_0)^j (tY_1)^{i-j} t^{n-(i-j)} \tag{5.3.16}$$

$$Q(xY_0, Y_1) = \textstyle\sum_{i,j,n} (xY_1)^i Y_0^j t^n = \sum_{i,j,n} x^i (Y_1 Y_0)^i (tY_1)^{j-i} t^{n-(j-i)} \tag{5.3.17}$$

By the allowed steps, we find $n > (j-i)$ and $n \geq (i-j)$. Let us first consider $Q(xY_1, Y_0)$. When $i < j$ the summation converges clearly. When $i > j$, $(Y_0 Y_1)^j$ is a function of $\frac{1}{x}$, while $(tY_1)^{i-j}$ can be written as a formal series of t with constant terms. Thus each term in the summation is a formal series of t. But notice that when $n = i - j$, each term in the sum has a constant term different from 0 and we have an infinite number of such terms. Their sum tends to infinity. Thus $Q(xY_1, Y_0)$ is not valid.

The same argument can be applied to $Q(xY_0, Y_1)$. But notice $n > j - i$, so the equality does not hold for $Q(xY_0, Y_1)$. Thus each term in the summation $Q(xY_0, Y_1)$ except $n = i = j = 0$ is a formal series of t without constant terms. Their sum converges when $t \to 0$. Thus, (xY_0, Y_1) is valid. We further find $(xY_0, x^2 Y_1)$ is also valid.

For simplicity, let

$$R(x) = t(Q(x,0) - Q(0,0)), \qquad S(y) = t(1+y)Q(0,y) \tag{5.3.18}$$

Substituting all four valid pairs into the functional equation, we get four equations

$$R(x) + S(Y_0) = xY_0 \tag{5.3.19}$$

$$R(xY_0) + S(Y_1) = \frac{1}{x} \tag{5.3.20}$$

$$R\left(\frac{1}{x}\right) + S(x^2 Y_0) = xY_0 \tag{5.3.21}$$

$$R(xY_0) + S(x^2 Y_1) = x \tag{5.3.22}$$

(5.3.19) + (5.3.20) − (5.3.21) − (5.3.22) gives,

$$R(x) - S(x^2 Y_0) - S(x^2 Y_1) + x = R\left(\frac{1}{x}\right) - S(Y_0) - S(Y_1) + \frac{1}{x} \tag{5.3.23}$$

The LHS is a symmetric function of x^2Y_0 and x^2Y_1. The RHS is a symmetric function Y_0 and Y_1. By (5.3.14), we find the LHS of (5.3.23) only has $[x^{\geq}]$ terms and the RHS only has $[x^{\leq}]$ terms. This implies both sides are independent of x and equal to their constant terms. By (5.3.18), we have

$$S(Y_0) + S(Y_1) = R\left(\frac{1}{x}\right) + \frac{1}{x} + S(0) \tag{5.3.24}$$

Combine (5.3.24) with (5.3.19) and (5.3.14), we have

$$S(Y_1) - xY_1 = R(x) + R\left(\frac{1}{x}\right) + \frac{2}{x} - \frac{1}{t} + x + S(0) \tag{5.3.25}$$

Until now we have used similar techniques as that in Kreweras walks to get (5.3.25). The next issue is that (5.3.25) still contains $S(Y_1)$. To get rid of it, we need some further steps. Now multiplying (5.3.25) with (5.3.19), we have

$$(S(Y_0) - xY_0)(S(Y_1) - xY_1) = -R(x)\left(R(x) + R\left(\frac{1}{x}\right) + \frac{2}{x} - \frac{1}{t} + x + S(0)\right) \tag{5.3.26}$$

The left hand side of (5.3.26) is again a symmetric function of Y_0, Y_1. So the $[x^{\geq}]$ part only contains finitely many terms. We have

$$\begin{aligned}
[x^{\geq}](S(Y_0)S(Y_1)) &= [x^{\geq}](t^2(1 + Y_0 + Y_1 + Y_0Y_1)Q(0, Y_0)Q(0, Y_1)) \\
&= [x^{\geq}]\left(\frac{t}{x}Q(0, Y_0)Q(0, Y_1)\right) \\
&= 0
\end{aligned} \tag{5.3.27}$$

and $x^2Y_0Y_1 = 1$ by (5.3.14). And

$$[x^{\geq}](-x(Y_0S(Y_1) + Y_1S(Y_0))) = [x^{\geq}](-x(Y_1 + Y_0)S(0)) = \left(x - \frac{1}{t}\right)S(0) \tag{5.3.28}$$

Now taking the $[x^{\geq}]$ part of (5.3.26), we have

$$1 + \left(x - \frac{1}{t}\right)S(0) = -R(x)^2 - [x^{\geq}]R(x)R\left(\frac{1}{x}\right) - \left(\frac{2}{x} - \frac{1}{t} + x + S(0)\right)R(x) \tag{5.3.29}$$

The $[x^0]$ part of (5.3.29) is

$$1 - \frac{S(0)}{t} + [x^0]R(x)R\left(\frac{1}{x}\right) + 2R'(0) = 0 \tag{5.3.30}$$

Next, we want to get rid of $[x^{\geq}]R(x)R\left(\frac{1}{x}\right)$. Since $R(x)R\left(\frac{1}{x}\right)$ is symmetric in x and $\frac{1}{x}$, we let $x \to \frac{1}{x}$ in (5.3.29) and get,

$$1 + \left(\frac{1}{x} - \frac{1}{t}\right)S(0) = -R\left(\frac{1}{x}\right)^2 - [x^{\leq}]R(x)R\left(\frac{1}{x}\right) - \left(2x - \frac{1}{t} + \frac{1}{x} + S(0)\right)R\left(\frac{1}{x}\right) \tag{5.3.31}$$

We may reconstruct $R(x)R\left(\frac{1}{x}\right)$ as

$$\begin{aligned}R(x)R\left(\frac{1}{x}\right) &= [x^{\geq}]R(x)R\left(\frac{1}{x}\right) + [x^{\leq}]R(x)R\left(\frac{1}{x}\right) - [x^0]R(x)R\left(\frac{1}{x}\right) \\ &= -R(x)^2 - \left(\frac{2}{x} - \frac{1}{t} + x + S(0)\right)R(x) - R\left(\frac{1}{x}\right)^2 \\ &\quad - \left(2x - \frac{1}{t} + \frac{1}{x} + S(0)\right)R\left(\frac{1}{x}\right) - 1 - \left(x + \frac{1}{x} - \frac{1}{t}\right)S(0) + 2R'(0)\end{aligned} \tag{5.3.32}$$

We finally get an equation with only $R(x)$ and $R\left(\frac{1}{x}\right)$,

$$\begin{aligned}R(x)^2 + R(x)R\left(\frac{1}{x}\right) + R\left(\frac{1}{x}\right)^2 + \left(\frac{2}{x} - \frac{1}{t} + x + S(0)\right)R(x) + \left(2x - \frac{1}{t} + \frac{1}{x} + S(0)\right)R\left(\frac{1}{x}\right) \\ = 2R'(0) - \left(x + \frac{1}{x} - \frac{1}{t}\right)S(0) - 1\end{aligned} \tag{5.3.33}$$

There is still a hybrid term $R(x)R\left(\frac{1}{x}\right)$ in (5.3.33). To decompose it, we notice that $(x-y)(x^2+xy+y^2) = x^3 - y^3$. More precisely, we multiply (5.3.33) by $R(x) - R\left(\frac{1}{x}\right) + \frac{1}{x} - x$, to get a decoupled equation

$$P(x) = P\left(\frac{1}{x}\right) \tag{5.3.34}$$

where

$$\begin{aligned}P(x) = R(x)^3 + \left(S(0) + \frac{3}{x} - \frac{1}{t}\right)R(x)^2 - x^2S(0) + x\left(2R'(0) + \frac{S(0)}{t} - 1\right) \\ + \left(\frac{2}{x^2} - \frac{1}{xt} + \frac{x}{t} - x^2 - 2R'(0) + \left(\frac{2}{x} - \frac{1}{t}\right)S(0)\right)R(x)\end{aligned} \tag{5.3.35}$$

Now we can separate the $[x^{\geq}]$ and $[x^{\leq}]$ terms of (5.3.34). Notice that (5.3.34) is symmetric by $x \to \frac{1}{x}$. The $[x^{\geq}]$ equation and $[x^{\leq}]$ equation give the same equation. By calculation, we find

$$\begin{aligned}[x^<]P(x) &= \frac{2}{x}R'(0) \\ [x^0]P(x) &= R'(0)\left(2S(0) - \frac{1}{t}\right) + R''(0) \\ [x^>]P\left(\frac{1}{x}\right) &= 2xR'(0)\end{aligned} \tag{5.3.36}$$

Thus, we finally have,

$$\begin{aligned}R(x)^3 + \left(S(0) + \frac{3}{x} - \frac{1}{t}\right)R(x)^2 \\ + \left(\frac{2}{x^2} - \frac{1}{xt} + \frac{x}{t} - x^2 - 2R'(0) + \left(\frac{2}{x} - \frac{1}{t}\right)S(0)\right)R(x) \\ = R''(0) + R'(0)\left(2S(0) + \frac{2}{x} - \frac{1}{t}\right) + x\left(x - \frac{1}{t}\right)S(0) + x\end{aligned} \tag{5.3.37}$$

where $S(0) = tQ(0,0)$, $R'(0) = tQ_{1,0}$ and $R''(0) = 2tQ_{2,0}$.

The equation (5.3.37) is not in a kernel form. We can solve the equation and write it as $R(x) = (\bullet)^{\frac{1}{3}} + (\bullet)$. Then we can solve it by the kernel method. In [17], the authors give a general form of the kernel method, which is called quadratic method. We will discuss this method later in Chapter 12. Here we show how to use it.

The general problem is to solve an equation,

$$P(Q(u), Q_1, Q_2 \dots Q_k, t, u) = 0 \tag{5.3.38}$$

where $Q(u)$ is a formal series of t with coefficients in $\mathbb{R}(u)$ and the Q_i are formal series of t. The general process is as follows.

1. Differentiate the equation with respect to u:

$$Q'(u)\frac{\partial P}{\partial x_0}(Q(u), Q_1, Q_2 \dots Q_k, t, u) + \frac{\partial P}{\partial x_{k+2}}(Q(u), Q_1, Q_2 \dots Q_k, t, u) = 0 \tag{5.3.39}$$

2. Find U_i which satisfy

$$\frac{\partial P}{\partial x_0}(Q(u), Q_1, Q_2 \dots Q_k, t, u) = 0 \tag{5.3.40}$$

   Then

$$\frac{\partial P}{\partial x_{k+2}}(Q(u), Q_1, Q_2 \dots Q_k, t, u) = 0 \tag{5.3.41}$$

   automatically holds.

3. If we find k distinct U_i such that (5.3.40) holds, then we get $3k$ polynomial equations

$$\begin{aligned} P(Q(U_i), Q_1, Q_2 \dots Q_k, t, U_i) &= 0 \\ \frac{\partial P}{\partial x_0}(Q(U_i), Q_1, Q_2 \dots Q_k, t, U_i) &= 0 \\ \frac{\partial P}{\partial x_{k+2}}(Q(U_i), Q_1, Q_2 \dots Q_k, t, U_i) &= 0 \end{aligned} \tag{5.3.42}$$

If these $3k$ equations are independent, we can solve Q_i and $Q(u)$ follows.

Now apply the process to (5.3.37). Our version of (5.3.40) reads

$$\begin{aligned} 3R(X)^2 + 2\left(S(0) + \frac{3}{X} - \frac{1}{t}\right) R(X) \\ + \frac{2}{X^2} - \frac{1}{tX} + \frac{X}{t} - X^2 - 2R'(0) + \left(\frac{2}{X} - \frac{1}{t}\right) S(0) = 0 \end{aligned} \tag{5.3.43}$$

Multiply (5.3.43) by tX, and we can write it as

$$X(1-X)(1+X) = tPol_1(Q(X,0), R'(0), R''(0), Q(0,0), t, X) \tag{5.3.44}$$

This equation shows that there are exactly three roots, denoted as X_0, X_1 and X_2. The series expansions of them have constant terms 0, 1 and -1. By further calculation, we have

$$tx^2\left(\frac{\partial Pol}{\partial x_0} + x^2\frac{\partial Pol}{\partial x}\right) = (1-x)(1+x)(2tx^2 + 2t - x)(xR(x) + xS(0) + 1) = 0 \tag{5.3.45}$$

Thus we have $X_1 = 1$, $X_2 = -1$ and X_0 is one of the roots of $2tx^2 + 2t - x = 0$. The fourth factor cannot vanish when x is a formal series of t. We find three roots. Next, we calculate the discriminant $Dis(x)$ of $Pol(R(x), S(0), R'(0), R''(0), t, x)$. It is obtained by eliminating $R(x)$ between this equation and its derivative with respect to x_0. Since (5.3.43) and its derivative are both 0 when $x = X_i$, $Dis(X_i) = 0$. Then we can find three polynomials of unknowns $Q(0,0), Q_{1,0}$ and $Q_{2,0}$. We solve this equation set and finally solve all generating functions of **Model 23**.

5.4 Approach of the algebraic kernel method for D-finite walks

The tactics of the obstinate kernel method differ from model to model. For the algebraic kernel method, everything works in a similar way. The basic idea of the algebraic kernel method was also introduced in [20]. Among all 23 models, all except Gessel's walks were solved by the algebraic kernel method. The general strategy is to take the full orbit sum of the symmetry group. The full orbit sum (**Proposition 5** in [20]) is,

$$\sum_{g\in G}\operatorname{sgn}(g)g(xyQ(x,y)) = \frac{1}{K(x,y)}\sum_{g\in G}\operatorname{sgn}(g)g(xy) \tag{5.4.1}$$

where the sum is over the symmetry group of the kernel, and $\operatorname{sgn}(g) = \pm 1$ according to whether g can be written as a composition of an even or odd number of involutions ϕ and ψ. We can check this proposition by calculations for all 23 models. The RHS is independent of $Q(x,0), Q(0,y)$ and $Q(0,0)$. The LHS is a sum of $g(Q(x,y))$. We can extract the $[x^{>}y^{>}]$ of LHS. Then **Proposition 8** in [20] tells:

Proposition 1. *For the 23 models associated with a finite group, except from the four cases,* $\{\leftarrow,\downarrow,\nearrow\}$, $\{\rightarrow,\uparrow,\swarrow\}$, $\{\rightarrow,\uparrow,\leftarrow,\downarrow,\swarrow,\nearrow\}$, $\{\leftarrow,\rightarrow,\nwarrow,\searrow\}$, *the following holds,*

$$R(x,y,t) = \frac{1}{K(x,y)}\sum_{g\in G}\operatorname{sgn}(g)g(xy) \tag{5.4.2}$$

is a power series in t with coefficients in $\mathbb{Q}[x,\frac{1}{x},y,\frac{1}{y}]$. Moreover, the $[y^{>}]$ part of $R(x,y,t)$ is a power series of t with coefficients in $\mathbb{Q}[x,\frac{1}{x},y]$. Further extracting the $[x^{>}]$ part gives $xyQ(x,y,t)$ in LHS. In brief, we have

$$xyQ(x,y) = [x^{>}y^{>}]R(x,y,t) \tag{5.4.3}$$

5.5 Approach of the algebraic kernel method for algebraic walks

The four exceptions to Proposition 1 are Kreweras walks, reverse Kreweras walks, double Kreweras walks and Gessel's walks. They all have $R(x,y,t) = 0$. The full orbit-sums all yield $0 = 0$. Gessel's walks were not solved by the algebraic kernel method. For the other three, we instead consider the half orbit-sum.

Proposition 2 (**Proposition 6** in [20])**.** *For the models with allowed steps* $\{\leftarrow,\downarrow,\nearrow\}$, $\{\rightarrow,\uparrow,\swarrow\}$, $\{\rightarrow,\uparrow,\leftarrow,\downarrow,\swarrow,\nearrow\}$, *we have*

$$xyQ(x,y) - \frac{1}{x}Q\left(\frac{1}{xy},y\right) + \frac{1}{y}Q\left(\frac{1}{xy},x\right) = \frac{xy - \frac{1}{x} + \frac{1}{y} - 2txA_{-1}(x)Q(x,0) + tC_{-1,-1}Q(0,0)}{K(x,y)} \tag{5.5.1}$$

These three models are all associated with the group D_3. Their symmetry group is

$$\left\{(x,y),\left(\frac{1}{xy},y\right),\left(y,\frac{1}{xy}\right),(y,x),\left(\frac{1}{xy},x\right),\left(x,\frac{1}{xy}\right)\right\}. \tag{5.5.2}$$

We can see that in (5.5.1), we only used half of the symmetry group. That is why (5.5.1) is called a half orbit sum. Following the general idea of the algebraic kernel method, we extract the $[y^0]$ term of both sides. Notice that $[y^0]\frac{1}{x}Q\left(\frac{x}{y},y\right)=\frac{1}{x}Q_0^{\mathrm{d}}(x)$, where $Q_0^{\mathrm{d}}(x)$ is the generating function of walks ending on the diagonal. Taking the $[y^0]$ term of (5.5.1) gives

$$-\frac{1}{x}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)=\frac{1}{\sqrt{\Delta}}\left(xY_0(x)-\frac{1}{x}+\frac{1}{Y_1(x)}-2txA_{-1}(x)Q(x,0)+tC_{-1,-1}Q(0,0)\right) \tag{5.5.3}$$

By the fact $Y_0Y_1=\frac{1}{x}$, we further have

$$-\frac{1}{x}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)=\frac{1}{\sqrt{\Delta}}\left(2xY_0(x)-\frac{1}{x}-2txA_{-1}(x)Q(x,0)+tC_{-1,-1}Q(0,0)\right) \tag{5.5.4}$$

To deal with (5.5.4), we again use the canonical factorization Lemma 7.

Now, we may multiply (5.5.4) by $A_1(x)\sqrt{\Delta_-}$ and we have,

$$\sqrt{\Delta_-\left(\frac{1}{x}\right)}\left(-\frac{1}{x}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)\right)=$$
$$\frac{1}{\sqrt{\Delta_0\Delta_+(x)}}\left(2xY_0(x)-\frac{1}{x}-2txA_{-1}(x)Q(x,0)+tC_{-1,-1}Q(0,0)\right) \tag{5.5.5}$$

Notice that in (5.5.5), the coefficient of $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$ is a formal series of t with coefficients in $\mathbb{R}(\frac{1}{x})$ and the coefficient of $Q(x,0)$ is a formal series of t with coefficients in $\mathbb{R}(x)$. Thus, we can separate the $[x^>]$, $[x^0]$ and $[x^<]$ terms of this equation. Further notice $\sqrt{\Delta_-}=1-\frac{X_i}{2x}+\ldots$ and $\frac{1}{\sqrt{\Delta_+}\sqrt{\Delta_0}}=1+\frac{x}{2X_i}+\ldots$. The series expansion of $\sqrt{\Delta_-\left(\frac{1}{x}\right)}\frac{1}{x}Q_0^{\mathrm{d}}(\frac{1}{x})$ only has $[x^{\leq}]$ terms and the series expansion of $\frac{1}{\sqrt{\Delta_0\Delta_+(x)}}2txA_{-1}(x)Q(x,0)$ only has $[x^>]$ terms. Thus, the only extra unknown function that appears in the $[x^>]$ and $[x^<]$ equation is $Q(0,0)$. And we can use the original kernel method to solve both $Q(x,0)$, $Q_0^{\mathrm{d}}(\frac{1}{x})$. Other generating functions follow directly.

5.6 Conclusion of quarter-plane lattice walks without interactions

All quarter-plane lattice walk models without interactions and whose symmetry groups are finite have been solved explicitly. By the previous discussion, we can see that not all models can be solved by both the obstinate kernel method and the algebraic kernel method. The algebraic kernel method and the obstinate kernel both have their own advantages. We currently cannot give a comment on which one is better or more essential. The relation and difference of these two methods will be discussed in Chapter 9 after we apply them to cases with interactions. Normally, the advantage of the obstinate kernel method is it is simpler in calculation since we only use half of the symmetry group. The advantage of the algebraic kernel method is that we can know the algebraic property of the generating functions before solving them. We observe that when the full orbit sum equals 0, the generating function of the walk is always algebraic. We may assert that the generating function of a model is algebraic before explicitly solving it.

5.7 History of solving quarter-plane lattice walks

Historically, the solutions to quarter-plane lattice walk problems via the kernel method begin with Gessel's conjecture (Personal communication stated in [43]). Before this, the enumeration of lattice paths was normally discussed in terms of combinatorial methods [35, 40] and the research focused more on the explicit $q_{i,j,n}$, the number of walks ending at (i,j) of step length n. There was little interest in the properties of the generating functions. The solutions always contain products of binomial coefficients. For quarter-plane models, there was a group of probabilists studying quarter-plane random walks via analytical methods based on Riemann surfaces and differential equations [53, 33, 27]. The solutions always contain integrals of elliptic functions.

Around 2001, Gessel conjectured that the generating function $Q(0,0)$ of walk with steps $\{\leftarrow\swarrow\rightarrow\nearrow\}$ ending at origin, is

$$Q(0,0) = \sum_{2n} 16^n \frac{(\frac{5}{6})_n(\frac{1}{2})_n}{(\frac{5}{3})_n(2)_n} t^{2n} \tag{5.7.1}$$

Here $(a)_n$ in (5.7.1) is defined as the Pochhammer symbol, $a(a+1)\dots(a+n-1)$. Gessel further conjectured that $Q(0,0)$ can be expressed by an algebraic function of t. Later, in 2008, Kauers, Koutschan and Zeilberger [43] proved (5.7.1) by a computer-aided method called Takayama's Algorithm [69]. Later in 2010, $Q(0,0)$ was proved to be algebraic in [12] by computer algebra. However, these computer based approaches are based on the conjecture that Gessel's model is D-finite [44]. Without this D-finite assumption, the computer based method does not work. This raised a general question: Do all quarter-plane lattice walk models have D-finite generating functions? Besides Gessel's walks, are there any other models having an algebraic generating function? If not, what are the properties of these generating functions? D-algebraic or something else?

We can see the obstacle if we want to answer these questions and prove Gessel's conjecture by combinatorial methods or analytic methods. For a combinatorial proof, we need to find the closed form of a sum of terms containing binomial coefficients. For an analytic approach, we need to calculate integrals of elliptic functions (shown in Chapter 11). The kernel method approach to quarter-plane lattice walk problems was developed at this time. A major goal was to find a human proof for Gessel's conjecture. The kernel method had already been proved to be effective for different enumeration problems, including some basic one-boundary lattice walk problems [2, 30, 13, 1, 21]. The advantage of the kernel method for proving the algebraicity is it only involves elementary operations (such as addition, subtraction, multiplication, division or taking certain parts of the equation). This means we can prove the algebraic or D-finite properties immediately after we solve the problem.

The breakthrough started from Kreweras walks. Kreweras walks are the most widely studied algebraic lattice walk model [16, 44, 34, 8, 38]. At first in [14], Bousquet-Mélou proved that Kreweras walks are algebraic by the obstinate kernel method. There was a series of work about solving different models via the kernel method, summarized in [20]. Some non-D-finite cases were detected by the kernel method in [55, 22]. However, Gessel's conjecture seems to be a bit harder. In 2016, Gessel's conjecture was finally proved by the kernel method in [17]. Before this Kurkova and Raschel [46] also gave an explicit solution for Gessel's walks by RBVP, which we will introduce in Chapter 11.

Chapter 6

Model description: quarter-plane lattice walks with boundary interactions

In this chapter, we begin to introduce interactions into the quarter-plane lattice walk problems. For interactions, we assign weights to each step ending on the boundaries. These are normally interpreted as surface energies in a physical model. Unlike probabilistic weights, these interactions depend on the end point but not the starting point of a step. This will lead to some special phenomena.

6.1 The derivation of the functional equation

Consider a walk starting from the origin with allowed steps $\mathcal{S} \subseteq \{-1, 0, 1\} \times \{-1, 0, 1\} \setminus \{(0, 0)\}$. The *step generator* S is

$$S(x, y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j. \tag{6.1.1}$$

It can be written as

$$S(x, y) = A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y = B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x \tag{6.1.2}$$

We denote $A(x, y) = A_{-1}(x)\frac{1}{y}$, $B(x, y) = B_{-1}(y)\frac{1}{x}$, and $C_{-1,-1} = [x^{-1}]A_{-1}(x) = [y^{-1}]B_{-1}(y)$.

Theorem 8. *For a lattice walk restricted to the quarter-plane, starting from the origin, with weight 'a' associated with steps ending on the x-axis (except the origin), weight 'b' associated with steps ending on y-axis (except the origin) and weight 'c' associated with steps ending at the origin, the generating function $Q(x, y)$ satisfies the following functional equation:*

$$\begin{aligned} K(x, y)Q(x, y) = \frac{1}{c} + \frac{1}{a}(a - 1 - taA(x, y))Q(x, 0) + \frac{1}{b}(b - 1 - tbB(x, y))Q(0, y) \\ + \left(\frac{1}{abc}(ac + bc - ab - abc) + \frac{tC_{-1,-1}}{xy}\right) Q(0, 0) \end{aligned} \tag{6.1.3}$$

where $K(x, y) = 1 - tS(x, y)$.

Proof. This equation can be constructed the same way as [7, Theorem 6], except the weight at the origin is c instead of ab. We introduce another method here.

Consider how one obtains the configuration by adding a single step onto a walk of arbitrary length. For a walk in the quarter-plane with boundary interactions, we have 7 situations,

(i) The empty walk with no steps. This gives 1.

(ii) Walks ending away from the boundary, which gives

$$\begin{aligned} tS(x,y)[Q(x,y) - Q_{-,1}(x) - Q_{1,-}(y) - Q_{-,0}(x) - Q_{0,-}(y) + Q_{1,1} + Q_{0,0} + Q_{0,1} + Q_{1,0}] \\ +t([x^{>}]S(x,y))[Q_{0,-}(y) - Q_{0,0} - Q_{0,1}] + t([x^{\geq}]S(x,y))[Q_{1,-}(y) - Q_{1,1} - Q_{1,0}] \\ +t([y^{>}]S(x,y))[Q_{-,0}(x) - Q_{0,0} - Q_{1,0}] + t([y^{\geq}]S(x,y))[Q_{-,1}(x) - Q_{1,1} - Q_{1,0}] \\ +t[x^{>}y^{>}]S(x,y)[Q_{0,0}] + t[x^{\geq}y^{\geq}]S(x,y)Q_{1,1}]. \end{aligned}$$

We need to subtract the steps that interact with boundary and those steps that step outside the quarter-plane (We will call these steps "illegal" steps).

(iii) Walks arriving at the positive x-axis, which gives

$$at\left(A(x,y)(Q_{-,1}(x) - Q_{0,1}) - \frac{C_{-1,-1}}{xy}Q_{1,1} + [x^1]A(x,y)Q_{0,1}\right)$$

(iv) Walks along the positive x-axis, which gives

$$at\left([y^0]S(x,y)(Q_{-,0}(x) - Q_{1,0} - Q_{0,0}) + [x^1y^0]S(x,y)(Q_{1,0} + Q_{0,0})\right)$$

(v) Walks arriving at the positive y-axis, which gives

$$bt\left(B(x,y)(Q_{1,-}(y) - Q_{1,0}) - \frac{C_{-1,-1}}{xy}Q_{1,1} + [y^1]B(x,y)Q_{0,1}\right)$$

(vi) Walks along the positive y-axis, which gives

$$bt\left([x^0]S(x,y)(Q_{0,-}(y) - Q_{0,1} - Q_{0,0}) + [x^0y^1]S(x,y)(Q_{0,1} + Q_{0,0})\right)$$

(vii) Walks ending at the origin, which gives

$$ct\left(\frac{C_{-1,-1}}{xy}Q_{1,1} + [x^{-1}y^0]S(x,y)Q_{1,0} + [x^0y^{-1}]S(x,y)Q_{0,1}\right)$$

Summing cases (i)– (vii) together, we get $Q(x,y)$. We have many unknowns, $Q_{-,1}(x)$, $Q_{0,0}$, $Q_{1,1}$ etc. Notice that case (i)+(iii)+(iv)+(vii) actually equals $Q(x,0)$ and case (i)+(v)+(vi)+(vii) equals $Q(0,y)$ by definition. This provides a way to eliminate $Q_{-,1}(x)$ and $Q_{1,-}(y)$. We further have the boundary condition,

$$Q_{0,0} = 1 + ct\left([x^0]A(x,y)Q_{0,1} + [y^0]B(x,y)Q_{1,0} + \frac{C_{-1,-1}}{xy}Q_{1,1}\right). \tag{6.1.4}$$

Substitute $Q_{1,1}$ by $Q_{0,1}, Q_{1,0}, Q_{0,0}$ into the sum, we find $Q_{0,1}$ and $Q_{1,0}$ are eliminated automatically. Thus, we eliminate all the extra unknowns and get (6.1.3). □

Comparing with the discussion in Section 3.5, the derivation of quarter-plane lattice walks with interactions is actually the same as the non-interacting case. The functional equation looks similar. The main difference is that the coefficients of $Q(x,0)$ and $Q(0,y)$ are not monomials in x,y. If we let $a = b = c = 1$, we get the functional equation for quarter-plane walks without interactions.

6.2 Analytic properties of the kernel

In (6.1.3), $K(x,y)$ is the kernel of the lattice walk problem. Normally, $xyK(x,y)$ is a quadratic function of both x and y. If $xyK(x,y)$ is either reducible or of degree 1 in at least one of the variables it will be a singular walk (See **Lemma 2.32** in [33] or Definition 19). All the 23 quarter-plane lattice walk models with finite groups are non-singular. As an irreducible quadratic function, the kernel can be written as

$$xyK(x,y) = a(x,t)y^2 + b(x,t)y + c(x,t) = \tilde{a}(y,t)x^2 + \tilde{b}(y,t)x + \tilde{c}(y,t) \tag{6.2.1}$$

Let us consider the discriminant $\Delta(x,t) = b(x,t)^2 - 4a(x,t)c(x,t)$. If the walk is non-singular, then for t in some small disc around 0, we can solve $\Delta = 0$. Δ has three or four roots, depending on the step generator $S(x,y)$. If we expand the roots as series of t, x_1, x_2 are formal series of t without constant terms and x_3, x_4 are Laurent series of t with $\frac{1}{t}$. And, we can choose suitable t small enough such that

1. $|x_1(t)| \le |x_2(t)| < 1 < |x_3(t)| \le |x_4(t)|$[1] when the degree of Δ is 4.
2. $|x_1(t)| < |x_2(t)| < 1 < |x_3(t)|$ and $|x_4(t)|$ is ∞ when the degree of Δ is 3.

The discriminant $\Delta(y,t)$ and its roots can be calculated similarly.

6.3 Previous results for quarter-plane lattice walks with interactions

Previous work in quarter-plane lattice walks with interactions was done in [7]. It contains

1. One-boundary interacting case. The walks only have interactions with either the x- or y-axis. This corresponds to $a = 1, c = b$ or $b = 1, c = a$.
2. Homogeneous boundary interacting case, which corresponds to $a = b, c = b^2$.
3. Two-boundary interacting case. The origin is treated as an interaction with both the x- and y-axes. This corresponds to $c = ab$.

Condition 3 is the generalized form of conditions 1 and 2. Many models have not been solved by the kernel method with condition 3. In the following tables, we list the property of $Q(0,0)$ over t. We abbreviate D-finite as DF, D-algebraic as DAlg, Algebraic as Alg. The ✓ means this result is fully proved. The ✗ means this result is a hypothesis, that is the property can be viewed by computer calculations but not rigorous proofs.

6.3.1 Group D_2

When the step set has a reflection symmetry about the y-axis (that is, when we have a $\leftarrow$, we also have a $\rightarrow$ in the step set, and likewise for $\nwarrow/\nearrow$ and $\swarrow/\searrow$), the kernel is associated with a group isomorphic to D_2. We will always have $(\frac{1}{x}, y)$ as an element in the group because of the symmetry. If the walk is further symmetric about both axes, the group is generated by $\{(\frac{1}{x}, y), (x, \frac{1}{y})\}$. In this case, all non-interacting or partial-interacting models can be solved explicitly via the kernel method. For other D_2 models that are only symmetric about the y-axis,

[1] We normally have $|x_1(t)| < |x_2(t)| < 1 < |x_3(t)| < |x_4(t)|$. For **Model 2**, we have $x_3(t) = -x_4(t)$ and $x_2(t) = -x_1(t)$, hence the $\le$ signs.

if we only add interactions on the x-axis, we are still able to solve them. Those models were not solved when $b \neq 1$.

All models associated with D_2 groups are D-finite without interactions. **Model 2** is D-finite with the condition $c = ab$ and a, b arbitrary. For **Models 1, 3** and **4**, $Q(0,0)$ is D-algebraic with interaction $c = ab$. $Q(0,0)$ becomes D-finite when $a = 1$ or $b = 1$. For **Models 5–16**, they are D-finite with $c = ab$ and $b = 1$. They were not solved with arbitrary a, b. The results are listed in Tables 6.1 and 6.2.

	Model	$(1,1)$	(a,a)	$(a,1)$	$(1,b)$	(a,b)	Group
1	↑←↓→	DF ✓	DAlg ✓	DF ✓	DF ✓	DAlg ✓	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
2	↖↙↘↗	DF ✓	DF ✓	DF ✓	DF ✓	DF ✓	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
3	↑↖↙↓↘↗	DF ✓	DAlg ✓	DF ✓	DF ✓	DAlg ✓	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
4	↑↖←↙↓↘↗→	DF ✓	DAlg ✓	DF ✓	DF ✓	DAlg ✓	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$

Table 6.1: The first 4 models are the walks whose symmetry group is D_2 and whose steps are both vertically and horizontally symmetric (D-finite → DF, D-algebraic → DAlg, Algebraic → Alg, ✓→ fully proved, ✗→ hypothesis).

	Model	$(1,1)$	(a,a)	$(a,1)$	$(1,b)$	(a,b)	Group
5	↖↓↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{1}{x+\frac{1}{x}}\frac{1}{y}\right)$
6	↖←↓→↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{1}{x+\frac{1}{x}}\frac{1}{y}\right)$
7	↑↖↓↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{1}{x+1+\frac{1}{x}}\frac{1}{y}\right)$
8	↑↖←↓→↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{1}{x+1+\frac{1}{x}}\frac{1}{y}\right)$
9	↑↖↙↘↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{x+\frac{1}{x}}{x+1+\frac{1}{x}}\frac{1}{y}\right)$
10	↖←↙↑↘→↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{x+\frac{1}{x}}{x+1+\frac{1}{x}}\frac{1}{y}\right)$
11	↑↙↓↘	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, (x+1+\frac{1}{x})\frac{1}{y}\right)$
12	↑←↙↓↘→	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, (x+1+\frac{1}{x})\frac{1}{y}\right)$
13	↖↙↓↘↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{x+1+\frac{1}{x}}{x+\frac{1}{x}}\frac{1}{y}\right)$
14	↖←↙↓↘→↗	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, \frac{x+1+\frac{1}{x}}{x+\frac{1}{x}}\frac{1}{y}\right)$
15	↑↙↘	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, (x+\frac{1}{x})\frac{1}{y}\right)$
16	↑←↙↘→	DF ✓	?	DF ✓	?	?	4: $(\frac{1}{x}, y), \left(x, (x+\frac{1}{x})\frac{1}{y}\right)$

Table 6.2: **Models 5–16** are walks whose symmetry groups are D_2 and whose steps are vertically symmetric but not horizontally symmetric. In this table, the results only contain non-interacting cases and $b = 1$ cases (D-finite → DF, D-algebraic → DAlg, Algebraic → Alg, ✓→ fully proved, ✗→ hypothesis).

6.3.2 Group D_3

The results of the D_3 cases are listed in Table 6.3. **Model 17** was solved when $c = ab$ and a, b arbitrary. **Models 18–20** were only solved with equal interaction $a = b$. For **Model 21**, only the non-interacting case has been solved.

The three algebraic models, Kreweras walks, reverse Kreweras walks and double Kreweras walks (**Models 19–21**) were only partly solved. In [7], the authors failed to solve double Kreweras walks with interactions. Reverse Kreweras and Kreweras walks were only solved when $a = b$. They are both algebraic with and without interactions.

	Model	$(1,1)$	(a,a)	$(a,1)$	$(1,b)$	(a,b)	Group
17	↑←↘	DF ✓	DAlg ✓	DF ✓	DF ✓	DAlg ✓	6: $(\frac{y}{x}, y), (x, \frac{x}{y})$
18	↑↖←↓↘→	DF ✓	DAlg ✓	DF ✗	DF ✗	?	6: $(\frac{y}{x}, y), (x, \frac{x}{y})$
19	←↓↗	Alg ✓	Alg ✓	Alg ✗	Alg ✗	Alg ✗	6: $(\frac{1}{xy}, y), (x, \frac{1}{xy})$
20	↑↙→	Alg ✓	Alg ✓	Alg ✗	Alg ✗	Alg ✗	6: $(\frac{1}{xy}, y), (x, \frac{1}{xy})$
21	↑←↙↓→↗	Alg ✓	?	DF ✗	DF ✗	?	6: $(\frac{1}{xy}, y), (x, \frac{1}{xy})$

Table 6.3: Step sets associated with D_3 groups. We have three algebraic cases here. There are many hypotheses that have not been proved (D-finite → DF, D-algebraic → DAlg, Algebraic → Alg, ✓→ fully proved, ✗→ hypothesis).

6.3.3 Group D_4

There are only two walks associated with D_4 groups. **Model 22** is the example we quoted in Chapter 1. It has a bijection with some physics models [66, 67]. Without interactions or with interaction $a = 1$ or $b = 1$, $Q(0, 0)$ is D-finite. When $a, b \neq 1$, the solution is D-algebraic. **Model 23** is the famous Gessel's walk. The solution is algebraic without interactions. The solution with interactions has not been solved explicitly.

	Model	$(1,1)$	(a,a)	$(a,1)$	$(1,b)$	(a,b)	Group
22	↖←↘→	DF ✓	DAlg ✓	DF ✓	DF ✓	DAlg ✓	8: $(\frac{y}{x}, y), (x, \frac{x^2}{y})$
23	←↙→↗	Alg ✓	?	Alg ✗	DF ✗	?	8: $(\frac{1}{xy}, y), (x, \frac{1}{x}^2 \frac{1}{y})$

Table 6.4: Step sets associated with the group isomorphic to D_4 (D-finite → DF, D-algebraic → DAlg, Algebraic → Alg, ✓→ fully proved, ✗→ hypothesis).

Chapter 7

Criteria for D-finite, algebraic, and D-algebraic generating functions

In Tables 3.1 to 3.3 and Tables 6.1 to 6.4, we distinguish the properties of solutions as algebraic, D-finite, and D-algebraic functions.

All the discussion in Tables 6.1 to 6.4 (in [7, 14]) are about the properties of $Q(0,0)$ over t. Normally, $Q(x,y), Q(x,0), Q(y,0)$ share the same property over t with $Q(0,0)$ because of their linear relations. Without interactions, if $Q(0,0)$ is D-finite (or algebraic) over t, then $Q(x,y), Q(x,0), Q(y,0)$ are D-finite (or algebraic) over x, y, t. Therefore we did not emphasise this in previous discussions. We will later see that with interactions, $Q(x,y)$ can be D-finite over x, y while being D-algebraic over t.

To understand the previous results and to make our future explanation easier, we give some criteria about algebraic, D-finite and D-algebraic generating functions. These criteria will be useful in the following derivation.

7.1 Criteria

Recall the process of the kernel method. We only use elementary operations such as addition, subtraction, multiplication, division, substitution and taking $[x^0], [x^>], [x^<]$ terms. We need to find out how these operations change the properties of certain functions.

Theorem 9. *Let $R(x_1, x_2 \ldots x_n)$ be a rational function. It can be expanded as*

$$R(x_1, \ldots, x_n) = \sum_{i \geq j} r_i(x_2, \ldots, x_n) x_1^i \tag{7.1.1}$$

for some finite (possibly negative) j. Then $r_i(x_2, \ldots, x_n)$ is algebraic over $x_2, \ldots, x_n$.

Proof. We treat $R(x_1, x_2 \ldots x_n)$ as a function of x_1 and all other x_k as parameters. Since R is rational, its denominator is a polynomial with roots $X_1, X_2 \ldots X_m$. These roots are algebraic functions of all other $x_2, \ldots, x_n$. When we expand a rational function $\frac{1}{f}$, the coefficients are formed by $a_i(X_l)^i$ or $b_i(X_l)^{-i}$, where a_i and b_i are constants. Combining with the polynomial in the numerator, we can see the coefficients of x_1^i of the series are still algebraic. Thus $r_i(x_2, \ldots, x_n)$ is algebraic. □

Naturally Theorem 9 then applies to any other x_k.

Theorem 10. *Let Δ be a rational function of x with different zeros and singularities. Suppose $\sqrt{\Delta}$ is analytic in some annulus $x_l < |x| < x_r$ in the complex plane, with at least one singularity in each of the regions $|x| \le x_l$ and $|x| \ge x_r$. Expand $\sqrt{\Delta}$ as a Laurent series of x,*

$$\sqrt{\Delta} = \sum_i a_i x^i \tag{7.1.2}$$

which is convergent in this annulus. This expansion must have infinite $[x^>]$ and $[x^<]$ parts. Now we write this series expansion as $\sqrt{\Delta} = A + B$, where

$$\begin{aligned} A &= [x^{\ge}]\sqrt{\Delta} = \sum_{i\ge 0} a_i x^i, \\ B &= [x^{<}]\sqrt{\Delta} = \sum_{i<0} a_i x^i. \end{aligned} \tag{7.1.3}$$

Then both A and B are D-finite but not algebraic over x.

Proof. First, we prove that they are not algebraic. Suppose A is algebraic, then B is also algebraic. By the definition of algebraic, A satisfies an irreducible polynomial equation of degree n with coefficients in $\mathbb{C}(x)$:

$$p_0 + p_1 A + p_2 A^2 \cdots + p_n A^n = 0. \tag{7.1.4}$$

We have $A = \sqrt{\Delta} - B$. Substituting this relation into (7.1.4) and collecting the terms with odd powers of $\sqrt{\Delta}$, we have

$$\sqrt{\Delta}(q_0 + q_1 B + q_2 B^2 \cdots + q_{n-1} B^{n-1}) + (u_0 + u_1 B + u_2 B^2 \dots u_n B^n) = 0. \tag{7.1.5}$$

Here q_i, u_i are all rational functions. We abbreviate (7.1.5) as

$$\sqrt{\Delta} g(B) + f(B) = 0. \tag{7.1.6}$$

Now we factor (7.1.6). Since Δ is a rational function of x, we can express it by its roots. We denote the roots and singularities larger than x_r as x_i, x_j, and the roots and singularities smaller than x_l as x_p, x_q. Then, we have

$$\Delta = x^k \Delta_0 \frac{\prod_i (x/x_i - 1)^{n_i+}}{\prod_j (x/x_j - 1)^{n_j-}} \times \frac{\prod_p (1 - x_p/x)^{n_p+}}{\prod_q (1 - x_q/x)^{n_q-}} = x^k \Delta_0 \times \Delta_+ \times \Delta_-. \tag{7.1.7}$$

Here Δ_+ contains the roots or poles in the region $|x| \ge x_r$ and Δ_- contains those in the region $|x| \le x_l$. Δ_0 is a constant. If $g(B) \ne 0$, we may rewrite (7.1.6) as

$$x^{\frac{k}{2}} \sqrt{\Delta_0} \sqrt{\Delta_+} + \frac{f(B)}{g(B)\sqrt{\Delta_-}} = 0 \tag{7.1.8}$$

where $\frac{f(B)}{g(B)}$ is a rational function of B. We still want to expand (7.1.8) in the annulus $x_l < |x| < x_r$. Now $\sqrt{\Delta_0}\sqrt{\Delta_+}$ is a formal series of x in this annulus. This implies that $\frac{f(B)}{g(B)\sqrt{\Delta_-}}$ is also a formal series of x. But this is impossible. Since $f(B)$ has degree n in B and $g(B)$ has degree $n-1$ in B, $\frac{f(B)}{g(B)}$ is always dependent on B. Then since B itself is a $\frac{1}{x}$ series, we will have a series of $\frac{1}{x}$ in the expansion unless all the terms with B are cancelled by $\frac{1}{\sqrt{\Delta_-}}$. Notice that $\frac{f(B)}{g(B)}$ only has rational functions in its coefficients. This implies that $\frac{1}{\sqrt{\Delta_-}}$ equals $\frac{g(B)}{f(B)}$ multiplied by some rational function of x. But, $B \neq \sqrt{\Delta_-}$ and a rational function cannot have the same series expansion as a function with square roots. This gives a contradiction.

Thus, $g(B) = 0$, so B satisfies a polynomial equation of degree $n-1$. Applying the previous discussion to B again we find A satisfies a polynomial equation of degree $n-2$. By induction, A or B satisfies a linear equation, and is thus a polynomial, contradicting the fact that they must have infinitely many terms. Thus, A and B are not algebraic.

The D-finite property is straightforward by Theorem 3.7 in [52]. It says $f(x_1, x_2 \dots x_n) = \sum a(i_1, i_2 \cdots {}_n) x^{i_1} x^{i_2} \dots x^{i_n}$ is D-finite if and only if $a(i_1, i_2 \dots i_n)$ is P-recursive.

Definition 11. *A sequence $a(i_1, i_2 \dots i_n)$ is P-recursive if there is a $k \in \mathbb{N}$ such that: for each $j = 1, 2, \dots n$ and each $v = (v_1, v_2, \dots v_n) \in \{0, 1, \dots, k\}^n$, there is a polynomial $p_v^{(j)}(i_j)$ (at least one of them is not 0) such that*

$$\sum_v p_v^{(j)}(i_j) a(i_1 - v_1, i_2 - v_2 \dots i_n - v_n) = 0 \tag{7.1.9}$$

for all $i_1, i_2, \dots i_n \geq k$.

Taking either the $[x^>]$ or $[x^<]$ terms of a D-finite function will not change the P-recursive relation. These operations only change the initial condition of the iteration. Thus A and B are both D-finite. This completes the proof. □

Remark 2. *Theorem 10 can also be extended to cases $\frac{1}{\sqrt{\Delta}}$ or any rational function of $\sqrt{\Delta}$.*

Lemma 12. *Let f be a D-finite function satisfying*

$$L(f, f' \dots f^{(n)}) + P(x) = 0 \tag{7.1.10}$$

where L is a homogeneous linear differential operator with polynomial coefficients and $P(x)$ is a polynomial. The function f is a Laurent series in x, convergent in some annulus of the complex plane. Then $[x^>]f$ (or $[x^<]f$) satisfies a linear differential equation whose homogeneous part is the same as L.

Proof. By the linear property of derivatives, $L(f) = L([x^>]f + [x^\leq]f) = L([x^>]f) + L([x^\leq]f)$. Then (7.1.10) becomes

$$L([x^>]f) + L([x^\leq]f) + P(x) = 0 \tag{7.1.11}$$

Take the $[x^>]$ terms of (7.1.11). Notice that $L([x^\leq]f)$ may contain $[x^>]$ terms and $L([x^>]f)$ may contain $[x^0]$ terms. Since L is a linear function with polynomial coefficients, $[x^>]L([x^\leq]f) = P_2(x)$ is a polynomial. We have

$$L([x^>]f) + P(x) + P_2(x) - C - P(0) = 0 \tag{7.1.12}$$
$$L([x^\leq]f) - P_2(x) + C + P(0) = 0 \tag{7.1.13}$$

where $C = [x^0]L([x^>]f)$. Since the coefficients in L are polynomials, the calculations of $P_2(x)$ and C are practicable. We may calculate them as follows. Suppose L is written as

$$p_0 f + p_1 \frac{df}{dx} + \frac{d^2 f}{dx^2} + \cdots + p_k \frac{d^k f}{dx^n} \tag{7.1.14}$$

and $p_i = \sum_j q_j x^j$ is a finite sum. Then the terms of $P_2(x)$ can be computed as

$$\begin{aligned}[x^>]p_i \frac{d^i [x^{\leq}]f}{dx^i} &= [x^>] \sum_j q_j x^j [x^{\leq}] f^{(i)} = [x^>] \sum_j q_j \sum_{m=-\infty}^{0} x^{m+j} [x^m] f^{(i)} \\ &= \sum_j q_j \sum_{m=-j+1}^{0} x^{m+j} [x^m] f^{(i)}\end{aligned} \tag{7.1.15}$$

and the terms of C can be computed as

$$[x^0]p_i \frac{d^i [x^>]f}{dx^i} = [x^0] \sum_j q_j x^j [x^>] f^{(i)} = q_0 [x^0] f^{(i)}. \tag{7.1.16}$$

Thus, we get the linear differential equation. $\square$

For quarter-plane lattice walk problems, $\sqrt{\Delta(x,t)}$ is a two variable function. We thus wish to extend Theorem 10 to two variable $\sqrt{\Delta(x,t)}$ cases and consider the properties with respect to t.

Firstly, a function is defined as *elementary* if it is a sum, product, and composition of finitely many polynomials, rational functions, trigonometric and exponential functions, and their inverse functions.

We then require the following lemma.

Lemma 13 (Laplace's Principle [41, Page 18])**.** *Let y be an algebraic function of x and consider the integral*

$$\int R(x,y)dx \tag{7.1.17}$$

where R denotes a rational function. If the integral is elementary, it is a sum of algebraic functions and logarithms of algebraic functions with constant coefficients. These algebraic functions are rational functions of x, y.

Theorem 14. *Let $\Delta(x,t)$ be a monic polynomial of x of degree 3 or 4 with coefficients in $\mathbb{C}(t)$. We may write it as $(x-x_1)(x-x_2)(x-x_3)$ or $(x-x_1)(x-x_2)(x-x_3)(x-x_4)$ and we require all these roots to be non-zero and distinct. Suppose that these four roots satisfy, when $t \to 0$, $x_1, x_2 \to 0$ and $x_3, x_4 \to \infty$. We may find some $\epsilon > 0$ such that $|x_1| \leq |x_2| < |x_3| \leq |x_4|$ when $t \in (0, \epsilon)$. Then $\sqrt{\Delta(x,t)}$ can be expanded as a formal series of t with coefficients in $\mathbb{C}[x, 1/x]$, or when $t \in (0,\epsilon)$ it can alternatively be viewed as a Laurent series in x, convergent for $|x_2| < |x| < |x_3|$. This series will have infinite $[x^>]$ and $[x^<]$ parts. We may write $\sqrt{\Delta} = A(x,t) + B(\frac{1}{x}, t) + C(t)$, where A, B, C are*

$$A(x,t) = [x^>]\sqrt{\Delta} \tag{7.1.18}$$

$$B\left(\frac{1}{x}, t\right) = [x^<]\sqrt{\Delta} \tag{7.1.19}$$

$$C(t) = [x^0]\sqrt{\Delta} \tag{7.1.20}$$

Then all three terms A, B and C are D-finite but not algebraic over t.

Proof. The series expansion of $\sqrt{\Delta}$ can be calculated directly by binomial expansion of $\sqrt{1 - x_i/x}$ and $\sqrt{x/x_j - 1}$. We find it is a formal series of t with coefficients in $\mathbb{C}[x, 1/x]$. We need to prove they are D-finite and not algebraic.

We should first prove that they are D-finite. First consider C. We may write $\sqrt{\Delta(x,t)}$ as a value of a three variable function $f(x,y,t)$,

$$
\begin{aligned}
\sqrt{\Delta(x,t)} &= \sqrt{(1-x_1/x)(1-x_2/x)(x/x_3-1)(x/x_4-1)x^2x_3x_4} \\
&= \sqrt{(1-x_1y)(1-x_2y)(x/x_3-1)(x/x_4-1)x^2x_3x_4}|_{y=\frac{1}{x}} \\
&= f(x,y,t)_{y=\frac{1}{x}} \\
&= \sum a_{i,j,n}x^iy^jt^n|_{y=\frac{1}{x}}
\end{aligned}
\tag{7.1.21}
$$

That is, we denote $\frac{1}{x}$ as y. The expansion of $f(x,y,t)$ in the corresponding convergent domain $U(x,y,t)$ is $\sum a_{i,j,n}x^iy^jt^n$. So $[x^0]\sqrt{\Delta} = [x^0]f\left(x,\frac{1}{x},t\right) = \sum a_{i,i,n}x^i(\frac{1}{x})^it^n$.

By the definition of the diagonal of a D-finite function in [51], $I_{12}f(x,y,t) = \sum a_{i,i,n}x^it^n$. We immediately notice $[x^0]\sqrt{\Delta}$ equals the value of $I_{12}f(x,y,t)$ when $x=1$. By [51], the diagonal of a D-finite series is D-finite. Thus $C(t)$ is D-finite over x,t. $[x^i]\sqrt{\Delta}$ follows similarly.

Now we go to A and B. Since $\sqrt{\Delta}$ is D-finite over both variables, we have

$$
p_c + p_0\sqrt{\Delta} + p_1\frac{d\sqrt{\Delta}}{dt} + \frac{d^2\sqrt{\Delta}}{dt^2}\cdots + p_k\frac{d^k\sqrt{\Delta}}{dt^k} = 0 \tag{7.1.22}
$$

All coefficients p_i are polynomials of x,t. Substituting $\sqrt{\Delta} = A+B+C$ into (7.1.22) and taking the $[x^>]$ and $[x^<]$ terms of this equation, we have

$$
p_{c+} + p_0A + p_1\frac{dA}{dt} + p_2\frac{d^2A}{dt^2}\cdots + p_n\frac{d^kA}{dt^k} = 0 \tag{7.1.23}
$$

$$
p_{c-} + p_0B + p_1\frac{dB}{dt} + p_2\frac{d^2B}{dt^2}\cdots + p_n\frac{d^kB}{dt^k} = 0 \tag{7.1.24}
$$

Similar to the discussion in Lemma 12, each p_i remains the same except the terms p_{c+}, p_{c-}. Considering p_{c+}, it first contains the $[x^>]$ part of p_c, which is a polynomial of x,t. Second, it contains $[x^>]p_i\frac{d^iB}{dt^i}$ and $[x^>]p_i\frac{d^iC}{dt^i}$. The latter term $[x^>]p_i\frac{d^iC}{dt^i}$ is polynomial over x and is D-finite over t. Since B is a formal series of $\frac{1}{x}$, $[x^>]p_i\frac{d^i(B)}{dt^i}$ is a polynomial of x.

We further need to describe the property of t in $[x^>]p_i\frac{d^iB}{dt^i}$. Let us first write $[x^>]p_i\frac{d^iB}{dt^i} = \sum_j \lambda_j(x)[x^{-j}]\frac{d^iB}{dt^i}$. Notice that we can change the order of $[x^{-j}]$ and $\frac{d}{dt}$, so our problem becomes checking the property of $\frac{d^i([x^{-j}]B)}{dt^i}$. But $[x^{-j}]B = [x^{-j}]\sqrt{\Delta}$ and we have already proved that it is D-finite over t. Thus $\frac{d^i([x^{-j}]B)}{dt^i}$ is D-finite over t. Then p_{c+} is a finite sum of these terms. By [64, Theorem 2.3], their sum is still D-finite. Thus p_{c+} is a polynomial of x but a D-finite series of t. The same arguments also apply to p_{c-}.

As p_{c+}, p_{c-} are D-finite, they satisfy linear differential equations of t. Substituting (7.1.23) and (7.1.24) in these linear differential equations, we get the linear differential equations of A and B,

$$
r_c + r_0A + r_1\frac{dA}{dt} + r_2\frac{d^2A}{dt^2}\cdots + r_n\frac{d^lA}{dt^l} = 0 \tag{7.1.25}
$$

$$
s_c + r_0B + s_1\frac{dB}{dt} + s_2\frac{d^2B}{dt^2}\cdots + s_n\frac{d^mB}{dt^m} = 0 \tag{7.1.26}
$$

Thus A,B are D-finite over t.

We now prove A,B and C are not algebraic. First assume Δ is a polynomial of degree 4. Let us consider $[x^{n-1}]\sqrt{\Delta}$. By the definition of Laurent series, the $[x^{n-1}]$ term of $\sqrt{\Delta}$ can be expressed by an integral

$$
\frac{1}{2\pi i}\int_\Gamma \frac{\sqrt{\Delta}}{x^n}\,dx \tag{7.1.27}
$$

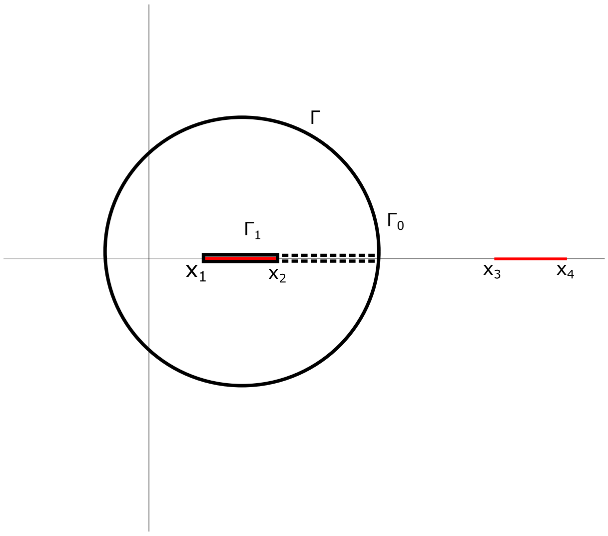


Figure 7.1: A schematic figure of the integral. The contour Γ of (7.1.27) should contain x_1, x_2 inside it and x_3, x_4 outside it. We may draw a contour Γ_1 clockwise just around $\overrightarrow{x_1x_2}$ and we join Γ and Γ_1 by two dash line and make it a contour Γ_0. The roots do not need to be all real. They only need to satisfy $x_1, x_2 \to 0$ and $x_3, x_4 \to \infty$ when $t \to 0$.

taken over some contour Γ in some analytic region. Since we expand $\sqrt{\Delta}$ as a formal series of t, it converges when $t \in (0, \epsilon)$. So the contour should contain $0, x_1, x_2$ inside it and leave x_3, x_4 outside. Since the integral contains a square root of a quartic polynomial, we may choose $\overrightarrow{x_1x_2}$ and $\overrightarrow{x_3x_4}$ or $(\overrightarrow{x_3\infty})$ as the branch cutting line (we assume they are on the real line for simplicity). We may further draw the contour Γ_0 as Γ joining a contour Γ_1 clockwise on $\overrightarrow{x_1x_2}$ by two lines just above and below the real axis in opposite directions (see the dashed lines in Figure 7.1). On the x-axis, the integrals of these two dashed lines cancel each other. So we can write the integral on Γ as

$$\int_\Gamma \frac{\sqrt{\Delta}}{x^n}\,dx = 2\int_{x_2(t)}^{x_1(t)} \frac{\sqrt{\Delta}}{x^n}\,dx + \int_{C_0} \frac{\sqrt{\Delta}}{x^n}\,dx. \tag{7.1.28}$$

Now Γ_0 is the boundary of some area where $\sqrt{\Delta}$ is single valued and holomorphic. So, the second integral on the right can be calculated by the residue theorem and is thus algebraic. Now consider the first term. By Laplace's principle Lemma 13, if $\int \frac{\sqrt{\Delta}}{x^n}\,dx$ is an elementary function, it can be written as $R_1(x, \sqrt{\Delta}) + m\log(R_2(x, \sqrt{\Delta}))$, where R_1 and R_2 are rational functions and m is independent of x. The coefficients are all in $\mathbb{C}(t)$. Then $\int_{x_1(t)}^{x_2(t)} \frac{\sqrt{\Delta}}{x^n} dx$ may be written as

$$\int_{x_1(t)}^{x_2(t)} \frac{\sqrt{\Delta}}{x^n} dx = (R_1(x_1, 0) - R_1(x_2, 0)) + m\log\left(\frac{R_2(x_1, 0)}{R_2(x_2, 0)}\right) \tag{7.1.29}$$

The first term is clearly algebraic since x_1 and x_2 are algebraic. If the second term vanishes or it is algebraic, we prove the theorem.

We first suppose the log term vanishes in the integral. We can write $\int \frac{\sqrt{\Delta}}{x^n}\,dx$ as

$$\int \frac{\sqrt{\Delta}}{x^n}\,dx = U + V\sqrt{\Delta} \tag{7.1.30}$$

where U, V are rational functions. (Notice any rational function of $\{x, \sqrt{\Delta}\}$ can be written in this form). Taking derivatives of both sides of (7.1.30), we have

$$U' + V'\sqrt{\Delta} + \frac{V\sqrt{\Delta}\Delta'}{2\Delta} = \frac{\sqrt{\Delta}}{x^n}. \tag{7.1.31}$$

Clearly U is constant and $V' + \frac{V\Delta'}{2\Delta} = \frac{1}{x^n}$. Now we calculate V by the singularities. The RHS only contains 0 as a singularity (or a zero when $n < 0$). Notice that Δ has roots different from 0, so $\frac{\Delta'}{\Delta}$ can be written as

$$\frac{\Delta'}{\Delta} = \left(\frac{1}{x - x_1} + \frac{1}{x - x_2} + \frac{1}{x - x_3} + \frac{1}{x - x_4}\right). \tag{7.1.32}$$

So the singularities at x_1, x_2, x_3, x_4 on the LHS must be cancelled. There are two ways to cancel these singularities. First, V and V' both have these singularities and the singularities cancel by addition. Second V is a multiple of Δ and the singularities cancel by multiplication. We show these two cases are both impossible. We may first suppose V has a degree g singularity at x_1,

$$V = P + \frac{a_1}{(x - x_1)^g} \tag{7.1.33}$$

where a_1 is constant, P is rational and the degree of the singularity at x_1 in P (if there is one) is less than g. Then

$$V' = P' - \frac{a_1}{g(x - x_1)^{g+1}}. \tag{7.1.34}$$

Substitute (7.1.34) into (7.1.31). By calculation, we find the coefficient of $\frac{1}{(x-x_1)^{g+1}}$ on the LHS is $-ga_1 + \frac{1}{2}a_1$, so we must have $a_1 = 0$. By induction, V is analytic at x_1. Similar proofs are applied to x_2, x_3, x_4. Thus, V is analytic at x_1, x_2, x_3, x_4.

Now (7.1.31) can be written as

$$V' + \frac{V}{2}\left(\frac{1}{x - x_1} + \frac{1}{x - x_2} + \frac{1}{x - x_3} + \frac{1}{x - x_4}\right) = \frac{1}{x^n} \tag{7.1.35}$$

Comparing the singularities of both sides, we notice

1. V must be a multiple of Δ.
2. V has a pole of order $n - 1$ at 0 if $n > 0$.
3. V has a zero of order $|n|$ at 0 if $n \leq 0$.
4. V does not have other poles.

Let us first consider the $n > 0$ case. We may write V as

$$V = \frac{\Delta Q}{x^{n-1}} \tag{7.1.36}$$

where Q is some polynomial. Then multiplying both sides by x^n, (7.1.35) is equivalent to

$$x\Delta' Q + x\Delta Q' - (n - 1)\Delta Q + \frac{1}{2}\Delta' Q x = 1. \tag{7.1.37}$$

Suppose $Q = b_i x^i + b_{i-1}x^{i-1} + \dots b_0$, and let us consider the $[x^{i+4}]$ part of (7.1.37) (the highest degree of x in (7.1.37)). It reads

$$4b_i + ib_i - (n-1)b_i + \frac{1}{2}4b_i = 0. \tag{7.1.38}$$

Since $b_i \neq 0$, we have $i = n - 7$. As Q is a polynomial, we have $i \geq 0$. We cannot find such Q when $0 < n \leq 6$. This implies we cannot find rational functions U, V for (7.1.30) when $0 < n \leq 6$. So the coefficients $[x^{0\sim 5}]\sqrt{\Delta}$ are not algebraic. This immediately gives $C(t)$ is not algebraic ($C(t)$ is the $n = 1$ case).

For the $n \leq 0$ cases, we may let $h = -n$ and $V = x^h \Delta Q$. Then, similarly, we have

$$x\Delta Q' + (h)\Delta Q + x\Delta' Q + \frac{1}{2}xQ\Delta' = x. \tag{7.1.39}$$

The $[x^{i+4}]$ term of this equation shows

$$ib_i + (h)b_i + 4b_i + \frac{1}{2}4b_i = 0. \tag{7.1.40}$$

Since i is positive, (7.1.40) does not have a solution. This implies $[x^{-h}]\sqrt{\Delta}$ are all not algebraic.

For A and B, we may expand $A = xA_1 + x^2A_2 + \dots$ and $B = x^{-1}B_1 + x^{-2}B_2 + \dots$ where $A_i = [x^i]A$ and $B_i = [x^{-i}]B$ for $i = 1, 2 \dots$. If A, B are algebraic over t, $\frac{A}{x}$ and Bx are algebraic over t when $x = 0$ and $x \to \infty$. And we proved $[x^i]B$ are all not algebraic. So $[x^1]A \sim [x^5]A$ are not algebraic. Thus, A and B are not algebraic over t.

If Δ has three roots, all the derivation will be the same until (7.1.38). Then (7.1.38) will become

$$3b_i + ib_i - (n-1)b_i + \frac{1}{2}3b_i = 0 \tag{7.1.41}$$

This equation clearly does not have a natural number solution because of the $\frac{1}{2}$ in the equation. We will find i is a half integer by (7.1.38) and it is a contradiction. Thus A, B, C are not algebraic.

Now consider the case that the log term does not vanish. This means the indefinite integral contains a $\log R_2(x, \Delta)$ term. The derivative of $\log(R_2(x, \sqrt{\Delta}))$ reads,

$$\frac{d}{dx}\log(R_2(x, \sqrt{\Delta})) = \frac{R_2'(x, \sqrt{\Delta})}{|R_2(x, \sqrt{\Delta})|}\frac{\Delta'\sqrt{\Delta}}{2\Delta} \tag{7.1.42}$$

We may suppose the integral is written as $m\log(R_2(x, \sqrt{\Delta})) + U + V\sqrt{\Delta}$ where $m \in \mathbb{C}$ and U, V are rational functions of x with coefficients in $\mathbb{C}(t)$. Then we have

$$m\frac{R_2'(x, \sqrt{\Delta})}{|R_2(x, \sqrt{\Delta})|}\frac{\sqrt{\Delta}\Delta'}{2\Delta} + V'\sqrt{\Delta} + \frac{V\sqrt{\Delta}\Delta'}{2\Delta} + U' = \frac{\sqrt{\Delta}}{x^n} \tag{7.1.43}$$

As before, we consider the poles of $\frac{d}{dx}\log R_2(x, \sqrt{\Delta})$ in (7.1.43) and try to cancel them by the remaining part.

Notice that $R_2(x, \sqrt{\Delta})$ always has zeros or poles different from that of $\Delta = 0$, otherwise $\log R_2(x, \sqrt{\Delta}) = \log((\sqrt{\Delta})^k)$ and its derivative is a polynomial in a fixed branch. This term does not contribute anything to $\frac{\sqrt{\Delta}}{x^n}$. By the previous discussion, we cannot find U, V, such that $\frac{d}{dx}(U + V\sqrt{\Delta}) = \frac{\sqrt{\Delta}}{x^n}$.

Now consider the poles different from that of $\Delta = 0$. If $f(x)$ is algebraic then $\frac{d}{dx}\log f(x)$ only has order 1 poles [63, Chapter 6-5]. Except for the roots of Δ itself, order 1 poles cannot be cancelled by $V'\sqrt{\Delta} + \frac{V\sqrt{\Delta}\Delta'}{2\Delta} + U'$ since U', V' will always have order 2 poles if U, V have order 1 poles for algebraic U, V. Since $R_2(x, \sqrt{\Delta})$ has a zero or pole different from that of $\Delta = 0$, the poles of $\frac{d}{dx}\log R_2(x, \sqrt{\Delta})$ are of order 1 and cannot be cancelled by $V'\sqrt{\Delta} + \frac{V\sqrt{\Delta}\Delta'}{2\Delta} + U'$. If $\frac{d}{dx}\log R_2(x, \sqrt{\Delta})$ only has a pole at $x = 0$, then $R_2(x, \sqrt{\Delta}) = x^{k(t)}$ for some function $k(t)$. So $\frac{d}{dx}\log R_2(x, \sqrt{\Delta}) = \frac{k(t)}{|x|}$ and it cannot be cancelled by $\frac{\sqrt{\Delta}}{x^n}$. So we cannot find $U, V, R_2(x, \sqrt{\Delta})$ such that (7.1.43) holds.

This finishes the proof. □

Remark 3. *Theorems 10 and 14 together give the property of* $[x^i]\sqrt{\Delta(x,t)}$. *Notice that* $[x^i]\sqrt{\Delta(x,t)}$ *is not algebraic if and only if some roots of* $\sqrt{\Delta(x,t)}$ *are formal series of* t *and some roots of* $\sqrt{\Delta(x,t)}$ *contain* $\frac{1}{t}$ *terms in the expansion. If* $\Delta(x,t)$ *only has roots which are formal series of* t*, then there is not a branch cutting line outside the contour and we calculate the residue of* ∞ *instead. Then* (7.1.27) *will be algebraic over* t.

Now consider a rational function in $\mathbb{C}(x, t, \sqrt{\Delta}(x,t))$. *We may write it in a standard form* $p + q\sqrt{\Delta}$*, where* p, q *are rational functions. We can expand* p, q *as a formal series in* t *with coefficients in* $\mathbb{C}[[x]][\frac{1}{x}]$ *(maybe with an extra* $\frac{1}{t^n}$*, in which case we can multiply the rational function by* t^n *to cancel it). Thus, if we relax our condition from a formal series in* t *with coefficients in* $\mathbb{C}[x, 1/x]$ *to a formal series in* t *with coefficients in* $\mathbb{C}[[x]][\frac{1}{x}]$*, then, the discussion of Theorem 14 can be applied to any rational function* $R(x, t, \sqrt{\Delta})$. *The* $[x^>], [x^<]$ *parts are D-finite but not algebraic. The* $[x^0]$ *part is generally D-finite and not algebraic except for some special* q. *We apply* (7.1.43) *as*

$$m\frac{R_2'(x, \sqrt{\Delta})}{|R_2(x, \sqrt{\Delta})|}\frac{\sqrt{\Delta}\Delta'}{2\Delta} + V'\sqrt{\Delta} + \frac{V\sqrt{\Delta}\Delta'}{2\Delta} + U' = \frac{q\sqrt{\Delta}}{x} \tag{7.1.44}$$

Only special q *satisfying* (7.1.44) *can make the* $[x^0]$ *algebraic.*

In some cases, we will need to take the $[x^>]$, $[x^0]$ and $[x^<]$ terms of $\log(p + q\sqrt{\Delta})$. The following theorems show that they are all D-finite but not algebraic.

Theorem 15 (Theorem 4.1 in [39])**.** *Let* f *be a function in* $\mathbb{C}[[x, \frac{t}{x}]]$ *with constant term* 1. *Then* f *has a unique decomposition* $f = f_- f_0 f_+$*, where* f_- *is of the form* $1 + \sum_{i>0,j>0} a_{ij}t^i x^{-j}$*,* f_0 *is of the form* $1 + \sum_{i>0} a_i t^i$*, and* f_+ *is of the form* $1 + \sum_{i\geq 0, j>0} a_{ij}t^i x^j$.

Proof. Let $\log(f) = \sum_{ij} b_{ij}t^i x^j$. Then $f_- = e^{\sum_{i\geq 0, j<0} b_{ij}t^i x^j}$. $f_0 = e^{\sum_{i\geq 0}^{\infty} b_{i0}t^i}$, and $f_+ = e^{\sum_{i\geq 0, j>0} b_{ij}t^i x^j}$. Expand the exponent function and the proof is finished. □

Remark 4. *Actually, if* f *can be expanded as a formal series of* t *with coefficients in* $\mathbb{C}[[x]][\frac{1}{x}]$, $\log(f(x,t))$ *can always be expanded as* $[x^<]\log f + [x^0]\log f + [x^>]\log f + k\log x + i\log t$ *in the convergent domain. Suppose* f *can be expanded as a series of* t

$$f = a_0(x) + a_1(x)t + a_2(x)t^2 + \dots. \tag{7.1.45}$$

Let $l = \min\{i : a_i \neq 0\}$. *Since each* a_i *is in* $\mathbb{C}[[x]][\frac{1}{x}]$*, we write* a_l *as*

$$a_l = b_k x^k + b_{k+1}x^{k+1} + \dots = b_k x^k(1 + r(x)) \tag{7.1.46}$$

where b_k *is the coefficient of the smallest degree of* x *in* a_l *and* $r(x)$ *is a formal series of* x *without constant terms. We take the* $x^k t^i$ *out and write* f *as*

$$f = (b_k x^k t^i) \times (1 + r(x) + g(x,t)) \tag{7.1.47}$$

where $g(x,t)$ converges to 0 when $t \to 0$. Then, by the Taylor expansion of $\log(1+x)$,

$$\begin{aligned}
\log(f(x,t)) &= \log(b_k x^k t^i) + \log(1 + r(x) + g(x,t)) \\
&= \log(b_k x^k t^i) + \log(1 + r(x)) + \log\left(1 + \frac{g(x,t)}{1+r(x)}\right) \\
&= \log(b_k x^k t^i) + \log(1 + r(x)) + 1 + \frac{g(x,t)}{1+r(x)} + \frac{1}{2}\left(\frac{g(x,t)}{1+r(x)}\right)^2 + \dots
\end{aligned} \tag{7.1.48}$$

We can further expand $\log(1+r(x))$ *as a Laurent series of x and* $\frac{g(x,t)}{1+r(x)}$ *as formal series of t to get the result. To simplify the calculation, we always divide f by x^k to eliminate the* $\log x$ *term in the expansion.*

Now we can separate the function $\log(p + q\sqrt{\Delta})$.

Theorem 16. *Let $\sqrt{\Delta}$ be the function defined in Theorem 14. Consider* $\log(p+q\sqrt{\Delta})$*, where p and q are polynomials of x, t. Suppose $p + q\sqrt{\Delta}$ has n roots and $|x_1| < |x_2| < \cdots < |x_k| < |x_{k+1}| < \cdots < |x_n|$ as $t \to 0$. We further require $x_1, x_2 \dots x_k \to 0$ and $x_{k+1}, x_{k+2} \dots, x_n \to \infty$ when $t \to 0$. Then when t is small enough,* $\log(p+q\sqrt{\Delta})$ *can be expanded in $|x_k| < |x| < |x_{k+1}|$, as a formal series of t with coefficients in $\mathbb{C}[[x]][\frac{1}{x}]$ with a special term of* $k \log x + i \log t$*. We can write* $\log(p + q\sqrt{\Delta}) = L_+ + L_0 + L_- + k\log x + i \log t$ *where*

$$\begin{aligned}
L_+ &= [x^>] \log(p + q\sqrt{\Delta}) \\
L_- &= [x^<] \log(p + q\sqrt{\Delta}) \\
L_0 &= [x^0] \log(p + q\sqrt{\Delta})
\end{aligned} \tag{7.1.49}$$

Then L_+ and L_- are both D-finite but not algebraic over x and all three terms are D-finite but not algebraic over t.

Proof. We prove that L_+, L_-, L_0 are D-finite but not algebraic over t. The property over x can be proved similarly. By Theorem 14 and Remark 3, $p+q\sqrt{\Delta}$ can be expanded as a formal series of t with coefficients in $\mathbb{C}[[x]][\frac{1}{x}]$. By Remark 4, $\log(p+q\sqrt{\Delta}) = L_+ + L_0 + L_- + k\log x + i\log t$. Taking derivatives of this equation,

$$\frac{(p' + (q\sqrt{\Delta})')(p - q\sqrt{\Delta})}{p^2 - q^2\Delta} = L'_+ + L'_0 + L'_- + \frac{k}{|x|}. \tag{7.1.50}$$

By Remark 2 the $[x^>]$ and $[x^<]$ series of $\frac{(p'+(q\sqrt{\Delta})')(p-q\sqrt{\Delta})}{p^2-q^2\Delta}$ are all D-finite but not algebraic over x. And by Remark 3, the $[x^>]$, $[x^0]$ and $[x^<]$ series of $\frac{(p'+(q\sqrt{\Delta})')(p-q\sqrt{\Delta})}{p^2-q^2\Delta}$ are all D-finite but not algebraic over t. Then, L'_+, L'_0, L'_- also have this property. Thus, L_+ and L_- are both D-finite but not algebraic over x and all three terms are D-finite but not algebraic over t. □

Theorem 17. *If $u(x)$ and $v(x)$ are D-finite but not algebraic over x, then $f(x) = \frac{u(x)}{v(x)}$ is D-algebraic. If $v(x)$ is algebraic, $f(x)$ is D-finite.*

Proof. We mimic the proof that the product of D-finite functions are D-finite in [64]. The only difference is that one of our multipliers is D-algebraic.

First we prove $\frac{1}{v(x)}$ is D-algebraic. This is clear, since for $v(x)$ D-finite, we have

$$l_0 + l_1 v'(x) + l_2 v''(x) + \cdots + l_n v^{(n)}(x) = 0 \tag{7.1.51}$$

where each l_i is a polynomial in x. Let $g(x) = 1/v(x)$ and we immediately get

$$l_0 + l_1 \left(\frac{1}{g(x)}\right)' + l_2 \left(\frac{1}{g(x)}\right)'' + \cdots + l_n \left(\frac{1}{g(x)}\right)^{(n)} = 0 \tag{7.1.52}$$

Each term in (7.1.52) is a rational function of $g(x)$ and its derivatives; by clearing denominators, we obtain a polynomial in $x, g(x)$ and the derivatives of $g(x)$. So $g(x)$ is D-algebraic.

Let us denote V_u and V_v as the vector spaces spanned by derivatives of u and v. Suppose $\dim(V_u) = m$. For $f^{(k)} = (u/v)^{(k)} = (ug)^{(k)}$, $(ug)^{(k)}$ can be written as a polynomial of $u, u' \dots u^{(m)}$ and $g, g' \dots g^{(n)}$. Notice that $g^{(k)}$ and $u^{(k)}$ can always be expressed as algebraic functions of $u, u' \dots u^m$ and $g, g' \dots g^{(n)}$. Thus, for $k > \max(n, m)$, $f^{(k)}$ can also be expressed by these $m + n$ derivatives. Now we choose a K large enough such that we can solve $u, u' \dots u^{(m)}, g, g', \dots g^{(n)}$ as a function of $f^{(k)}$, $k = 0, 1, 2, \dots, K$. Since u satisfies a linear equation, $f^{(k)}$ satisfies an algebraic equation and is thus D-algebraic.

If v is algebraic, then $1/v$ is algebraic and thus D-finite. Then u/v is D-finite immediately by [64, Theorem 2.3]. □

Theorem 17 tells us D-algebraic functions form a field. One can also find a proof in [61].

Chapter 8

Exact solutions of quarter-plane walks with boundary interactions

From previous work, we have derived all the solutions of non-interacting quarter-plane walks with finite groups. For the interacting cases, there are still many unknowns and conjectures. The aim of this section is to:

1. Introduce the use of the kernel method to solve quarter-plane lattice walk with interactions.

2. Show how the point boundary interaction c affects the final solutions.

3. Add the solutions of the D-finite models which were only solved with condition $b = 1$.

4. Add the solutions of interacting Kreweras and reverse Kreweras walks which were only solved with condition $a = b$.

We will also show the properties of the generating functions by the solutions. Later in Section 12.3, we will give a condition about when the generating functions become algebraic.

In the following calculations, we assume $a, b, c \geq 1$. Then $\sqrt{(a-1)^2}$, $\sqrt{(b-1)^2}$ and $\sqrt{(c-1)^2}$ will always give $a-1$, $b-1$ and $c-1$. This assumption does not affect the final results. In Section 8.2, we will discuss in detail what happens when $a, b, c \leq 1$. For all the models, we use the same notation to define the intermediate variables. The assistant variables A, B, T, R, S, P refer to similar meanings. So we only define them once. One can find the definitions in Section 8.1.1. The names of the models follow the numbering in Tables 6.1 to 6.4. Section 8.1.1 will demonstrate in detail the general process of the obstinate kernel method. Section 8.1.2 will demonstrate in detail the general process of the algebraic kernel method. In the other sections, we will shorten our solutions.

8.1 Exactly solved D-algebraic cases

In this section, we first go through all 6 models that were exactly solved in [7] with $c = ab$. We introduce interaction c into these models and see how c affects previous results.

8.1.1 Model 1: $\{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$

Let us begin with the simplest case (**Model 1**), which is D-finite without interactions. The allowed steps of this walk are $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$, so we have

$$S(x,y) = x + y + \frac{1}{x} + \frac{1}{y} \tag{8.1.1}$$

and the functional equation (6.1.3) reads

$$\begin{aligned} K(x,y)Q(x,y) = \frac{1}{abc}\Bigg((bc - a((b-1)c+b))Q(0,0) + bc\left(-\frac{at}{y} + a - 1\right)Q(x,0) \\ +ac\left(-\frac{bt}{x} + b - 1\right)Q(0,y) + ab\Bigg) \end{aligned} \tag{8.1.2}$$

where

$$K(x,y) = 1 - t\left(x + y + \frac{1}{x} + \frac{1}{y}\right). \tag{8.1.3}$$

The two roots of the kernel are

$$\begin{aligned} Y_0 &= -\frac{\sqrt{(-tx^2 - t + x)^2 - 4t^2x^2} + tx^2 + t - x}{2tx} \\ Y_1 &= \frac{\sqrt{(-tx^2 - t + x)^2 - 4t^2x^2} - tx^2 - t + x}{2tx} \\ Y_0 &= t + t^2\left(x + \frac{1}{x}\right) + t^3\left(x^2 + \frac{1}{x^2} + 3\right) + \mathrm{O}(t^4) \\ Y_1 &= \frac{1}{t} + \left(-x - \frac{1}{x}\right) - t + t^2\left(-x - \frac{1}{x}\right) + t^3\left(-x^2 - \frac{1}{x^2} - 3\right) + \mathrm{O}(t^4) \end{aligned} \tag{8.1.4}$$

The two generators of the symmetry group are

$$\phi : (x,y) \mapsto \left(\frac{1}{x}, y\right) \quad \text{and} \quad \psi : (x,y) \mapsto \left(x, \frac{1}{y}\right). \tag{8.1.5}$$

which generate a group isomorphic to D_2,

$$\left\{(x,y), \left(\frac{1}{x}, y\right), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right)\right\} \tag{8.1.6}$$

We demonstrate the solution of this model by the obstinate kernel method. Following the ideas in Section 4.4, we consider the kernel as an algebraic function of y and substitute root Y_i into (8.1.2) to eliminate the LHS. For $Q(x,y)$ to be a valid power series, we want $Y_i \to 0$ when $t \to 0$. This implies Y_i should be a formal series of t without constant terms. So we choose $Y = Y_0$. This gives our first equation.

$$\frac{bcQ(x,0)\left(-\frac{at}{Y_0} + a - 1\right) + acQ(0,Y_0)\left(-\frac{bt}{x} + b - 1\right) + Q(0,0)(bc - a((b-1)c+b)) + ab}{abc} = 0. \tag{8.1.7}$$

Notice that in the symmetry group, only $(\frac{1}{x}, y)$ is valid since $Q\left(x, \frac{1}{Y_0}\right)$ and $Q\left(\frac{1}{x}, \frac{1}{Y_0}\right)$ do not converge when $t \to 0$. So we apply $(x, Y_0) \to (\frac{1}{x}, Y_0)$ to (8.1.7) and get

$$\frac{bcQ\left(\frac{1}{x}, 0\right)\left(-\frac{at}{Y_0} + a - 1\right) + acQ\left(0, Y_0\right)\left(-btx + b - 1\right) + Q(0,0)(bc - a((b-1)c + b)) + ab}{abc} = 0. \tag{8.1.8}$$

By linear combination of (8.1.7) and (8.1.8), we may eliminate $Q(0, Y_0)$ and get an equation with unknowns $Q(x, 0)$ and $Q\left(\frac{1}{x}, 0\right)$

$$\begin{aligned} -\frac{t(x^2-1)Y_0Q(0,0)(abc+ab-ac-bc)}{c\left(at - aY_0 + Y_0\right)} + Q\left(\frac{1}{x}, 0\right)(bt - bx + x) \\ - xQ(x,0)(btx - b + 1) + \frac{abt(x^2-1)Y_0}{c\left(at - aY_0 + Y_0\right)} = 0 \end{aligned} \tag{8.1.9}$$

Taking the $[x^0]$ part of (8.1.9), we have

$$btQ(0,0) + (1-b)Q_{1,0} + T_2Q(0,0) + T_1 = 0 \tag{8.1.10}$$

where

$$\begin{aligned} T_1 &= [x^0]\frac{abt(x^2-1)Y_0}{c\left(at - aY_0 + Y_0\right)} \\ T_2 &= [x^0] - \frac{t(x^2-1)Y_0(abc+ab-ac-bc)}{c\left(at - aY_0 + Y_0\right)}. \end{aligned} \tag{8.1.11}$$

Note that (8.1.10) holds independently of x, y.

In (8.1.10), we have introduced a new unknown $Q_{1,0}$. By boundary conditions, we know

$$Q(0,0) = 1 + ct(Q_{1,0} + Q_{0,1}). \tag{8.1.12}$$

This implies that if we find another equation involving $Q_{0,1}$ and $Q(0,0)$, we can solve $Q(0,0)$ by linear calculations. Since the walk is symmetric by the diagonal, an equation of $Q_{1,0}$ will always have a corresponding equation of $Q_{0,1}$.

To get the corresponding equation, we consider the kernel as an algebraic function of x. As in the previous discussion, we have two roots

$$\begin{aligned} X_0 &= -\frac{\sqrt{\left(-ty^2 - t + y\right)^2 - 4t^2y^2} + ty^2 + t - y}{2ty} \\ X_1 &= \frac{\sqrt{\left(-ty^2 - t + y\right)^2 - 4t^2y^2} - ty^2 - t + y}{2ty} \\ X_0 &= t + t^2\left(y + \frac{1}{y}\right) + t^3\left(y^2 + \frac{1}{y^2} + 3\right) + \mathrm{O}(t^4) \\ X_1 &= \frac{1}{t} + \left(-y - \frac{1}{y}\right) - t + t^2\left(-y - \frac{1}{y}\right) + t^3\left(-y^2 - \frac{1}{y^2} - 3\right) + \mathrm{O}(t^4). \end{aligned} \tag{8.1.13}$$

We may substitute X_0 into (8.1.2). We have

$$\frac{bcQ\left(X_0, 0\right)\left(-\frac{at}{y} + a - 1\right) + acQ(0, y)\left(-\frac{bt}{X_0} + b - 1\right) + Q(0,0)\left(bc - a((b-1)c + b)\right) + ab}{abc} = 0. \tag{8.1.14}$$

This time, only $(x, \frac{1}{y})$ is valid in the symmetry group. Apply this symmetric transformation to (8.1.2) and we have

$$\frac{bcQ\left(X_0,0\right)\left(-aty+a-1\right)+acQ\left(0,\frac{1}{y}\right)\left(-\frac{bt}{X_0}+b-1\right)+Q(0,0)\left(bc-a((b-1)c+b)\right)+ab}{abc}=0. \tag{8.1.15}$$

Combining (8.1.14) and (8.1.15) together, we eliminate $Q(X_0, 0)$ and get

$$\begin{aligned}-\frac{tX_0(y-1)(y+1)Q(0,0)(abc+ab-ac-bc)}{c\left(bt-bX_0+X_0\right)}+Q\left(0,\frac{1}{y}\right)(at-ay+y)\\ -yQ(0,y)(aty-a+1)+\frac{abtX_0(y-1)(y+1)}{c\left(bt-bX_0+X_0\right)}=0.\end{aligned} \tag{8.1.16}$$

Taking the $[y^0]$ part of (8.1.16) we have

$$atQ(0,0)+(1-a)Q_{0,1}+P_2Q(0,0)+P_1=0 \tag{8.1.17}$$

where

$$\begin{aligned}P_1&=[y^0]\frac{abtX_0(y-1)(y+1)}{c\left(bt-bX_0+X_0\right)}\\ P_2&=[y^0]\frac{tX_0(y-1)(y+1)Q(0,0)(abc+ab-ac-bc)}{c\left(bt-bX_0+X_0\right)}.\end{aligned} \tag{8.1.18}$$

By (8.1.10), (8.1.12), (8.1.17), we immediately have

$$Q(0,0)=\frac{ab+actT_1-a+bcP_1t-b-cP_1t-ctT_1+1}{2abct^2-ab-act^2+actT_2+a+bcP_2t-bct^2+b-cP_2t-ctT_2-1}. \tag{8.1.19}$$

We can see that $Q(0,0)$ is a rational function of T_1, T_2, P_1, P_2. Since T_1, T_2, P_1, P_2 are $[x^0]$ terms of rationals function of $x, t, \sqrt{\Delta}$, they are D-finite by Remark 3. $Q(0,0)$ is a rational function of D-finite terms, thus it is D-algebraic. Further, if a, b, c obey certain relations such that the denominator of $Q(0,0)$ does not contain T_2 or P_2, or certain relations that makes T_2 and P_2 algebraic, then $Q(0,0)$ is a linear sum of D-finite terms and is thus D-finite. By (8.1.19) we first check that T_2, P_2 are cancelled in the denominator if and only if $a = b = 1$. And by the expression of T_2, P_2, we find T_2 and P_2 become algebraic if and only if $abc + ab - ac - bc = 0$. Thus we conclude $Q(0,0)$ is D-finite if and only if

$$abc+ab-ac-bc=0 \quad \text{or} \quad a=b=1. \tag{8.1.20}$$

If we let $c = ab$ we get the previous result in Table 6.1. We may further discuss the effect of point boundary interactions. When $a = b = 1$, the interaction at the origin does not affect the property of $Q(0,0)$. When $a, b \neq 1$, normally $Q(0,0)$ is D-algebraic but not D-finite. However, we can choose $c = ab/(a + b - ab)$ to make $Q(0,0)$ D-finite again.

Substitute $Q(0,0)$ back to (8.1.9) and (8.1.16). Separate the $[x^>]$, $[x^<]$, $[y^>]$, $[y^<]$ parts and we get the solutions of $Q(x,0)$, $Q(0,y)$. Substitute these results back into the functional equation (8.1.2) and we get $Q(x,y)$. They are all D-algebraic over t generally. Notice that (8.1.9) is a linear function of $Q(x,0)$, so taking the $[x^>]$ part of this equation will only introduce D-finite terms into the equation. Thus $Q(x,0)$ is D-finite over x, and $Q(0,y)$ is D-finite over y by the same reason. Then $Q(x,y)$ is D-finite over x, y. This criteria is valid for all previously exactly solved D-algebraic cases (**Models 1, 2, 3, 4, 17, 22**). We will not mention it again in the discussion of each model.

Remark 5. *The way to solve* ***Model 1*** *here is a bit different from that in [7]. In [7], the authors get an equation of $Q(0,0)$ directly. But here, we have both $Q(0,0)$ and $Q_{1,0}$ in (8.1.10) and we use the symmetry of the walk to get an equation of $Q(0,0)$. The difference is due to the different choice of coefficients of $Q(x,0)$ and $Q\left(\frac{1}{x},0\right)$ in (8.1.9). We can multiply (8.1.9) by some rational functions and then take the $[y^0]$ part of the equation. This will give different results. It is better to choose the coefficients carefully such that we get an equation of $Q(0,0)$ only. But in some cases, there will always be other unknown functions such as $Q_{1,0}$ or $Q_{0,1}$ if we take $[y^0]$ part of the equation. Here we demonstrate a way to deal with such problems.*

8.1.2 Model 2: $\{\mathbf{NE},\mathbf{NW},\mathbf{SE},\mathbf{SW}\}$

The second case is walk with allowed steps $\{\nearrow,\nwarrow,\searrow,\swarrow\}$. This model is also D-finite without interactions. We solve this model by the algebraic kernel method.

The functional equation of this model is

$$K(x,y)Q(x,y) = \frac{1}{abc}\left(bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}\right)+a-1\right)Q(x,0)\right.$$
$$\left.+ac\left(-bt\left(\frac{y}{x}+\frac{1}{xy}\right)+b-1\right)Q(0,y)+\left(bc-a((b-1)c+b)\frac{abct}{xy}\right)Q(0,0)+ab\right) \quad (8.1.21)$$

where the kernel is

$$K(x,y) = 1-t\left(xy+\frac{x}{y}+\frac{1}{xy}+\frac{y}{x}\right) \quad (8.1.22)$$

and the formal series root is

$$\begin{aligned} Y_0 &= \frac{x-\sqrt{x^2-4\left(-tx^2-t\right)^2}}{2\left(tx^2+t\right)} \\ &= \frac{t\left(x^2+1\right)}{x}+\frac{t^3\left(x^2+1\right)^3}{x^3}+\mathrm{O}(t^4). \end{aligned} \quad (8.1.23)$$

The two generators are

$$\phi:(x,y)\mapsto\left(\frac{1}{x},y\right) \quad \text{and} \quad \psi:(x,y)\mapsto\left(x,\frac{1}{y}\right) \quad (8.1.24)$$

and the symmetry group is

$$\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{1}{y}\right),\left(\frac{1}{x},\frac{1}{y}\right)\right\} \quad (8.1.25)$$

which is exactly the same as **Model 1**. Following the ideas in Section 4.5, we directly apply the symmetry group to the functional equation and get four different equations. For simplicity, we prefer to write them in a matrix form

$$K(x,y)\mathbf{Q} = \mathbf{MV}+\mathbf{C}. \quad (8.1.26)$$

where $\mathbf{Q}$ is the column vector of all transformed $Q(x,y)$,

$$\mathbf{Q} = \begin{pmatrix} Q(x,y) \\ Q\left(\frac{1}{x},y\right) \\ Q\left(x,\frac{1}{y}\right) \\ Q\left(\frac{1}{x},\frac{1}{y}\right) \end{pmatrix}, \quad (8.1.27)$$

and $\mathbf{V}$ is the transformed line boundary terms

$$\mathbf{V} = \begin{pmatrix} Q(x,0) \\ Q(0,y) \\ Q\left(\frac{1}{x},0\right) \\ Q\left(0,\frac{1}{y}\right) \end{pmatrix}, \tag{8.1.28}$$

$\mathbf{M}$ is the coefficient matrix

$$\mathbf{M} = \begin{pmatrix} A'(x,y) & B'(x,y) & 0 & 0 \\ 0 & B'\left(\frac{1}{x},y\right) & A'\left(\frac{1}{x},y\right) & 0 \\ A'\left(x,\frac{1}{y}\right) & 0 & 0 & B'\left(x,\frac{1}{y}\right) \\ 0 & 0 & A'\left(\frac{1}{x},\frac{1}{y}\right) & B'\left(\frac{1}{x},\frac{1}{y}\right) \end{pmatrix} \tag{8.1.29}$$

where $A'(x,y) = bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}\right)+a-1\right)$ and $B'(x,y) = ac(-bt\left(\frac{y}{x}+\frac{1}{xy}\right)+b-1)$. The column vector $\mathbf{C}$ contains the point boundary terms and some known terms in $\mathbb{R}(x,y,t)[Q(0,0)]$.

$$\mathbf{C} = \begin{pmatrix} \frac{Q(0,0)(abct-abcxy-abxy+acxy+bcxy)}{abcxy} + \frac{1}{c} \\ \frac{Q(0,0)(abctx-abcy-aby+acy+bcy)}{abcy} + \frac{1}{c} \\ \frac{Q(0,0)(abcty-abcx-abx+acx+bcx)}{abcx} + \frac{1}{c} \\ \frac{Q(0,0)(abctxy-abc-ab+ac+bc)}{abc} + \frac{1}{c} \end{pmatrix}. \tag{8.1.30}$$

It is not hard to find that the determinant of $\mathbf{M}$ equals zero. This means the coefficient matrix is not full rank and we can find a vector $\mathbf{N}$ in the nullspace such that

$$K(x,y)\mathbf{NQ} = \mathbf{NMV} + \mathbf{NC} = \mathbf{NC}. \tag{8.1.31}$$

We can multiply the vector by any scalar. We choose a good scalar such that the coefficients of all transformed $Q(x,y)$ are in $\mathbb{R}\left(x,1/x,y,1/y,t\right)$ and the degree and valuation of x,y are small enough. We may choose

$$\mathbf{N}^T = \begin{pmatrix} -\frac{\left(x^2+1\right)^2\left(atx^2y+aty-ax+x\right)\left(btxy^2+btx-by+y\right)}{x^2} \\ -\frac{\left(x^2+1\right)^2\left(atx^2y+aty-ax+x\right)\left(-bty^2-bt+bxy-xy\right)}{x^3} \\ \frac{\left(x^2+1\right)^2\left(atx^2+at-axy+xy\right)\left(btxy^2+btx-by+y\right)}{x^2y} \\ \frac{\left(x^2+1\right)^2\left(atx^2+at-axy+xy\right)\left(-bty^2-bt+bxy-xy\right)}{x^3y} \end{pmatrix}. \tag{8.1.32}$$

Now, we first divide the RHS by $K(x,y)$. Then the RHS reads,

$$\begin{aligned} \frac{\mathbf{NC}}{K(x,y)} = \frac{1}{K(x,y)} \Bigg(& -\frac{abt^2(x^2-1)\left(x^2+1\right)^3(y-1)(y+1)\left(y^2+1\right)}{cx^3y} \\ & +\frac{1}{cx^3y}t(x^2-1)\left(x^2+1\right)^2(y-1)(y+1) \\ & Q(0,0)\left(abctx^2y^2+abctx^2+abcty^2+abct-abcxy+abtx^2y^2+abtx^2\right. \\ & +abty^2+abt-actx^2y^2-actx^2-acty^2-act+acxy \\ & \left.-bctx^2y^2-bctx^2-bcty^2-bct+bcxy-cxy\right)\Bigg) \\ & = R_1 + R_2Q(0,0) \end{aligned} \tag{8.1.33}$$

Since the only unknown function in $\mathbf{NC}/K(x,y)$ is $Q(0,0)$ and it is independent of x, y, we can take the $[y^>]$, $[y^<]$, $[y^i]$, $[x^>]$, $[x^<]$, $[x^i]$ parts of (8.1.33) without introducing any new unknown function of x. By Theorems 10 and 14 and Proposition 2.5 in [52], we know they are all D-finite over x, y, t.

Now, let us consider the LHS. The LHS is a linear sum of transformed $Q(x,y)$ with coefficients in $\mathbb{R}[x, 1/x, y, 1/y, t]$. This means we can take the $[y^i]$ part of the LHS without introducing an infinite number of $Q_{i,j}$, for example

$$[y^0]\left(-\frac{\left(x^2+1\right)^2\left(atx^2y+aty-ax+x\right)\left(btxy^2+btx-by+y\right)}{x^2}Q(x,y)\right) \\ = (a-1)bt\left(x^2+1\right)^2 Q(x,0) \quad (8.1.34)$$

Thus, we have

$$\begin{aligned}
[y^0]\mathbf{NQ} = {} & \frac{a(b-1)t\left(x^2+1\right)^3 Q\left(\frac{1}{x},0\right)}{x^2} \\
& - \frac{\left(x^2+1\right)^2 Q_{-,1}\left(\frac{1}{x}\right)\left(abt^2x^2+abt^2+abx^2-ax^2-bx^2+x^2\right)}{x^3} \\
& + \frac{(a-1)bt\left(x^2+1\right)^2 Q_{-,2}\left(\frac{1}{x}\right)}{x^2} - \frac{a(b-1)t\left(x^2+1\right)^3 Q(x,0)}{x^2} \\
& \frac{\left(x^2+1\right)^2 Q_{-,1}(x)\left(abt^2x^2+abt^2+ab-a-b+1\right)}{x} \\
& - (a-1)bt\left(x^2+1\right)^2 Q_{-,2}(x)
\end{aligned} \quad (8.1.35)$$

We have introduced some new unknown boundary functions. However, by the geometric properties of this model,

$$\begin{aligned}
Q_{-,2}(x) &= \frac{-btx^2Q(x,0)-btQ(x,0)+btQ(0,0)+btQ_{0,2}-bxQ_{0,1}+bxQ_{-,1}(x)+xQ_{0,1}}{bt\left(x^2+1\right)} \\
Q_{-,1}(x) &= \frac{actQ_{0,1}+axQ(0,0)-cxQ(0,0)+cxQ(x,0)-ax}{act\left(x^2+1\right)},
\end{aligned} \quad (8.1.36)$$

so we can express $Q_{-,2}(x)$, $Q_{-,1}(x)$ by linear functions of $Q(x,0)$. We may thus eliminate $Q_{-,2}(x)$, $Q_{-,1}(x)$ and finally get,

$$\begin{aligned}
[y^0]\mathbf{NQ} = {} & -\frac{b(x^2-1)\left(at^2x^4+2at^2x^2+at^2-ax^2+x^2\right)}{ctx^2} \\
& + \frac{b(x^2-1)Q(0,0)(a-c)\left(at^2x^4+2at^2x^2+at^2-ax^2+x^2\right)}{actx^2} \\
& + \frac{\left(bx^2-x^2-1\right)Q\left(\frac{1}{x},0\right)\left(a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2\right)}{atx^2} \\
& + \frac{\left(-b+x^2+1\right)Q(x,0)\left(a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2\right)}{atx^2}
\end{aligned} \quad (8.1.37)$$

Now rearranging the equation $[y^0]\mathbf{NQ} = [y^0]\frac{\mathbf{NC}}{K(x,y)}$, we have

$$
\begin{aligned}
Q(x,0) - &\frac{\left(bx^2 - x^2 - 1\right) Q\left(\frac{1}{x}, 0\right)}{b - x^2 - 1} \\
&+ \left(- \frac{b(x^2-1)(a-c)\left(at^2x^4 + 2at^2x^2 + at^2 - ax^2 + x^2\right)}{c\left(b - x^2 - 1\right)\left(a^2t^2x^4 + 2a^2t^2x^2 + a^2t^2 - ax^2 + x^2\right)} - [y^0]R_2 \right) Q(0,0) \\
&\qquad + \frac{ab(x^2-1)\left(at^2x^4 + 2at^2x^2 + at^2 - ax^2 + x^2\right)}{c\left(b - x^2 - 1\right)\left(a^2t^2x^4 + 2a^2t^2x^2 + a^2t^2 - ax^2 + x^2\right)} - [y^0]R_1 = 0. \quad (8.1.38)
\end{aligned}
$$

We want to take the $[x^0]$ part of (8.1.38). Several terms in this equation are not polynomials. We need to expand them into series for further calculations. First, notice that the coefficient of $Q\left(\frac{1}{x}, 0\right)$ can be written as

$$
-\frac{bx^2 - x^2 - 1}{b - x^2 - 1} = (b-1) + \frac{b^2 - 2b}{x^2} + \frac{b^3 - 3b^2 + 2b}{x^4} + \mathrm{O}\left(\left(\frac{1}{x}\right)^6\right) \quad (8.1.39)
$$

for $|x| > \sqrt{b-1}$. The term $b - x^2 - 1$ also appears in the denominator of other terms. They all have to be expanded as a series of $\frac{1}{x}$. This is because if we expand one of them as series in x, the convergent area is $|x| < \sqrt{b-1}$ which does not intersect with $|x| > \sqrt{b-1}$. Next, we expand $\frac{1}{a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2}$ as a formal series of t. The idea is the same as that in Section 4.1. We can write $\frac{1}{a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2}$ as $\frac{1}{a^2t^2(x-X_1)(x-X_2)(x-X_3)(x-X_4)}$. The polynomial in the denominator has four roots. Their series expansions can be written as

$$
\begin{aligned}
X_1 &= -\frac{at}{\sqrt{a-1}} - \frac{a^3t^3}{(a-1)^{3/2}} + \mathrm{O}(t^5) \\
X_2 &= \frac{at}{\sqrt{a-1}} + \frac{a^3t^3}{(a-1)^{3/2}} + \mathrm{O}(t^5) \\
X_3 &= -\frac{\sqrt{a-1}}{at} + \frac{at}{\sqrt{a-1}} + \frac{a^3t^3}{(a-1)^{3/2}} + \mathrm{O}(t^5) \\
X_4 &= -\frac{\sqrt{a-1}}{at} + \frac{at}{\sqrt{a-1}} + \frac{a^3t^3}{(a-1)^{3/2}} + \mathrm{O}(t^5).
\end{aligned} \quad (8.1.40)
$$

Now X_1 and X_2 are two formal series of t without constant terms. The corresponding factors should be expanded as $\sum(\frac{X_1}{x})^i$ and $\sum(\frac{X_2}{x})^i$. The convergent domain is $\max(X_1, X_2) < x$. On the other hand, X_3 and X_4 have $\frac{1}{t}$ in the expansion. Further notice

$$
\begin{aligned}
\frac{1}{X_3} &= -\frac{at}{\sqrt{a-1}} - \frac{a^3t^3}{(a-1)^{3/2}} + \mathrm{O}(t^5) \\
\frac{1}{X_4} &= \frac{at}{\sqrt{a-1}} + \frac{a^3t^3}{(a-1)^{3/2}} + \mathrm{O}(t^5).
\end{aligned} \quad (8.1.41)
$$

Their corresponding factors should be expanded as $\sum(\frac{x}{X_3})^i$ and $\sum(\frac{x}{X_4})^i$. The convergent domain is $x < \min(|X_3|, |X_4|)$. Thus the convergent domain of $\frac{1}{a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2}$ is some annulus around 0. By similar calculations, we find the convergent domains of $[y^0]R_2$, $[y^0]R_1$ are also annuli with similar properties. We further require these convergent domains to intersect. Notice that X_1, X_2 are formal series of t without constant terms and X_3, X_4 contain $\frac{1}{t}$ in their expansion. We may choose t small enough such that the inner circle of the annulus tends to 0 and the outer circle terms to infinity. Then this annulus always intersects with $|x| < \sqrt{b-1}$. Now take the $[x^0]$ part of (8.1.38), we have

$$Q(0,0) = \frac{T_1}{b + [x^0](-\frac{b(x^2-1)(a-c)(at^2x^4+2at^2x^2+at^2-ax^2+x^2)}{c(b-x^2-1)(a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2)} - T_2)} \tag{8.1.42}$$

where

$$\begin{aligned} T_1 &= [x^0]\left(\frac{ab(x^2-1)\left(at^2x^4+2at^2x^2+at^2-ax^2+x^2\right)}{c\left(b-x^2-1\right)\left(a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2\right)} - [y^0]R_1\right) \\ T_2 &= [x^0y^0]R_2. \end{aligned} \tag{8.1.43}$$

The expressions of R_1 and R_2 can be found in (8.1.33). By (4.5.2), $[y^0]R_1, [y^0]R_2$ contain square roots. So T_1, T_2 are D-finite.

Similar to **Model 1**, this $Q(0,0)$ is a rational function of D-finite terms and is thus D-algebraic. For $Q(0,0)$ to be D-finite, we first consider $[x^0]\left(-\frac{b(x^2-1)(a-c)\left(at^2x^4+2at^2x^2+at^2-ax^2+x^2\right)}{c(b-x^2-1)(a^2t^2x^4+2a^2t^2x^2+a^2t^2-ax^2+x^2)}\right)$. It is algebraic by Theorem 9. Then we consider T_2.

To calculate T_2, we recall (4.5.2) and (4.5.3). We have

$$\begin{aligned} [y^0]R_2 = {} & \frac{1}{c\sqrt{\Delta}x^3Y_1}t(x^2-1)\left(x^2+1\right)^2 \\ & \big(abctx^2Y_0^3Y_1 - abctx^2 + abctY_0^3Y_1 - abct - abcxY_0^2Y_1 + abcxY_1 + abtx^2Y_0^3Y_1 - abtx^2 \\ & +abtY_0^3Y_1 - abt - actx^2Y_0^3Y_1 + actx^2 - actY_0^3Y_1 + act + acxY_0^2Y_1 - acxY_1 - bctx^2Y_0^3Y_1 \\ & +bctx^2 - bctY_0^3Y_1 + bct + bcxY_0^2Y_1 - bcxY_1 - cxY_0^2Y_1 + cxY_1\big) \end{aligned} \tag{8.1.44}$$

Substitute Y_0 and Y_1 in and by calculation, we finally have

$$\begin{aligned} [y^0]R_2 = {} & (ab-c)x^2\sqrt{1-4t^2\left(x+\frac{1}{x}\right)^2} + 2abct^2x^4 + 4abct^2x^2 \\ & + 2abct^2 + 2abt^2x^4 + 4abt^2x^2 \\ & + 2abt^2 - abx^2 - 2act^2x^4 - 4act^2x^2 \\ & - 2act^2 - 2bct^2x^4 - 4bct^2x^2 - 2bct^2 + cx^2. \end{aligned} \tag{8.1.45}$$

If $[y^0]R_2$ is a polynomial, T_2 is algebraic and $Q(0,0)$ becomes D-finite. Thus, we have $Q(0,0)$ is D-finite if $ab = c$, which is accidentally the result in Table 6.1.

8.1.3 Model 3: $\{\mathbf{N}, \mathbf{NE}, \mathbf{SE}, \mathbf{S}, \mathbf{SW}, \mathbf{NW}\}$

The allowed steps of the third model are $\{\uparrow, \nearrow, \searrow, \downarrow, \swarrow, \nwarrow\}$. The functional equation of this model is

$$\begin{aligned} K(x,y)Q(x,y) = {} & \frac{1}{abc}\left(bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}+\frac{1}{y}\right)+a-1\right)Q(x,0)\right. \\ & \left. +ac\left(-bt\left(\frac{y}{x}+\frac{1}{xy}\right)+b-1\right)Q(0,y) + \left(bc - a((b-1)c+b) + \frac{abct}{xy}\right)Q(0,0) + ab\right) \end{aligned} \tag{8.1.46}$$

where

$$K(x,y) = 1 - t\left(xy + \frac{x}{y} + \frac{1}{xy} + \frac{y}{x} + y + \frac{1}{y}\right). \tag{8.1.47}$$

The formal series root is

$$\begin{aligned} Y_0 &= \frac{x - \sqrt{x^2 - 4\left(-tx^2 - tx - t\right)^2}}{2\left(tx^2 + tx + t\right)} \\ Y_0 &= \frac{t\left(x^2+x+1\right)}{x} + \frac{t^3\left(x^2+x+1\right)^3}{x^3} + \frac{2t^5\left(x^2+x+1\right)^5}{x^5} + \mathrm{O}(t^7). \end{aligned} \tag{8.1.48}$$

The symmetry group is still

$$\left\{(x,y), \left(\frac{1}{x}, y\right), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right)\right\} \tag{8.1.49}$$

We can use either the obstinate kernel method or the algebraic kernel method to obtain the following equation,

$$-Q\left(\frac{1}{x}, 0\right) - \frac{-bx - b + x^2 + x + 1}{bx^2 + bx - x^2 - x - 1} Q(x,0) + S_1 + S_2 Q(0,0) = 0 \tag{8.1.50}$$

where

$$\begin{aligned} S_1 &= + \frac{abtx(x^2-1)Y_0\left(Y_0^2+1\right)}{c\left(atx^2 + atx + at - axY_0 + xY_0\right)\left(-btY_0^2 - bt + bxY_0 - xY_0\right)} \\ S_2 &= - \frac{tx(x^2-1)Y_0\left(abcY_0^2 + abY_0^2 + ab - acY_0^2 - bcY_0^2 - bc\right)}{c\left(atx^2 + atx + at - axY_0 + xY_0\right)\left(-btY_0^2 - bt + bxY_0 - xY_0\right)}. \end{aligned} \tag{8.1.51}$$

The solution is written as

$$Q(0,0) = \frac{T_1}{1 - [x^0]\left(-\frac{-bx-b+x^2+x+1}{bx^2+bx-x^2-x-1}\right) - T_2} \tag{8.1.52}$$

where $T_1 = [x^0]S_1$ and $T_2 = [x^0]S_2$.

For $Q(0,0)$ to be D-finite, T_2 should be algebraic. By simple calculation, we find

$$T_2 = [x^0]\left(A + \frac{-8act^4x(x^2-1)\left(x^2+x+1\right)^4(c-ab)}{B}\sqrt{x^2 - 4\left(-tx^2 - tx - t\right)^2}\right) \tag{8.1.53}$$

where A, B are polynomials. Thus when $c = ab$, $Q(0,0)$ is D-finite.

8.1.4 Model 4: $\{\mathbf{N}, \mathbf{NE}, \mathbf{E}, \mathbf{SE}, \mathbf{S}, \mathbf{SW}, \mathbf{W}, \mathbf{NW}\}$

The allowed steps of fourth model are $\{\uparrow, \nearrow, \leftarrow, \searrow, \downarrow, \swarrow, \rightarrow, \nwarrow\}$. The functional equation is

$$\begin{aligned} K(x,y)Q(x,y) &= \frac{1}{abc}\left(bc\left(-at\left(\frac{x}{y} + \frac{1}{xy} + \frac{1}{y}\right) + a - 1\right)Q(x,0)\right. \\ &\left. + ac\left(-bt\left(\frac{y}{x} + \frac{1}{xy} + \frac{1}{x}\right) + b - 1\right)Q(0,y) + \left(bc - a((b-1)c + b) + \frac{abct}{xy}\right)Q(0,0) + ab\right) \end{aligned} \tag{8.1.54}$$

where

$$K(x,y) = 1 - t\left(xy + \frac{x}{y} + \frac{1}{xy} + \frac{y}{x} + x + \frac{1}{x} + y + \frac{1}{y}\right). \tag{8.1.55}$$

The formal series root is

$$\begin{aligned} Y_0 &= \frac{tx^2 + \sqrt{(-tx^2 - t + x)^2 - 4(-tx^2 - tx - t)^2} + t - x}{2(-tx^2 - tx - t)} \\ Y_0 &= \frac{t(x^2+x+1)}{x} + \frac{t^2(x^2+1)(x^2+x+1)}{x^2} \\ &\quad + \frac{t^3(x^2+x+1)(2x^4+2x^3+5x^2+2x+2)}{x^3} + \mathrm{O}(t^4). \end{aligned} \tag{8.1.56}$$

The symmetry group is still

$$\left\{(x,y), \left(\frac{1}{x}, y\right), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right)\right\} \tag{8.1.57}$$

We can use either the obstinate kernel method or the algebraic kernel method to obtain the following equation,

$$-Q\left(\frac{1}{x}, 0\right) + \frac{btx^2 - bx - b + x^2 + x + 1}{bt - bx^2 - bx + x^2 + x + 1} Q(x,0) + S_1 + S_2 Q(0,0) = 0 \tag{8.1.58}$$

where

$$\begin{aligned} S_1 &= \frac{abtx(x^2-1)Y_0\left(Y_0^2 + Y_0 + 1\right)}{c\left(atx^2 + atx + at - axY_0 + xY_0\right)\left(-btY_0^2 - btY_0 - bt + bxY_0 - xY_0\right)} \\ S_2 &= -\frac{tx(x^2-1)Y_0\left(abcY_0^2 + abcY_0 + abY_0^2 + abY_0 + ab - acY_0^2 - acY_0 - bcY_0^2 - bcY_0 - bc\right)}{c\left(atx^2 + atx + at - axY_0 + xY_0\right)\left(-btY_0^2 - btY_0 - bt + bxY_0 - xY_0\right)}. \end{aligned} \tag{8.1.59}$$

The solution can be written as

$$Q(0,0) = \frac{T_1}{1 - [x^0]\left(\frac{btx^2 - bx - b + x^2 + x + 1}{bt - bx^2 - bx + x^2 + x + 1}\right) - T_2} \tag{8.1.60}$$

where $T_1 = [x^0]S_1$, $T_2 = [x^0]S_2$ as before. For $Q(0,0)$ to be D-finite, we check the condition for T_2 to become algebraic. By simple calculation, we find

$$\begin{aligned} T_2 = [x^0]\Bigg(A &+ \frac{ax(x^2-1)(abct + abt + ab - act - bct - c)}{B} \\ &\times \sqrt{(-tx^2 - t + x)^2 - 4(-tx^2 - tx - t)^2}\Bigg) \end{aligned} \tag{8.1.61}$$

where A, B are polynomials. The coefficient of $\sqrt{(-tx^2 - t + x)^2 - 4(-tx^2 - tx - t)^2}$ equals 0 if and only if

$$c = ab \text{ and } ab - a - b + 1 = 0 \tag{8.1.62}$$

These are the criteria for the D-finite property of $Q(0,0)$.

8.1.5 Model 17: $\{\mathbf{N}, \mathbf{W}, \mathbf{SE}\}$

The next exactly solved model is the **Model 17**. The allowed steps are $\{\uparrow, \leftarrow, \searrow\}$. The functional equation is

$$\begin{aligned}K(x,y)Q(x,y) = \frac{1}{abc}\Bigg(&bc\left(-\frac{atx}{y}+a-1\right)Q(x,0)\\&+ac\left(-\frac{bt}{x}+b-1\right)Q(0,y)+(bc-a((b-1)c+b))\,Q(0,0)+ab\Bigg)\end{aligned} \tag{8.1.63}$$

where

$$K(x,y) = 1-t\left(\frac{x}{y}+\frac{1}{x}+y\right) \tag{8.1.64}$$

and the formal series root is

$$\begin{aligned}Y_0 &= \frac{-\sqrt{-4t^2x^3+t^2-2tx+x^2}-t+x}{2tx}\\ Y_0 &= tx+t^2+t^3\left(x^2+\frac{1}{x}\right)+t^4\left(\frac{1}{x^2}+3x\right)+t^5\left(2x^3+\frac{1}{x^3}+6\right)+\mathrm{O}(t^6).\end{aligned} \tag{8.1.65}$$

The symmetry group is

$$\left\{(x,y),\left(\frac{y}{x},y\right),\left(x,\frac{x}{y}\right),\left(\frac{1}{y},\frac{x}{y}\right),\left(\frac{y}{x},\frac{1}{x}\right),\left(\frac{1}{y},\frac{1}{x}\right)\right\}. \tag{8.1.66}$$

Unlike the previous cases, this group is isomorphic to D_3. For the obstinate kernel method, we find $Q(x,Y_0), Q\left(\frac{Y_0}{x},Y_0\right), Q\left(\frac{Y_0}{x},\frac{1}{x}\right)$ are three valid pairs. We can then use a linear combination to try to solve it. However, for D_3 group cases, we prefer the algebraic kernel method which follows a more computational procedure.

By either the obstinate kernel method or the algebraic kernel method, we get

$$\frac{aQ\left(0,\frac{1}{x}\right)(-at+ax-x)(-bt+bx-x)}{bx\left(a^2t^2x^2+a^2t-at-ax+x\right)}+Q(x,0)+S_1+S_2Q(0,0)=0 \tag{8.1.67}$$

where

$$\begin{aligned}S_1 = &\frac{-a}{2cx\left(a^2t^2x^2+a^2t-at-ax+x\right)\left(b^2t^2x^2+b^2t-bt-bx+x\right)}\big(ab^2t^3x^3-2ab^2t^3\\&-3ab^2t^2x^4+ab^2t^2x-ab^2tx^2+2abt^2x^4+2abt^2x-2abtx^2+2abx^3+2atx^2-2ax^3+2b^2t^2x^4\\&+2b^2tx^2-2btx^2-2bx^3+2x^3-ab^2t\left(tx^3+2t-x\right)\sqrt{-4t^2x^3+t^2-2tx+x^2}\big)\\ S_2 = &\frac{abc+ab-ac-bc}{2bcx\left(a^2t^2x^2+a^2t-at-ax+x\right)\left(b^2t^2x^2+b^2t-bt-bx+x\right)}\\&\big(ab^2t^3x^3-2ab^2t^3-3ab^2t^2x^4+ab^2t^2x\\&-ab^2tx^2+2abt^2x^4+2abt^2x-2abtx^2+2abx^3+2atx^2-2ax^3+2b^2t^2x^4\\&+2b^2tx^2-2btx^2-2bx^3+2x^3-ab^2t\left(tx^3+2t-x\right)\sqrt{-4t^2x^3+t^2-2tx+x^2}\big).\end{aligned} \tag{8.1.68}$$

We want to take the $[x^0]$ term of this equation. Consider the coefficient of $Q\left(\frac{1}{x},0\right)$. The denominator is a quadratic function $a^2t^2x^2+a^2t-at-ax+x$. It has two roots X_1, X_2. Using the Newton polygon method (or just solving these two roots and expanding them as series of t), we find one root has degree 1 (X_1) in t and another has valuation -1 (X_2) in t. By Theorem 6, the coefficient of $Q(1/x, 0)$ cannot be expanded as a series of $\frac{1}{x}$ at the same time as a formal series of t. To deal with this situation, we multiply (8.1.67) by $x - X_2$. Then the denominator of coefficient of $Q\left(\frac{1}{x}, 0\right)$ only has a root X_0 and the coefficient of $Q\left(\frac{1}{x}, 0\right)$ can be expanded as a formal series of $\frac{1}{x}$. The multiplication does not bring any problems to other terms since $x - X_2$ is just a polynomial in x. After some calculation, we find

$$Q(0,0) = \frac{2bT_1}{b\sqrt{-(a-1)\left(4a^3t^3 - a + 1\right)} - 2a^2b + 2a^2 + 3ab - 2a - 2bT_2 - b} \tag{8.1.69}$$

where

$$\begin{aligned} T_1 &= [x^0]S_1(x - X_2) \\ T_2 &= [x^0]S_2(x - X_2). \end{aligned} \tag{8.1.70}$$

For T_2 to be algebraic, the coefficient of $\sqrt{-4t^2x^3 + t^2 - 2tx + x^2}$ in S_2 should be 0. Clearly, it implies

$$(ab - ac - bc + abc) = 0. \tag{8.1.71}$$

8.1.6 Model 22: $\{\mathbf{NW}, \mathbf{W}, \mathbf{SE}, \mathbf{E}\}$

The last D-algebraic model that was solved with $c = ab$ before is **Model 22**. The allowed steps are $\{\nwarrow, \leftarrow, \searrow, \rightarrow\}$. With arbitrary a, b, c, the functional equation is written as

$$\begin{aligned} K(x,y)Q(x,y) = \frac{1}{abc}\Big(bc\left(-\frac{atx}{y} + a - 1\right)Q(x,0) \\ + ac\left(-bt\left(\frac{y}{x} + \frac{1}{x}\right) + b - 1\right)Q(0,y) + (bc - a((b-1)c + b))Q(0,0) + ab\Big) \end{aligned} \tag{8.1.72}$$

where

$$K(x,y) = 1 - t\left(\frac{x}{y} + \frac{y}{x} + x + \frac{1}{x}\right). \tag{8.1.73}$$

The formal series root is

$$\begin{aligned} Y_0 &= -\frac{\sqrt{\left(-tx^2 - t + x\right)^2 - 4t^2x^2} + tx^2 + t - x}{2t} \\ Y_0 &= tx + t^2\left(x^2 + 1\right) + t^3\left(x^3 + 3x + \frac{1}{x}\right) + t^4\left(x^4 + 6x^2 + \frac{1}{x^2} + 6\right) + \mathrm{O}(t^5) \end{aligned} \tag{8.1.74}$$

This walk has a symmetry group isomorphic to D_4, and the group elements are

$$\left\{(x,y), \left(\frac{y}{x}, y\right), \left(x, \frac{x^2}{y}\right), \left(\frac{x}{y}, \frac{x^2}{y}\right), \left(\frac{y}{x}, \frac{y}{x^2}\right), \left(\frac{x}{y}, \frac{1}{y}\right), \left(\frac{1}{y}, \frac{y}{x^2}\right), \left(\frac{1}{x}, \frac{1}{y}\right)\right\} \tag{8.1.75}$$

By the obstinate kernel method or the algebraic kernel method, we have

$$\frac{Q\left(\frac{1}{x}, 0\right)(-at + ax - x)}{x} + \frac{Q(x,0)(atx - a + 1)\left(b^2t + btx^2 - bt - bx + x\right)}{b^2tx^2 - btx^2 + bt - bx + x} + S_1 + S_2Q(0,0) = 0. \tag{8.1.76}$$

where

$$
\begin{aligned}
S_1 = &\frac{abt(x^2-1)}{2cx\left(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x\right)\left(b^2tx^2-btx^2+bt-bx+x\right)}\\
&\Big(2a^2bt^2x^2+2a^2btx^3+2a^2btx+2a^2t^2x^4+2a^2t^2-3a^2tx^3\\
&-3a^2tx-a^2x^2-2abtx^3-2abtx-2abx^2+6ax^2+2bx^2-4x^2\\
&+a^2\left(2tx^2+2t-x\right)\sqrt{\left(-tx^2-t+x\right)^2-4t^2x^2}\Big)\\
S_2 = &\frac{t(x^2-1)(abc+ab-ac-bc)}{2cx\left(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x\right)\left(b^2tx^2-btx^2+bt-bx+x\right)}\\
&\Big(2a^2bt^2x^2+2a^2btx^3+2a^2btx+2a^2t^2x^4+2a^2t^2-3a^2tx^3\\
&-3a^2tx-a^2x^2-2abtx^3-2abtx-2abx^2+6ax^2+2bx^2-4x^2\\
&+a^2\left(2tx^2+2t-x\right)\sqrt{\left(-tx^2-t+x\right)^2-4t^2x^2}\Big).
\end{aligned}
\tag{8.1.77}
$$

The denominator of the $Q(x,0)$ coefficient again has a polynomial factor $b^2tx^2-btx^2+bt-bx+x$. It has a root (X_1) of degree 1 in t and a root (X_2) of valuation -1 in t. Since we need to expand the coefficient as a series of x, we multiply both sides by $b^2t(x-X_1)$, then (8.1.76) becomes

$$
\begin{aligned}
&\frac{Q\left(\frac{1}{x},0\right)b^2t(x-X_1)(-at+ax-x)}{x}+\frac{Q(x,0)(atx-a+1)\left(b^2t+btx^2-bt-bx+x\right)}{(x-X_2)}\\
&\qquad+S_1b^2t(x-X_1)+S_2(x-X_1)b^2tQ(0,0)=0
\end{aligned}
\tag{8.1.78}
$$

Then we take the $[x^0]$ part of (8.1.78). When taking the $[x^0]$ part of $\frac{Q\left(\frac{1}{x},0\right)b^2t(x-X_1)(-at+ax-x)}{x}$, we introduce another unknown function, $Q_{1,0}$.

$$
[x^0]\frac{Q\left(\frac{1}{x},0\right)b^2t(x-X_1)(-at+ax-x)}{x}=b^2t[(1-a)Q_{1,0}+(at-(1-a)X_1)Q(0,0)]. \tag{8.1.79}
$$

For **Model 22**, we fortunately have $Q(0,0)=1+ctQ_{1,0}$ by the geometric property. We can eliminate $Q_{1,0}$ by substitution. Further,

$$
[x^0]\frac{Q(x,0)(atx-a+1)\left(b^2t+btx^2-bt-bx+x\right)}{(x-X_2)}=\frac{-b^2t(1-a)}{X_2}Q(0,0). \tag{8.1.80}
$$

Then we can solve $Q(0,0)$:

$$
Q(0,0)=\frac{\frac{1-a}{ct}-T_1}{\frac{1-a}{ct}+at+(1-a)X_1-\frac{1-a}{X_2}+T_2} \tag{8.1.81}
$$

where

$$
\begin{aligned}
T_1&=[x^0]S_1b^2t(x-X_1)\\
T_2&=[x^0]S_2b^2t(x-X_1).
\end{aligned}
\tag{8.1.82}
$$

For $Q(0,0)$ to be D-finite, we require T_2 to be algebraic. By calculation, S_2 reads

$$
-\frac{a^2bt(x^2-1)\left(2tx^2+2t-x\right)(abc+ab-ac-bc)\left(\sqrt{-(b-1)\left(4b^2t^2-b+1\right)}+2b^2tx-2btx-b+1\right)}{4(b-1)cx\left(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x\right)\left(b^2tx^2-btx^2+bt-bx+x\right)} \tag{8.1.83}
$$

There are two cases for (8.1.83) to be 0. First,

$$abc + ab - ac - bc = 0. \tag{8.1.84}$$

Second,

$$\left(\sqrt{-(b-1)\left(4b^2t^2 - b + 1\right)} + 2b^2tx - 2btx - b + 1\right) = 0 \tag{8.1.85}$$

for arbitrary t and x. The condition (8.1.85) holds if and only if $b = 1$. Thus our conclusion is $Q(0,0)$ is D-finite if $abc + ab - ac - bc = 0$ or $b = 1$.

8.2 Algebraic cases

The previous models are all D-algebraic with arbitrary a, b, c and D-finite with $a = b = c = 1$. They were all solved with $c = ab$. Now we come to two special cases, Kreweras and reverse Kreweras walks. These two models have algebraic generating function when $a = b$ and $c = ab$. We will first solve reverse Kreweras walks and prove that the generating function is algebraic. Then we will solve Kreweras walks and show the special qualities of this model. We will clearly go through the algebraic kernel method twice again to solve these two problems.

8.2.1 Reverse Kreweras walk

For reverse Kreweras walks, the allowed steps are $\{\swarrow, \uparrow, \rightarrow\}$. So we have

$$S(x, y) = x + y + \frac{1}{xy}. \tag{8.2.1}$$

The functional equation (6.1.3) becomes

$$\begin{aligned}\left(1 - t\left(x + y + \frac{1}{xy}\right)\right) Q(x, y) = \frac{1}{c} + \frac{1}{a}\left(a - 1 - \frac{at}{xy}\right) Q(x, 0) + \frac{1}{b}\left(b - 1 - \frac{bt}{xy}\right) Q(0, y) \\ + \left(\frac{1}{abc}(ac + bc - ab - abc) + \frac{t}{xy}\right) Q(0, 0).\end{aligned} \tag{8.2.2}$$

The symmetry group

The two generators of symmetry group are

$$\phi : (x, y) \mapsto \left(\frac{1}{xy}, y\right) \quad \text{and} \quad \psi : (x, y) \mapsto \left(x, \frac{1}{xy}\right). \tag{8.2.3}$$

The resulting group is isomorphic to D_3:

$$\left\{(x, y), \left(\frac{1}{xy}, y\right), \left(y, \frac{1}{xy}\right), (y, x), \left(\frac{1}{xy}, x\right) \left(x, \frac{1}{xy}\right)\right\} \tag{8.2.4}$$

Full-orbit sum

We use the algebraic kernel method. Applying all these symmetries to the functional equation (8.2.2), we have,

$$K(x, y)\mathbf{Q} = \mathbf{M}\mathbf{V} + \mathbf{C}. \tag{8.2.5}$$

$\mathbf{Q}$, $\mathbf{M}$,$\mathbf{V}$ and $\mathbf{C}$ are defined similarly as that in Section 8.1.2,

$$\mathbf{Q} = \begin{pmatrix} Q(x,y) \\ Q\left(\frac{1}{xy}, y\right) \\ Q\left(y, \frac{1}{xy}\right) \\ Q(y,x) \\ Q\left(\frac{1}{xy}, x\right) \\ Q\left(x, \frac{1}{xy}\right) \end{pmatrix}, \tag{8.2.6}$$

and

$$\mathbf{V} = \begin{pmatrix} Q(x,0) \\ Q(0,y) \\ Q\left(\frac{1}{xy}, 0\right) \\ Q\left(0, \frac{1}{xy}\right) \\ Q(0,x) \\ Q(y,0) \end{pmatrix}, \tag{8.2.7}$$

and

$$\mathbf{M} = \begin{pmatrix} A'(x,y) & B'(x,y) & 0 & 0 & 0 & 0 \\ 0 & B'\left(\frac{1}{xy}, y\right) & A'\left(\frac{1}{xy}, y\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & B'\left(y, \frac{1}{xy}\right) & 0 & A'\left(y, \frac{1}{xy}\right) \\ 0 & 0 & 0 & 0 & B'(y,x) & A'(y,x) \\ 0 & 0 & A'\left(\frac{1}{xy}, x\right) & 0 & B'\left(\frac{1}{xy}, x\right) & 0 \\ A'\left(x, \frac{1}{xy}\right) & 0 & 0 & B'\left(x, \frac{1}{xy}\right) & 0 & 0 \end{pmatrix} \tag{8.2.8}$$

where $A'(x,y) = \frac{1}{a}(a - 1 - \frac{ta}{xy})$ and $B'(x,y) = \frac{1}{b}(b - 1 - \frac{tb}{xy})$. The column vector $\mathbf{C}$ contains the point boundary terms $Q(0,0)$ and some known terms in $\mathbb{R}(x,y,t)$:

$$\mathbf{C} = \frac{1}{abc} \begin{pmatrix} \left[-ab + ac + bc - abc + tabc\frac{1}{xy}\right] Q(0,0) + ab \\ \left[-ab + ac + bc - abc + tabcx\right] Q(0,0) + ab \\ \left[-ab + ac + bc - abc + tabcx\right] Q(0,0) + ab \\ \left[-ab + ac + bc - abc + tabc\frac{1}{xy}\right] Q(0,0) + ab \\ \left[-ab + ac + bc - abc + tabcy\right] Q(0,0) + ab \\ \left[-ab + ac + bc - abc + tabcy\right] Q(0,0) + ab \end{pmatrix}. \tag{8.2.9}$$

It is straightforward to show that the determinant of $\mathbf{M}$ equals zero. Thus, there exists a linear combination of these six equations that cancel all variables in $\mathbf{V}$. We choose a vector $\mathbf{N}$ that spans the nullspace:

$$\mathbf{N} = \begin{pmatrix} -y(1-b+tbx)(1-a+tay) \\ \frac{1}{x}(1-a+tay)(tb+xy-bxy) \\ -\frac{1}{x}(1-b+tby)(ta+xy-axy) \\ y(1-a+tax)(1-b+tby) \\ -\frac{1}{x}(1-a+tax)(tb+xy-bxy) \\ -\frac{1}{x}(1-b+tbx)(ta+xy-axy) \end{pmatrix}^{\mathsf{T}} \tag{8.2.10}$$

Multiplying both sides of (8.2.5) by $\mathbf{N}$, we have

$$K(x,y)\mathbf{NQ} = \mathbf{NMV} + \mathbf{NC} = \mathbf{NC}. \tag{8.2.11}$$

The last equality holds since $\mathbf{NM} = \mathbf{0}$.

We also have $\mathbf{NC} = \mathbf{0}$ in (8.2.11), which is different from (8.1.33). The phenomenon $\mathbf{NC} = 0$ happens in all non-interacting algebraic models [20], including Kreweras and reverse Kreweras walks.

Next, divide both sides by $K(x,y)$ and then extract the $[y^0]$ term of (8.2.11). Some new boundary terms will appear when performing the extraction.

By geometric properties, we can eliminate some of them and find an equation of the form

$$[y^0]\mathbf{NQ} = 0 = \alpha + \alpha_{0,0}Q(0,0) + \alpha_{0,x}Q(0,x) + \alpha_{x,0}Q(x,0) + \alpha_0^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + \alpha_1^{\mathrm{d}}Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) \tag{8.2.12}$$

where the α coefficients are Laurent polynomials in x and t.

One can see we have the diagonal terms here. This comes from the fact $[x^0]Q\left(\frac{1}{xy}, y\right) = Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$. This is the term that never appears in the obstinate kernel method. We then take the $[x^>]$ and $[x^<]$ parts of (8.2.12). After some simplifications by boundary conditions, we obtain

$$[x^>y^0]\mathbf{NQ} = 0 = \beta + \beta_{0,0}Q(0,0) + \beta_{0,1}Q_{0,1} + \beta_{1,0}Q_{1,0} + \beta_{x,0}Q(x,0) + \beta_{0,x}Q(0,x) \tag{8.2.13}$$

and

$$[x^<y^0]\mathbf{NQ} = 0 = \gamma + \gamma_{0,0}Q(0,0) + \gamma_{0,1}Q_{0,1} + \gamma_{1,0}Q_{1,0} + \gamma_0^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + \gamma_1^{\mathrm{d}}Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) \tag{8.2.14}$$

where the β and γ coefficients are Laurent polynomials in x and t.

These two equations describe the reflection symmetry of walks on two axes and the translation symmetry of walks around the diagonal. When $a = b$, (8.2.13) and (8.2.14) give $Q(x,0) = Q(0,x)$ and $0 = 0$. This step is omitted when solving the non-interacting or equally-interacting cases [16, 7] since the geometric symmetry is obvious.

Half-orbit sum

We have now obtained some geometric symmetries of reverse Kreweras walks from the full-orbit sum. However, we lost the information contained in the kernel. This is because the RHS of the full-orbit sum was 0. We will now "regain" this information by taking a half-orbit sum.

Recall the six equations obtained by applying the symmetry group. We rewrite the matrix equation (8.2.5) slightly differently:

$$K(x,y)\mathbf{Q} = \mathbf{M}_2\mathbf{V}_2 + \mathbf{C}_2 \tag{8.2.15}$$

Unlike the full-orbit sum, $Q(x,0)$ and $Q(0,x)$ are not included in the vector $\mathbf{V}_2$. Instead, $\mathbf{V}_2$ is

$$V_2 = \begin{pmatrix} Q(0,y) \\ Q\left(\frac{1}{xy}, 0\right) \\ Q\left(0, \frac{1}{xy}\right) \\ Q(y,0) \end{pmatrix} \tag{8.2.16}$$

The terms $Q(x,0)$ and $Q(0,x)$ are treated as constant and we put them in $\mathbf{C}_2$. $\mathbf{M}_2$ contains the corresponding coefficients of $\mathbf{V}_2$ in $\mathbf{M}$. So $\mathbf{M}_2$ is a 6×4 rectangular matrix. The dimension of the nullspace is 2. Two orthogonal basis vectors span $\mathbf{M_2}^\mathsf{T}$. We may choose either of them in the following derivation. The vector we choose is

$$\mathbf{N}_2 = \begin{pmatrix} -x(1-b+tbx)(1-a+tay) \\ \frac{1}{y}(1-a+tay)(tb+xy-bxy) \\ 0 \\ 0 \\ -\frac{1}{y}(1-a+tax)(tb+xy-bxy) \\ 0 \end{pmatrix}^\mathsf{T}. \tag{8.2.17}$$

Notice that $\mathbf{N}_2$ has three zeros and three non-zeros, corresponding to a half-orbit sum.

Then, multiply (8.2.15) by $\mathbf{N}_2$ and divide by $K(x,y)$. We find

$$\mathbf{N}_2\mathbf{Q} = \frac{\mathbf{N_2C_2}}{K(x,y)}, \tag{8.2.18}$$

since $\mathbf{N}_2\mathbf{M}_2 = 0$.

As we did for the full-orbit sum case, we extract the $[y^0]$ term of (8.2.18) to get an equation without the variable y. We can again use boundary conditions to simplify this equation, eventually finding an equation of the form

$$[y^0]\mathbf{N_2Q} = \delta + \delta_{0,0}Q(0,0) + \delta_{x,0}Q(x,0) + \delta_{0,x}Q(0,x) + \delta_0^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + \delta_1^{\mathrm{d}}Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) \tag{8.2.19}$$

where the δ coefficients are polynomials in x and t.

Now, consider the RHS. The RHS contains two parts, $\mathbf{N}_2\mathbf{C}_2$ and $1/K(x,y)$. $\mathbf{N}_2\mathbf{C}_2$ is a linear combination of $Q(x,0)$, $Q(0,x)$ and $Q(0,0)$ with coefficients in $\mathbb{R}(x,t)((y))$. It can be treated as a function of y whose coefficients are linear terms of $Q(x,0)$, $Q(0,x)$, $Q(0,0)$ and $\mathbb{R}(x,t)$. By Theorem 6, we can take the $[y^0]$ of $\mathbf{N}_2\mathbf{C}_2/K(x,y)$. It is an algebraic function of x with $\sqrt{\Delta}$ appearing.

Now equating (8.2.19) and (4.5.3), using (8.2.13) and (8.2.14) to eliminate $Q(0,x)$ and $Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)$, the half orbit sum gives

$$\begin{aligned}\mu_{x,0}Q(x,0) + \nu_0^{\mathrm{d}}\sqrt{\Delta}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) &= (\mu+\nu\sqrt{\Delta}) + (\mu_{0,0}+\nu_{0,0}\sqrt{\Delta})Q(0,0) \\ &\quad + (\mu_{0,1}+\nu_{0,1}\sqrt{\Delta})Q_{0,1} + (\mu_{1,0}+\nu_{1,0}\sqrt{\Delta})Q_{1,0},\end{aligned} \tag{8.2.20}$$

where the μ and ν are all *polynomials*, and

$$\sqrt{\Delta} = \sqrt{\frac{t^2}{x}(x^3-4) - 2xt + 1}. \tag{8.2.21}$$

Unfortunately (8.2.20) still cannot be separated directly, as $\sqrt{\Delta}$ is a Laurent series of x and $Q_0^{\mathrm{d}}(\frac{1}{x})$ is a series of $\frac{1}{x}$ with unknown coefficients. Then the $[x^>]$ part of $\sqrt{\Delta(x)}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$ involves an infinite number of unknowns. This also happens for non-interacting Kreweras walks. As before, we use the canonical factorization as per Lemma 7. In our case, Δ has one root which is a power series in t and two which are not:

$$X_1 = 4t^2 + 32t^5 + 448t^8 + \mathrm{O}(t^{11}) \tag{8.2.22}$$

$$X_2 = t^{-1} + 2t^{1/2} - 2t^2 + 5t^{7/2} - 16t^5 + \tfrac{231}{4}t^{13/2} - 224t^8 + \tfrac{7293}{8}t^{19/2} + \mathrm{O}(t^{11}) \tag{8.2.23}$$

$$X_3 = t^{-1} - 2t^{1/2} - 2t^2 - 5t^{7/2} - 16t^5 - \tfrac{231}{4}t^{13/2} - 224t^8 - \tfrac{7293}{8}t^{19/2} + \mathrm{O}(t^{11}) \tag{8.2.24}$$

with relations

$$X_1X_2X_3 = 4 \tag{8.2.25}$$
$$X_1X_2 + X_2X_3 + X_1X_3 = t^{-2} \tag{8.2.26}$$
$$X_1 + X_2 + X_3 = 2t^{-1}. \tag{8.2.27}$$

The canonical factorization is then

$$\Delta_0 = t^2X_2X_3 \tag{8.2.28}$$
$$\Delta_+ = (1 - x/X_2)(1 - x/X_3) \tag{8.2.29}$$
$$\Delta_- = 1 - X_1\frac{1}{x} \tag{8.2.30}$$

so that

$$\frac{1}{\sqrt{\Delta_+}} = 1 + tx + t^2x^2 + t^3x^3 + t^4(6x + x^4) + \mathrm{O}(t^5) \tag{8.2.31}$$
$$\sqrt{\Delta_0\Delta_-} = 1 - 2t^2\frac{1}{x} - 4t^3 - 2t^4\frac{1}{x}^2 + \mathrm{O}(t^5). \tag{8.2.32}$$

The algebraic solution of reverse Kreweras walks

We now take (8.2.20) and divide by $\sqrt{\Delta_+}$. Each μ term, divided by $\sqrt{\Delta_+}$, produces only non-negative powers of x. Each ν term, now multiplied by $\sqrt{\Delta_0\Delta_-}$ only, produces only finitely many non-negative powers of x. It is thus possible to compute the positive and negative parts with respect to x. Doing so, rearranging a bit and sending $x \mapsto \frac{1}{x}$ in the second equation gives the expressions

$$\sigma_{x,0}P_{x,0}Q(x,0) = \sigma + \sigma_{0,0}Q(0,0) + \sigma_{0,1}Q_{0,1} + \sigma_{1,0}Q_{1,0} \tag{8.2.33}$$
$$\tau_0^{\mathrm{d}}P_0^{\mathrm{d}}Q_0^{\mathrm{d}}(x) = \tau + \tau_{0,0}Q(0,0) + \tau_{0,1}Q_{0,1} + \tau_{1,0}Q_{1,0}, \tag{8.2.34}$$

where

$$P_{x,0} = (a^2t^2 + x - ax - atx^2 + a^2tx^2)(2abt^2 + 2x - ax - bx - atx^2 - btx^2 + 2abtx^2) \tag{8.2.35}$$
$$P_0^{\mathrm{d}} = (-at + a^2t + x - ax + a^2t^2x^2)(-bt + b^2t + x - bx + b^2t^2x^2) \tag{8.2.36}$$

and the τ and σ coefficients are algebraic functions of t and x (and (a, b, c)).

Now observe that (8.2.33) is in the form of a kernel equation. The unknown function $Q(x, 0)$ is a formal power series of t with polynomial coefficients in x according to its definition. We can thus again apply the kernel method [60].

The kernel $P_{x,0}$ has two quadratic factors. The roots are

$$x_{1,2} = \frac{\mp\sqrt{-4a^4t^3 + 4a^3t^3 + a^2 - 2a + 1} + a - 1}{2at(a-1)} \tag{8.2.37}$$
$$x_{3,4} = \frac{\mp\sqrt{(a + b - 2)^2 - 8abt^3(2ab - a - b)} + a + b - 2}{2t(2ab - a - b)}. \tag{8.2.38}$$

If $a > 1$ (resp. $0 < a < 1$) then x_1 (resp. x_2) is a formal power series of t and converges as $t \to 0$. Similarly, if $a + b > 2$ (resp. $0 < a + b < 2$) then x_3 (resp. x_4) is a formal power series of t and converges as $t \to 0$. For given a, b we thus get two equations with unknowns $Q(0, 0)$, $Q_{0,1}$ and

$Q_{1,0}$. The values of a and b only affect which roots we are choosing. Unless $a=1$ or $a+b=2$, they will not affect the number of roots we find here.

We can likewise view (8.2.34) as a kernel equation. The kernel P_0^{d} again has two quadratic factors, leading to two pairs of roots

$$x_{5,6} = \frac{\mp\sqrt{-4a^4t^3+4a^3t^3+a^2-2a+1}+a-1}{2a^2t^2} \tag{8.2.39}$$

$$x_{7,8} = \frac{\mp\sqrt{-4b^4t^3+4b^3t^3+b^2-2b+1}+b-1}{2b^2t^2}. \tag{8.2.40}$$

If $a>1$ (resp. $0<a<1$) then x_5 (resp. x_6) is a formal power series of t and converges as $t\to 0$. Similarly, if $b>1$ (resp. $0<b<1$) then x_7 (resp. x_8) is a formal power series of t and converges as $t\to 0$. So for given a,b, (8.2.34) also provides us two equations with unknowns $Q(0,0)$, $Q_{0,1}$ and $Q_{1,0}$.

For any $a,b>0$ with $a\neq 1, b\neq 1$ and $a+b\neq 2$, we thus get a set of four equations of the form

$$0 = \zeta^{(i)} + \zeta^{(i)}_{0,0}Q(0,0) + \zeta^{(i)}_{0,1}Q_{0,1} + \zeta^{(i)}_{1,0}Q_{1,0} \equiv H_i, \tag{8.2.41}$$

where i indicates which x_i root has been used in (8.2.33) or (8.2.34).

For simplicity assume $a,b>1$, so that $i\in\{1,3,5,7\}$. The final question, then, is which combination(s) of these equations, if any, give a solution. That is, for which i,j,k is the determinant

$$D_{i,j,k} = \zeta^{(i)}_{0,0}\zeta^{(j)}_{0,1}\zeta^{(k)}_{1,0} - \zeta^{(i)}_{0,0}\zeta^{(j)}_{1,0}\zeta^{(k)}_{0,1} - \zeta^{(i)}_{0,1}\zeta^{(j)}_{0,0}\zeta^{(k)}_{1,0} + \zeta^{(i)}_{0,1}\zeta^{(j)}_{1,0}\zeta^{(k)}_{0,0} + \zeta^{(i)}_{1,0}\zeta^{(j)}_{0,0}\zeta^{(k)}_{0,1} - \zeta^{(i)}_{1,0}\zeta^{(j)}_{0,1}\zeta^{(k)}_{0,0} \tag{8.2.42}$$

non-zero? It appears that $D_{1,3,5}=0$ while

$$D_{1,3,7} = \frac{16a^6b^4c^2(a-1)(a-2)(a-b)^2(ab-1)t^{10}}{a+b-2} + \mathrm{O}(t^{11}) \tag{8.2.43}$$

$$D_{1,5,7} = -8a^8b^3c^2(a-1)^2(b-1)^2(a-b)^2(ab-a+1)t^{10} + \mathrm{O}(t^{11}) \tag{8.2.44}$$

$$D_{3,5,7} = -\frac{16a^7b^4c^2(a-1)^2(b-1)^2(a-b)^2(ab-1)t^{10}}{a+b-2} + \mathrm{O}(t^{11}) \tag{8.2.45}$$

(These determinants depend on the choice of $\sigma_{x,0}$ and τ_0^{d} from (8.2.33) and (8.2.34). What is important is whether they are 0 or not.) Later in Section 12.3, we will prove that for the determinant obtained from an equation in the form of (8.2.20), we at least have $D_{1,3,7}\neq 0$.

Choosing one of the valid combinations gives algebraic solutions to $Q(0,0)$, $Q_{0,1}$ and $Q_{1,0}$. By back-substitution into (8.2.33) we then get the solution to $Q(x,0)$, and then by symmetry $Q(0,x)$ (and hence $Q(0,y)$). Finally, the original equation (8.2.2) yields $Q(x,y)$.

Some special cases

There are some special cases where things appear to break down, namely $a=1$, $b=1$ and $a+b=2$. When $a+b=2$ the second factor in $P_{x,0}$ loses its x term. As a result, neither x_3 nor x_4 are valid power series roots of $P_{x,0}$, and we lose one of our equations in $Q(0,0)$, $Q_{0,1}$ and $Q_{1,0}$. However, if $(a,b)\neq(1,1)$, the remaining three equations are still valid, and the solution still emerges.

When $b=1$, the system simplifies dramatically. One finds that the coefficient $\sigma_{1,0}$ of $Q_{1,0}$ vanishes. The two equations then obtained (either H_1 and H_3, or H_2 and H_4, depending on

a) are linearly independent, and the solution follows. Naturally the $a = 1$ case is then just a reflection.

In addition to using the $[x^>]$ and $[x^<]$ parts of (8.2.20), the $[x^0]$ part also provides an equation (say, H_0) with unknowns $Q(0,0), Q_{1,0}, Q_{0,1}$. This is not in a kernel form (it has no dependence on x or y), but it can be combined with the other equations discussed above. It turns out that the equation sets formed by $\{H_5, H_7, H_0\}$ or $\{H_1, H_5, H_0\}$ have non-zero determinants, and can thus also be used to generate the solutions.

8.2.2 Kreweras walks

We now turn our attention to Kreweras walks, and attempt to obtain the solution using a similar method. We still use the algebraic kernel method, but the process is different because some symmetries have changed.

The functional equation and the symmetry group

For Kreweras walks, the allowed steps are $\{\nearrow, \downarrow, \leftarrow\}$. The functional equation (6.1.3) becomes

$$K(x,y)Q(x,y) = \frac{1}{c} + \frac{1}{a}\left(a - 1 - \frac{at}{y}\right)Q(x,0) + \frac{1}{b}\left(b - 1 - \frac{bt}{x}\right)Q(0,y) + \frac{1}{abc}(ac + bc - ab - abc)Q(0,0). \quad (8.2.46)$$

where

$$K(x,y) = \left(1 - t\left(xy + \frac{1}{x} + \frac{1}{y}\right)\right) \quad (8.2.47)$$

The symmetry group is

$$\left\{(x,y), \left(\frac{1}{xy}, y\right), \left(y, \frac{1}{xy}\right), (y,x), \left(\frac{1}{xy}, x\right), \left(x, \frac{1}{xy}\right)\right\}. \quad (8.2.48)$$

It is the same as for reverse Kreweras walks.

Full-orbit sum

Again by the algebraic kernel method, we have

$$K(x,y)\mathbf{Q} = \mathbf{MV} + \mathbf{C}. \quad (8.2.49)$$

Here $\mathbf{Q}$ and $\mathbf{V}$ are as per (8.2.6) and (8.2.7), while $\mathbf{C}$ is just the constant vector of

$$\frac{(-ab + ac + bc - abc)Q(0,0)}{abc} + \frac{1}{c}, \quad (8.2.50)$$

and $\mathbf{M}$ is as per (8.2.8) but now with $A'(x,y) = \frac{1}{a}(a - 1 - ta\frac{1}{y})$ and $B'(x,y) = \frac{1}{b}(b - 1 - tb\frac{1}{x})$.

For Kreweras walks, we still have $\det(\mathbf{M}) = 0$, which means we can find a linear combination of these equations that cancels all variables in $\mathbf{V}$. The nullspace is spanned by the vector

$$\mathbf{N} = \begin{pmatrix} (at + x - ax)(bt + y - by)(1 - a + atxy)(1 - b + btxy) \\ -\frac{1}{x}(at + x - ax)(bt + x - bx)(bt + y - by)(1 - a + atxy) \\ \frac{1}{x}(at + x - ax)(bt + x - bx)(at + y - ay)(1 - b + btxy) \\ -(bt + x - bx)(at + y - ay)(1 - a + atxy)(1 - b + btxy) \\ \frac{1}{y}(bt + x - bx)(at + y - ay)(bt + y - by)(1 - a + atxy) \\ -\frac{1}{y}(at + x - ax)(at + y - ay)(bt + y - by)(1 - b + btxy) \end{pmatrix}^{\mathsf{T}} \quad (8.2.51)$$

As with reverse Kreweras, left-multiplying by $\mathbf{N}$ gives

$$K(x,y)\mathbf{NQ} = \mathbf{NC}. \tag{8.2.52}$$

But now, the difference becomes apparent. For reverse Kreweras, we had $\mathbf{NC} = 0$, but for Kreweras walks, we have

$$\mathbf{NC} = \frac{t^3 \frac{1}{xy}(a-b)(x-y)\left(x^2y-1\right)\left(xy^2-1\right)\left[ab-(ab-ac-bc+abc)Q(0,0)\right]}{c}. \tag{8.2.53}$$

This is no longer zero. We may recall the similar situation we met in Section 8.1.2. Following the same process, we divide (8.2.52) by the kernel and take the $[y^0]$ part. After simplification by boundary relations, the left hand side is a linear equation of the form

$$[y^0]\mathbf{NQ} = \alpha + \alpha_{0,0}Q(0,0) + \alpha_{0,x}Q(0,x) + \alpha_{x,0}Q(x,0) + \alpha_0^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + \alpha_1^{\mathrm{d}}Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) \tag{8.2.54}$$

where the α coefficients are Laurent polynomials in x and t. For the RHS, since $\mathbf{NC}$ does not contain $Q(0,y)$, it can be regarded as a polynomial of y. Thus, by Theorem 6 again, we may write it as

$$[y^0]\left(\frac{\mathbf{NC}}{K(x,y)}\right) = (v_{-1}/Y_1 + v_0 + v_1Y_0 + v_2Y_0^2 + v_3Y_0^3) \tag{8.2.55}$$

$$= \eta + \eta_{0,0}Q(0,0) \tag{8.2.56}$$

where the η coefficients contains $\sqrt{\Delta}$, which is

$$\Delta = \left(1-\frac{t}{x}\right)^2 - 4t^2x. \tag{8.2.57}$$

We next take the $[x^>]$ parts of (8.2.54) and (8.2.56). The former is straightforward, and results in an expression of the form

$$[x^>y^0]\mathbf{NQ} = \beta + \beta_{x,0}Q(x,0) + \beta_{0,x}Q(0,x) + \beta_{0,0}Q(0,0) + \beta_{1,0}Q_{1,0} + \beta_{2,0}Q_{2,0} + \beta_{3,0}Q_{3,0} \tag{8.2.58}$$

where the β coefficients are Laurent polynomials in t, x.

For the latter, we simply write them as

$$[x^>y^0]\left(\frac{\mathbf{NC}}{K(x,y)}\right) = \theta + \theta_{0,0}Q(0,0). \tag{8.2.59}$$

Here $\theta = [x^>]\eta$ involves the calculation of $[x^>]\sqrt{\Delta}$. by Theorem 10, θ and $\theta_{0,0}$ are D-finite but not algebraic. This means we introduce a non-algebraic terms into the full orbit-sum. This is similar to what we have met in Section 8.1.2. However, the full orbit-sum still has not given us the solution. Equating (8.2.58) and (8.2.59) gives an equation relating $Q(x,0)$ and $Q(0,x)$ – the equivalent of (8.2.13). Unlike reverse Kreweras walks, the $[x^<y^0]$ part of the full-orbit sum will give the same result as (8.2.58) and (8.2.59), not a new relation around the diagonal.

Half-orbit sum

Following the same process as for reverse Kreweras walks, we now take the half-orbit sum. We will also have equations similar to (8.2.15), and (8.2.16) in this case. The half-orbit sum is

$$K(x,y)\mathbf{Q} = \mathbf{M}_2\mathbf{V}_2 + \mathbf{C}_2 \tag{8.2.60}$$

where

$$\mathbf{V}_2 = \begin{pmatrix} Q(0,y) \\ Q\left(\frac{1}{xy},0\right) \\ Q\left(0,\frac{1}{xy}\right) \\ Q(y,0) \end{pmatrix}. \tag{8.2.61}$$

We have two vectors spanning the nullspace. The one we choose is

$$\mathbf{N_2} = \begin{pmatrix} (at+x-ax)(1-b+btxy) \\ -\frac{1}{x}(at+x-ax)(bt+x-bx) \\ 0 \\ 0 \\ \frac{1}{y}(bt+x-bx)(at+y-ay) \\ 0 \end{pmatrix} \tag{8.2.62}$$

and then

$$\mathbf{N}_2\mathbf{Q}_2 = \frac{\mathbf{N}_2\mathbf{C}_2}{K(x,y)}. \tag{8.2.63}$$

We take the $[y^0]$ term of (8.2.63) analogously to the reverse Kreweras case, and end up with the equivalents of (8.2.19).

$$[y^0]\mathbf{N_2Q} = \delta + \delta_{0,0}Q(0,0) + \delta_{x,0}Q(x,0) + \delta_{0,x}Q(0,x) + \delta_0^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) \tag{8.2.64}$$

$$[y^0]\left(\frac{\mathbf{N}_2\mathbf{C}_2}{K(x,y)}\right) = \epsilon + \epsilon_{0,0}Q(0,0) + \epsilon_{x,0}Q(x,0) + \epsilon_{0,x}Q(0,x). \tag{8.2.65}$$

As with reverse Kreweras walks, the δ coefficients are rational functions of x, t and ϵ contains $\sqrt{\Delta}$. (The absence of $Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)$ in (8.2.64) is why we did not need to take the $[x^{<}y^0]$ part of the full-orbit sum.)

We then equate (8.2.64) and (8.2.65) and use (8.2.58) and (8.2.59) to eliminate $Q(0,x)$. This leads to an equation of the form

$$\begin{aligned}\mu_{x,0}Q(x,0) + \nu_0^{\mathrm{d}}\sqrt{\Delta}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) &= (\hat{\mu} + \hat{\nu}\sqrt{\Delta}) + (\hat{\mu}_{0,0} + \hat{\nu}_{0,0}\sqrt{\Delta})Q(0,0) \\ &+ (\mu_{1,0} + \nu_{1,0}\sqrt{\Delta})Q_{1,0} + (\mu_{2,0} + \nu_{2,0}\sqrt{\Delta})Q_{2,0} + (\mu_{3,0} + \nu_{3,0}\sqrt{\Delta})Q_{3,0},\end{aligned} \tag{8.2.66}$$

where the μ and ν coefficients are polynomials in t, x and the $\hat{\mu}$ and $\hat{\nu}$ coefficients are D-finite functions, being polynomials in t, x, and θ or $\theta_{0,0}$ respectively.

We next apply the canonical factorization to Δ. There are three roots, of which two are Puiseux series which converge at 0 and one is not:

$$X_1 = t + 2t^{5/2} + 6t^4 + 21t^{11/2} + 80t^7 + \tfrac{1287}{4}t^{17/2} + 1344t^{10} + O(t^{21/2}) \tag{8.2.67}$$

$$X_2 = t - 2t^{5/2} + 6t^4 - 21t^{11/2} + 80t^7 - \tfrac{1287}{4}t^{17/2} + 1344t^{10} + O(t^{21/2}) \tag{8.2.68}$$

$$X_3 = \tfrac{1}{4}t^{-2} - 2t - 12t^4 - 160t^7 - 2688t^{10} + O(t^{11}) \tag{8.2.69}$$

Following Lemma 7 we then define

$$\Delta_0 = 4t^2X_3 \tag{8.2.70}$$

$$\Delta_+ = 1 - x/X_3 \tag{8.2.71}$$

$$\Delta_- = (1 - X_1/x)(1 - X_2/x) \tag{8.2.72}$$

so that

$$\frac{1}{\sqrt{\Delta_+}} = 1 + 2t^2x + 6t^4x^2 + 16t^5x + \mathrm{O}(t^6) \tag{8.2.73}$$

$$\sqrt{\Delta_0\Delta_-} = 1 - t\frac{1}{x} - 4t^3 - 2t^4\frac{1}{x} - 2t^5\frac{1}{x}^2 + \mathrm{O}(t^6). \tag{8.2.74}$$

As with reverse Kreweras walks, we next divide (8.2.66) by $\sqrt{\Delta_+}$ and extract the $[x^>]$ and $[x^<]$ parts. Unfortunately this has the side effect of introducing $Q_{4,0}$ as another unknown. We get two equations of the form

$$\sigma_{x,0}P_{x,0}Q(x,0) = \hat{\sigma} + \hat{\sigma}_{0,0}Q(0,0) + \sigma_{1,0}Q_{1,0} + \sigma_{2,0}Q_{2,0} + \sigma_{3,0}Q_{3,0} + \sigma_{4,0}Q_{4,0} \tag{8.2.75}$$

$$\tau_0^{\mathrm{d}}P_0^{\mathrm{d}}Q_0^{\mathrm{d}}(x) = \hat{\tau} + \hat{\tau}_{0,0}Q(0,0) + \tau_{1,0}Q_{1,0} + \tau_{2,0}Q_{2,0} + \tau_{3,0}Q_{3,0} + \tau_{4,0}Q_{4,0} \tag{8.2.76}$$

where

$$\begin{aligned} P_{x,0} &= (ta - ta^2 - x + ax - t^2a^2x^2)(tb - tb^2 - x + bx - t^2b^2x^2) \\ &\qquad \times (ta + tb - 2tab - 2x + ax + bx - 2t^2abx^2) \end{aligned} \tag{8.2.77}$$

$$P_0^{\mathrm{d}} = (t^2a^2 + x - ax - tax^2 + ta^2x^2)(t^2b^2 + x - bx - tbx^2 + tb^2x^2), \tag{8.2.78}$$

the σ and τ coefficients are algebraic functions of t, x (in fact $\sigma_{4,0}$ and $\tau_{4,0}$ are polynomials), and the $\hat{\sigma}$ and $\hat{\tau}$ coefficients are D-finite functions of t with non-negative powers of x.

The roots of $P_{x,0}$ are

$$x_{1,2} = \frac{\mp\sqrt{-4a^4t^3 + 4a^3t^3 + a^2 - 2a + 1} + a - 1}{2a^2t^2} \tag{8.2.79}$$

$$x_{3,4} = \frac{\mp\sqrt{-4b^4t^3 + 4b^3t^3 + b^2 - 2b + 1} + b - 1}{2b^2t^2} \tag{8.2.80}$$

$$x_{5,6} = \frac{\mp\sqrt{(a + b - 2)^2 - 8abt^2(2abt - at - bt)} + a + b - 2}{4abt^2} \tag{8.2.81}$$

and the roots of P_0^{d} are

$$x_{7,8} = \frac{\mp\sqrt{-4a^4t^3 + 4a^3t^3 + a^2 - 2a + 1} + a - 1}{2at\,(a - 1)} \tag{8.2.82}$$

$$x_{9,10} = \frac{\mp\sqrt{-4b^4t^3 + 4b^3t^3 + b^2 - 2b + 1} + b - 1}{2bt\,(b - 1)}. \tag{8.2.83}$$

If $a > 1$ (resp. $0 < a < 1$) then x_1 and x_7 (resp. x_2 and x_8) are formal series in t; if $b > 1$ (resp. $0 < b < 1$) then x_3 and x_9 (resp. x_4 and x_{10}) are formal series in t; and if $a + b > 2$ (resp. $0 < a + b < 2$) then x_5 (resp. x_6) is a formal series in t.

For any $a, b > 0$ with $a \neq 1, b \neq 1$ and $a + b \neq 2$, we thus get a set of five equations of the form

$$0 = \zeta^{(i)} + \zeta_{0,0}^{(i)}Q(0,0) + \zeta_{1,0}^{(i)}Q_{1,0} + \zeta_{2,0}^{(i)}Q_{2,0} + \zeta_{3,0}^{(i)}Q_{3,0} + \zeta_{4,0}^{(i)}Q_{4,0} \equiv H_i, \tag{8.2.84}$$

where i indicates which x_i root has been used in (8.2.75) or (8.2.76).

Unfortunately, the determinant of the corresponding 5×5 matrix of coefficients appears to be 0. We can even include a sixth equation (say, H_0), by taking the $[x^0]$ part of (8.2.66), but there is still no set of five linearly independent equations.

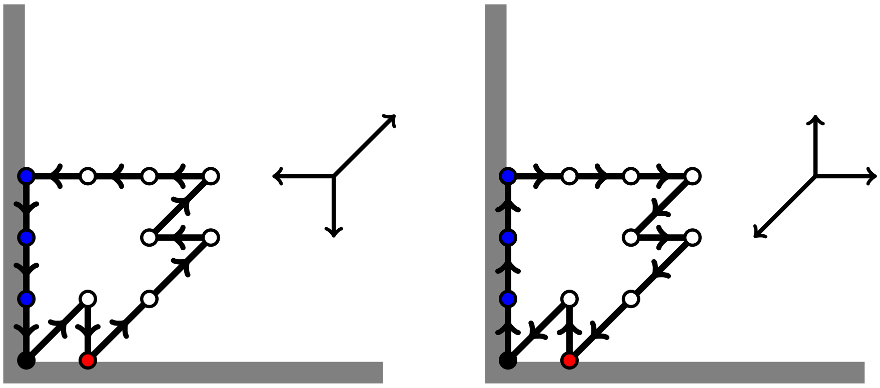


Figure 8.1: Examples of reverse Kreweras and Kreweras walks ending at the origin.

Incorporating the solution to reverse Kreweras walks

To get the final solution of Kreweras walk, we appeal to the solution of reverse Kreweras walks. Reverse Kreweras walks and Kreweras walks have a kind of geometric symmetry,

Lemma 18. *For Kreweras walks, the generating function $Q_{0,0} \equiv Q(0,0)$ is algebraic for all (a, b, c).*

Proof. If we just reverse the direction of each step of a Kreweras walk starting from the origin and ending at the origin, we get a reverse Kreweras walk. So $Q(0,0)$ is the same for these two models. Thus $Q(0,0)$ for Kreweras walks is algebraic. See Figure 8.1. □

This property does not hold for general $Q_{i,j}$. But it is already enough for our purposes. By Lemma 18, the generating function $Q(0,0)$ is the same for Kreweras and reverse Kreweras walks. Since we have solved reverse Kreweras walks, $Q(0,0)$ is actually known. We substitute it into (8.2.84) and then we have six equations with four unknowns, and hence 15 different equation sets.

As with reverse Kreweras, some of the equation sets yield a non-zero determinant and some do not. Those that do are

$$D_{0,1,3,7} = 16a^{12}b^5c^4(a-1)^3(a-2)(b-1)^5(a-b)^4(ab-a-b)^2t^{22} + \mathrm{O}(t^{23}) \tag{8.2.85}$$

$$D_{0,1,7,9} = 16a^{12}b^6c^4(a-1)^3(a-2)(b-1)^2(a-b)^4(ab-a-b)^2t^{22} + \mathrm{O}(t^{23}) \tag{8.2.86}$$

$$D_{1,3,5,7} = \frac{16a^{12}b^5c^4(a-1)^3(a-2)(b-1)^5(a-b)^4(ab-a-b)(2ab-a-b)^5t^{26}}{(a+b-2)^4} + \mathrm{O}(t^{23}) \tag{8.2.87}$$

$$D_{1,3,7,9} = -16a^{12}b^9c^4(a-1)^3(a-2)(b-1)^2(a-b)^4(ab-a-b)(2ab-a-b)t^{26} + \mathrm{O}(t^{27}) \tag{8.2.88}$$

$$D_{1,5,7,9} = -\frac{16a^{12}b^6c^4(a-1)^3(a-2)(b-1)^2(a-b)^4(ab-a-b)(2ab-a-b)^5t^{26}}{(a+b-2)^4} + \mathrm{O}(t^{27}) \tag{8.2.89}$$

(As with reverse Kreweras walks, these determinants depend on the the choice of $\sigma_{x,0}$ and τ_0^{d}.)

Any set of equations with a non-zero determinant then leads to a solution for $Q_{1,0}$, $Q_{2,0}$, $Q_{3,0}$ and $Q_{4,0}$. Back-substitution yields $Q(x,0)$, and by reflective symmetry, $Q(0,y)$. The original functional equation then gives $Q(x,y)$. If $a < 1$ or $b < 1$ then the roots which are not power series in t can be replaced as required, and the resulting determinants remain non-zero.

Since $\zeta_{1,0}^{(i)}, \zeta_{2,0}^{(i)}, \zeta_{3,0}^{(i)}$ and $\zeta_{4,0}^{(i)}$ are algebraic while $\zeta^{(i)} + \zeta_{0,0}^{(i)} Q(0,0)$ is D-finite, the resulting solutions to $Q_{1,0}, Q_{2,0}, Q_{3,0}$ and $Q_{4,0}$ are D-finite. By back-substituting we find that the same is true for $Q(x,0), Q(0,y)$ and $Q(x,y)$. So we finally get a special result for Kreweras walks with interactions. The functions $Q(x,y), Q(x,0), Q(0,y)$ are D-finite but not algebraic over x, y, t. All $Q_{i,j}$ are also D-finite but not algebraic, except $i = j = 0$. In that case, $Q(0,0)$ is an algebraic function.

Special cases

The special values of a, b work in much the same way as for reverse Kreweras. When $a + b = 2$ the third factor in $P_{x,0}$ loses its x term, and as a result $x_{5,6}$ are not longer valid kernel roots. However, there remain three combinations which do not use these roots, and the solution can be obtained from any of those.

If $b = 1$ then $\sigma_{3,0} = \sigma_{4,0} = \tau_{3,0} = \tau_{4,0} = 0$, and we thus only need two independent equations to solve for the two unknowns $Q_{1,0}$ and $Q_{2,0}$. Using H_1 and H_7 (or H_2 and H_8) suffices. Naturally $a = 1$ is just a reflection of the $b = 1$ case.

When $a = b$, note that (8.2.53) vanishes, and hence $\eta = \eta_{0,0} = \theta = \theta_{0,0} = 0$. As was the case for reverse Kreweras walks, all coefficients in (8.2.66) are then algebraic, and it follows that the resulting solution is too. Note that this has already been established for the $c = ab$ case in [7].

8.3 Half symmetry D_2 Models

Now we go to **Models 5–16**. In previous results of D-algebraic models, only those models which are simultaneously symmetric about x, y axes were solved exactly with boundary interactions. For the remaining D-algebraic models, they were only partly solved with the condition $a = 1, c = b$. We will show these models are all solvable in this section.

8.3.1 Models 5 and 15

Let us begin with the model with simplest allowed steps (**Model 5**). **Model 15** will be simultaneously considered. These two models are the reverses of each other. Without interactions, the generating functions of these two models are both D-finite. Let us first consider **Model 5**. The allowed steps are $\{\nearrow, \nwarrow, \downarrow\}$, The functional equation can be written as

$$K(x,y)Q(x,y) = \frac{1}{abc}\Bigg(bc\left(-\frac{at}{y} + a - 1\right)Q(x,0) + ac\left(-\frac{bty}{x} + b - 1\right)Q(0,y) \\ + (bc - a((b-1)c + b))Q(0,0) + ab\Bigg) \quad (8.3.1)$$

where

$$K(x,y) = 1 - t\left(xy + \frac{y}{x} + \frac{1}{y}\right). \quad (8.3.2)$$

The roots are

$$
\begin{aligned}
Y_0 &= \frac{x-\sqrt{-4t^2x^3-4t^2x+x^2}}{2\left(tx^2+t\right)} \\
Y_1 &= \frac{x+\sqrt{-4t^2x^3-4t^2x+x^2}}{2\left(tx^2+t\right)} \\
Y_0 &= t+\frac{t^3\left(x^2+1\right)}{x}+\frac{2t^5\left(x^2+1\right)^2}{x^2}+\mathrm{O}(t^6) \\
Y_1 &= \frac{x}{t\left(x^2+1\right)}-t-\frac{t^3\left(x^3+x\right)^2}{x^3\left(x^2+1\right)}-\frac{2t^5\left(x^3+x\right)^3}{x^5\left(x^2+1\right)}+\mathrm{O}(t^6).
\end{aligned}
\tag{8.3.3}
$$

The symmetry group is

$$
\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{1}{(x+\frac{1}{x})y}\right),\left(\frac{1}{x},\frac{1}{(x+\frac{1}{x})y}\right)\right\}.
\tag{8.3.4}
$$

We first try the algebraic kernel method. Apply the symmetric transformation and write the equations into a matrix form

$$
K(x,y)\mathbf{Q}=\mathbf{MV}+\mathbf{C}
\tag{8.3.5}
$$

where

$$
\mathbf{Q}=\begin{pmatrix} Q(x,y) \\ Q\left(\frac{1}{x},y\right) \\ Q\left(x,\frac{1}{y(x+\frac{1}{x})}\right) \\ Q\left(\frac{1}{x},\frac{1}{y(x+\frac{1}{x})}\right) \end{pmatrix}
\tag{8.3.6}
$$

and

$$
\mathbf{V}=\begin{pmatrix} Q(x,0) \\ Q\left(\frac{1}{x},0\right) \\ Q(0,y) \\ Q\left(0,\frac{1}{y(x+\frac{1}{x})}\right) \end{pmatrix}
\tag{8.3.7}
$$

and

$$
\mathbf{M}=\begin{pmatrix} A'(x,y) & 0 & B'(x,y) & 0 \\ 0 & A'(\frac{1}{x},y) & B'(\frac{1}{x},y) & 0 \\ A'(x,\frac{1}{y(x+\frac{1}{x})}) & 0 & 0 & B'(x,\frac{1}{y(x+\frac{1}{x})}) \\ 0 & A'(\frac{1}{x},\frac{1}{y(x+\frac{1}{x})}) & 0 & B'(\frac{1}{x},\frac{1}{y(x+\frac{1}{x})}) \end{pmatrix},
\tag{8.3.8}
$$

where $A'(x,y)=\frac{1}{a}(-1+a-(at)/y)$ and $B'(x,y)=\frac{1}{b}(-1+b-(bty)/x)$. We find $\det\mathbf{M}\neq 0$. This means we cannot find a linear combination that eliminates all line boundary terms. Imitating what we did for the half orbit-sum of reverse Kreweras walks in Section 8.2.1, we leave $Q(x,0)$ in $\mathbf{C_2}$. Then, the equation is again written as

$$
K(x,y)\mathbf{Q}=\mathbf{M}_2\mathbf{V}_2+\mathbf{C}_2.
\tag{8.3.9}
$$

Here $\mathbf{M_2}$ is a 4×3 matrix. We can always find an $\mathbf{N_2}$ in the nullspace such that

$$\mathbf{N}_2\mathbf{Q} = \frac{\mathbf{N}_2\mathbf{M}_2}{K(x,y)} + \frac{\mathbf{N}_2\mathbf{C}_2}{K(x,y)} = \frac{\mathbf{N}_2\mathbf{C}_2}{K(x,y)} \tag{8.3.10}$$

where

$$\begin{aligned}\mathbf{N}_2\mathbf{C}_2 = &-\frac{1}{cx}[bt(x^2-1)\left(x^2y^2-x+y^2\right)\\ &\left(abt^2y-abtx^2y^2-abtx-abty^2+abxy+atx^2y^2+atx+aty^2-axy-bxy+xy\right)]\\ &+\frac{1}{acx}t(x^2-1)\left(x^2y^2-x+y^2\right)(abc+ab-ac-bc)\\ &\left(abt^2y-abtx^2y^2-abtx-abty^2+abxy+atx^2y^2+atx+aty^2-axy-bxy+xy\right)Q(0,0)\\ &-\frac{(b-1)bt(x^2-1)\left(x^2y^2-x+y^2\right)(at-ay+y)\left(atx^2y+aty-ax+x\right)}{ax}\\ &+-\frac{(b-1)bt(x^2-1)\left(x^2y^2-x+y^2\right)Q(x,0)(at-ay+y)\left(atx^2y+aty-ax+x\right)}{ax}.\end{aligned} \tag{8.3.11}$$

We take the $[y^0]$ part of (8.3.10), simplify the boundary terms by boundary conditions and rearrange the terms in the equation. We finally have

$$\begin{aligned}0 = {}&S_1 + S_2Q(0,0) + \frac{Q\left(\frac{1}{x},0\right)\left(abt^2-bx+x\right)}{at} - \frac{b\left(ax^2+a-x^2\right)Q(x,0)}{a\left(x^2+1\right)}\\ -&\frac{(b-1)bt^2(x^2-1)\left(a^2t^2x^3+a^2t^2x+at^4x^4+2at^4x^2+at^4-ax^2-t^4x^4-2t^4x^2-t^4+x^2\right)Q(x,0)}{ax\sqrt{-4t^2x^3-4t^2x+x^2}}\end{aligned} \tag{8.3.12}$$

where S_1 and S_2 are functions in $R(x,t,\sqrt{\Delta})$. Δ is defined as per (5.1.2). It is related to the discriminant of $xyK(x,y)$. Notice that the coefficient of $Q(x,0)$ is in the form $p+q\sqrt{\Delta}$. Taking the $[x^i]$ terms of (8.3.12) is no longer useful. This is because, if $1/\sqrt{\Delta}$ is expanded as series of t, it is the same time a Laurent series of x with infinite $[x^>]$ and $[x^<]$ part. Thus, if we expand $(p+q\sqrt{\Delta})Q(x,0)$ as a formal series of t, we have

$$[x^i]\left(\sum_{-\infty}^{\infty} a_n x^n\right)Q(x,0) = \sum a_n q_{i+n}. \tag{8.3.13}$$

Here a_n and q_{i+n} are both unknowns and we have introduced an infinite number of them. The algebraic kernel method stops here.

For the obstinate kernel method, things are even simpler. We apply the group elements (x,y) and $(1/x,y)$ to the functional equation and eliminate $Q(0,Y_0)$, obtaining

$$\begin{aligned}0 = \frac{Q\left(\frac{1}{x},0\right)}{x} &- \frac{abt(x^2-1)Y_0^2}{cx\left(-at+aY_0-Y_0\right)\left(btY_0-bx+x\right)}\\ &\frac{t(x^2-1)Y_0^2(abc+ab-ac-bc)Q(0,0)}{cx\left(-at+aY_0-Y_0\right)\left(btY_0-bx+x\right)} - \frac{\left(btxY_0-b+1\right)Q(x,0)}{btY_0-bx+x}.\end{aligned} \tag{8.3.14}$$

The equation (8.3.11) is actually the same as (8.3.12). The obstinate kernel method also stops here.

Now we can see that the problem is the coefficient $p+q\sqrt{\Delta}$ is not a formal series of x or $\frac{1}{x}$. This is actually the same problem we met when dealing with Kreweras and reverse Kreweras walks. $\sqrt{\Delta}$ is a series with infinite $[x^>]$ and $[x^<]$ part. We can factor $\sqrt{\Delta}$ as $\sqrt{\Delta_-}\sqrt{\Delta_0}\sqrt{\Delta_+}$ and separate the $[x^>]$ series and $[x^<]$ apart. But $p+q\sqrt{\Delta}$ cannot be factored directly. However, we have Theorem 15. By some suitable multiplication by $x^k t^i$, a function f can be factored as $f_- f_0 f_+$ and $\log(f) = \log(f_-) + \log(f_0) + \log(f_+)$.

Now we can factor the coefficient of (8.3.14). By simplification (8.3.14) can be written as

$$P_c + P_{00}Q(0,0) + P_{x,0}Q(x,0) + P_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) = 0, \tag{8.3.15}$$

where

$$\begin{aligned} P_{x,0} &= -\frac{2b^2t^2x + b^2x^2 + b^2 - 3bx^2 - 3b + 2x^2 + 2}{2\left(b^2t^2 + b^2x^3 - 2bx^3 - bx + x^3 + x\right)} - \frac{(b-1)b(x^2-1)\sqrt{-x\left(4t^2x^2 + 4t^2 - x\right)}}{2x\left(b^2t^2 + b^2x^3 - 2bx^3 - bx + x^3 + x\right)} \\ P_{\frac{1}{x},0} &= \frac{1}{x}. \end{aligned} \tag{8.3.16}$$

And $P_c, P_{0,0}$ have the form $p + q\sqrt{\Delta}$. If we let

$$L = \log\left(-(2b^2t^2x + b^2x^2 + b^2 - 3bx^2 - 3b + 2x^2 + 2) - \frac{(b-1)b(x^2-1)\sqrt{-x\left(4t^2x^2 + 4t^2 - x\right)}}{x}\right) \tag{8.3.17}$$

and write L as $L_- + L_0 + L_+$, then

$$P_{x,0} = -\frac{e^{L_+}e^{L_0}e^{L_-}}{2(b^2t^2 + x - bx + x^3 - 2bx^3 + b^2x^3)}. \tag{8.3.18}$$

Besides e^L, the cubic factor in the denominator is also important since it provides singularities. We denote the roots of $b^2t^2 + x - bx + x^3 - 2bx^3 + b^2x^3 = 0$ as X_1, X_2, X_3, and their series expansions are

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{b-1}} + \frac{b^2t^2}{2-2b} - \frac{3b^4t^4}{8(b-1)^{3/2}} - \frac{b^6t^6}{2(b-1)^2} - \frac{105b^8t^8}{128(b-1)^{5/2}} + \mathrm{O}(t^9) \\ X_2 &= -\frac{1}{\sqrt{b-1}} + \frac{b^2t^2}{2-2b} + \frac{3b^4t^4}{8(b-1)^{3/2}} - \frac{b^6t^6}{2(b-1)^2} + \frac{105b^8t^8}{128(b-1)^{5/2}} - \frac{3b^{10}t^{10}}{2(b-1)^3} + \mathrm{O}(t^{11}) \\ X_3 &= \frac{b^2t^2}{b-1} + \frac{b^6t^6}{(b-1)^2} + \frac{3b^{10}t^{10}}{(b-1)^3} + \mathrm{O}(t^{11}). \end{aligned} \tag{8.3.19}$$

Notice that the series expansions of X_1, X_2 both contain constant terms. Thus $\frac{1}{X_1}$ and $\frac{1}{X_2}$ can also be expanded as formal series of t. This implies

$$\frac{1}{x - X_{1,2}} = \frac{-1}{X_{1,2}(1 - \frac{x}{X_{1,2}})} = \frac{-1}{X_{1,2}}\sum_n \left(\frac{x}{X_{1,2}}\right)^n \tag{8.3.20}$$

is still a formal series of t. So $\frac{1}{x-X_2}$ and $\frac{1}{x-X_1}$ can be expanded as formal series of t with coefficients in $\mathbb{R}(x)$. X_3 does not contain constant terms, so it cannot be expanded as a formal series of t. Thus

$$\frac{1}{x - X_3} = \frac{1}{x(1 - X_3/x)} = \frac{1}{x}\sum_n \left(\frac{X_3}{x}\right)^n. \tag{8.3.21}$$

It can only be expanded as a formal series of t with coefficients in $\mathbb{R}[\frac{1}{x}]$. Now we can see $\frac{1}{(x-X_3)}$ and e^{L_-} in the coefficient of $Q(x,0)$ cannot be expanded as series of t with coefficient in $\mathbb{R}[x]$, Thus, if we multiply (8.3.15) by $\frac{(x-X_3)}{e^{L_-}}$, then each term in the equation can be separated into $[x^>]$ and $[x^<]$ and $[x^0]$ terms. We have

$$Q(0,0) - \frac{e^{L_+}e^{L_0}(x-X_3)Q(x,0)}{2(b^2t^2+x-bx+x^3-2bx^3+b^2x^3)} + [x^>]\frac{(x-X_3)}{e^{L_-}}P_{00}Q(0,0) + [x^>]\frac{(x-X_3)}{e^{L_-}}P_c = 0 \tag{8.3.22}$$

$$\frac{(x-X_3)\,Q\left(\frac{1}{x},0\right)}{e^{L_-}x} - Q(0,0) + [x^<]\frac{(x-X_3)}{e^{L_-}}P_{00}Q(0,0) + [x^<]\frac{(x-X_3)}{e^{L_-}}P_c = 0 \tag{8.3.23}$$

$$Q(0,0) - Q(0,0) = 0. \tag{8.3.24}$$

Now, we have obtained the equation of $Q(x,0)$ and $Q(0,0)$. This is a kernel form equation. If we can find X_i that cancel the coefficient of $Q(x,0)$, we can solve $Q(0,0)$. However, the coefficient of $Q(x,0)$ is $\frac{e^{L_0}e^{L_+}}{(x-X_1)(x-X_2)}$. There are two x values that make this factor equal 0. First, $x=\infty$. Clearly, this root is of no use. Second, the root that $e^{L_+}=0$. But e^{L_+} cannot be 0 as well. This is because if $e^{L_+}\to 0$, $L_+\to-\infty$. Then the expansion $L=L^+L_0L_-$ is illegal. Thus, we cannot find the root.

This also happens to the $Q\left(\frac{1}{x},0\right)$ equation. The coefficient of $Q\left(\frac{1}{x},0\right)$ equals 0 when $x=X_3$ or $\frac{1}{e^{L_-}}=0$. But $Q\left(\frac{1}{X_3},0\right)$ does not converge when $t\to 0$, while there is no valid solution to $\frac{1}{e^{L_-}}=0$. The $Q\left(\frac{1}{x},0\right)$ equation cannot give the solution either. The $[x^0]$ equation is $0=0$. Currently, we have gone through all three parts of (8.3.18) and we still cannot get the solution.

Furthermore, if we take the $[x^1]$ part of (8.3.18), the equation will contain $Q_{1,0}$ as an extra unknown. We cannot find a relation between $Q(0,0)$ and $Q_{1,0}$ by boundary conditions.

To solve (8.3.18), we again apply the same tactics as we did when solving Kreweras walks. Let us consider **Model 15**, the model with allowed steps $\{\swarrow,\searrow,\uparrow\}$. The functional equation reads

$$K(x,y)Q(x,y) = \frac{1}{abc}\left(+bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}\right)+a-1\right)Q(x,0)\right.$$
$$\left.+ac\left(-\frac{bt}{xy}+b-1\right)Q(0,y)+\left(bc-a((b-1)c+b)\frac{abct}{xy}\right)Q(0,0)+ab\right), \tag{8.3.25}$$

where

$$K(x,y) = 1-t\left(\frac{1}{xy}+\frac{x}{y}+y\right). \tag{8.3.26}$$

The formal series root of $K(x,y)$[1] is

$$Y_0 = -\frac{\sqrt{-4t^2x^3-4t^2x+x^2}-x}{2tx} \tag{8.3.27}$$

$$Y_0 = t\left(x+\frac{1}{x}\right)+t^3\left(x^2+\frac{1}{x^2}+2\right)+t^5\left(2x^3+\frac{2}{x^3}+6x+\frac{6}{x}\right)+\mathrm{O}(t^7). \tag{8.3.28}$$

The symmetry group is the same as **Model 5**,

$$\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{(x+\frac{1}{x})}{y}\right),\left(\frac{1}{x},\frac{(x+\frac{1}{x})}{y}\right)\right\}. \tag{8.3.29}$$

[1] We do not use the other root Y_1 in our calculation so we do not write it here or in the remaining models.

By the algebraic or obstinate kernel method, we can get an equation

$$P'_c + P'_{00}Q(0,0) + P'_{x,0}Q(x,0) + P'_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) = 0, \tag{8.3.30}$$

where

$$\begin{aligned} P'_{x,0} &= \frac{2b^2t^2x + b^2x^2 + b^2 - 3bx^2 - 3b + 2x^2 + 2}{2ab\left(b^2t^2 + b^2x^3 - 2bx^3 - bx + x^3 + x\right)} - \frac{(b-1)(x^2-1)\sqrt{-x\left(4t^2x^2 + 4t^2 - x\right)}}{2ax\left(b^2t^2 + b^2x^3 - 2bx^3 - bx + x^3 + x\right)} \\ P'_{\frac{1}{x},0} &= -\frac{1}{abx}, \end{aligned} \tag{8.3.31}$$

and P'_c, P_{00} have the $p + q\sqrt{\Delta}$ form. If we let

$$L_2 = \log\left(\frac{2b^2t^2x + b^2x^2 + b^2 - 3bx^2 - 3b + 2x^2 + 2}{b} - \frac{(b-1)(x^2-1)\sqrt{-x\left(4t^2x^2 + 4t^2 - x\right)}}{x}\right), \tag{8.3.32}$$

then

$$P'_{x,0} = -\frac{e^{L_{2+}}e^{L_{20}}e^{L_{2-}}}{2a(b^2t^2 + x - bx + x^3 - 2bx^3 + b^2x^3)}. \tag{8.3.33}$$

We can see that if $P_{x,0}$ is written as $p + q\sqrt{\Delta}$, then $P'_{x,0}$ is $-p + q\sqrt{\Delta}$, the conjugate of $P_{x,0}$. The denominator of $P'_{x,0}$ is still $b^2t^2 + b^2x^3 - 2bx^3 - bx + x^3 + x$. So the roots are still X_1, X_2, X_3 as we saw for **Model 5**. Just as what we did in **Model 5**, we multiply both sides of (8.3.30) by $\frac{x - X_3}{e^{L_{2-}}}$ and separate the $[x^>]$, $[x^<]$ and $[x^0]$ terms. This time, the $[x^0]$ term is no longer $0 = 0$ but a useful equation. We have

$$T_1 + T_2Q(0,0) + T_{00}Q(0,0) + T_3Q(0,0) = 0, \tag{8.3.34}$$

where

$$T_1 = [x^0]\frac{x - X_3}{e^{L_{2-}}}P'_c \tag{8.3.35}$$

$$T_2 = [x^0]\frac{x - X_3}{e^{L_{2-}}}P'_{00} \tag{8.3.36}$$

$$T_{00} = [x^0]\frac{x - X_3}{e^{L_{2-}}}P'_{\frac{1}{x},0} = -\frac{1}{ab} \tag{8.3.37}$$

$$T_3 = [x^0]\frac{x - X_3}{e^{L_{2-}}}P'_{x,0}. \tag{8.3.38}$$

Fortunately the $[x^0]$ equation only has one unknown, $Q(0,0)$. We can solve $Q(0,0)$ directly:

$$Q(0,0) = -\frac{T_1}{T_{00} + T_2 + T_3}. \tag{8.3.39}$$

Substituting $Q(0,0)$ back into the equation of $[x^>]$, we solve $Q(x,0)$. Now recall that the allowed steps of **Model 15** are just the reverse of those of **Model 5**. Thus, each path starting from $(0,0)$ and ending at $(0,0)$ in **Model 15** corresponds to such a path in **Model 5**. That is, $Q(0,0)$ of these two models are the same. Substituting (8.3.39) into (8.3.22)–(8.3.24), we solve $Q(x,0)$ of **Model 5**. We can also multiply (8.3.30) by all factors in the denominator and use the technique in solving reverse Kreweras and Kreweras walks to solve $Q(0,0)$.

As before, we discuss the properties of $Q(x,0)$ and $Q(0,0)$ of **Model 15**. **Model 5** can be discussed similarly. $P'_{\frac{1}{x},0}$ is rational. $P'_c, P'_{00}, P'_{x,0}$ are in the form $p+q\sqrt{\Delta}$ and they are algebraic. L_2 is in the form $\log(p+q\sqrt{\Delta})$. By Theorem 16, L_{2+}, L_{2-}, L_{20} are all D-finite over x, t in the convergent domain. Notice that $\frac{\partial e^{L_{2+}}}{\partial x} = e^{L_{2+}}\frac{\partial L_{2+}}{\partial x}$ and $\frac{\partial e^{L_{2+}}}{\partial t} = e^{L_{2+}}\frac{\partial L_{2+}}{\partial t}$. By induction, $\frac{\partial^n e^{L_{2+}}}{\partial x^n}$ and $\frac{\partial^n e^{L_{2+}}}{\partial t^n}$ are polynomials of derivatives of L_{2+} times e^{L}_{2+}. Thus $e^{L_{2+}}$ is D-algebraic. The same idea can be applied to $e^{L_{2-}}$ and $e^{L_{20}}$. By definition, T_1, T_2, T_3 and T_{00} are all D-algebraic over t since they are the $[x^0]$ terms of D-algebraic functions. Thus, $Q(0,0)$ is D-algebraic over t. Similar consideration can be applied to all terms in (8.3.14) and (8.3.30). Thus $Q(x,0)$ and $Q(x,y)$ are D-algebraic.

For $Q(0,0)$ to be D-finite, T_{00}, T_2, T_3 must be algebraic. This implies $e^{L_{2-}}, X_3, P'_{00}, P'_{\frac{1}{x},0}, P'_{x,0}$ must be rational. $e^{L_{2-}}$ being rational implies $P'_{x,0}$ does not have a $\sqrt{\Delta}$ term. This implies $b=1$ and $x-X_3$ vanishes. Under this condition, $Q(x,0)$ is D-finite over x by (8.3.30). $P'_{00}=0$ requires $a=c$. Thus, for $Q(0,0)$ and $Q(x,0)$ to be D-finite over t, we require $b=1$ and $c=a$.

8.3.2 Models 6 and 16

The following few sections are all about the other D_2 models with half symmetry. They can be solved in the same way as **Models 5** and **15**. So we will go through them quickly.

For **Models 6** and **16**, first consider **Model 16**. The allowed steps are $\{\leftarrow, \swarrow, \searrow, \rightarrow, \uparrow\}$. The functional equation reads,

$$\begin{aligned}K(x,y)Q(x,y) = &\frac{1}{abc}\left(bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}\right)+a-1\right)Q(x,0)\right.\\ &\left.+ac\left(-bt\left(\frac{1}{xy}+\frac{1}{x}\right)+b-1\right)Q(0,y)+\left(bc-a((b-1)c+b)+\frac{abct}{xy}\right)Q(0,0)+ab\right),\end{aligned} \tag{8.3.40}$$

where

$$K(x,y) = 1-t\left(\frac{x}{y}+\frac{1}{xy}+x+\frac{1}{x}+y\right). \tag{8.3.41}$$

The formal series root is

$$\begin{aligned}Y_0 &= -\frac{tx^2+\sqrt{(-tx^2-t+x)^2+4tx(-tx^2-t)}+t-x}{2tx}\\ Y_0 &= \frac{t(x^2+1)}{x}+\frac{t^2(x^2+1)^2}{x^2}+\frac{t^3(x^2+1)^2(x^2+x+1)}{x^3}+\frac{t^4(x^2+1)^3(x^2+3x+1)}{x^4}\\ &+\frac{t^5(x^2+1)^3(x^4+6x^3+4x^2+6x+1)}{x^5}+\mathrm{O}(t^6).\end{aligned} \tag{8.3.42}$$

The symmetry group is

$$\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{x+\frac{1}{x}}{y}\right),\left(\frac{1}{x},\frac{x+\frac{1}{x}}{y}\right)\right\} \tag{8.3.43}$$

By the algebraic or obstinate kernel method, we get the following equation

$$P_c+P_{00}Q(0,0)+P_{x,0}Q(x,0)+P_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right)=0, \tag{8.3.44}$$

where

$$
\begin{aligned}
P_{x,0} = &-\frac{(b-1)(x^2-1)\sqrt{\left(-tx^2-t+x\right)^2+4tx\left(-tx^2-t\right)}}{2ax\left(b^2t^2-b^2tx^2+b^2x^3+btx^2+bt-2bx^3-bx+x^3+x\right)} \\
&-\frac{-2b^2t^2x^2+b^2tx^4+b^2t-b^2x^3-b^2x-btx^4-2btx^2-bt+3bx^3+3bx-2x^3-2x}{2abx\left(b^2t^2-b^2tx^2+b^2x^3+btx^2+bt-2bx^3-bx+x^3+x\right)} \\
P_{\frac{1}{x},0} = &\frac{1}{-abx}.
\end{aligned} \tag{8.3.45}
$$

And P_c, P_{00} are in the form $p+q\sqrt{\Delta}$. By previous experience, we next consider the polynomial factor in the denominator of $P_{x,0}$. It is $(b^2t^2-b^2tx^2+b^2x^3+btx^2+bt-2bx^3-bx+x^3+x)$. The series expansions of its roots of are

$$
\begin{aligned}
X_1 = &\frac{1}{\sqrt{b-1}}+\frac{b^2t^2}{2-2b}-\frac{b^3t^3}{2(b-1)^{3/2}}-\frac{\left(\left(3\sqrt{b-1}+4\right)b^4\right)t^4}{8(b-1)^2}-\frac{\left(\left(2\sqrt{b-1}+1\right)b^5\right)t^5}{2(b-1)^{5/2}} \\
&-\frac{\left(b^6\left(4b+15\sqrt{b-1}\right)\right)t^6}{8(b-1)^3}+\mathrm{O}(t^7) \\
X_2 = &-\frac{1}{\sqrt{b-1}}+\frac{b^2t^2}{2-2b}+\frac{b^3t^3}{2(b-1)^{3/2}}+\frac{\left(3\sqrt{b-1}-4\right)b^4t^4}{8(b-1)^2}+\frac{\left(1-2\sqrt{b-1}\right)b^5t^5}{2(b-1)^{5/2}} \\
&+\frac{\left(15\sqrt{b-1}-4b\right)b^6t^6}{8(b-1)^3}+\mathrm{O}(t^7) \\
X_3 = &\frac{bt}{b-1}+\frac{b^2t^2}{b-1}+\frac{b^4t^4}{(b-1)^2}+\frac{2b^5t^5}{(b-1)^2}+\frac{b^7t^6}{(b-1)^3}+\mathrm{O}(t^7).
\end{aligned} \tag{8.3.46}
$$

Let

$$
\begin{aligned}
L = \log\Bigg(&-\frac{-2b^2t^2x^2+b^2tx^4+b^2t-b^2x^3-b^2x-btx^4-2btx^2-bt+3bx^3+3bx-2x^3-2x}{2abx} \\
&-\frac{(b-1)(x^2-1)\sqrt{\left(-tx^2-t+x\right)^2+4tx\left(-tx^2-t\right)}}{2ax}\Bigg).
\end{aligned} \tag{8.3.47}
$$

Then

$$
P_{x,0} = \frac{e^{L_+}e^{L_-}e^{L_0}}{(b-1)^2(x-X_1)(x-X-2)(x-X_3)}. \tag{8.3.48}
$$

Multiply (8.3.44) by $\frac{x-X_3}{e^{L_-}}$, and then take the $[x^0]$ term of this equation. We get

$$
Q(0,0) = \frac{T_1}{T_{00}+T_2+T_3}, \tag{8.3.49}
$$

where

$$
\begin{aligned}
T_1 &= [x^0]\frac{x-X_3}{e^{L_-}}P_c \\
T_2 &= [x^0]\frac{x-X_3}{e^{L_-}}P_{00} \\
T_3 &= [x^0]\frac{x-X_3}{e^{L_-}}P_{x,0} \\
T_{00} &= [x^0]\frac{x-X_3}{e^{L_-}}P_{\frac{1}{x},0}.
\end{aligned} \tag{8.3.50}
$$

Next we consider **Model 6**. The allowed steps are $\{\nwarrow, \leftarrow, \downarrow, \rightarrow, \nearrow\}$. The functional equation reads

$$
\begin{aligned}
K(x,y)Q(x,y) &= \frac{1}{abc}\left(bc\left(-\frac{at}{y}+a-1\right)Q(x,0)\right.\\
&\left.+ac\left(-bt\left(\frac{y}{x}+\frac{1}{x}\right)+b-1\right)Q(0,y)+(bc-a((b-1)c+b))Q(0,0)+ab\right), \quad (8.3.51)
\end{aligned}
$$

where

$$
K(x,y) = 1 - t\left(\frac{x}{y}+\frac{1}{xy}+x+\frac{1}{x}+y\right). \quad (8.3.52)
$$

And the formal series root is

$$
\begin{aligned}
Y_0 &= \frac{-tx^2-\sqrt{\left(-tx^2-t+x\right)^2+4tx\left(-tx^2-t\right)}-t+x}{2\left(tx^2+t\right)}\\
Y_0 &= t+\frac{t^2\left(x^2+1\right)}{x}+\frac{t^3\left(x^2+1\right)\left(x^2+x+1\right)}{x^2}+\frac{t^4\left(x^2+1\right)^2\left(x^2+3x+1\right)}{x^3}\\
&+\frac{t^5\left(x^2+1\right)^2\left(x^4+6x^3+4x^2+6x+1\right)}{x^4}+\frac{t^6\left(x^2+1\right)^3\left(x^4+10x^3+12x^2+10x+1\right)}{x^5}\\
&+\frac{t^7\left(x^2+1\right)^3\left(x^6+15x^5+33x^4+35x^3+33x^2+15x+1\right)}{x^6}+\mathrm{O}(t^7).
\end{aligned}
\quad (8.3.53)
$$

The symmetry group is

$$
\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{1}{y(x+\frac{1}{x})}\right),\left(\frac{1}{x},\frac{1}{y(x+\frac{1}{x})}\right)\right\}. \quad (8.3.54)
$$

By the algebraic or obstinate kernel method, we have

$$
P'_c+P'_{00}Q(0,0)+P'_{x,0}Q(x,0)+P'_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right)=0, \quad (8.3.55)
$$

where

$$
\begin{aligned}
P'_{x,0} &= \frac{-2b^2t^2x^2+b^2tx^4+b^2t-b^2x^3-b^2x-btx^4-2btx^2-bt+3bx^3+3bx-2x^3-2x}{2x\left(b^2t^2-b^2tx^2+b^2x^3+btx^2+bt-2bx^3-bx+x^3+x\right)}\\
&\qquad -\frac{(b-1)b(x^2-1)\sqrt{\left(-tx^2-t+x\right)^2+4tx\left(-tx^2-t\right)}}{2x\left(b^2t^2-b^2tx^2+b^2x^3+btx^2+bt-2bx^3-bx+x^3+x\right)}\\
P'_{\frac{1}{x},0} &= \frac{1}{x}.
\end{aligned}
\quad (8.3.56)
$$

And P'_c, P'_{00} are in the form $p+q\sqrt{\Delta}$. We find $P'_{x,0}$ is still the conjugate of $P_{x,0}$. Let

$$
\begin{aligned}
L_2 = \log\left(\frac{-2b^2t^2x^2+b^2tx^4+b^2t-b^2x^3-b^2x-btx^4-2btx^2-bt+3bx^3+3bx-2x^3-2x}{2x}\right.\\
\left.-\frac{(b-1)b(x^2-1)\sqrt{\left(-tx^2-t+x\right)^2+4tx\left(-tx^2-t\right)}}{2x}\right). \quad (8.3.57)
\end{aligned}
$$

Then

$$P_{x,0} = \frac{e^{L_{2+}} e^{L_{20}} e^{L_{2-}}}{(b-1)^2(x-X_1)(x-X_2)(x-X_3)} \tag{8.3.58}$$

where X_1, X_2, X_3 are the same as (8.3.46). Now we multiply both sides of (8.3.55) by $(x - X_3)/e^{L_{2-}}$ and separate the $[x^>]$, $[x^<]$, $[x^0]$ terms. We get three equations similar to (8.3.22)–(8.3.24). The $[x^0]$ part of (8.3.55) is $0 = 0$. The roots of the coefficient of $Q(x, 0)$ in the $[x^>]$ equation are not formal series of t. And the roots of the coefficient of $Q\left(\frac{1}{x}, 0\right)$ are not series with $\frac{1}{t}$ terms. So we cannot solve (8.3.55) directly. We must again use $Q(0, 0)$ of **Model 16** to solve it.

For the properties of $Q(0, 0)$ and $Q(x, 0)$, we again assert they are D-algebraic by the same discussion in Section 8.3.1. For $Q(0, 0)$ to be D-finite, we first require $P_{x,0}$ to not have $\sqrt{\Delta}$ terms, which implies $b = 1$. Then T_{00}, T_2, T_3 need to be algebraic. This further require $a = c$. Thus, we conclude $Q(0, 0)$ is D-finite when $b = 1$ and $a = c$.

8.3.3 Models 7 and 11

For **Models 7** and **11** we first consider **Model 11**. The allowed steps are $\{\swarrow, \downarrow, \searrow, \uparrow\}$. The functional equation is

$$\begin{aligned} K(x,y)Q(x,y) = \frac{1}{abc}\Bigg(bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}+\frac{1}{y}\right)+a-1\right)Q(x,0) \\ +ac\left(-\frac{bt}{xy}+b-1\right)Q(0,y) + \left(bc - a((b-1)c+b)\frac{abct}{xy}\right)Q(0,0)+ab\Bigg), \end{aligned} \tag{8.3.59}$$

where

$$K(x,y) = 1 - t\left(\frac{x}{y}+\frac{1}{xy}+y+\frac{1}{y}\right). \tag{8.3.60}$$

The formal series root is

$$\begin{aligned} Y_0 &= -\frac{\sqrt{-4t^2x^3-4t^2x^2-4t^2x+x^2}-x}{2tx} \\ Y_0 &= \frac{t\left(x^2+x+1\right)}{x} + \frac{t^3\left(x^2+x+1\right)^2}{x^2} + \frac{2t^5\left(x^2+x+1\right)^3}{x^3} + \frac{5t^7\left(x^2+x+1\right)^4}{x^4} + O(t^8). \end{aligned} \tag{8.3.61}$$

The symmetry group is

$$\left\{(x,y), \left(\frac{1}{x}, y\right), \left(x, \frac{x+1+\frac{1}{x}}{y}\right), \left(\frac{1}{x}, \frac{x+1+\frac{1}{x}}{y}\right)\right\}. \tag{8.3.62}$$

By the algebraic or obstinate kernel method, we get the following equation

$$P_c + P_{00}Q(0,0) + P_{x,0}Q(x,0) + P_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) = 0, \tag{8.3.63}$$

where

$$\begin{aligned} P_{x,0} &= \frac{(b-1)b(x^2-1)\sqrt{-x\left(4t^2x^2+4t^2x+4t^2-x\right)}}{2x\left(b^2t^2+b^2x^3+b^2x^2-2bx^3-2bx^2-bx+x^3+x^2+x\right)} \\ &\quad - \frac{2b^2t^2x+b^2x^2+2b^2x+b^2-3bx^2-4bx-3b+2x^2+2x+2}{2\left(b^2t^2+b^2x^3+b^2x^2-2bx^3-2bx^2-bx+x^3+x^2+x\right)} \\ P_{\frac{1}{x},0} &= \frac{1}{x}, \end{aligned} \tag{8.3.64}$$

and P_c, P_{00} are in the form $p+q\sqrt{\Delta}$. The factor $(b^2t^2+b^2x^3+b^2x^2-2bx^3-2bx^2-bx+x^3+x^2+x)$ in the denominator of $P_{x,0}$ has three roots. We can check that X_1 and X_2 are formal series of t with constant terms. X_3 is a formal series of t without constants. Thus $\frac{1}{(x-X_3)}$ cannot be expanded in $R(x)[[t]]$. (The series are too complicated so we do not write them here. See the Mathematica file.)

Let

$$L = \log\left(\frac{(b-1)b(x^2-1)\sqrt{-x\left(4t^2x^2+4t^2x+4t^2-x\right)}}{2x} - \frac{1}{2}\left(2b^2t^2x+b^2x^2+2b^2x+b^2-3bx^2-4bx-3b+2x^2+2x+2\right)\right). \quad (8.3.65)$$

Then

$$P_{x,0} = \frac{e^{L_+}e^{L_-}e^{L_0}}{(b-1)^2(x-X_1)(x-X_2)(x-X_3)} \quad (8.3.66)$$

Multiplying (8.3.63) by $\frac{x-X_3}{e^{L_-}}$, we can then take the $[x^0]$ part of this equation and get

$$Q(0,0) = \frac{T_1}{T_{00}+T_2+T_3} \quad (8.3.67)$$

where T_1, T_2, T_3 and T_{00} are defined as in Section 8.3.1.

Now we consider **Model 7**. The allowed steps are $\{\downarrow, \nearrow, \uparrow, \nwarrow\}$. The functional equation reads

$$K(x,y)Q(x,y) = \frac{1}{abc}\left(bc\left(-\frac{at}{y}+a-1\right)Q(x,0)+ac\left(-\frac{bty}{x}+b-1\right)Q(0,y) + (bc-a((b-1)c+b))Q(0,0)+ab\right), \quad (8.3.68)$$

where

$$K(x,y) = 1-t\left(xy+\frac{y}{x}+y+\frac{1}{y}\right). \quad (8.3.69)$$

The formal series root is

$$Y_0 = \frac{x-\sqrt{-4t^2x^3-4t^2x^2-4t^2x+x^2}}{2\left(tx^2+tx+t\right)}$$
$$Y_0 = t+\frac{t^3\left(x^2+x+1\right)}{x}+\frac{2t^5\left(x^2+x+1\right)^2}{x^2}+\frac{5t^7\left(x^2+x+1\right)^3}{x^3}+\mathrm{O}(t^8). \quad (8.3.70)$$

The symmetry group is

$$\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{1}{y\left(x+1+\frac{1}{x}\right)}\right),\left(\frac{1}{x},\frac{1}{y\left(x+1+\frac{1}{x}\right)}\right)\right\}. \quad (8.3.71)$$

By the algebraic or obstinate kernel method, we have

$$P_c'+P_{00}'Q(0,0)+P_{x,0}'Q(x,0)+P_{\frac{1}{x},0}'Q\left(\frac{1}{x},0\right)=0, \quad (8.3.72)$$

where

$$
\begin{aligned}
P'_{x,0} &= -\frac{2b^2t^2x+b^2x^2+2b^2x+b^2-3bx^2-4bx-3b+2x^2+2x+2}{2\left(b^2t^2+b^2x^3+b^2x^2-2bx^3-2bx^2-bx+x^3+x^2+x\right)} \\
&\quad -\frac{(b-1)b(x^2-1)\sqrt{-x\left(4t^2x^2+4t^2x+4t^2-x\right)}}{2x\left(b^2t^2+b^2x^3+b^2x^2-2bx^3-2bx^2-bx+x^3+x^2+x\right)} \\
P'_{\frac{1}{x},0} &= \frac{1}{x},
\end{aligned} \tag{8.3.73}
$$

and P'_c, P'_{00} are in the form $p+q\sqrt{\Delta}$. We find $P'_{x,0}$ is still the conjugate of $P_{x,0}$. Let

$$
\begin{aligned}
L_2 = \log\Big[&-\frac{1}{2}\left(2b^2t^2x+b^2x^2+2b^2x+b^2-3bx^2-4bx-3b+2x^2+2x+2\right) \\
&-\frac{(b-1)b(x^2-1)\sqrt{-x\left(4t^2x^2+4t^2x+4t^2-x\right)}}{2x}\Big].
\end{aligned} \tag{8.3.74}
$$

Then

$$
P'_{x,0} = \frac{e^{L_{2+}}e^{L_{20}}e^{L_{2-}}}{(b-1)^2(x-X_1)(x-X_2)(x-X_3)} \tag{8.3.75}
$$

where X_1, X_2, X_3 are the same as in **Model 11**. The same process shows that the $[x^0]$ part of (8.3.72) is $0=0$ and the $[x^>]$, $[x^<]$ equations do not have the valid roots. So we cannot solve (8.3.72) directly. We again use $Q(0,0)$ of **Model 11** to solve it.

By the same discussion as per in Section 8.3.1, $Q(0,0)$, $Q(x,0)$ are still D-algebraic. And $Q(0,0)$ is D-finite when $b=1$ and $a=c$.

8.3.4 Models 8 and 12

For **Models 8** and **12** we first consider **Model 12**. The allowed steps are $\{\leftarrow, \swarrow, \downarrow, \searrow, \rightarrow, \uparrow\}$. The functional equation is

$$
\begin{aligned}
K(x,y)Q(x,y) = &\frac{1}{abc}\left(+bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}+\frac{1}{y}\right)+a-1\right)Q(x,0)\right. \\
&\left.+ac\left(-bt\left(\frac{1}{xy}+\frac{1}{x}\right)+b-1\right)Q(0,y)+\left(bc-a((b-1)c+b)+\frac{abct}{xy}\right)Q(0,0)+ab\right),
\end{aligned} \tag{8.3.76}
$$

where

$$
K(x,y) = 1-t\left(\frac{x}{y}+\frac{1}{xy}+x+\frac{1}{x}+y+\frac{1}{y}\right). \tag{8.3.77}
$$

The formal series root is

$$
\begin{aligned}
Y_0 &= -\frac{tx^2+\sqrt{\left(-tx^2-t+x\right)^2+4tx\left(-tx^2-tx-t\right)}+t-x}{2tx} \\
Y_0 &= t\left(x+\frac{1}{x}+1\right)+t^2\left(x^2+\frac{1}{x^2}+x+\frac{1}{x}+2\right)+t^3\left(x^3+\frac{1}{x^3}+2x^2+\frac{2}{x^2}+5x+\frac{5}{x}+5\right) \\
&\quad +t^4\left(x^4+\frac{1}{x^4}+4x^3+\frac{4}{x^3}+10x^2+\frac{10}{x^2}+15x+\frac{15}{x}+18\right)+O(t^5).
\end{aligned} \tag{8.3.78}
$$

The symmetry group is

$$\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{x+1+\frac{1}{x}}{y}\right),\left(\frac{1}{x},\frac{x+1+\frac{1}{x}}{y}\right)\right\}. \tag{8.3.79}$$

By the algebraic or obstinate kernel method, we get the following equation

$$P_c+P_{00}Q(0,0)+P_{x,0}Q(x,0)+P_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right)=0, \tag{8.3.80}$$

where

$$\begin{aligned}
P_{x,0}=&\frac{1}{2x\left(2b^2t^2-b^2tx^2-2b^2tx+b^2x^3+b^2x^2+btx^2+2btx+bt-2bx^3-2bx^2-bx+x^3+x^2+x\right)}\\
&\times\left(-4b^2t^2x^2+b^2tx^4+2b^2tx^3+2b^2tx+b^2t-b^2x^3-2b^2x^2-b^2x-btx^4\right.\\
&\left.-2btx^3-2btx^2-2btx-bt+3bx^3+4bx^2+3bx-2x^3-2x^2-2x\right)\\
&+\frac{(b-1)b(x^2-1)\sqrt{\left(-tx^2-t+x\right)^2+4tx\left(-tx^2-tx-t\right)}}{2x\left(2b^2t^2-b^2tx^2-2b^2tx+b^2x^3+b^2x^2+btx^2+2btx+bt-2bx^3-2bx^2-bx+x^3+x^2+x\right)}\\
P_{\frac{1}{x},0}=&\frac{1}{x}.
\end{aligned} \tag{8.3.81}$$

$P_c, P_{00}, P_{x,0}, P_{\frac{1}{x},0}$ are defined similarly as before in Section 8.3.1. When the kernel becomes more and more complicated, the coefficients $P_c, P_{00}, P_{x,0}, P_{\frac{1}{x},0}$ also become more complicated. So we do not list them all here. See the Mathematica file for checking. We still bear in mind that $P_{\frac{1}{x},0}$ is normally set to constant and $P_{x,0}, P_{00}, P_c$ are in the form $p+q\sqrt{\Delta}$. The polynomial factor in the denominator of $P_{x,0}$ is $(b^2t^2-b^2tx^2+b^2x^3+btx^2+bt-2bx^3-bx+x^3+x)$. One of its factors $\frac{1}{x-X_3}$ cannot be expanded as formal series in $\mathbb{R}[x,t]$. We further let

$$L=\log\left(P_{x,0}(b^2t^2-b^2tx^2+b^2x^3+btx^2+bt-2bx^3-bx+x^3+x)\right). \tag{8.3.82}$$

Then

$$P_{x,0}=\frac{e^{L_+}e^{L_-}e^{L_0}}{(b-1)^2(x-X_1)(x-X_2)(x-X_3)} \tag{8.3.83}$$

Multiplying (8.3.80) by $\frac{x-X_3}{e^{L_-}}$, we can then take the $[x^0]$ part of this equation and get

$$Q(0,0)=\frac{T_1}{T_{00}+T_2+T_3} \tag{8.3.84}$$

where T_1, T_2, T_3, T_{00} are defined as before.

Then we consider **Model 8**. The allowed steps are $\{\nwarrow,\leftarrow,\downarrow,\rightarrow,\nearrow\}$. The functional equation reads

$$\begin{aligned}
K(x,y)Q(x,y)=&\frac{1}{abc}\left(bc\left(-\frac{at}{y}+a-1\right)Q(x,0)\right.\\
&\left.+ac\left(b(-t)\left(\frac{y}{x}+\frac{1}{x}\right)+b-1\right)Q(0,y)+(bc-a((b-1)c+b))Q(0,0)+ab\right),
\end{aligned} \tag{8.3.85}$$

where

$$K(x,y)=1-t\left(xy+\frac{y}{x}+x+\frac{1}{x}+y+\frac{1}{y}\right), \tag{8.3.86}$$

and the formal series root is

$$Y_0 = \frac{-tx^2 - \sqrt{\left(-tx^2-t+x\right)^2 + 4tx\left(-tx^2-tx-t\right)} - t + x}{2\left(tx^2+tx+t\right)}$$
$$Y_0 = t + \frac{t^2\left(x^2+1\right)}{x} + \frac{t^3\left(x^4+x^3+3x^2+x+1\right)}{x^2} + \frac{t^4\left(x^6+3x^5+6x^4+6x^3+6x^2+3x+1\right)}{x^3} + \mathrm{O}(t^5). \tag{8.3.87}$$

The symmetry group is

$$\left\{(x,y), \left(\frac{1}{x}, y\right), \left(x, \frac{1}{y(x+1+\frac{1}{x})}\right), \left(\frac{1}{x}, \frac{1}{y(x+1+\frac{1}{x})}\right)\right\}. \tag{8.3.88}$$

By the algebraic or obstinate kernel method, we have

$$P'_c + P'_{00}Q(0,0) + P'_{x,0}Q(x,0) + P'_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) = 0. \tag{8.3.89}$$

Here $P'_{x,0}$ is the conjugate of $P_{x,0}$ and the polynomial factor in denominator of $P'_{x,0}$ is still $(bt + 2b^2t^2 + x - bx + 2btx - 2b^2tx + x^2 - 2bx^2 + b^2x^2 + btx^2 - b^2tx^2 + x^3 - 2bx^3 + b^2x^3)$, the same as **Model 12**. And L_2 is still written as

$$L_2 = \log\left((bt + 2b^2t^2 + x - bx + 2btx - 2b^2tx + x^2 - 2bx^2 + b^2x^2 + btx^2 - b^2tx^2 + x^3 - 2bx^3 + b^2x^3)P'_{x,0}\right). \tag{8.3.90}$$

Then

$$P'_{x,0} = \frac{e^{L_{2+}}e^{L_{20}}e^{L_{2-}}}{(b-1)^2(x-X_1)(x-X_2)(x-X_3)}, \tag{8.3.91}$$

where X_1, X_2, X_3 are the roots of $(bt + 2b^2t^2 + x - bx + 2btx - 2b^2tx + x^2 - 2bx^2 + b^2x^2 + btx^2 - b^2tx^2 + x^3 - 2bx^3 + b^2x^3)$. Then the same argument as before will show that we cannot solve $Q(0,0)$ directly. We still use $Q(0,0)$ of **Model 12** to solve **Model 8**.

$Q(0,0)$ and $Q(x,0)$ are still D-algebraic by the same discussion in Section 8.3.1. The condition for $Q(0,0)$ to be D-finite is still $b = 1$ and $a = c$.

8.3.5 Models 9 and 13

We next consider **Models 9** and **13**. For these two models, first consider **Model 13**. The allowed steps are $\{\swarrow, \searrow, \downarrow, \nearrow, \nwarrow\}$. The functional equation reads

$$K(x,y)Q(x,y) = \frac{1}{abc}\left(bc\left(-at\left(\frac{x}{y} + \frac{1}{xy} + \frac{1}{y}\right) + a - 1\right)Q(x,0) + ac\left(-bt\left(\frac{y}{x} + \frac{1}{xy}\right) + b - 1\right)Q(0,y) + \left(bc - a((b-1)c+b) + \frac{abct}{xy}\right)Q(0,0) + ab\right), \tag{8.3.92}$$

where

$$K(x,y) = 1 - t\left(xy + \frac{x}{y} + \frac{1}{xy} + \frac{y}{x} + \frac{1}{y}\right). \tag{8.3.93}$$

The formal series root is

$$
\begin{aligned}
Y_0 &= \frac{x-\sqrt{x^2-4\left(-tx^2-t\right)\left(-tx^2-tx-t\right)}}{2\left(tx^2+t\right)} \\
Y_0 &= \frac{t\left(x^2+x+1\right)}{x}+\frac{t^3\left(x^2+1\right)\left(x^2+x+1\right)^2}{x^3} \\
&\quad +\frac{2t^5\left(x^2+1\right)^2\left(x^2+x+1\right)^3}{x^5}+\frac{5t^7\left(x^2+1\right)^3\left(x^2+x+1\right)^4}{x^7}+\mathrm{O}(t^8) \\
&\qquad\qquad +\frac{t^5\left(x^2+1\right)^3\left(x^4+6x^3+4x^2+6x+1\right)}{x^5}+\mathrm{O}(t^6).
\end{aligned}
\tag{8.3.94}
$$

The symmetry group is

$$
\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{x+1+\frac{1}{x}}{\left(x+\frac{1}{x}\right)y}\right),\left(\frac{1}{x},\frac{x+1+\frac{1}{x}}{\left(x+\frac{1}{x}\right)y}\right)\right\}. \tag{8.3.95}
$$

By the algebraic or obstinate kernel method, we get the following equation

$$
P_c+P_{00}Q(0,0)+P_{x,0}Q(x,0)+P_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right)=0, \tag{8.3.96}
$$

where

$$
\begin{aligned}
&P_{x,0}= \\
&-\frac{2b^2t^2x^2+b^2x^3+2b^2x^2+b^2x-2bx^4-3bx^3-4bx^2-3bx-2b+2x^4+2x^3+4x^2+2x+2}{2\left(b^2t^2+b^2x^4+b^2x^3-2bx^4-2bx^3-2bx^2-bx+x^4+x^3+2x^2+x+1\right)} \\
&-\frac{(b-1)b(x^2-1)\sqrt{-4t^2x^4-4t^2x^3-8t^2x^2-4t^2x-4t^2+x^2}}{2\left(b^2t^2+b^2x^4+b^2x^3-2bx^4-2bx^3-2bx^2-bx+x^4+x^3+2x^2+x+1\right)} \\
&P_{\frac{1}{x},0}=1.
\end{aligned}
\tag{8.3.97}
$$

The polynomial in the denominator of coefficient of $P_{x,0}$ is

$$
(b^2t^2+b^2x^4+b^2x^3-2bx^4-2bx^3-2bx^2-bx+x^4+x^3+2x^2+x+1) \tag{8.3.98}
$$

so let

$$
L=\log\left((b^2t^2+b^2x^4+b^2x^3-2bx^4-2bx^3-2bx^2-bx+x^4+x^3+2x^2+x+1)P_{x,0}\right). \tag{8.3.99}
$$

This time the polynomial has 4 roots. By calculation, we find that the series expansions of all of the roots contain constant terms. This means $\frac{1}{(b^2t^2+b^2x^4+b^2x^3-2bx^4-2bx^3-2bx^2-bx+x^4+x^3+2x^2+x+1)}$ can be expanded as a formal series of t with coefficients in $\mathbb{R}[x]$. Thus, we only need to divide (8.3.96) by e^{L_-}. The coefficients $P_{x,0}$ and $P_{\frac{1}{x},0}$ can be chosen so that we do not have $Q_{1,0}$ in the equation when we take the $[x^0]$ part of the equation. Then dividing (8.3.96) by e^{L_-} and taking the $[x^0]$ part of this equation, we have

$$
Q(0,0)=\frac{T_1}{T_{00}+T_2+T_3}, \tag{8.3.100}
$$

where

$$\begin{aligned}T_1 &= [x^0]\frac{1}{e^{L_-}}P_c\\ T_2 &= [x^0]\frac{1}{e^{L_-}}P_{00}\\ T_3 &= [x^0]\frac{1}{e^{L_-}}P_{x,0}\\ T_4 &= [x^0]\frac{1}{e^{L_-}}P_{\frac{1}{x},0}.\end{aligned} \tag{8.3.101}$$

Next we consider **Model 9**. The allowed steps are $\{\nearrow, \nwarrow, \uparrow, \swarrow, \searrow\}$. The functional equation reads

$$\begin{aligned}K(x,y)Q(x,y) = \frac{1}{abc}&\left(bc\left(-at\left(\frac{x}{y}+\frac{1}{xy}\right)+a-1\right)Q(x,0)\right.\\ &\left.+ac\left(-bt\left(\frac{y}{x}+\frac{1}{xy}\right)+b-1\right)Q(0,y)+\left(bc-a((b-1)c+b)+\frac{abct}{xy}\right)Q(0,0)+ab\right),\end{aligned} \tag{8.3.102}$$

where

$$K(x,y) = 1-t\left(xy+\frac{x}{y}+\frac{1}{xy}+\frac{y}{x}+y\right). \tag{8.3.103}$$

The formal series root is

$$\begin{aligned}Y_0 &= \frac{x-\sqrt{x^2-4\left(-tx^2-t\right)\left(-tx^2-tx-t\right)}}{2\left(tx^2+tx+t\right)}\\ Y_0 &= \frac{t\left(x^2+1\right)}{x}+\frac{t^3\left(x^2+1\right)^2\left(x^2+x+1\right)}{x^3}+\frac{2t^5\left(x^2+1\right)^3\left(x^2+x+1\right)^2}{x^5}+\mathrm{O}(t^6).\end{aligned} \tag{8.3.104}$$

The symmetry group is

$$\left\{(x,y),\left(\frac{1}{x},y\right),\left(x,\frac{(x+\frac{1}{x})}{y\left(x+1+\frac{1}{x}\right)}\right),\left(\frac{1}{x},\frac{(x+\frac{1}{x})}{y\left(x+1+\frac{1}{x}\right)}\right)\right\}. \tag{8.3.105}$$

By the algebraic or obstinate kernel method, we have

$$P'_c+P'_{00}Q(0,0)+P'_{x,0}Q(x,0)+P'_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right)=0. \tag{8.3.106}$$

Here $P'_{x,0}$ equals the conjugate of $P_{x,0}$. The polynomial in the denominator of $P'_{x,0}$ is the same as the previous case. It has four roots all of which can be expanded as formal series of t with coefficients in $\mathbb{R}[x]$. Let

$$\begin{aligned}L_2 = \log&\left(\frac{1}{2}(b-1)b(x^2-1)\sqrt{-4t^2x^4-4t^2x^3-8t^2x^2-4t^2x-4t^2+x^2}\right.\\ &\left.-\frac{1}{2}\left(2b^2t^2x^2+b^2x^3+2b^2x^2+b^2x-2bx^4-3bx^3-4bx^2-3bx-2b+2x^4+2x^3+4x^2+2x+2\right)\right).\end{aligned} \tag{8.3.107}$$

Then

$$P'_{x,0} = \frac{e^{L_{2+}}e^{L_{20}}e^{L_{2-}}}{(b-1)^2(x-X_1)(x-X_2)(x-X_3)(x-X_4)}. \tag{8.3.108}$$

As we did for **Model 13**, we divide (8.3.106) by $e^{L_{2-}}$ and separate the $[x^{>}]$, $[x^0]$, $[x^{<}]$ terms. This time, the $[x^0]$ term of (8.3.106) will directly give the result,

$$Q(0,0) = \frac{T_1'}{T_{00}' + T_2' + T_3'}, \tag{8.3.109}$$

where

$$\begin{aligned}
T_1' &= [x^0]\frac{1}{e^{L_{2-}}}P_c' \\
T_2' &= [x^0]\frac{1}{e^{L_{2-}}}P_{00}' \\
T_3' &= [x^0]\frac{1}{e^{L_{2-}}}P_{x,0}' \\
T_4' &= [x^0]\frac{1}{e^{L_{2-}}}P_{\frac{1}{x},0}'.
\end{aligned} \tag{8.3.110}$$

Here $Q(0,0)$ and $Q(x,0)$ are D-algebraic as before in Section 8.3.1. For $Q(0,0)$ to be D-finite, the condition is still $b=1$ and $c=a$.

8.3.6 Models 10 and 14

For **Models 10** and **14** we first consider **Model 14**. The allowed steps are $\{\swarrow, \searrow, \downarrow, \nearrow, \nwarrow, \leftarrow, \rightarrow\}$. The functional equation is

$$\begin{aligned}
K(x,y)Q(x,y) &= \frac{1}{abc}\left(+bc\left(a(-t)\left(\frac{x}{y}+\frac{1}{xy}+\frac{1}{y}\right)+a-1\right)Q(x,0)\right. \\
&\left. +ac\left(-bt\left(\frac{y}{x}+\frac{1}{xy}+\frac{1}{x}\right)+b-1\right)Q(0,y)+(bc-a((b-1)c+b)+\frac{abct}{xy})Q(0,0)+ab\right),
\end{aligned} \tag{8.3.111}$$

where

$$K(x,y) = 1-t\left(xy+\frac{x}{y}+\frac{1}{xy}+\frac{y}{x}+x+\frac{1}{x}+\frac{1}{y}\right). \tag{8.3.112}$$

The power series root is

$$\begin{aligned}
Y_0 &= \frac{-tx^2-t+x-\sqrt{(-tx^2-t+x)^2-4\left(-tx^2-t\right)\left(-tx^2-tx-t\right)}}{2\left(tx^2+t\right)} \\
Y_0 &= \frac{t\left(x^2+x+1\right)}{x}+\frac{t^2\left(x^2+1\right)\left(x^2+x+1\right)}{x^2}+\frac{t^3\left(x^2+1\right)\left(x^2+x+1\right)\left(2x^2+x+2\right)}{x^3} \\
&\quad +\frac{t^4\left(x^2+1\right)^2\left(x^2+x+1\right)\left(4x^2+3x+4\right)}{x^4}+\mathrm{O}(t^5).
\end{aligned} \tag{8.3.113}$$

The symmetry group is

$$\left\{(x,y), \left(\frac{1}{x},y\right), \left(x, \frac{x+1+\frac{1}{x}}{\left(x+\frac{1}{x}\right)y}\right), \left(\frac{1}{x}, \frac{x+1+\frac{1}{x}}{\left(x+\frac{1}{x}\right)y}\right)\right\}. \tag{8.3.114}$$

By the algebraic or obstinate kernel method, we get the following equation

$$P_c + P_{00}Q(0,0) + P_{x,0}Q(x,0) + P_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) = 0, \tag{8.3.115}$$

where

$$\begin{aligned}
&P_{x,0}\\
&= -\frac{1}{2\left(b^2t^2 - b^2tx^2 + b^2x^4 + b^2x^3 + btx^2 + bt - 2bx^4 - 2bx^3 - 2bx^2 - bx + x^4 + x^3 + 2x^2 + x + 1\right)}\\
&\times \left(2b^2t^2x^2 - b^2tx^4 - b^2t + b^2x^3 + 2b^2x^2 + b^2x + btx^4 + 2btx^2 + bt - 2bx^4\right.\\
&\left.- 3bx^3 - 4bx^2 - 3bx - 2b + 2x^4 + 2x^3 + 4x^2 + 2x + 2\right)\\
&- \frac{(b-1)b(x^2-1)\sqrt{-3t^2x^4 - 4t^2x^3 - 6t^2x^2 - 4t^2x - 3t^2 - 2tx^3 - 2tx + x^2}}{2\left(b^2t^2 - b^2tx^2 + b^2x^4 + b^2x^3 + btx^2 + bt - 2bx^4 - 2bx^3 - 2bx^2 - bx + x^4 + x^3 + 2x^2 + x + 1\right)}\\
&P_{\frac{1}{x},0} = 1.
\end{aligned} \tag{8.3.116}$$

The polynomial in the denominator of $P_{x,0}$ is

$$(b^2t^2 - b^2tx^2 + b^2x^4 + b^2x^3 + btx^2 + bt - 2bx^4 - 2bx^3 - 2bx^2 - bx + x^4 + x^3 + 2x^2 + x + 1). \tag{8.3.117}$$

It is quartic and the series expansions of all of the roots contain constant terms. Thus, let

$$\begin{aligned}
L = \log\Bigg(&-\frac{1}{2}(2b^2t^2x^2 - b^2tx^4 - b^2t + b^2x^3 + 2b^2x^2 + b^2x + btx^4\\
&+ 2btx^2 + bt - 2bx^4 - 3bx^3 - 4bx^2 - 3bx - 2b + 2x^4 + 2x^3 + 4x^2 + 2x + 2)\\
&-\frac{1}{2}(b-1)b(x^2-1)\sqrt{-3t^2x^4 - 4t^2x^3 - 6t^2x^2 - 4t^2x - 3t^2 - 2tx^3 - 2tx + x^2}\Bigg).
\end{aligned} \tag{8.3.118}$$

Dividing (8.3.115) by e^{L_-}, and taking the $[x^0]$ part of the equation, we have

$$Q(0,0) = \frac{T_1}{T_{00} + T_2 + T_3}, \tag{8.3.119}$$

where

$$\begin{aligned}
T_1 &= [x^0]\frac{1}{e^{L_-}}P_c\\
T_2 &= [x^0]\frac{1}{e^{L_-}}P_{00}\\
T_3 &= [x^0]\frac{1}{e^{L_-}}P_{x,0}\\
T_{00} &= [x^0]\frac{1}{e^{L_-}}P_{\frac{1}{x},0}.
\end{aligned} \tag{8.3.120}$$

Model 10 can be studied in the same way. The allowed steps are $\{\nearrow, \nwarrow, \uparrow, \rightarrow, \leftarrow, \searrow, \swarrow\}$. The functional equation reads

$$\begin{aligned}
K(x,y)Q(x,y) = \frac{1}{abc}\Bigg(&bc\left(-at\left(\frac{x}{y} + \frac{1}{xy}\right) + a - 1\right)Q(x,0)\\
&+ac\left(-bt\left(\frac{y}{x} + \frac{1}{xy} + \frac{1}{x}\right)Q(0,y) + b - 1\right) + (bc - a((b-1)c + b) + \frac{abct}{xy})Q(0,0) + ab\Bigg),
\end{aligned} \tag{8.3.121}$$

where

$$K(x,y) = 1 - t\left(xy + \frac{x}{y} + \frac{1}{xy} + \frac{y}{x} + x + \frac{1}{x} + y\right). \tag{8.3.122}$$

The formal series root is

$$\begin{aligned}
Y_0 &= \frac{-tx^2 - \sqrt{\left(-tx^2 - t + x\right)^2 - 4\left(-tx^2 - t\right)\left(-tx^2 - tx - t\right)} - t + x}{2\left(tx^2 + tx + t\right)} \\
Y_0 &= \frac{t\left(x^2+1\right)}{x} + \frac{t^2\left(x^2+1\right)^2}{x^2} + \frac{t^3\left(x^2+1\right)^2\left(2x^2+x+2\right)}{x^3} \\
&\quad + \frac{t^4\left(x^2+1\right)^3\left(4x^2+3x+4\right)}{x^4} + \frac{t^5\left(x^2+1\right)^3\left(9x^4+10x^3+20x^2+10x+9\right)}{x^5} + \mathrm{O}(t^6).
\end{aligned} \tag{8.3.123}$$

The symmetry group is

$$\left\{(x,y), \left(\frac{1}{x}, y\right), \left(x, \frac{\left(x+\frac{1}{x}\right)}{y\left(x+1+\frac{1}{x}\right)}\right), \left(\frac{1}{x}, \frac{\left(x+\frac{1}{x}\right)}{y\left(x+1+\frac{1}{x}\right)}\right)\right\}. \tag{8.3.124}$$

By the algebraic or obstinate kernel method, we have

$$P'_c + P'_{00}Q(0,0) + P'_{x,0}Q(x,0) + P'_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) = 0. \tag{8.3.125}$$

Let

$$\begin{aligned}
L_2 = \log\Big[&\frac{1}{2}(b-1)b(x^2-1)\sqrt{-3t^2x^4 - 4t^2x^3 - 6t^2x^2 - 4t^2x - 3t^2 - 2tx^3 - 2tx + x^2} \\
&- \frac{1}{2}(2b^2t^2x^2 - b^2tx^4 - b^2t + b^2x^3 + 2b^2x^2 + b^2x + btx^4 + 2btx^2 + bt \\
&\qquad - 2bx^4 - 3bx^3 - 4bx^2 - 3bx - 2b + 2x^4 + 2x^3 + 4x^2 + 2x + 2)\Big].
\end{aligned} \tag{8.3.126}$$

Divide (8.3.125) by $e^{L_{2-}}$ and separate the $[x^>]$, $[x^<]$ and $[x^0]$ terms. The $[x^0]$ terms will directly gives the result, still written as

$$Q(0,0) = \frac{T'_1}{T'_{00} + T'_2 + T'_3}, \tag{8.3.127}$$

where $T'_1, T'_2, T'_3, T'_{00}$ are defined as before in Section 8.3.1. $Q(0,0)$ and $Q(x,0)$ are still D-algebraic over x, t. For $Q(0,0)$ to be D-finite, the condition is still $b = 1$ and $a = c$.

8.4 D-algebraic D_3 models

There are two models associated with the group D_3 that were not solved in previous work, **Models 18** and **21**. **Model 21** is the double Kreweras model. It is algebraic without interactions and was only solved with $a = b = 1$. Double Kreweras walks need further techniques so we postpone it to Section 10.2. Here we solve **Model 18**. **Model 18** is D-finite without interactions.

8.4.1 Model 18

The allowed step of **Model 18** is $\{\uparrow, \nwarrow, \leftarrow, \downarrow, \searrow, \rightarrow\}$. The functional equation reads

$$\begin{aligned} K(x,y)Q(x,y) = {} & \frac{1}{abc}\Big(bc\left(-at\left(\frac{x}{y}+\frac{1}{y}\right)+a-1\right)Q(x,0) \\ & + ac\left(-bt\left(\frac{y}{x}+\frac{1}{x}\right)+b-1\right)Q(0,y)+(bc-a((b-1)c+b))Q(0,0)+ab\Big), \end{aligned} \tag{8.4.1}$$

where the kernel is

$$K(x,y) = 1 - t\left(\frac{x}{y}+\frac{y}{x}+x+\frac{1}{x}+y+\frac{1}{y}\right). \tag{8.4.2}$$

The formal series root is,

$$\begin{aligned} Y_0 &= \frac{-tx^2-\sqrt{(-tx^2-t+x)^2-4(t(-x)-t)\left(-tx^2-tx\right)}-t+x}{2(tx+t)} \\ Y_0 &= t(x+1)+\frac{t^2(x+1)\left(x^2+1\right)}{x}+\frac{t^3(x+1)\left(x^4+x^3+4x^2+x+1\right)}{x^2} \\ &\quad + \frac{t^4(x+1)\left(x^6+3x^5+9x^4+6x^3+9x^2+3x+1\right)}{x^3}+\mathrm{O}(t^5). \end{aligned} \tag{8.4.3}$$

The symmetry group is

$$\left\{(x,y),\left(\frac{y}{x},y\right),\left(x,\frac{x}{y}\right),\left(\frac{1}{y},\frac{x}{y}\right),\left(\frac{y}{x},\frac{1}{x}\right),\left(\frac{1}{y},\frac{1}{x}\right)\right\}. \tag{8.4.4}$$

By the algebraic kernel method or the obstinate kernel method, we can find an equation

$$P_c + P_{00}Q(0,0) + P_{x,0}Q(x,0) + P_{0,\frac{1}{x}}Q\left(0,\frac{1}{x}\right) = 0. \tag{8.4.5}$$

Here we choose $P_{\frac{1}{x},0} = \frac{1}{x^3}$. Then as usual, let us consider $P_{x,0}$.

$$\begin{aligned} P_{x,0} = {} & \Big(-\frac{b(a-b)\left(tx^4+2tx^3+2tx+t-x^2\right)}{2ax^2} \\ & \times \left(2a^2b^2t^3+a^2b^2t^2+a^2b^2-2a^2b+a^2-2ab^2+3ab-a+b^2-b\right) \\ & \times \sqrt{t^2x^4-4t^2x^3-6t^2x^2-4t^2x+t^2-2tx^3-2tx+x^2}+C_2(x,t)\Big) \\ & \times \frac{1}{(2a^2t^2+a^2tx^2-2a^2tx+a^2x^2-atx^2+2atx+at-2ax^2-ax+x^2+x)} \\ & \times \frac{1}{(2b^2t^2+b^2tx^2-2b^2tx+b^2x^2-btx^2+2btx+bt-2bx^2-bx+x^2+x)} \\ & \times \frac{1}{(b^2t^2x^2+2b^2t^2x+b^2t^2+b^2tx^2+b^2t-btx^2-bt-bx+x)}, \end{aligned} \tag{8.4.6}$$

where $C_2(x,t)$ is a rational function. $P_{x,0}$ has three polynomial factors in the denominator (ignore the monomial factor). They are

$$\begin{aligned} &(2a^2t^2+a^2tx^2-2a^2tx+a^2x^2-atx^2+2atx+at-2ax^2-ax+x^2+x) \\ &(2b^2t^2+b^2tx^2-2b^2tx+b^2x^2-btx^2+2btx+bt-2bx^2-bx+x^2+x) \\ &(b^2t^2x^2+2b^2t^2x+b^2t^2+b^2tx^2+b^2t-btx^2-bt-bx+x). \end{aligned} \tag{8.4.7}$$

We list the series expansions of all their roots in order:

$$
\begin{aligned}
X_1 &= \frac{at}{a-1} + \frac{a^2t^2}{a-1} + \frac{a^3t^3}{(a-1)^2} + \frac{a^4(a+1)t^4}{(a-1)^2} + \frac{a^5(3a-1)t^5}{(a-1)^3} + \mathrm{O}(t^6) \\
X_2 &= \frac{1}{a-1} + \frac{(a-2)at}{(a-1)^2} - \frac{a^2\left(a^2-2\right)t^2}{(a-1)^3} - \frac{(a-2)^2a^3t^3}{(a-1)^4} - \frac{a^4\left(a^4-2a^3+4a-4\right)t^4}{(a-1)^5} \\
&\qquad - \frac{a^5\left(3a^4-10a^3+12a^2-8a+4\right)t^5}{(a-1)^6} + \mathrm{O}(t^6) \\
X_3 &= \frac{bt}{b-1} + \frac{b^2t^2}{b-1} + \frac{b^3t^3}{(b-1)^2} + \frac{b^4(b+1)t^4}{(b-1)^2} + \frac{b^5(3b-1)t^5}{(b-1)^3} + \mathrm{O}(t^6) \\
X_4 &= \frac{1}{b-1} + \frac{(b-2)bt}{(b-1)^2} - \frac{b^2\left(b^2-2\right)t^2}{(b-1)^3} - \frac{(b-2)^2b^3t^3}{(b-1)^4} - \frac{b^4\left(b^4-2b^3+4b-4\right)t^4}{(b-1)^5} \\
&\qquad - \frac{b^5\left(3b^4-10b^3+12b^2-8b+4\right)t^5}{(b-1)^6} + \mathrm{O}(t^6) \\
X_5 &= bt + \frac{b^2t^2}{b-1} + \frac{b^3(b+1)t^3}{b-1} + \frac{b^4(3b-1)t^4}{(b-1)^2} + \frac{b^5\left(2b^2+2b+3\right)t^5}{(b-1)^2} + \mathrm{O}(t^6) \\
X_6 &= \frac{1}{bt} + \frac{1}{1-b} - \frac{b\left(b^2-2\right)t}{(b-1)^2} - \frac{(b-2)^2b^2t^2}{(b-1)^3} - \frac{b^3\left(b^4-2b^3+4b-4\right)t^3}{(b-1)^4} \\
&\qquad - \frac{b^4\left(3b^4-10b^3+12b^2-8b+4\right)t^4}{(b-1)^5} - \frac{b^6\left(2b^5-6b^4+7b^3-8b^2+12b-8\right)t^5}{(b-1)^6} + \mathrm{O}(t^6).
\end{aligned}
\tag{8.4.8}
$$

Among all these roots, we see X_1, X_3, X_5 are formal series of t without constant terms. Thus, we let

$$
\begin{aligned}
L = \log[P_{x,0}&(2a^2t^2 + a^2tx^2 - 2a^2tx + a^2x^2 - atx^2 + 2atx + at - 2ax^2 - ax + x^2 + x) \\
&\times (2b^2t^2 + b^2tx^2 - 2b^2tx + b^2x^2 - btx^2 + 2btx + bt - 2bx^2 - bx + x^2 + x) \\
&\times (b^2t^2x^2 + 2b^2t^2x + b^2t^2 + b^2tx^2 + b^2t - btx^2 - bt - bx + x)]
\end{aligned}
\tag{8.4.9}
$$

and multiply (8.4.5) by $\frac{(x-X_1)(x-X_3)(x-X_5)}{e^{L_-}}$. Then the coefficient of $Q(x,0)$ is a formal series of t with coefficients in $\mathbb{R}(x)$. The $[x^>]$, $[x^0]$ and $[x^<]$ terms of this equation can be separated. We finally have

$$
Q(0,0) = \frac{T_1}{T_{00} + T_2 + T_3}, \tag{8.4.10}
$$

where

$$
\begin{aligned}
T_1 &= [x^0]\frac{(x-X_1)(x-X_3)(x-X_5)}{e^{L_-}}P_c \\
T_2 &= [x^0]\frac{(x-X_1)(x-X_3)(x-X_5)}{e^{L_-}}P_{00} \\
T_3 &= [x^0]\frac{(x-X_1)(x-X_3)(x-X_5)}{e^{L_-}}P_{x,0} \\
T_{00} &= [x^0]\frac{(x-X_1)(x-X_3)(x-X_5)}{e^{L_-}}P_{\frac{1}{x},0}.
\end{aligned}
\tag{8.4.11}
$$

The generating functions are still D-algebraic over x, y, t. For $Q(0,0)$ to be D-finite, first L should not contain $\sqrt{\Delta}$, which implies $a = b$. Secondly $P_{0,0}$ should not contain $\sqrt{\Delta}$. This further requires $b - 2c + bc = 0$. Thus $Q(0,0)$ is D-finite if $a = b$ and $c = \frac{b}{2-b}$.

8.5 Results for all models with a finite symmetry group

Currently, **Models 21** and **23** have not been discussed. These two algebraic walks (without interactions) have special properties. We will discuss them in Chapter 10. Now we list the results of all 23 models associated with a finite group, which is a refinement of Table 6.1. The results of **Models 21** and **23** will also be included. We also list the properties of functions $Q(x,0), Q(0,y)$ and $Q(x,y)$ over x, y and $Q(0,0)$ over t. The DF Condition and Alg Condition refer to the conditions that make $Q(0,0)$ D-finite and algebraic over t.

	Model	x, y	t	DF Condition	Alg Condition	Group
1	↑←↓→	DF	DAlg	$abc + ab - ac - bc = 0$ or $a = b = 1$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
2	↖↙↘↗	DF	DAlg	$c = ab$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
3	↑↖↙↓↘↗	DF	DAlg	$c = ab$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
4	↑↖←↙↓↘↗	DF	DAlg	$c = ab$ and $1 - a - b - ab = 0$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$

Table 8.1: These four models are the symmetric D_2 models. The generating functions are D-algebraic over t and further D-finite over x, y. Comparing Table 6.1 and Table 8.1, we can see $Q(0,0)$ of **Model 2** being D-finite in Table 6.1 is just a special case of the result in Table 8.1 (D-finite → DF, D-algebraic → DAlg, Algebraic → Alg, ✓→ fully proved, ✗→ hypothesis).

	Model	x, y	t	DF Condition	Alg Condition	Group
5	↖↓↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{1}{(x+\frac{1}{x})y}\right)$
6	↖←↓→↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{1}{(x+\frac{1}{x})y}\right)$
7	↑↖↓↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{1}{(x+1+\frac{1}{x})y}\right)$
8	↑↖←↓→↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{1}{(x+1+\frac{1}{x})y}\right)$
9	↑↖↙↘↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{x+\frac{1}{x}}{(x+1+\frac{1}{x})y}\right)$
10	↖←↙↘→↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{x+\frac{1}{x}}{(x+1+\frac{1}{x})y}\right)$
11	↑↙↓↘	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, (x+1+\frac{1}{x})\frac{1}{y}\right)$
12	↑←↙↓↘→	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, (x+1+\frac{1}{x})\frac{1}{y}\right)$
13	↖↙↓↘↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{x+1+\frac{1}{x}}{(x+\frac{1}{x})y}\right)$
14	↖←↙↓↘→↗	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, \frac{x+1+\frac{1}{x}}{(x+\frac{1}{x})y}\right)$
15	↑↙↘	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, (x+\frac{1}{x})\frac{1}{y}\right)$
16	↑←↙↘→	DAlg	DAlg	$c = a$ and $b = 1$	–	4: $(\frac{1}{x}, y), \left(x, (x+\frac{1}{x})\frac{1}{y}\right)$

Table 8.2: **Models 5** to **16** are the generally D-algebraic D_2 cases. The generating function is D-algebraic over x, y, t (D-finite → DF, D-algebraic → DAlg, Algebraic → Alg, ✓→ fully proved, ✗→ hypothesis).

	Model	x,y	t	DF Condition	Alg Condition	Group
17	$\uparrow\leftarrow\searrow$	DF	DAlg	$ab-ac-bc+abc=0$	–	6: $(\frac{y}{x},y),(x,\frac{x}{y})$
18	$\uparrow\nwarrow\leftarrow\downarrow\searrow\rightarrow$	DAlg	DAlg	$b-2c+bc=0$ and $a=b$	–	6: $(\frac{y}{x},y),(x,\frac{x}{y})$
19	$\leftarrow\downarrow\nearrow$	DF	Alg	–	All	6: $(\frac{1}{xy},y),(x,\frac{1}{xy})$
20	$\uparrow\swarrow\rightarrow$	Alg	Alg	–	All	6: $(\frac{1}{xy},y),(x,\frac{1}{xy})$
21	$\uparrow\leftarrow\swarrow\downarrow\rightarrow\nearrow$	DAlg?	DAlg?	–	$a=b=1$	6: $(\frac{1}{xy},y),(x,\frac{1}{xy})$

Table 8.3: The D_3 cases. $Q(0,0)$ of Kreweras and reverse Kreweras are still algebraic. When considering $Q(x,0), Q(0,y)$ and $Q(x,y)$, reverse Kreweras walks are still algebraic, but Kreweras walks are D-finite over x,y,t. The other special case, double Kreweras walks, will be discussed later in Section 10.2. We will solve it with condition $a=b$ (D-finite $\rightarrow$ DF, D-algebraic $\rightarrow$ DAlg, Algebraic $\rightarrow$ Alg, ✓$\rightarrow$ fully proved, ✗$\rightarrow$ hypothesis).

	Model	x,y	t	DF Condition	Alg Condition	Group
22	$\nwarrow\leftarrow\searrow\rightarrow$	DF	DAlg	$ab-ac-bc+abc=0$ or $b=1$	–	8: $(\frac{y}{x},y),(x,\frac{x^2}{y})$
23	$\leftarrow\swarrow\rightarrow\nearrow$	DAlg	DAlg?	–	$b=1$	8: $(\frac{1}{xy},y),(x,\frac{1}{x^2y})$

Table 8.4: The final two cases associated with D_4 group. We will prove that the generating function $Q(x,y)$, $Q(x,0)$ and $Q(0,y)$ of Gessel's walks is D-algebraic over x,y in Section 10.1 (D-finite $\rightarrow$ DF, D-algebraic $\rightarrow$ DAlg, Algebraic $\rightarrow$ Alg, ✓$\rightarrow$ fully proved, ✗$\rightarrow$ hypothesis).

Something interesting in the solutions of quarter-plane lattice walk models with interactions is that when $Q(x,y)$ is D-algebraic over x,y, the models cannot be solved directly if it does not have a $\swarrow$ step. This happens for all D_2 models as well as Kreweras walks. It brings a special property for Kreweras walks. $Q(0,0)$ is algebraic but all other $Q_{i,j}$ are D-finite. We currently cannot give an explanation for this phenomenon.

8.6 An alternative way to derive the results

Besides the kernel method introduced in this chapter, We have an alternative way to derive the results of the models with interactions a,b,c via previous results of models with interactions a,b,ab. This method works for the models which were completely solved with interactions a,b,ab, for example, **Models 1-4**. For **Models 5-16**, we only have the results with interactions $a,1,a$ previously. We may extend the previous results to models with interactions $a,1,c$, but for general results with arbitrary a,b,c, we still need to use the kernel method discussed in this chapter.

Here in this section, we specify $Q(0,0), Q(x,y)$ of the models with interactions a,b,c as $Q(0,0,c)$ and $Q(x,y,c)$. $Q(0,0), Q(x,y)$ of the models with interactions a,b,ab are denoted as $Q(0,0,ab)$ and $Q(x,y,ab)$.

Consider the generating function of first-return $F(0,0)$. We may define $F(0,0)$ as walks starting from the origin, ending at $(0,0)$ and only arriving at $(0,0)$ once. Further, we do not count the interaction of the last step ending at $(0,0)$. Thus, $F(0,0)$ is independent of the interactions at the origin. We have the following two equations

$$Q(0,0,ab) = 1 + abF(0,0) + (abF(0,0))^2 + (abF(0,0))^3 \dots \tag{8.6.1}$$

$$Q(0,0,c) = 1 + cF(0,0) + (cF(0,0))^2 + (cF(0,0))^3 \dots . \tag{8.6.2}$$

These two equations show that the generating functions of walks ending at the origin are composed by walks returning once, twice and n times etc. Now we have

$$Q(0,0,c) = \frac{1}{1 - \frac{c}{ab}\left(1 - \frac{1}{Q(0,0,ab)}\right)}. \tag{8.6.3}$$

A similar idea can also be applied to derive $Q(x,y)$. We denote $R(x,y)$ as the generating function of excursions, which are the walks starting from the origin and never returning to $(0,0)$. $R(x,y)$ is also independent of the interaction at the origin. We have

$$Q(x,y,ab) = Q(0,0,ab)R(x,y) \tag{8.6.4}$$

$$Q(x,y,c) = Q(0,0,c)R(x,y) \tag{8.6.5}$$

$$\tag{8.6.6}$$

Thus, we have

$$Q(x,y,c) = \frac{Q(0,0,c)}{Q(0,0,ab)Q(x,y,ab)} \tag{8.6.7}$$

Chapter 9

A general discussion of the kernel method

For most of the models we discussed above, we may apply either version of the kernel method to solve them. In this chapter, we discuss the difference and relation between these two methods. We will also discuss the limitations of the kernel method.

9.1 The general process of the kernel method

Obstinate kernel method

1. Substitute the root Y_0 which is a formal series of t in to the functional equation.

2. Apply the group elements to the functional equation. Choose those equations in which $Q(g(x), g(Y_0))$ converges when $t \to 0$.

3. Use a linear combination of the equations. Find an equation only containing unknown functions of x (or y).

4. Use the canonical factorization to factor the coefficients $\sqrt{\Delta}$ or $p+q\sqrt{\Delta}$. Then, find a way to separate the $[x^0]$, $[x^>]$, $[x^<]$ parts of this equation and use the original kernel method to solve it.

Algebraic kernel method

1. Apply the group elements to the functional equation. Find a linear combination (full orbit-sum) that eliminates the line boundary terms.

2. Take the $[y^0]$ (or $[x^0]$) part of the full orbit sum. The full orbit sum may give trivial information $0 = 0$ or some information about reflection symmetry or directly the same result as Step 3 in **Obstinate kernel method**.

3. If **Step 2** does not give the same result as **Step 3** in **Obstinate kernel method**, we then take the half-orbit sum and substitute the reflection symmetry into it. We get an equation only with unknown functions of x (or y).

4. Find a way to separate the $[x^0]$, $[x^>]$, $[x^<]$ parts of this equation and use the original kernel method to solve it.

Step 4 of both methods are the same. For different models, we apply different tactics. We have seen that for some models, separating the $[x^>]$, $[x^0]$ and $[x^<]$ terms of the equation will directly give the solutions. For some models, this operation will introduce more unknowns and we need to use the original kernel method to solve the problems. For half symmetric models, we sometimes need to use the $Q(0,0)$ of the reverse walk to solve $Q(x,0)$.

We can see that in every step described above, we only use elementary operations on the functional equation. It contains substitution, factorization, linear operations and taking certain exponents of the functional equation. These operations are closed in some field. The field is generated by a transcendental extension of some non-algebraic terms. This implies that if the solution is not in this field, we cannot solve it by the kernel method. And the solution is rational over this field and is thus D-algebraic.

9.2 Comparison of the obstinate kernel method and the algebraic kernel method

We have demonstrated that most solvable models can be solved by both methods. However the obstinate kernel method and the algebraic kernel method sometimes give different information. Here we want to discuss the relation and difference between these two methods. To compare these two methods, we use Kreweras walks with interactions as an example.

Recall the functional equation (8.2.46),

$$\left(1-t\left(xy+\frac{1}{x}+\frac{1}{y}\right)\right)Q(x,y)=\frac{1}{c}+\frac{1}{a}\left(a-1-\frac{ta}{y}\right)Q(x,0)+\frac{1}{b}\left(b-1-\frac{tb}{x}\right)Q(0,y)$$
$$+\frac{1}{abc}(ac+bc-ab-abc)Q(0,0). \quad (9.2.1)$$

We denote this equation as E_{Kr}. Recall the symmetry group (8.2.48)

$$\left\{(x,y),\left(\frac{1}{xy},y\right),\left(y,\frac{1}{xy}\right),(y,x),\left(\frac{1}{xy},x\right),\left(x,\frac{1}{xy}\right)\right\}. \quad (9.2.2)$$

The two roots of kernel are

$$\begin{aligned}
Y_0&=\frac{-\sqrt{-4t^2x^3+t^2-2tx+x^2}-t+x}{2tx^2}=t+\frac{t^2}{x}+t^3\left(\frac{1}{x^2}+x\right)+\mathrm{O}(t^4)\\
Y_1&=\frac{\sqrt{-4t^2x^3+t^2-2tx+x^2}-t+x}{2tx^2}=\frac{1}{tx}-\frac{1}{x^2}-t-\frac{t^2}{x}+t^3\left(-\frac{1}{x^2}-x\right)+\mathrm{O}(t^4).
\end{aligned} \quad (9.2.3)$$

Now, let us first apply the obstinate kernel method to E_{Kr}. We substitute Y_0 into this equation and the LHS cancels. This gives

$$0=\frac{bc\left(-\frac{at}{Y_0}+a-1\right)Q(x,0)+ac\left(-\frac{bt}{x}+b-1\right)Q(0,Y_0)+(bc-a((b-1)c+b))Q(0,0)+ab}{abc}. \quad (9.2.4)$$

Then let us directly apply the algebraic kernel method to E_{Kr}: divide the RHS by the kernel and take the $[y^0]$ part of this equation. This gives:

$$\begin{aligned}&\sqrt{\Delta}Q(x,0)\\&=\frac{ac\left(-\frac{bt}{x}+b-1\right)Q(0,Y_0)+bc\left(-\frac{at}{Y_1}+a-1\right)Q(x,0)+(bc-a((b-1)c+b))Q(0,0)+ab}{abc}.\end{aligned}\tag{9.2.5}$$

Notice that (9.2.4) and (9.2.5) both give a relation between $Q(0,Y_0)$ and $Q(x,0)$. They are actually the same because of (5.2.9). Therefore, in this case, the obstinate method and the algebraic kernel method are equivalent. The equations $\delta_{(x,y)}E_{Kr}, \delta_{(y,x)}E_{Kr}, \delta_{(\frac{1}{xy},x)}E_{Kr}, \delta_{(x,\frac{1}{xy})}E_{Kr}$ have the same properties as E_{Kr}. The equations obtained from the algebraic kernel method and the obstinate kernel method are the same.

Let us have a look at another case. Apply the group element $(\frac{1}{xy}, y)$ to E_{Kr} first and then apply both methods. The obstinate kernel method gives us:

$$\begin{aligned}&0=\\&\frac{bc\left(-\frac{at}{Y_0}+a-1\right)Q\left(\frac{1}{xY_0},0\right)+acQ(0,Y_0)(-btxY_0+b-1)+Q(0,0)(bc-a((b-1)c+b))+ab}{abc}.\end{aligned}\tag{9.2.6}$$

We substitute Y_0 in to the transformed functional equation. Note that although $\frac{1}{xY_0}$ is not a formal series of t, $Q\left(\frac{1}{xY_0},Y_0\right)$ actually is by the same discussion in Section 5.2.2.

By the relation between Y_0 and Y_1, we can write (9.2.6) as

$$\begin{aligned}&0=\\&\frac{bc\left(-\frac{at}{Y_0}+a-1\right)Q\left(Y_1,0\right)+acQ(0,Y_0)(-btxY_0+b-1)+Q(0,0)(bc-a((b-1)c+b))+ab}{abc}.\end{aligned}\tag{9.2.7}$$

The obstinate kernel method gives a relation between $Q(Y_1,0)$ and $Q(0,Y_0)$.

Next, consider the algebraic kernel method. Divide the RHS by the kernel and take the $[y^0]$ part of the equation, by Theorem 6 and Lemma 5, we have,

$$\begin{aligned}&\sqrt{\Delta}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)=\\&\frac{bc\left(-\frac{at}{Y_1}+a-1\right)Q\left(\frac{1}{xY_1},0\right)+ac\left(-btxY_0+b-1\right)Q\left(0,Y_0\right)+(bc-a((b-1)c+b))Q(0,0)+ab}{abc}.\end{aligned}\tag{9.2.8}$$

Notice that $[y^0]\frac{Q\left(\frac{1}{xy},0\right)}{K(x,y)}=\frac{1}{\sqrt{\Delta}}Q(\frac{1}{xY_1},0)$. By the relation between the roots, we have

$$\begin{aligned}&\sqrt{\Delta}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)=\\&\frac{bc\left(-\frac{at}{Y_1}+a-1\right)Q\left(Y_0,0\right)+ac\left(-btxY_0+b-1\right)Q\left(0,Y_0\right)+(bc-a((b-1)c+b))Q(0,0)+ab}{abc}.\end{aligned}\tag{9.2.9}$$

The algebraic kernel method gives us a relation between $Q(Y_0, 0)$, $Q(0, Y_0)$ and $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$. For group elements $(\frac{1}{xy}, y), (y, \frac{1}{xy})$, we have similar results.

This is the first difference. For the obstinate kernel method, $Q(0, Y_1)$ appears in the equations. We may use the relation between two roots to get a relation between $Q(0, Y_0)$ and $Q(0, Y_1)$ and simplify the final equation. For example, for Kreweras walks without interactions, we have

$$\frac{Y_0 tQ(Y_0, 0) - Y_1 tQ(Y_1, 0)}{Y_0 - Y_1} - x = tx \frac{2xtQ(x, 0) + \frac{2}{x} - \frac{1}{t}}{\sqrt{\Delta}}, \tag{9.2.10}$$

where the LHS of (9.2.10) is a symmetric function of Y_0, Y_1. It can be expressed by a polynomial of $Y_0 + Y_1$ and $Y_0 Y_1$. By (5.2.9), they are both functions of $\frac{1}{x}$.

However, this idea is not effective for reverse Kreweras walks. Since for reverse Kreweras walks,

$$\begin{aligned} Y_0 + Y_1 &= -\left(x - \frac{1}{t}\right) \\ Y_0 Y_1 &= \frac{1}{x}. \end{aligned} \tag{9.2.11}$$

So $P(Y_0) + P(Y_1)$ cannot be simplified to a series of x with finite $\frac{1}{x}$ part or series of $\frac{1}{x}$ with finite x part. This makes it hard to solve reverse Kreweras walks by the obstinate kernel method.

For the algebraic kernel method, we introduce a new unknown function $Q_0^{\mathrm{d}}(\frac{1}{x})$. By calculation (linear combination of (9.2.7) and (9.2.9)), we find

$$\sqrt{\Delta} Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) = xtY_0 Q(0, Y_0) - xtY_1 Q(0, Y_1). \tag{9.2.12}$$

The series $Q_0^{\mathrm{d}}(\frac{1}{x})$ for Kreweras walks is a function of $Q(0, Y_1)$ and $Q(0, Y_0)$. We can substitute $Q_0^{\mathrm{d}}(\frac{1}{x})$ as a function of $Q(0, Y_1)$ and $Q(0, Y_0)$. And any process through the obstinate kernel method so far can be changed into a process of the algebraic kernel method. In reverse, we can also change the process from the algebraic kernel method to the obstinate kernel method. For example, for reverse Kreweras walks without interactions, we actually have that

$$\frac{1 - txQ(Y_0, 0) - txQ(0, Y_0) + txQ(0, 0)}{\sqrt{\Delta}} \tag{9.2.13}$$

is a formal series of $\frac{1}{x}$. But normally, one can not get this result without the help of the algebraic kernel method.

Until now, the algebraic kernel method and the obstinate method do the same thing. Now the following equation from the algebraic kernel method provides a relation that does not have any correspondence in the obstinate kernel method:

$$\begin{aligned} Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) = \frac{1}{cY_1} + \frac{1}{aY_1}\left(a - 1 - \frac{ta}{Y_1}\right) Q\left(\frac{1}{xY_1}, 0\right) \\ + \frac{1}{bY_1}(b - 1 - tbxY_0)Q(0, Y_0) + \frac{1}{b}(b-1)\left(\frac{Q(0,0)}{Y_1} - \frac{Q(0,0)}{Y_0}\right) \\ + \frac{1}{abcY_1}(ac + bc - ab - abc)Q(0, 0). \end{aligned} \tag{9.2.14}$$

This relation comes from the transformed equation $\delta_{(\frac{1}{xy}, y)} E_{Kr}$. We multiply $\delta_{(\frac{1}{xy}, y)} E_{Kr}$ by $\frac{1}{y}$ and then take the $[y^0]$ part. We get an extra relation between the off diagonal $Q_1^{\mathrm{d}}(\frac{1}{x})$ and $Q(Y_0, 0)$, $Q(0, Y_0)$.

This operation is not valid for the equation $\delta_{(x,y)}E_{Kr}$. Suppose we analogously do this operation to $\delta_{(x,y)}E_{Kr}$. The LHS becomes $[y^0]\frac{1}{y}Q(x,y) = [y^1]Q(x,y)$. We get a relation between $[y^1]Q(x,y)$ and $Q(x,0), Q(0,Y_0)$. However, this relation does not contain new information. We can get the relation between $Q(x,0)$ and $[y^1]Q(x,y)$ by the geometric property of the walk. And combining the relation obtained from $\delta_{(x,y)}E_{Kr}$ with the geometric relation, we get (9.2.4) again.

In our previous calculations, this new relation appears as a part of the orbit sum. $Q_1^{\mathrm{d}}(x)$ appears in the orbit sums of both reverse Kreweras walks and Kreweras walks with interactions. For reverse Kreweras walks, we use the relation between $Q_0^{\mathrm{d}}(x)$ and $Q_1^{\mathrm{d}}(x)$ obtained from the full orbit-sum to simplify the half-orbit sum. For Kreweras walks, this extra relation is not pivotal. In the following calculation for Gessel's walks, we will demonstrate how important this new relation is.

Chapter 10

Application of the general form of the kernel method

In this chapter, we apply the discussion in Chapter 9 to discuss the two unsolved cases, Gessel's walks and double Kreweras walks. Because of the complexity of the calculation, we currently cannot solve them explicitly. Instead, we find the polynomial equation that $Q(x,0)$ satisfies. This will show that $Q(x,0)$ is D-algebraic with respect to the variable x.

10.1 An alternative method for solving Gessel's walks with boundary interactions

For the solution of Gessel's walks without boundary interactions, one can recall Section 5.3.2 or [17]. The idea for solving Gessel's walks with interactions is still to find a polynomial equation $P(Q(x,0), Q_i, x, t) = 0$ and apply Bousquet-Mélou's theorem [19][1]. We will also attempt to follow this strategy.

10.1.1 The functional equation and symmetry group

For Gessel's walks, the allowed steps are $\{\nearrow, \swarrow, \rightarrow, \leftarrow\}$. The kernel is

$$K(x,y) = 1 - t\left(\frac{1}{x} + x + xy + \frac{1}{xy}\right), \tag{10.1.1}$$

with discriminant

$$\Delta = \left(t\left(x + \frac{1}{x}\right) - 1\right)^2 - 4t^2. \tag{10.1.2}$$

The roots of the kernel are

$$\begin{aligned} Y_0 &= \frac{x - t - tx^2 - x\sqrt{\Delta}}{2tx^2} && (10.1.3)\\ &= \frac{t}{x} + t^2\left(\frac{1}{x^2} + 1\right) + t^3\left(\frac{1}{x^3} + x + \frac{3}{x}\right) + t^4\left(\frac{1}{x^4} + x^2 + \frac{6}{x^2} + 6\right) + \mathrm{O}(t^5) && (10.1.4) \end{aligned}$$

[1]We will discuss this theorem in Chapter 12.

and

$$Y_1 = \frac{x - t - tx^2 + x\sqrt{\Delta}}{2tx^2} \tag{10.1.5}$$
$$= \frac{1}{tx} + \left(-\frac{1}{x^2} - 1\right) - \frac{t}{x} + t^2\left(-\frac{1}{x^2} - 1\right) + t^3\left(-\frac{1}{x^3} - x - \frac{3}{x}\right) \tag{10.1.6}$$
$$+ t^4\left(-\frac{1}{x^4} - x^2 - \frac{6}{x^2} - 6\right) + \mathrm{O}(t^5).$$

The relation between the two roots is

$$Y_0 + Y_1 = -\left(1 - \frac{1}{tx} + \frac{1}{x^2}\right) \tag{10.1.7}$$
$$Y_0 Y_1 = \frac{1}{x^2}. \tag{10.1.8}$$

By (6.1.3), the functional equation is

$$K(x,y)Q(x,y) = \frac{1}{c} + \frac{1}{a}\left(a - 1 - \frac{ta}{xy}\right)Q(x,0) + \frac{1}{b}\left(b - 1 - tb\left(\frac{1}{x} + \frac{1}{xy}\right)\right)Q(0,y)$$
$$+ \left(\frac{1}{abc}(ac + bc - ab - abc) + \frac{t}{xy}\right)Q(0,0). \tag{10.1.9}$$

The symmetry group for Gessel's walks is

$$\left\{(x,y), \left(\frac{1}{xy}, y\right), \left(x, \frac{1}{x^2y}\right), \left(xy, \frac{1}{x^2y}\right), \left(\frac{1}{x}, x^2y\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(xy, \frac{1}{y}\right), \left(\frac{1}{xy}, x^2y\right)\right\} \tag{10.1.10}$$

which is isomorphic to D_4.

10.1.2 Full-orbit sum

The first step is the same as Kreweras and reverse Kreweras. We apply the symmetries to the functional equation and write them in a matrix form,

$$K(x,y)\mathbf{Q} = \mathbf{MV} + \mathbf{C}, \tag{10.1.11}$$

where $\mathbf{Q}$, $\mathbf{V}$ and $\mathbf{C}$ are

$$\mathbf{Q} = \begin{pmatrix} Q(x,y) \\ Q\left(\frac{1}{xy}, y\right) \\ Q\left(x, \frac{1}{x^2y}\right) \\ Q\left(xy, \frac{1}{x^2y}\right) \\ Q\left(\frac{1}{x}, x^2y\right) \\ Q\left(\frac{1}{x}, \frac{1}{y}\right) \\ Q\left(xy, \frac{1}{y}\right) \\ Q\left(\frac{1}{xy}, x^2y\right) \end{pmatrix}, \tag{10.1.12}$$

$$\mathbf{V} = \begin{pmatrix} Q(0,y) \\ Q(1/x,0) \\ Q\left(0,\frac{1}{x^2y}\right) \\ Q(0,x^2y) \\ Q\left(0,\frac{1}{y}\right) \\ Q\left(\frac{1}{xy},0\right) \\ Q(xy,0) \\ Q\left(\frac{1}{x},0\right) \\ Q(x,0) \end{pmatrix}, \tag{10.1.13}$$

$$\mathbf{C} = \frac{1}{abc}\begin{pmatrix} [\frac{abctQ(0,0)}{xy} + Q(0,0)(bc-a((b-1)c+b)) + ab] \\ [abctQ(0,0)x + Q(0,0)(bc-a((b-1)c+b)) + ab] \\ [abctQ(0,0)xy + Q(0,0)(bc-a((b-1)c+b)) + ab] \\ [abctQ(0,0)x + Q(0,0)(bc-a((b-1)c+b)) + ab] \\ [\frac{abctQ(0,0)}{xy} + Q(0,0)(bc-a((b-1)c+b)) + ab] \\ [abctQ(0,0)xy + Q(0,0)(bc-a((b-1)c+b)) + ab] \\ [\frac{abctQ(0,0)}{x} + Q(0,0)(bc-a((b-1)c+b)) + ab] \\ [\frac{abctQ(0,0)}{x} + Q(0,0)(bc-a((b-1)c+b)) + ab] \end{pmatrix}. \tag{10.1.14}$$

Here $\mathbf{M}$ is the coefficient matrix. For Gessel's walks, we still have $\det \mathbf{M} = \mathbf{0}$. The nullspace is spanned by the vector,

$$\mathbf{N} = \frac{1}{x^3}\begin{pmatrix} \frac{(at-ax+x)(atxy-a+1)(btxy+btx-b+1)\left(btx^2y+bt-bxy+xy\right)}{y^2} \\ -\frac{(-at+ax-x)(atxy-a+1)(-bty-bt+bxy-xy)\left(btx^2y+bt-bxy+xy\right)}{xy^3} \\ -\frac{(-at+ax-x)(-at+axy-xy)(btxy+btx-b+1)\left(btx^2y+bt-bxy+xy\right)}{xy^3} \\ \frac{(-at+ax-x)(-at+axy-xy)(btxy+btx-b+1)\left(btx^2y+bt-bx+x\right)}{xy^2} \\ -\frac{(atx-a+1)(atxy-a+1)(bty+bt-bxy+xy)\left(btx^2y+bt-bx+x\right)}{y^2} \\ \frac{(atx-a+1)(-at+axy-xy)(-bty-bt+bxy-xy)\left(btx^2y+bt-bx+x\right)}{xy^3} \\ -\frac{(atx-a+1)(at-axy+xy)(btxy+btx-b+1)\left(btx^2y+bt-bx+x\right)}{y^2} \\ \frac{(atx-a+1)(atxy-a+1)(bty+bt-bxy+xy)\left(btx^2y+bt-bxy+xy\right)}{y^3} \end{pmatrix}^{\mathsf{T}}. \tag{10.1.15}$$

We still have

$$K(x,y)\mathbf{NQ} = \mathbf{NC}, \tag{10.1.16}$$

where

$$\mathbf{NC} = -\frac{a^2(b-1)t^3(x^2-1)(y-1)(x^2y^2-1)\left(x^2y-1\right)((b-c)Q(0,0)-b)}{cx^4y^3}. \tag{10.1.17}$$

Since $\mathbf{NC} \neq 0$, this is the same case as Kreweras walks. Following the process of the algebraic kernel method we divide the RHS of (10.1.16) by $K(x,y)$ and take the $[y^0]$ term. We find

$$\begin{aligned}[y^0]\frac{\mathbf{NC}}{K(x,y)} &= -\frac{a^2(b-1)(x^2-1)\left(2tx^2+2t-x\right)\sqrt{\Delta}((b-c)Q(0,0)-b)}{2cx^2} \\ &- \frac{a^2(b-1)(x^2-1)\left(2t^2x^4+2t^2x^2+2t^2-3tx^3-3tx+x^2\right)((b-c)Q(0,0)-b)}{2cx^3}. \end{aligned} \tag{10.1.18}$$

We observe that $[y^0]\frac{\mathbf{NC}}{K(x,y)}$ can be written as a function $f(x+\frac{1}{x})$ times $(x-\frac{1}{x})$. The coefficients $[x^i][y^0]\frac{\mathbf{NC}}{K(x,y)}$ and $[x^{-i}][y^0]\frac{\mathbf{NC}}{K(x,y)}$ have the same absolute value with opposite signs. Thus, we write

$$[y^0]\frac{\mathbf{NC}}{K(x,y)} = PR(x) - PR\left(\frac{1}{x}\right). \tag{10.1.19}$$

Here $PR(x)$ and $PR(\frac{1}{x})$ are the $[x^>]$ and $[x^<]$ series of $[y^0]\frac{\mathbf{NC}}{K(x,y)}$. If we write $PR(x) = f(x) + g(x)Q(0,0)$ then f and g are D-finite and non-algebraic by Theorems 10 and 14.

Remark 6. *For brevity, for the remainder of this section we will keep $PR(x)$ as defined above, and in particular, dependent on $Q(0,0)$. Since $Q(0,0)$ appears separately in virtually every equation anyway, this does not conceal any complexity.*

After some calculations (take the $[y^0x^>]$ terms of the full orbit-sum and use boundary relations to simplify the equation), we find the LHS of the full-orbit sum contains $Q(x,0), Q_0^{\mathrm{d}}(x), Q_1^{\mathrm{d}}(x)$, $Q(0,0), Q_{1,0}, Q_{2,0}, Q_{3,0}, Q_{0,1}$. The $[y^0x^>]$ part of (10.1.16) can be written as

$$\alpha_{x,d,1}Q_1^{\mathrm{d}}(x) + \alpha_{x,d}Q_0^{\mathrm{d}}(x) + \alpha_{x,0}Q(x,0) + \alpha_{other} = 0. \tag{10.1.20}$$

Since we have too many boundary terms $Q_{i,j}$ (and we will introduce more in later calculations), we put all the unknowns which are not related to x into α_{other}. When we are dealing with x, such as taking the $[x^>]$ or $[x^<]$ terms, the unknown functions of t, Q_{ij} are not involved. So we treat them all as constants until Section 10.1.6. In Section 10.1.6, we will discuss these Q_{ij}. Notice that α_{other} also contains $PR(x)$, the non-algebraic term of x.

If we apply the transformation $x \to \frac{1}{x}$ to (10.1.20), we will also get the relation between $Q_1^{\mathrm{d}}(\frac{1}{x}), Q_0^{\mathrm{d}}(\frac{1}{x})$ and $Q\left(\frac{1}{x}, 0\right)$. This is actually the $[y^0x^<]$ part of (10.1.16). The full orbit sum thus provides two equations with $x \to \frac{1}{x}$ symmetry.

10.1.3 The diagonal relation

Unlike the procedure for Kreweras and reverse Kreweras walks, the next step is not taking the half-orbit sum. We first give an explanation for why the half-orbit sum is not effective.

Consider the half-orbit sum. By calculation, we get an equation with $Q_0^{\mathrm{d}}(x)$, $Q_1^{\mathrm{d}}(x)$, $Q(x,0)$ and $Q_0^{\mathrm{d}}(\frac{1}{x})$, $Q_1^{\mathrm{d}}(\frac{1}{x})$, $Q\left(\frac{1}{x}, 0\right)$. The coefficients of $Q_0^{\mathrm{d}}(x), Q_0^{\mathrm{d}}(\frac{1}{x})$ and $Q_1^{\mathrm{d}}(x), Q_1^{\mathrm{d}}(\frac{1}{x})$ are polynomials and the coefficients of $Q(x,0)$ and $Q\left(\frac{1}{x}, 0\right)$ are in the form $p + q\sqrt{\Delta}$. If we substitute the equations we obtained from the full orbit-sum and eliminate $Q_1^{\mathrm{d}}(x)$ and $Q_1^{\mathrm{d}}(\frac{1}{x})$, the half orbit sum will provide us an equation with unknown functions $Q_0^{\mathrm{d}}(x), Q_0^{\mathrm{d}}(\frac{1}{x})$ and $Q(x,0), Q\left(\frac{1}{x}, 0\right)$. Notice that in (10.1.20), all α except α_{other} are polynomials. So when we do the substitution, we cannot eliminate the $\sqrt{\Delta}$ in the coefficients of $Q(x,0)$ and $Q\left(\frac{1}{x}, 0\right)$. Thus in the final equation, we have $\sqrt{\Delta}$ in the coefficients of both an unknown formal series of x ($Q(x,0)$) and an unknown formal series of $\frac{1}{x}$ ($Q\left(\frac{1}{x}, 0\right)$). Meanwhile, $Q_0^{\mathrm{d}}(x)$ and $Q_0^{\mathrm{d}}(\frac{1}{x})$ are still in the equation and their coefficients are polynomials without $\sqrt{\Delta}$. We cannot take the $[x^>]$ or $[x^<]$ of this equation.

The problem with the half orbit-sum is that it is too symmetric. Even if we got rid of the $\sqrt{\Delta}$ in the coefficient of $Q(x,0)$, we still cannot solve the equation. If we simultaneously eliminated $\sqrt{\Delta}$ in the coefficients of $Q(x,0)$ and $Q\left(\frac{1}{x}, 0\right)$ but leave both $Q_0^{\mathrm{d}}(x)$ and $Q_0^{\mathrm{d}}(\frac{1}{x})$ in the equation, when we take the $[x^>]$, $Q_0^{\mathrm{d}}(x)$ and $Q(x,0)$ are still both in the equation and they are both formal series of x. We still need some equations that help us to separate these two functions.

Thus, we want to find an equation that is not symmetric. If we do something to the $[x^<]$ part, the $[x^>]$ part must not behave the same way. We want an equation that has $Q_0^{\mathrm{d}}(x)$ in the $[x^>]$ part but does not have $Q_0^{\mathrm{d}}(\frac{1}{x})$ in the $[x^<]$ part. Then we can separate $Q_0^{\mathrm{d}}(x)$ and $Q(x,0)$

apart. We will first get a relation between $Q^{\mathrm{d}}(x)$ $Q(x,0)$, and $Q(\frac{1}{x},0)$. Then in Section 10.1.4, we get rid of $Q^{\mathrm{d}}(x)$ and find an equation of $Q(x,0)$ and $Q(\frac{1}{x},0)$. Finally in Section 10.1.5, we will find an equation with $Q(x,0)$ only.

The first step is to consider the diagonal relation. Let

$$\delta_{(x',y')}(F(x,y)) = F(x',y') \tag{10.1.21}$$

and continue to denote the functional equation as E. First taking the $[y^0]$ part of $\sqrt{\Delta}\delta_{(\frac{1}{xy},y)}(\frac{E}{y})$[2], the diagonal relation reads

$$\begin{aligned}\sqrt{\Delta}Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) &= \frac{x^2Y_0}{c} + \frac{(ac-abc+abctx-bx^2(a-c)Y_0^2)Q(0,0)}{abcY_0}\\ &\quad - \frac{(1-b+btx+btxY_0)Q(0,Y_0)}{bY_0} - \frac{x^2(1-a+atx)Y_0Q(xY_0,0)}{a}.\end{aligned} \tag{10.1.22}$$

We have three unknown functions $Q_1^{\mathrm{d}}(\frac{1}{x}), Q(0,Y_0), Q(xY_0,0)$. The $[y^0]$ parts of $\sqrt{\Delta}E$ and $\sqrt{\Delta}\delta_{(\frac{1}{xy},y)}(E)$ can help us to eliminate $Q(0,Y_0), Q(xY_0,0)$. They are respectively

$$\begin{aligned}\left(\sqrt{\Delta}+\frac{1-a+atxY_0}{a}\right)Q(x,0) &= \frac{1}{c} + \frac{(abct-(ab-ac-bc+abc)xY_0)Q(0,0)}{abcxY_0}\\ &\quad - \frac{(bt+(bt+x-bx)Y_0)Q(0,Y_0)}{bxY_0}\end{aligned} \tag{10.1.23}$$

and

$$\begin{aligned}\sqrt{\Delta}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) &= \frac{1}{c} - \frac{(ab-ac-bc+abc-abctx)Q(0,0)}{abc}\\ &\quad - \frac{(1-b+btx+btxY_0)Q(0,Y_0)}{b} - \frac{(1-a+atx)Q(xY_0,0)}{a}.\end{aligned} \tag{10.1.24}$$

A simple linear combination of (10.1.22), (10.1.23), (10.1.24) will give us a relation between $Q_1^{\mathrm{d}}(\frac{1}{x}), Q_0^{\mathrm{d}}(\frac{1}{x})$ and $Q(x,0)$. Applying the transformation $x \to \frac{1}{x}$ we get the relation between $Q_1^{\mathrm{d}}(x), Q_0^{\mathrm{d}}(x)$ and $Q\left(\frac{1}{x},0\right)$:

$$\gamma_{x,d}Q_0^{\mathrm{d}}(x) + \gamma_{x,d,1}Q_1^{\mathrm{d}}(x) + \gamma_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) + \gamma_c = 0. \tag{10.1.25}$$

Combining (10.1.20) and (10.1.25) together, we eliminate $Q_1^{\mathrm{d}}(x)$ and finally get an equation with respect to $Q_0^{\mathrm{d}}(x), Q(x,0)$ and $Q\left(\frac{1}{x},0\right)$:

$$\theta_x^{\mathrm{d}}Q_0^{\mathrm{d}}(x) + \theta_{x,0}Q(x,0) + \theta_{\frac{1}{x},0}Q\left(\frac{1}{x},0\right) + \theta_c = 0. \tag{10.1.26}$$

By calculation, these coefficients are

$$\begin{aligned}\theta_x^{\mathrm{d}} &= -2ax\left(t(x-1)^2-x\right)\left(t(x+1)^2-x\right)\left(a(tx-1)+1\right)\\ &\quad \times \left(b^2t+bt\left(x^2-1\right)-bx+x\right)\left(b^2tx^2-b\left(t\left(x^2-1\right)+x\right)+x\right)\end{aligned} \tag{10.1.27}$$

[2]We apply the relation (10.1.8) between the two roots in the following calculations to simplify the equations.

$$\theta_{\frac{1}{x},0} = 2x^2(atx - a + 1)\left(b^2tx^2 - btx^2 + bt - bx + x\right)$$
$$\times \left(ab^2t^2x^4 + ab^2t^2 - ab^2tx^3 - ab^2tx - abt^2x^4 + 2abt^2x^2 - abt^2 + 3abtx^3\right.$$
$$\left. + 3abtx - atx^3 - atx - ax^2 - b^2tx^3 - b^2tx + b^2x^2 - 2bx^2 + 2x^2\right)\sqrt{\Delta} \quad (10.1.28)$$

$$\theta_{x,0} = \left(tx^2 - 2tx + t - x\right)\left(tx^2 + 2tx + t - x\right)(at - ax + x)\left(b^2t + btx^2 - bt - bx + x\right)$$
$$\times \left(ab^2tx^2 + ab^2t - abtx^2 - abt - ax - b^2x + 2bx\right)$$
$$+ x(at - ax + x)\left(b^2t + btx^2 - bt - bx + x\right)$$
$$\times \left(ab^2t^2x^4 + ab^2t^2 - ab^2tx^3 - ab^2tx - abt^2x^4 + 2abt^2x^2 - abt^2\right.$$
$$\left. + 3abtx^3 + 3abtx - atx^3 - atx - ax^2 - b^2tx^3 - b^2tx + b^2x^2 - 2bx^2 + 2x^2\right)\sqrt{\Delta}. \quad (10.1.29)$$

The remaining term θ_c is a linear function of $PR(x)$ with coefficients in $\mathbb{R}(x, t, Q_{i,j})$. The expression is too complicated so we do not write it here. One can check it in the Mathematica file.

This equation is not symmetric. It only has $Q(\frac{1}{x}, 0)$ in the $[x^<]$ side. However, we cannot directly separate it, because the coefficient of $Q(x, 0)$ in (10.1.26) is $p+q\sqrt{\Delta}$ and the coefficient of $Q\left(\frac{1}{x}, 0\right)$ is $q\sqrt{\Delta}$. The coefficient of $Q_0^{\mathrm{d}}(x)$ is a polynomial.

Applying the transformation $x \to \frac{1}{x}$, we get the corresponding $\frac{1}{x}$ form of (10.1.26):

$$\eta^{\mathrm{d}}_{\frac{1}{x}} Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + \eta_{\frac{1}{x},0} Q\left(\frac{1}{x}, 0\right) + \eta_{x,0} Q(x, 0) + \eta_c = 0. \quad (10.1.30)$$

Here $\eta^{\mathrm{d}}_{\frac{1}{x}}$ is a polynomial, $\eta_{x,0}$ is a polynomial times $\sqrt{\Delta}$ and $\eta_{\frac{1}{x},0}$ is in the form of $p + q\sqrt{\Delta}$.

We have achieved half of our objective. The equations (10.1.26) and (10.1.30) are not symmetric. But their $[x^>]$ and $[x^<]$ parts cannot be separated directly. Our next step is to find a way to separate $Q_0^{\mathrm{d}}(x)$ and $Q\left(\frac{1}{x}, 0\right)$ with the help of these two equations.

10.1.4 An equation with $Q(x, 0)$ and $Q(\frac{1}{x}, 0)$ only

The general idea

The idea of eliminating $Q_0^{\mathrm{d}}(x)$ from (10.1.26) is very similar to that in [17], which is a common technique to deal with these kind of symmetries. The idea is to square the equation and eliminate the $\sqrt{\Delta}$. Then use the $x \to \frac{1}{x}$ symmetry to deal with the $Q(x, 0)Q\left(\frac{1}{x}, 0\right)$ terms.

First notice that $\sqrt{\Delta}$ only appears in θ_c and the coefficients of $Q(x, 0)$ and $Q\left(\frac{1}{x}, 0\right)$ in (10.1.26) and (10.1.30). Let us first write (10.1.26) as

$$\theta_x^{\mathrm{d}} Q_0^{\mathrm{d}}(x) + \left(\theta_{x,0,c} + \theta_{x,0,\sqrt{\Delta}}\sqrt{\Delta}\right) Q(x, 0) + \theta_{\frac{1}{x},0,\sqrt{\Delta}}\sqrt{\Delta} Q\left(\frac{1}{x}, 0\right) + \theta_{c,c} + \theta_{c,\sqrt{\Delta}}\sqrt{\Delta} = 0. \quad (10.1.31)$$

Then move the $\sqrt{\Delta}$ to the RHS and square the equation, we have

$$\left(\theta_x^{\mathrm{d}} Q_0^{\mathrm{d}}(x) + \theta_{x,0,c} Q(x, 0) + \theta_{c,c}\right)^2 = \Delta\left(\theta_{x,0,\sqrt{\Delta}} Q(x, 0) + \theta_{\frac{1}{x},0,\sqrt{\Delta}} Q\left(\frac{1}{x}, 0\right) + \theta_{c,\sqrt{\Delta}}\right)^2. \quad (10.1.32)$$

Expand (10.1.32). The terms on the LHS are all products of formal series of x, including products of $Q_0^{\mathrm{d}}(x)$, $Q(x, 0)$ and $PR(x)$. The terms on the RHS are products of $Q(x, 0)$, $Q\left(\frac{1}{x}, 0\right)$,

$PR(x)$. Let us denote the coefficient of each of these terms as h, (10.1.32) can be written as (note that this time we explicitly write the $PR(x)$ terms too)

$$\begin{aligned}
&h_{Q(\frac{1}{x},0)^2}Q\left(\frac{1}{x},0\right)^2 + h_{Q(\frac{1}{x},0)Q(x,0)}Q\left(\frac{1}{x},0\right)Q(x,0) + h_{PR(x)Q(\frac{1}{x},0)}PR(x)Q\left(\frac{1}{x},0\right)\\
&+ h_{Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right) + h_{Q(x,0)^2}Q(x,0)^2 + h_{Q_0^{\mathrm{d}}(x)Q(x,0)}Q_0^{\mathrm{d}}(x)Q(x,0)\\
&+ h_{PR(x)Q(x,0)}PR(x)Q(x,0) + h_{Q(x,0)}Q(x,0) + h_{Q_0^{\mathrm{d}}(x)^2}Q_0^{\mathrm{d}}(x)^2\\
&+ h_{PR(x)Q_0^{\mathrm{d}}(x)}PR(x)Q_0^{\mathrm{d}}(x) + h_{Q_0^{\mathrm{d}}(x)}Q_0^{\mathrm{d}}(x) + h_{PR(x)^2}PR(x)^2 + h_{PR(x)}PR(x) + h_c = 0.
\end{aligned} \tag{10.1.33}$$

Notice that in (10.1.33), $Q_0^{\mathrm{d}}(x)$ only appears in the $[x^>]$ part. Thus if we take the $[x^<]$ part of this equation, $Q_0^{\mathrm{d}}(x)$ disappears and we achieve our objective.

Separating the $[x^>]$, $[x^0]$ and $[x^<]$ series

Our aim is clear. However, there are two terms preventing us from separating the positive and negative parts. The first is $h_{Q(\frac{1}{x},0)Q(x,0)}Q\left(\frac{1}{x},0\right)Q(x,0)$. It is a term with $x \to \frac{1}{x}$ symmetry if we have suitable $h_{Q(\frac{1}{x},0)Q(x,0)}$. Currently, we put this term aside. We will use the $x \to \frac{1}{x}$ symmetry to deal with it later.

The second problem is $h_{PR(x)Q(\frac{1}{x},0)}PR(x)Q\left(\frac{1}{x},0\right)$. Here $PR(x)$ is a series of x but $Q\left(\frac{1}{x},0\right)$ is a series of $\frac{1}{x}$. The $[x^>]$, $[x^0]$ and $[x^<]$ parts cannot be separated directly. However, by the full-orbit sum we have (10.1.19), and so we may write

$$PR(x)Q\left(\frac{1}{x},0\right) = \left([y^0]\frac{\mathbf{NC}}{K(x,y)} + PR\left(\frac{1}{x}\right)\right)Q\left(\frac{1}{x},0\right). \tag{10.1.34}$$

Substituting (10.1.34), (10.1.33) becomes

$$\begin{aligned}
&h_{Q(\frac{1}{x},0)Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right)^2 + h_{Q(\frac{1}{x},0)Q(x,0)}Q\left(\frac{1}{x},0\right)Q(x,0) + \sqrt{\Delta}h_{\sqrt{\Delta}Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right)\\
&+ h'_{PR(\frac{1}{x})Q(\frac{1}{x},0)}PR\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right) + h'_{Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right) + h_{Q(x,0)^2}Q(x,0)^2\\
&+ h_{Q_0^{\mathrm{d}}(x)Q(x,0)}Q_0^{\mathrm{d}}(x)Q(x,0) + h_{PR(x)Q(x,0)}PR(x)Q(x,0) + h_{Q(x,0)}Q(x,0) + h_{Q_0^{\mathrm{d}}(x)^2}Q_0^{\mathrm{d}}(x)^2\\
&+ h_{PR(x)Q_0^{\mathrm{d}}(x)}PR(x)Q_0^{\mathrm{d}}(x) + h_{Q_0^{\mathrm{d}}(x)}Q_0^{\mathrm{d}}(x) + h_{PR(x)^2}PR(x)^2 + h_{PR(x)}PR(x) + h_c = 0.
\end{aligned} \tag{10.1.35}$$

We have introduced a new $PR\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)$ term into this equation, and we have $h'_{PR(\frac{1}{x})Q(\frac{1}{x},0)} = h_{PR(x)Q(\frac{1}{x},0)}$. Notice also that the coefficient of $Q\left(\frac{1}{x},0\right)$ now becomes $h'_{Q(\frac{1}{x},0)}$, different from $h_{Q(\frac{1}{x},0)}$ in (10.1.33), since $[y^0]\frac{\mathbf{NC}}{K(x,y)}$ is in the form of $p + q\sqrt{\Delta}$. We eliminate $PR(x)Q\left(\frac{1}{x},0\right)$ in (10.1.35) but another problem $\sqrt{\Delta}h_{\sqrt{\Delta}Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right)$ appears. The factor $\sqrt{\Delta}$ cannot be separated directly.

We calculate $\sqrt{\Delta}h_{\sqrt{\Delta}Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right)$ as follows: Recall (10.1.26) and (10.1.30). We treat $\sqrt{\Delta}Q(x,0)$ as a single term in these two equations and eliminate it by linear combination. This gives

$$ps_x^{\mathrm{d}}Q_0^{\mathrm{d}}(x) + ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + ps_{x,0}Q(x,0) + ps_{\frac{1}{x},0,\sqrt{\Delta}}\sqrt{\Delta}Q\left(\frac{1}{x},0\right) + ps_{\frac{1}{x},0}Q(x,0)$$
$$+ ps_{PR(x)}PR(x) + ps_{PR(\frac{1}{x})}PR\left(\frac{1}{x}\right) + ps_c = 0. \quad (10.1.36)$$

In (10.1.36), besides $ps_{\frac{1}{x},0,\sqrt{\Delta}}\sqrt{\Delta}Q\left(\frac{1}{x},0\right)$, we do not have $\sqrt{\Delta}$ multiplying an unknown function of x any more. All other terms can be separated directly. Take the $[x^<]$ part of (10.1.36), we have

$$[x^<]ps_{\frac{1}{x},0,\sqrt{\Delta}}\sqrt{\Delta}Q\left(\frac{1}{x},0\right) = -[x^<]\left(ps_x^{\mathrm{d}}Q_0^{\mathrm{d}}(x) + ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + ps_{x,0}Q(x,0)\right.$$
$$\left. + ps_{\frac{1}{x},0}Q(x,0) + ps_{PR(x)}PR(x) + ps_{PR(\frac{1}{x})}PR\left(\frac{1}{x}\right) + ps_c\right). \quad (10.1.37)$$

We express $[x^<]ps_{\frac{1}{x},0,\sqrt{\Delta}}\sqrt{\Delta}Q\left(\frac{1}{x},0\right)$ by calculating other terms in the $[x^<]$ part of (10.1.36). Now, by multiplying (10.1.33) and (10.1.36) by suitable scalars, such that

1. $ps_{\frac{1}{x},0,\sqrt{\Delta}} = h_{\sqrt{\Delta}Q(\frac{1}{x},0)}$, and
2. $h_{Q(\frac{1}{x},0)Q(x,0)}$ has the $x \to \frac{1}{x}$ symmetry.

Then we can calculate the $[x^<]h_{\sqrt{\Delta}Q(\frac{1}{x},0)}\sqrt{\Delta}Q\left(\frac{1}{x},0\right)$ by (10.1.37). The second condition will be used in Section 10.1.4.

For the RHS of (10.1.37), the calculation of $[x^<]ps_x^{\mathrm{d}}Q_0^{\mathrm{d}}(x)$, $[x^<]ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$, and $[x^<]ps_{x,0}Q(x,0)$ is straightforward. However, the remaining terms involving $PR(x)$ and $PR\left(\frac{1}{x}\right)$ are more troublesome. Although $PR(x)$ does not involve unknown functions of x, it is not algebraic. The coefficient of this term, $ps_{PR(x)}$ contains $\sqrt{\Delta}$ and we can not explicitly separate $[x^>]\sqrt{\Delta}PR(x)$. To deal with this situation, we rewrite (10.1.18) as

$$\sqrt{\Delta} = -\frac{2cx^2}{a^2(b-1)(x^2-1)\left(2tx^2+2t-x\right)\left((b-c)Q(0,0)-b\right)}\left[PR(x) - PR\left(\frac{1}{x}\right)\right.$$
$$\left. - \frac{a^2(b-1)(x^2-1)\left(2t^2x^4+2t^2x^2+2t^2-3tx^3-3tx+x^2\right)\left((b-c)Q(0,0)-b\right)}{2cx^3}\right]. \quad (10.1.38)$$

Then we can express $\sqrt{\Delta}$ by $PR(x) - PR\left(\frac{1}{x}\right)$. Now using (10.1.38) to rewrite $\sqrt{\Delta}$ in $ps_{PR(x)}, ps_{PR(\frac{1}{x})}$ and ps_c, we find that (10.1.36) becomes

$$ps_x^{\mathrm{d}}Q_0^{\mathrm{d}}(x) + ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + ps_{x,0}Q(x,0) + ps_{\frac{1}{x},0,\sqrt{\Delta}}\sqrt{\Delta}Q\left(\frac{1}{x},0\right) + ps_{\frac{1}{x},0}Q(x,0)$$
$$+ ps'_{PR(x)}PR(x) + ps'_{PR(\frac{1}{x})}PR\left(\frac{1}{x}\right) + ps'_{PR(x)^2}PR(x)^2 + ps'_{PR(\frac{1}{x})^2}PR\left(\frac{1}{x}\right)^2$$
$$+ ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right) + ps'_c = 0. \quad (10.1.39)$$

The new coefficients with $'$ are all polynomials without $\sqrt{\Delta}$ now. The $[x^<]ps'_{PR(x)}PR(x)$ is just a polynomial and $[x^<]ps'_{PR(\frac{1}{x})}PR\left(\frac{1}{x}\right)$ is $ps'_{PR(\frac{1}{x})}PR\left(\frac{1}{x}\right)$ minus some polynomial. The only term that we cannot express explicitly is $ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)$. However, we find it is a term with $x \to \frac{1}{x}$ symmetry. It can be handled later together with $h_{Q(\frac{1}{x},0)Q(x,0)}Q\left(\frac{1}{x},0\right)Q(x,0)$. We just leave it as $[x^<]ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)$ for now.

Using the $x \to \frac{1}{x}$ symmetry

By the previous calculations, we can obtain the $[x^<]$ part of (10.1.35). We write it as $G_1+G_2+G_3=G_4$. These four parts G_1, G_2, G_3, G_4 are,

$$
\begin{aligned}
G_1 &= [x^<]\left(h_{Q(\frac{1}{x},0)Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right)^2 + h_{PR(\frac{1}{x})Q(\frac{1}{x},0)}PR\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right) + h_{Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right)\right) \\
G_2 &= [x^<]\Big(h_{Q(x,0)^2}Q(x,0)^2 + h_{PR(x)Q(x,0)}PR(x)Q(x,0) + h_{Q(x,0)}Q(x,0) \\
&\quad + h_{Q_0^{\mathrm{d}}(x)Q(x,0)}Q_0^{\mathrm{d}}(x)Q(x,0) + h_{PR(x)Q_0^{\mathrm{d}}(x)}PR(x)Q_0^{\mathrm{d}}(x) + h_{Q_0^{\mathrm{d}}(x)^2}Q_0^{\mathrm{d}}(x)^2 \\
&\quad + h_{Q_0^{\mathrm{d}}(x)}Q_0^{\mathrm{d}}(x) + h_{PR(x)^2}PR(x)^2 + h_{PR(x)}PR(x) + h_c\Big) \\
G_3 &= -[x^<]\left(ps_x^{\mathrm{d}}Q_0^{\mathrm{d}}(x) + ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) + ps_{x,0}Q(x,0) + ps_{\frac{1}{x},0}Q(x,0)\right. \\
&\quad \left. + ps'_{PR(x)}PR(x) + ps'_{PR(\frac{1}{x})}PR\left(\frac{1}{x}\right) + ps'_{PR(x)^2}PR(x)^2 + ps'_{PR(\frac{1}{x})^2}PR\left(\frac{1}{x}\right)^2 + ps'_c\right) \\
G_4 &= -[x^<]\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right) - ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right),
\end{aligned}
\tag{10.1.40}
$$

where

1. G_1 contains $Q\left(\frac{1}{x},0\right)$ as an unknown function.
2. G_2 is just polynomial, since the $[x^<]$ of a series of x has only finitely many terms.
3. G_3 comes from $h_{\sqrt{\Delta}Q(\frac{1}{x},0)}\sqrt{\Delta}Q\left(\frac{1}{x},0\right)$. We expressed it by (10.1.37). Since $[x^<]ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$ is $ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$ minus some polynomial, we still have $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$ in this part. Fortunately, it only contains linear terms of $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$ and we can eliminate them by (10.1.30). Then this part only contains $Q\left(\frac{1}{x},0\right)$ as the unknown function of x now. The disadvantage is we again introduce $\sqrt{\Delta}$ back into the coefficient of the equation because of (10.1.30).
4. G_4 contains the unknowns with $x \to \frac{1}{x}$ symmetry.

Thus, except G_4, (10.1.40) has only one unknown function of x, $Q\left(\frac{1}{x},0\right)$. We denote

(10.1.35) as

$$r_1\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)^2+r_2\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)+r_3\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)$$
$$+r_4\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)^2+r_5\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)+r_6\left(\frac{1}{x}\right)$$
$$=-[x^<]\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right). \quad (10.1.41)$$

Applying transformation $x\to\frac{1}{x}$ to (10.1.41), we get the corresponding equation

$$r_1(x)Q(x,0)^2+r_2(x)PR(x)Q(x,0)+r_3(x)Q(x,0)+r_4(x)PR(x)^2+r_5(x)PR(x)+r_6(x)$$
$$=-[x^>]\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right). \quad (10.1.42)$$

Similarly, the $[x^0]$ part of (10.1.35) is written as

$$[x^0]\left(h_{Q(\frac{1}{x},0)^2}Q\left(\frac{1}{x},0\right)^2+h_{PR(\frac{1}{x})Q(\frac{1}{x},0)}PR\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)+h_{Q(\frac{1}{x},0)}Q\left(\frac{1}{x},0\right)\right)$$

$$+[x^0]\Big(h_{Q(x,0)^2}Q(x,0)^2+h_{PR(x)Q(x,0)}PR(x)Q(x,0)+h_{Q(x,0)}Q(x,0)$$
$$+h_{Q_0^{\mathrm{d}}(x)Q(x,0)}Q_0^{\mathrm{d}}(x)Q(x,0)+h_{PR(x)Q_0^{\mathrm{d}}(x)}PR(x)Q_0^{\mathrm{d}}(x)+h_{Q_0^{\mathrm{d}}(x)^2}Q_0^{\mathrm{d}}(x)^2$$
$$+h_{Q_0^{\mathrm{d}}(x)}Q_0^{\mathrm{d}}(x)+h_{PR(x)^2}PR(x)^2+h_{PR(x)}PR(x)+h_c\Big)$$

$$-[x^0]\Bigg(ps_x^{\mathrm{d}}Q_0^{\mathrm{d}}(x)+ps_{\frac{1}{x}}^{\mathrm{d}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)+ps_{x,0}Q(x,0) \quad (10.1.43)$$
$$+ps_{\frac{1}{x},0,\sqrt{\Delta}}\sqrt{\Delta}Q\left(\frac{1}{x},0\right)+ps_{\frac{1}{x},0}Q(x,0)$$
$$+ps'_{PR(x)}PR(x)+ps'_{PR(\frac{1}{x})}PR\left(\frac{1}{x}\right)$$
$$+ps'_{PR(x)^2}PR(x)^2+ps'_{PR(\frac{1}{x})^2}PR\left(\frac{1}{x}\right)^2+ps'_c\Bigg)$$

$$=-[x^0]\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right).$$

All terms in the LHS can be calculated directly. We denote it as $l(t)$.

We know that

$$\begin{aligned}
&[x^0]\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right)\\
&+[x^>]\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right)\\
&[x^<]\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right)\\
&=\left(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right)\right).
\end{aligned}\tag{10.1.44}$$

For the LHS of (10.1.44), the $[x^>]$, $[x^<]$ and $[x^0]$ terms are calculated by (10.1.41), (10.1.42) and (10.1.43). So (10.1.41) + (10.1.42) + (10.1.43) finally gives an equation only with unknowns $Q(x,0)$ and $Q\left(\frac{1}{x},0\right)$. We denote this equation as,

$$\begin{aligned}
&l(t)+r_1(x)Q(x,0)^2+r_2(x)PR(x)Q(x,0)+r_3(x)Q(x,0)+r_4(x)PR(x)^2+r_5(x)PR(x)+r_6(x)\\
&+r_1\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)^2+r_2\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)+r_3\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)\\
&+r_4\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)^2+r_5\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)+r_6\left(\frac{1}{x}\right)\\
&=h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right).
\end{aligned}\tag{10.1.45}$$

We have now successfully eliminated $Q_0^{\mathrm{d}}(x)$ and $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$. We do not need to worry about how to separate $Q_0^{\mathrm{d}}(x)$ and $Q(x,0)$ any more. The only problem in (10.1.45) is $h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)$. We will deal with it in next section.

10.1.5 An equation with $Q(x,0)$ only

By the previous calculations, we obtained an equation with unknowns $Q(x,0)$ and $Q\left(\frac{1}{x},0\right)$, We also have $\sqrt{\Delta}$ ($\sqrt{\Delta}$ is introduced into the equation by the third part of (10.1.40)), $PR(x)$ and $PR\left(\frac{1}{x}\right)$ in this equation. The following calculation largely depends on the explicit expression of $r_i(x)$ in (10.1.45).

To simplify (10.1.45), we first want to get rid of $\sqrt{\Delta}$. We still use the same trick as we did to (10.1.37).

We express $\sqrt{\Delta}$ as a function of $PR(x)-PR\left(\frac{1}{x}\right)$. Substitute it into (10.1.45), and eliminate $\sqrt{\Delta}$. Then we get a new equation in the same form of (10.1.45). We still use $r_i(x)$ to denote the coefficients. The new equation is written as

$$
\begin{aligned}
&l(t)+r_1(x)Q(x,0)^2+r_2(x)PR(x)Q(x,0)+r_3(x)Q(x,0)+r_4(x)PR(x)^2+r_5(x)PR(x)+r_6(x)\\
&+r_1\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)^2+r_2\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)+r_3\left(\frac{1}{x}\right)Q\left(\frac{1}{x},0\right)\\
&+r_4\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)^2+r_5\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)+r_6\left(\frac{1}{x}\right)\\
&+r_7(x)PR\left(\frac{1}{x}\right)Q(x,0)+r_7\left(\frac{1}{x}\right)PR(x)Q\left(\frac{1}{x},0\right)\\
&=(h_{Q(x,0)Q(\frac{1}{x},0)}Q(x,0)Q\left(\frac{1}{x},0\right)-ps'_{PR(x)PR(\frac{1}{x})}PR(x)PR\left(\frac{1}{x}\right).
\end{aligned}
\tag{10.1.46}
$$

We again introduce $PR\left(\frac{1}{x}\right)Q(x,0)$ and $PR(x)Q\left(\frac{1}{x},0\right)$ into the equation. Notice that,

$$r_1(x)=\frac{(at+x-ax)^2(bt-b^2t-x+bx-btx^2)^2}{x^4} \tag{10.1.47}$$

$$r_1\left(\frac{1}{x}\right)=\frac{(atx-a+1)^2\left(b^2tx^2-btx^2+bt-bx+x\right)^2}{x^2} \tag{10.1.48}$$

$$r_2(x)=-\frac{2at(at-ax+x)\left(b^2t+btx^2-bt-bx+x\right)}{3x\left(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x\right)} \tag{10.1.49}$$

$$r_2\left(\frac{1}{x}\right)=-\frac{2at(atx-a+1)\left(b^2tx^2-btx^2+bt-bx+x\right)}{3(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x)}. \tag{10.1.50}$$

We also have

$$r_1(x)r_1\left(\frac{1}{x}\right)=\left(h_{Q(x,0)Q(\frac{1}{x},0)}\right)^2. \tag{10.1.51}$$

We let

$$
\begin{aligned}
Q(x,0)&=\frac{x^2}{(at-ax+x)\left(b^2t+btx^2-bt-bx+x\right)} && (10.1.52)\\
&\quad\times\left(R(x,0)+\frac{atxC(x)PR(x)}{3\left(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x\right)}\right)\\
Q\left(\frac{1}{x},0\right)&=\frac{x}{(atx-a+1)\left(b^2tx^2-btx^2+bt-bx+x\right)} && (10.1.53)\\
&\quad\times\left(R\left(\frac{1}{x},0\right)+\frac{atxC\left(\frac{1}{x}\right)PR\left(\frac{1}{x}\right)}{3\left(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x\right)}\right)
\end{aligned}
$$

for some functions R and C. Thus after substitution, (10.1.45) becomes

$$R(x)^2+R(x)R\left(\frac{1}{x}\right)+R\left(\frac{1}{x}\right)^2+f_xR(x)+f_{\frac{1}{x}}R\left(\frac{1}{x}\right)+f\left(PR(x),PR\left(\frac{1}{x}\right),x,t\right)=0, \tag{10.1.54}$$

where $f(PR(x),PR(\frac{1}{x}),x,t)$ are all the other terms without $R(x)$. The coefficients f_x and $f_{\frac{1}{x}}$ are

$$f_x=\frac{atx\left(\left(C\left(\frac{1}{x}\right)-1\right)PR\left(\frac{1}{x}\right)+2\left(C(x)-1\right)PR(x)\right)}{3(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x)}+P_1 \tag{10.1.55}$$

$$f_x=\frac{atx\left(2\left(C\left(\frac{1}{x}\right)-1\right)PR\left(\frac{1}{x}\right)+\left(C(x)-1\right)PR(x)\right)}{3(a^2t^2x+a^2tx^2+a^2t-atx^2-at-ax+x)}+P_2, \tag{10.1.56}$$

where P_1 and P_2 are polynomials of x, t, Q_{ij}. We can see that if $C(x) = C\left(\frac{1}{x}\right) = 1$, $PR(x)R\left(\frac{1}{x}\right)$ and $PR\left(\frac{1}{x}\right)R(x)$ disappear in f_x and $f_{\frac{1}{x}}$.

To decompose $R(x)R(\frac{1}{x})$ in (10.1.54), we multiply (10.1.54) by $R(x) - R(\frac{1}{x}) + f_x - f_{\frac{1}{x}}$. We have,

$$R(x^3) - R\left(\frac{1}{x}\right)^3 + p_{x^2}R(x)^2 - p_{\frac{1}{x}^2}R\left(\frac{1}{x}\right)^2 + p_xR(x) - p_{\frac{1}{x}}R\left(\frac{1}{x}\right) + f'(PR(x), PR\left(\frac{1}{x}\right), x, t) = 0. \tag{10.1.57}$$

If p_x, p_{x^2} are polynomials or series of x and $p_{\frac{1}{x}}, p_{\frac{1}{x}^2}$ are polynomials or series of $\frac{1}{x}$, we can separate the $[x^>]$ and $[x^<]$ series of (10.1.57). So we check their expressions. Firstly, p_{x^2} satisfies our requirement. For p_x, we have

$$\begin{aligned}
p_x = & \frac{a^2t^2x^2\left(3C(x)^2 - 6C(x) + 2\right)}{9\left(a^2t^2x + a^2tx^2 + a^2t - atx^2 - at - ax + x\right)^2}PR(x)^2 \\
& + \frac{2a^2t^2x^2}{9\left(a^2t^2x + a^2tx^2 + a^2t - atx^2 - at - ax + x\right)^2}PR(x)PR\left(\frac{1}{x}\right) \\
& - \frac{a^2t^2x^2}{9\left(a^2t^2x + a^2tx^2 + a^2t - atx^2 - at - ax + x\right)^2}PR\left(\frac{1}{x}\right)^2 \\
& + \frac{a^4(b-1)t^2(x^2-1)\left(2t^2x^4 + 2t^2x^2 + 2t^2 - 3tx^3 - 3tx + x^2\right)\left((b-c)Q(0,0) - b\right)}{9cx\left(a^2t^2x + a^2tx^2 + a^2t - atx^2 - at - ax + x\right)^2}PR\left(\frac{1}{x}\right) \\
& + P_3PR(x) + P_4.
\end{aligned} \tag{10.1.58}$$

Here P_3 is the coefficient of $PR(x)$; it is a rational function of x, t. We denote all the remaining terms without $PR(x)$ and $PR\left(\frac{1}{x}\right)$ as P_4. We also have that P_4 is a rational function of x, t. We have $PR\left(\frac{1}{x}\right)$, $PR(x)PR\left(\frac{1}{x}\right)$ and $PR\left(\frac{1}{x}\right)^2$ in p_x. To get rid of these terms, we apply (10.1.19) for the third time. Since (10.1.18) provides us a way to express $\sqrt{\Delta}$ as a function of $PR(x) - PR\left(\frac{1}{x}\right)$, we can square the equation and get a relation between Δ and $PR\left(\frac{1}{x}\right)PR(x)$. Therefore, we express $PR\left(\frac{1}{x}\right)PR(x)$ by $PR(x)^2$, $PR\left(\frac{1}{x}\right)^2$, $PR(x)$ and $PR\left(\frac{1}{x}\right)$. Notice that in (10.1.58), the coefficients of $PR\left(\frac{1}{x}\right)^2$ and $PR(x)PR\left(\frac{1}{x}\right)$ are in a square form. When we substitute $PR(x)PR\left(\frac{1}{x}\right)$ as a function of $PR(x)^2$, $PR\left(\frac{1}{x}\right)^2$, $PR(x)$ and $PR\left(\frac{1}{x}\right)$, we simultaneously eliminate $PR\left(\frac{1}{x}\right)^2$ and $PR\left(\frac{1}{x}\right)$ in (10.1.58). Thus, p_x reads

$$p_x = \frac{a^2t^2x^2\left(C(x) - 1\right)^2}{3\left(a^2t^2x + a^2tx^2 + a^2t - atx^2 - at - ax + x\right)^2}PR(x)^2 + P_3'PR(x) + P_4'. \tag{10.1.59}$$

We do not have $PR\left(\frac{1}{x}\right)$ any more. We further find $C(x) = 1$ will also eliminate $PR(x)^2$ and make p_x simpler. Then $p_{\frac{1}{x}}$ can be handled in the same way. The term $f'(PR(x), PR\left(\frac{1}{x}\right), x, t)$ does not have unknown functions of x so it can be separated. We have $PR(x)PR\left(\frac{1}{x}\right)$ terms in f as well. We use same techniques as above and change $PR(x)PR\left(\frac{1}{x}\right)$ to a linear function of $PR(x)^2, PR(x), PR\left(\frac{1}{x}\right)^2, PR\left(\frac{1}{x}\right)$. Then, we can explicitly write down the $[x^>]$ series using positive powers of $PR(x)$.

Now $p_x, p_{x^2}, p_{\frac{1}{x}}$ and $p_{\frac{1}{x^2}}$ all satisfy our requirement. We substitute $Q(x, 0)$ back to (10.1.57) and multiply (10.1.57) by a suitable scalar to clear the denominator. Then every unknown function in (10.1.57) only contains polynomial coefficients. The $[x^>]$ and $[x^<]$ terms of (10.1.57) can finally be separated. The $[x^>]$ terms of (10.1.57) form a polynomial equation of $Q(x, 0)$, $PR(x)$, x, t, $Q_{i,j}$, written as

$$Pos(Q(x,0), PR(x), x, t, Q_{i,j}) = 0. \tag{10.1.60}$$

Recall the definition of $PR(x)$. $PR(x)$ is the $[x^>]$ terms of (10.1.18). We may write it as $f(x) + g(x)Q(0,0)$, where $f(x)$ and $g(x)$ are two D-finite functions. Thus, $PR(x)$ is a linear function of $Q(0,0)$ and $Pos(Q(x,0), PR(x), x, t, Q_{i,j})$ is a polynomial of $Q(0,0)$. We finally have

$$Pol(Q(x,0), x, t, Q_{i,j}) = 0, \tag{10.1.61}$$

a polynomial equation of $Q(x,0)$ and its boundary terms $Q_{i,j}$. By Bousquet-Mélou's theorem in [19] (which we will discuss in Chapter 12), we can solve $Q(x,0)$ and Q_{ij} if we can find enough roots x_i of this equation. And the solutions are algebraic. However, the expression of the polynomial is too complicated and we cannot handle it. Our calculations currently have to stop here. We can only have the conclusion that $Q(x,0)$ is D-algebraic over x since all coefficients are D-finite.

10.1.6 Algebraic specialization

For Gessel's walks the D-finite term that appears in (10.1.61) is $PR(x)$. $PR(x)$ depends on a, b, c. If $PR(x) = 0$, then Gessel's model becomes algebraic. By (10.1.18), we notice that if $b = 1$, the full orbit-sum equals 0. Then we let $b = 1$ and still run through Sections 10.1.4 and 10.1.5. We again get (10.1.57). By calculation, we find this equation contains $Q(0,0), Q_{1,0}, Q_{2,0}, Q_{3,0}, Q_{4,0}, Q_{5,0}, Q_{0,1}, Q_{0,2}$ as unknown series of t. The equation is still too complicated and we can not solve it. We may conclude that $Q(x,0)$ is algebraic over x in current stage.

10.2 An alternative way of discussing double Kreweras walks with interactions

As with Gessel's walks, we only give a polynomial equation for $Q(x,0)$ and do not attempt to find an explicit solution. The allowed steps for double Kreweras walks are $\{\leftarrow, \swarrow, \downarrow, \rightarrow, \nearrow, \uparrow\}$. The functional equation reads

$$\begin{aligned} K(x,y)Q(x,y) = \frac{1}{c} + \frac{1}{a}\left(a - 1 - ta\left(\frac{1}{xy} + \frac{1}{y}\right)\right)Q(x,0) + \frac{1}{b}\left(b - 1 - tb\left(\frac{1}{x} + \frac{1}{xy}\right)\right)Q(0,y) \\ + \left(\frac{1}{abc}(ac + bc - ab - abc) + t\frac{1}{xy}\right)Q(0,0), \end{aligned} \tag{10.2.1}$$

where

$$K(x,y) = 1 - t\left(xy + \frac{1}{xy} + x + \frac{1}{x} + y + \frac{1}{y}\right). \tag{10.2.2}$$

As usual we denote the main functional equation (10.2.2) by E_{dKr}. The discriminant of $K(x,y)$ is

$$\Delta = \left(t\left(x + \frac{1}{x}\right) - (1 + 2t)\right)^2 - 4t(1 + 3t) \tag{10.2.3}$$

and the roots of the kernel are

$$Y_0 = \frac{x - t - tx^2 - x\sqrt{\Delta}}{2tx(1+x)} \tag{10.2.4}$$

$$= \frac{t(x+1)}{x} + \frac{t^2(x+1)\left(x^2+1\right)}{x^2} + \frac{t^3(x+1)\left(x^4+x^3+4x^2+x+1\right)}{x^3} + \frac{t^4(x+1)\left(x^6+3x^5+9x^4+6x^3+9x^2+3x+1\right)}{x^4} + \mathrm{O}(t^5) \tag{10.2.5}$$

$$Y_1 = \frac{x - t - tx^2 + x\sqrt{\Delta}}{2tx(1+x)} \tag{10.2.6}$$

$$= \frac{1}{t(x+1)} + \frac{-x^2-1}{x(x+1)} + \frac{t(-x-1)}{x} - \frac{t^2(x+1)\left(x^2+1\right)}{x^2} - \frac{t^3(x+1)\left(x^4+x^3+4x^2+x+1\right)}{x^3} - \frac{t^4(x+1)\left(x^6+3x^5+9x^4+6x^3+9x^2+3x+1\right)}{x^4} + \mathrm{O}(t^5). \tag{10.2.7}$$

The relation between the two roots is

$$Y_0 + Y_1 = -\frac{x + \frac{1}{x} - \frac{1}{t}}{1+x} \tag{10.2.8}$$

$$Y_0 Y_1 = \frac{1}{x}. \tag{10.2.9}$$

The symmetry group is

$$\left\{(x,y), \left(\frac{1}{xy}, y\right), \left(y, \frac{1}{xy}\right), (y,x), \left(\frac{1}{xy}, x\right), \left(x, \frac{1}{xy}\right)\right\}. \tag{10.2.10}$$

10.2.1 The obstacle of the obstinate kernel method and algebraic kernel method

It is natural to try the algebraic kernel method first. The equation set is written as

$$K(x,y)\mathbf{Q} = \mathbf{M}\mathbf{V} + \mathbf{C}, \tag{10.2.11}$$

where

$$\mathbf{Q} = \left(Q(x,y), Q\left(\frac{1}{xy}, y\right), Q\left(y, \frac{1}{xy}\right), Q(y,x), Q\left(\frac{1}{xy}, x\right), Q\left(x, \frac{1}{xy}\right)\right)^{\mathsf{T}} \tag{10.2.12}$$

and

$$\mathbf{V} = \left(Q(x,0), Q(0,y), Q(0,x), Q(y,0), Q\left(\frac{1}{xy}, 0\right), Q\left(0, \frac{1}{xy}\right)\right)^{\mathsf{T}} \tag{10.2.13}$$

and

$$\mathbf{M} = \begin{pmatrix} A'(x,y) & B'(x,y) & 0 & 0 & 0 & 0 \\ 0 & B'\left(\frac{1}{xy}, y\right) & 0 & 0 & A'\left(\frac{1}{xy}, y\right) & 0 \\ 0 & 0 & 0 & A'\left(y, \frac{1}{xy}\right) & 0 & B'\left(y, \frac{1}{xy}\right) \\ 0 & 0 & B'(y,x) & A'(y,x) & 0 & 0 \\ 0 & 0 & B'\left(\frac{1}{xy}, x\right) & 0 & A'\left(\frac{1}{xy}, x\right) & 0 \\ A'\left(x, \frac{1}{xy}\right) & 0 & 0 & 0 & 0 & B'\left(x, \frac{1}{xy}\right) \end{pmatrix}, \tag{10.2.14}$$

where $A'(x,y) = \frac{1}{a}(a - 1 - ta(\frac{1}{y} + \frac{1}{xy}))$ and $B'(x,y) = \frac{1}{b}(b - 1 - tb(\frac{1}{x} + \frac{1}{xy}))$.

By calculation, we unfortunately find $\det \mathbf{M} \neq 0$. We cannot find a linear combination of these equations such that all the unknown functions of x on the RHS are cancelled. There is always one unknown left. Let us choose this unknown function to be $Q(x,0)$. From the full orbit sum, we divide both sides by $K(x,y)$ and take the $[y^0]$ term. After simplifying the LHS has the form

$$P_{x,0}Q(x,0) + P_{0,x}Q(0,x) + P^{\mathrm{d}}_{\frac{1}{x}}Q^{\mathrm{d}}_0\left(\frac{1}{x}\right) + P^{d}_{1,\frac{1}{x}}Q^{\mathrm{d}}_1\left(\frac{1}{x}\right) + P_c = 0, \tag{10.2.15}$$

where $P_{0,x}$, $P^{\mathrm{d}}_{\frac{1}{x}}$, and $P^{\mathrm{d}}_{1,\frac{1}{x}}$ are polynomials while $P_{x,0}$ and P_c are in a form of $A + B\sqrt{\Delta}$. The only unknown in P_c is $Q(0,0)$. We may try to factor $P_{x,0}$, divide by the negative series and separate the $[x^>]$, $[x^0]$ and $[x^<]$ terms. However, this is not practicable. This is because we have $P_{0,x}Q(0,x)$ in the equation, and $P_{0,x}$ is a polynomial ($Q(0,x)$ only appears in LHS). This term belongs to $[x^>]$ series. If we divide it by a $[x^<]$ series, then it cannot be separated. The full orbit sum gives no further information.

The obstinate kernel method also fails. The pair $(\frac{1}{xY_0}, Y_0)$ is not valid, since

$$Q\left(\frac{1}{xY_0}, Y_0\right) = \sum_{i,j,n} q_{i,j,n} Y_1^i Y_0^j t^n = \sum_{i,j,n} q_{i,j,n} (Y_0Y_1)^j (tY_1)^{i-j} t^{n-(i-j)}. \tag{10.2.16}$$

Notice that double Kreweras walks contain paths formed by only $\rightarrow$ or $\uparrow$ steps, thus $n \geq |i-j|$. We can find an infinite number of i,j such that $n = |i-j|$. Thus $Q\left(\frac{1}{xY_0}, Y_0\right)$ is not convergent. This is also true for $Q\left(Y_0, \frac{1}{xY_0}\right)$. The only available pairs are (x, Y_0) and (Y_0, x). These two pairs are insufficient.

10.2.2 Diagonal relations

We need to find some more equations another way. We initially proceed as before, dividing the functional equation by $K(x,y)$ and taking the $[y^0]$ terms. We find three different equations. The first comes from E_{dKr} or $\delta_{(x,\frac{1}{xy})}(E_{dKr})$:

$$\begin{aligned}&\frac{Q(0,0)\left(abct - abcxY_0 - abxY_0 + acxY_0 + bcxY_0\right)}{abcxY_0} \\ &\quad - \frac{Q(x,0)\left(atx + at - axY_0 + xY_0\right)}{axY_0} - \frac{Q\left(0,Y_0\right)\left(btY_0 + bt - bxY_0 + xY_0\right)}{bxY_0} + \frac{1}{c} = 0.\end{aligned} \tag{10.2.17}$$

We can also get this equation by substituting Y_0 into E_{dKr} (or Y_1 into $\delta_{(x,\frac{1}{xy})}(E_{dKr})$).

The second equation comes from $\delta_{(y,x)}(E_{dKr})$ or $\delta_{(\frac{1}{xy},x)}(E_{dKr})$:

$$\begin{aligned}&\frac{Q(0,0)\left(abct - abcxY_0 - abxY_0 + acxY_0 + bcxY_0\right)}{abcxY_0} \\ &\quad - \frac{Q\left(Y_0,0\right)\left(atY_0 + at - axY_0 + xY_0\right)}{axY_0} - \frac{Q(0,x)\left(btx + bt - bxY_0 + xY_0\right)}{bxY_0} + \frac{1}{c} = 0.\end{aligned} \tag{10.2.18}$$

We can also get this equation by substituting Y_0 into $\delta_{(y,x)}(E_{dKr})$ or Y_1 into $\delta_{(\frac{1}{xy},x)}(E_{dKr})$.

The third equation comes from $\delta_{(y,\frac{1}{xy})}(E_{dKr})$ or $\delta_{(\frac{1}{xy},y)}(E_{dKr})$:

$$\begin{aligned}\frac{Q(0,0)(abctx-abc-ab+ac+bc)}{abc}-\frac{Q\left(Y_0,0\right)\left(atxY_0+atx-a+1\right)}{a}\\ -\frac{Q\left(0,Y_0\right)\left(btxY_0+btx-b+1\right)}{b}+\frac{t(x+1)\left(xY_0^2-1\right)Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{xY_0}+\frac{1}{c}=0.\end{aligned}\tag{10.2.19}$$

This gives a relation between $Q_0^{\mathrm{d}}(\frac{1}{x})$ and $Q(0,Y_0), Q(Y_0,0)$. Notice that normally we have $\sqrt{\Delta}$ in the equation. Since $\sqrt{\Delta}=\frac{Y_1-Y_0}{2(1+x)}$, we substitute this in to make the equation more symmetric. Combining these three equations, we get a relation between $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right), Q(x,0), Q(0,x)$,

$$\begin{aligned}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)=&-\frac{xY_0Q(0,x)\left(atxY_0+atx-a+1\right)\left(btx+bt-bxY_0+xY_0\right)}{bt(x+1)\left(xY_0^2-1\right)\left(atY_0+at-axY_0+xY_0\right)}\\ &-\frac{xY_0Q(x,0)\left(atx+at-axY_0+xY_0\right)\left(btxY_0+btx-b+1\right)}{at(x+1)\left(xY_0^2-1\right)\left(btY_0+bt-bxY_0+xY_0\right)}\\ &+L_1(Q(0,0);Y_0,x,t).\end{aligned}\tag{10.2.20}$$

We use $L_1(Q(0,0);Y_0,x,t)$ to describe a linear function of $Q(0,0)$ with coefficients in $\mathbb{R}(Y_0,x,t)$. The coefficients of $Q(x,0)$ and $Q(0,x)$ are both in the form of $A+B\sqrt{\Delta}$. And they are different. We currently cannot do anything to this equation. Note that we can also derive (10.2.20) by a combination of the full orbit-sum and the half orbit-sum via the algebraic kernel method.

Now we consider the off-diagonal $Q_1^{\mathrm{d}}(\frac{1}{x})$. Take $\delta_{(\frac{1}{xy},y)}(E_{dKr})$ and divide by $yK(x,y)$, then extract the $[y^0]$ term. This gives

$$\begin{aligned}-\frac{Q(0,0)\left(-abctx+abc+abxY_0^2-ac-bcxY_0^2\right)}{abcY_0}-\frac{xY_0Q\left(Y_0,0\right)\left(atxY_0+atx-a+1\right)}{a}\\ -\frac{Q\left(0,Y_0\right)\left(btxY_0+btx-b+1\right)}{bY_0}+\frac{t(x+1)\left(xY_0^2-1\right)Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)}{xY_0}+\frac{xY_0}{c}=0.\end{aligned}\tag{10.2.21}$$

A linear combination of this equation together with (10.2.17) and (10.2.18) gives something similar to (10.2.20):

$$\begin{aligned}Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)=&-\frac{x^2Y_0^2Q(0,x)\left(atxY_0+atx-a+1\right)\left(btx+bt-bxY_0+xY_0\right)}{bt(x+1)\left(xY_0^2-1\right)\left(atY_0+at-axY_0+xY_0\right)}\\ &-\frac{xQ(x,0)\left(atx+at-axY_0+xY_0\right)\left(btxY_0+btx-b+1\right)}{at(x+1)\left(xY_0^2-1\right)\left(btY_0+bt-bxY_0+xY_0\right)}\\ &+L_2(Q(0,0);Y_0,x,t).\end{aligned}\tag{10.2.22}$$

As before, $L_2(Q(0,0);Y_0,x,t)$ is a linear function of $Q(0,0)$ with coefficients in $\mathbb{R}(Y_0,x,t)$.

From the above equations we obtain relations between $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$, $Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)$ and $Q(x,0)$ or $Q(0,x)$:

$$\begin{aligned}Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)-\frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_1}=&\frac{xQ(0,0)}{act(x+1)Y_1\left(btxY_1+bt-bx+x\right)}\\ &\times\left(abctx-abctY_1+abtx^2Y_1+abtx-abxY_1\right.\\ &\left.-actx+actY_1+axY_1-bctx^2Y_1-bctx+bcxY_1-cxY_1\right)\\ &+\frac{x^2Q(x,0)\left(atxY_1+atY_1-a+1\right)\left(btxY_1+bt-bY_1+Y_1\right)}{at(x+1)Y_1\left(btxY_1+bt-bx+x\right)}\\ &-\frac{x^2\left(btxY_1+bt-bY_1+Y_1\right)}{ct(x+1)Y_1\left(btxY_1+bt-bx+x\right)}\end{aligned}\tag{10.2.23}$$

and

$$
\begin{aligned}
&Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)-\frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_0}= \\
&\frac{xQ(0,0)}{bct(x+1)\left(atY_0+at-axY_0+xY_0\right)} \\
&\times\left(abct^2xY_0+abct^2x-abctx^2Y_0^2-abctx^2Y_0-abct+abcxY_0-abtx^2Y_0^2-abtx^2Y_0\right. \\
&\left.+abtY_0+abt+actx^2Y_0^2+actx^2Y_0-acxY_0+bctx^2Y_0^2+bctx^2Y_0-bctY_0-bcxY_0+cxY_0\right) \\
&-\frac{xQ(0,x)\left(atxY_0+atx-a+1\right)\left(btx+bt-bxY_0+xY_0\right)}{bt(x+1)\left(atY_0+at-axY_0+xY_0\right)} \\
&+\frac{ax\left(Y_0+1\right)\left(x^2Y_0-1\right)}{c(x+1)\left(atY_0+at-axY_0+xY_0\right)}.
\end{aligned}
\tag{10.2.24}
$$

10.2.3 Application of diagonal relation

To make use of (10.2.23), (10.2.24), (10.2.20) and (10.2.22), we check their symmetries. The equation (10.2.23) is a relation between $Q_1^{\mathrm{d}}\left(\frac{1}{x}\right), Q_0^{\mathrm{d}}\left(\frac{1}{x}\right), Q(x,0)$. We use the relation $Y_0Y_1=\frac{1}{x}$ and change Y_0 in (10.2.23) to be Y_1. Similarly, the equation (10.2.24) is a relation between $Q_1^{\mathrm{d}}\left(\frac{1}{x}\right), Q_0^{\mathrm{d}}\left(\frac{1}{x}\right), Q(0,x)$. We can see in the RHS, the coefficient of $Q(x,0)$ in (10.2.23) and the coefficient of $Q(0,x)$ in (10.2.24) are partly symmetric by the transformation $a\to b, Y_0\to Y_1$, while other terms do not have symmetric properties.

By geometric properties, we know that double Kreweras walks have reflection symmetry about the line $y=x$. If we reflect the $\uparrow$ step by $\to$, $\leftarrow$ step by $\downarrow$ and exchange the x,y axes, we get the same walk. If we do this to the generating function $Q(x,y)$, we are exchanging x and y and exchanging a and b. The generating function $Q(x,0)$ does not change. This implies the asymptotic behavior, or the singularities do not change. The generating function should have paired singularities which are also symmetric about $a\leftrightarrow b$. By the previous discussions of other models, we notice that the singularities come from the zeros of coefficients of $Q(x,0)$ and $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$ in the equation when we separate the $[x^>]$ and $[x^<]$ terms. So if we want to separate the $[x^>]$ and $[x^<]$ terms of the equation, we need to have a symmetric coefficient before $Q(x,0)$. A simple idea is the product of (10.2.23) and (10.2.24). We have a symmetric term $Q(x,0)Q(0,x)$

and the coefficient is symmetric about $a \leftrightarrow b$. The equation reads

$$
\begin{aligned}
&\left(Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_1}\right)\left(Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_0}\right) = \\
&\frac{x^3\left(atxY_0 + atx - a + 1\right)\left(atxY_1 + atY_1 - a + 1\right)\left(btx + bt - bxY_0 + xY_0\right)\left(btxY_1 + bt - bY_1 + Y_1\right)}{abt^2(x+1)^2Y_1\left(atY_0 + at - axY_0 + xY_0\right)\left(btxY_1 + bt - bx + x\right)}Q(0,x)Q(x,0) \\
&+ \frac{x^3Q(0,x)\left(atxY_0 + atx - a + 1\right)\left(btx + bt - bxY_0 + xY_0\right)\left(btxY_1 + bt - bY_1 + Y_1\right)}{bct^2(x+1)^2Y_1\left(atY_0 + at - axY_0 + xY_0\right)\left(btxY_1 + bt - bx + x\right)} \\
&+ \frac{x^3\left(Y_0 + 1\right)\left(x^2Y_0 - 1\right)Q(x,0)\left(atxY_1 + atY_1 - a + 1\right)\left(btxY_1 + bt - bY_1 + Y_1\right)}{ct(x+1)^2Y_1\left(atY_0 + at - axY_0 + xY_0\right)\left(btxY_1 + bt - bx + x\right)} \\
&+ \frac{x^2Q(0,0)Q(0,x)\left(atxY_0 + atx - a + 1\right)\left(btx + bt - bxY_0 + xY_0\right)}{abct^2(x+1)^2Y_1\left(-atY_0 - at + axY_0 - xY_0\right)\left(btxY_1 + bt - bx + x\right)} \\
&\times \left(abctx - abctY_1 + abtx^2Y_1 + abtx - abxY_1 - actx + actY_1\right. \\
&\left. + axY_1 - bctx^2Y_1 - bctx + bcxY_1 - cxY_1\right) \\
&\frac{x^3Q(0,0)Q(x,0)\left(atxY_1 + atY_1 - a + 1\right)\left(btxY_1 + bt - bY_1 + Y_1\right)}{abct^2(x+1)^2Y_1\left(atY_0 + at - axY_0 + xY_0\right)\left(btxY_1 + bt - bx + x\right)} \\
&\times \left(abct^2xY_0 + abct^2x - abctx^2Y_0^2 - abctx^2Y_0 - abct + abcxY_0 - abtx^2Y_0^2 - abtx^2Y_0 + abtY_0 + abt\right. \\
&\left. + actx^2Y_0^2 + actx^2Y_0 - acxY_0 + bctx^2Y_0^2 + bctx^2Y_0 - bctY_0 - bcxY_0 + cxY_0\right) \\
&+ P_1(Q(0,0); Y_0, Y_1, x, t).
\end{aligned} \tag{10.2.25}
$$

Here $P_1(Q(0,0);, Y_0, Y_1, x, t)$ is a polynomial of $Q(0,0)$ with coefficients in $\mathbb{R}(Y_0, Y_1, x, t)$. To study (10.2.25), first consider the LHS. Expanding the product in the LHS, we find three $[x^<]$ unknown terms, $Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)^2$, $Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)^2$ and $Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)$. Notice that the LHS is a symmetric function of Y_0Y_1. By (10.2.8), the coefficients of these terms are rational functions of x, t.

Then consider the RHS. By expansion, we find five $[x^>]$ unknown terms, $Q(x,0)Q(0,x)$, $Q(x,0)Q(0,0)$, $Q(0,x)Q(0,0)$, $Q(x,0)$ and $Q(0,x)$. The coefficients of these terms are not symmetric about Y_0 and Y_1 so we have $\sqrt{\Delta}$ in these coefficients. Except $Q(x,0)Q(0,x)$, the coefficients of the other terms are not symmetric about $(a, Y_0) \leftrightarrow (b, Y_1)$ and they do not equal to the coefficient of $Q(x,0)Q(0,x)$. We cannot directly factor the coefficient of $Q(x,0)Q(0,x)$.

For brevity we introduce some more notation. Denote (10.2.23) as R_1 and (10.2.24) as R_2. Then denote $T_1 = Y_1R_1$ and $T_2 = Y_0R_2$. The product (10.2.25) is denoted as $R_{1\times 2}$. We can write

$$R_1 = Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_1} = (m_{R_1,x,0} + n_{R_1,x,0}\sqrt{\Delta})Q(x,0) + C_{R_1,00}Q(0,0) + C_{R_1,c}$$
$$R_2 = Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_0} = (m_{R_2,0,x} + n_{R_2,0,x}\sqrt{\Delta})Q(0,x) + C_{R_2,00}Q(0,0) + C_{R_2,c}$$
$$T_1 = Y_1 Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) = (m_{T_1,x,0} + n_{T_1,x,0}\sqrt{\Delta})Q(x,0) + C_{T_1,00}Q(0,0) + C_{T_1,c}$$
$$T_2 = Y_0 Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) = (m_{T_2,0,x} + n_{T_2,0,x}\sqrt{\Delta})Q(0,x) + C_{T_2,00}Q(0,0) + C_{T_2,c}$$

$$\begin{aligned} R_{1\times 2} &= \left(Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_1}\right)\left(Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_0}\right) \\ &= (m_{R_{1\times 2},x,0,0,x} + n_{R_{1\times 2},x,0,0,x}\sqrt{\Delta})Q(x,0)Q(0,x) \\ &+ (m_{R_{1\times 2},x,0,00} + n_{R_{1\times 2},x,0,00}\sqrt{\Delta})Q(x,0)Q(0,0) \\ &+ (m_{R_{1\times 2}0,x,00} + n_{R_{1\times 2}0,x,00}\sqrt{\Delta})Q(0,x)Q(0,0) \\ &+ (m_{R_{1\times 2},x,0} + n_{R_{1\times 2},x,0}\sqrt{\Delta})Q(x,0) + (m_{R_{1\times 2}0,x} + n_{R_{1\times 2}0,x}\sqrt{\Delta})Q(0,x) \\ &+ C_{R_{1\times 2},c}. \end{aligned} \tag{10.2.26}$$

Now consider the following linear combination

$$R_{1\times 2} + \lambda_R(R_1 + R_2)Q(0,0) + \lambda_T(T_1 + T_2)Q(0,0). \tag{10.2.27}$$

In (10.2.27), the coefficients of $Q(x,0)Q(0,x)$, $Q(x,0)$ and $Q(0,x)$ are not affected by λ_R or λ_T. We can adjust λ_R and λ_T to modify the coefficients of $Q(x,0)Q(0,0)$ and $Q(0,x)Q(0,0)$. Our aim is to find λ_R and λ_T such that

$$\begin{aligned} \lambda_R(m_{R_1,x,0} + n_{R_1,x,0}\sqrt{\Delta}) + \lambda_T(m_{T_1,x,0} + n_{T_1,x,0}\sqrt{\Delta}) + (m_{R_{1\times 2},x,0,00} + n_{R_{1\times 2},x,0,00}\sqrt{\Delta}) \\ = \mu_1(m_{R_{1\times 2},x,0,0,x} + n_{R_{1\times 2},x,0,0,x}\sqrt{\Delta}) \end{aligned} \tag{10.2.28}$$

and

$$\begin{aligned} \lambda_R(m_{R_2,0,x} + n_{R_2,0,x}\sqrt{\Delta}) + \lambda_T(m_{T_2,0,x} + n_{T_2,0,x}\sqrt{\Delta}) + (m_{R_{1\times 2}0,x,00} + n_{R_{1\times 2}0,x,00}\sqrt{\Delta}) \\ = \nu_1(m_{R_{1\times 2},x,0,0,x} + n_{R_{1\times 2},x,0,0,x}\sqrt{\Delta}) \end{aligned} \tag{10.2.29}$$

for some μ_1 and ν_1.

If we require λ_R and λ_T to be rational functions of x, t, (10.2.28) and (10.2.29) are equivalent to the following equation set,

$$\begin{aligned} \lambda_R m_{R_1,x,0} + \lambda_T m_{T_1,x,0} + m_{R_{1\times 2},x,0,00} &= \mu_1 m_{R_{1\times 2},x,0,0,x} \\ \lambda_R n_{R_1,x,0} + \lambda_T n_{T_1,x,0} + n_{R_{1\times 2},x,0,00} &= \mu_1 n_{R_{1\times 2},x,0,0,x} \\ \lambda_R m_{R_2,0,x} + \lambda_T m_{T_2,0,x} + m_{R_{1\times 2},0,x,00} &= \nu_1 m_{R_{1\times 2},x,0,0,x} \\ \lambda_R n_{R_2,0,x} + \lambda_T n_{T_2,0,x} + n_{R_{1\times 2},0,x,00} &= \nu_1 n_{R_{1\times 2},x,0,0,x}. \end{aligned} \tag{10.2.30}$$

To calculate this equation set, we consider the determinant,

$$\begin{vmatrix} m_{R_1,x,0} & m_{T_1,x,0} & m_{R_{1\times 2},x,0,0,x} & 0 \\ n_{R_1,x,0} & n_{T_1,x,0} & n_{R_{1\times 2},x,0,0,x} & 0 \\ m_{R_2,0,x} & m_{T_2,0,x} & 0 & m_{R_{1\times 2},x,0,0,x} \\ n_{R_2,0,x} & n_{T_2,0,x} & 0 & n_{R_{1\times 2},x,0,0,x} \end{vmatrix}. \tag{10.2.31}$$

Suppose $Y_1 = p + q\sqrt{\Delta}$ and $Y_0 = p - q\sqrt{\Delta}$. By definition,

$$m_{T_1,x,0} = pm_{R_1,x,0} + q\Delta n_{R_1,x,0} \qquad m_{T_2,0,x} = pm_{R_2,0,x} - q\Delta n_{R_2,0,x} \tag{10.2.32}$$

$$n_{T_1,x,0} = qm_{R_1,x,0} + pn_{R_1,x,0} \qquad n_{T_2,0,x} = qm_{R_2,0,x} - pn_{R_2,0,x} \tag{10.2.33}$$

$$m_{R_{1\times 2},x,0,0,x} = m_{R_1,x,0}m_{R_2,0,x} + \Delta n_{R_1,x,0}n_{R_2,0,x} \tag{10.2.34}$$

$$n_{R_{1\times 2},x,0,0,x} = m_{R_1,x,0}n_{R_2,0,x} + n_{R_1,x,0}m_{R_2,0,x}. \tag{10.2.35}$$

So the determinant can further be written as

$$\begin{vmatrix} m_{R_1} & pm_{R_1} + q\Delta n_{R_1} & m_{R_1}m_{R_2} + \Delta n_{R_1}n_{R_2} & 0 \\ n_{R_1} & qm_{R_1} + pn_{R_1} & m_{R_1}n_{R_2} + n_{R_1}m_{R_2} & 0 \\ m_{R_2} & m_{R_2}p - q\Delta n_{R_2} & 0 & m_{R_1}m_{R_2} + \Delta n_{R_1}n_{R_2} \\ n_{R_2} & pn_{R_2} - qm_{R_2} & 0 & m_{R_1}n_{R_2} + n_{R_1}m_{R_2} \end{vmatrix}, \tag{10.2.36}$$

where we have suppressed some subscripts.

This determinant equals $q\left(m_{R_2}n_{R_1} + m_{R_1}n_{R_2}\right)\left(\Delta n_{R_1}^2 - m_{R_1}^2\right)\left(m_{R_2}^2 - \Delta n_{R_2}^2\right)$.

If $(m_{R_2}n_{R_1} + m_{R_1}n_{R_2}) = 0$, $R_{1,x,0}$ is the conjugate of $R_{2,0,x}$. This cannot be true since the coefficient of $Q(x,0)$ in R_1 and the coefficient of $Q(0,x)$ in R_2 are not completely symmetric about $Y_0 \leftrightarrow Y_1$. If $\left(\Delta n_{R_1}^2 - m_{R_1}^2\right) = 0$, then $\left(\sqrt{\Delta}n_{R_1} - m_{R_1}\right)$ or $\left(\sqrt{\Delta}n_{R_1} + m_{R_1}\right)$ equals 0. This is not true since m, n are rational functions. Thus the determinant does not equal 0 and the equation set always has solutions.

Similarly to (10.2.27), consider the combination

$$R_{1\times 2} + \eta_R(R_1 + R_2) + \eta_T(T_1 + T_2) \tag{10.2.37}$$

Similar consideration holds for (10.2.37). We want the coefficient of $Q(x,0)$ and $Q(0,x)$ in (10.2.25) to be proportional to $m_{1\times 2,x,0,0,x} + n_{1\times 2,x,0,0,x}\sqrt{\Delta}$, which implies

$$\begin{aligned} \eta_R m_{R_1,x,0} + \eta_T m_{T_1,x,0} + m_{R_{1\times 2},x,0,00} &= \mu_2 m_{R_{1\times 2},x,0,0,x} \\ \eta_R n_{R_1,x,0} + \eta_T n_{T_1,x,0} + n_{R_{1\times 2},x,0,00} &= \mu_2 n_{R_{1\times 2},x,0,0,x} \\ \eta_R m_{R_2,0,x} + \eta_T m_{T_2,0,x} + m_{R_{1\times 2},0,x,00} &= \nu_2 m_{R_{1\times 2},x,0,0,x} \\ \eta_R n_{R_2,0,x} + \eta_T n_{T_2,0,x} + n_{R_{1\times 2},0,x,00} &= \nu_2 n_{R_{1\times 2},x,0,0,x}. \end{aligned} \tag{10.2.38}$$

The equation set (10.2.38) has the same determinant as (10.2.30). It always has solutions.

Now we can combine (10.2.27) and (10.2.37), and we have

$$\begin{aligned} &R_{1\times 2} + \lambda_R(R_1 + R_2)Q(0,0) + \lambda_T(T_1 + T_2)Q(0,0) + \eta_R(R_1 + R_2) + \eta_T(T_1 + T_2) \\ &= (m_{R_{1\times 2},x,0,0,x} + n_{R_{1\times 2},x,0,0,x}\sqrt{\Delta}) \\ &\times [Q(x,0)Q(0,x) + \mu_1 Q(x,0)Q(0,0) + \nu_1 Q(0,x)Q(0,0) + \mu_2 Q(x,0) + \nu_2 Q(0,x)] \\ &+ C_{R,T,c}, \end{aligned} \tag{10.2.39}$$

where $C_{R,T,c}$ are the terms independent of $Q(x,0)$ and $Q(0,x)$. The $[x^>]$ unknown series of the RHS now all have coefficient $(m_{R_{1\times 2},x,0,0,x} + n_{R_{1\times 2},x,0,0,x}\sqrt{\Delta})$. Other coefficients $\mu_1, \mu_2, \nu_1, \nu_2$

are rational functions and can be easily changed to polynomials. We factor $(m_{R_{1\times 2},x,0,0,x} + n_{R_{1\times 2},x,0,0,x}\sqrt{\Delta})$ as $e^{L_+}e^{L_0}e^{L_-}$ as per Section 8.3 and divide both sides by e^{L_-}. Then we can separate the $[x^>]$, $[x^0]$ and $[x^<]$ parts of the RHS.

Now consider the LHS of (10.2.39). The LHS is a symmetric function of R_1, R_2. Since the LHS of R_1 and R_2 are symmetric by $Y_0 \leftrightarrow Y_1$, the LHS of (10.2.39) is a symmetric function of Y_0, Y_1. So the coefficients are rational functions of x, t. To write it explicitly, the LHS reads

$$\begin{aligned}
&\left(Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_1}\right)\left(Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \frac{Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)}{Y_0}\right)\\
&+ \lambda_R\left[2Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \left(\frac{1}{Y_0}+\frac{1}{Y_1}\right)Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)\right]Q(0,0)\\
&- \lambda_T\left[2Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) - (Y_0+Y_1)Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)\right]Q(0,0)\\
&+ \eta_R\left[2Q_1^{\mathrm{d}}\left(\frac{1}{x}\right) - \left(\frac{1}{Y_0}+\frac{1}{Y_1}\right)Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)\right]\\
&- \eta_T\left[2Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) - (Y_0+Y_1)Q_1^{\mathrm{d}}\left(\frac{1}{x}\right)\right].
\end{aligned} \tag{10.2.40}$$

The unknown series are all $[x^<]$ terms and the coefficients are rational functions of x, t and can be changed to polynomials. After dividing e^{L_-}, we can separate the $[x^>]$, $[x^<]$ and $[x^0]$ parts of the LHS.

Thus, we can separate the $[x^<]$, $[x^0]$ and $[x^>]$ parts of (10.2.39). This gives two equations involving x. Write them as

$$Pos(Q(x,0), Q(0,x); Q_i, x, t) = 0 \tag{10.2.41}$$

and

$$Neg\left(Q_1^{\mathrm{d}}\left(\frac{1}{x}\right), Q_0^{\mathrm{d}}\left(\frac{1}{x}\right); Q_i, x, t\right) = 0. \tag{10.2.42}$$

Here Pos refers to the $[x^>]$ terms and Neg refers to the $[x^<]$ terms. Q_i refers to $Q(0,0)$, $Q_{0,1}$ or any function of point boundary terms that may appear during the separation. The coefficients may contain some D-algebraic terms of x, t such as $[x^>]\frac{p+q\sqrt{\Delta}}{e^{L_-}}$. The LHS of (10.2.42) is a polynomial function of $Q_1^{\mathrm{d}}(\frac{1}{x})$ and $Q_0^{\mathrm{d}}(\frac{1}{x})$. We find a relation between $Q(x,0)$ and $Q(0,x)$ and a relation between $Q_0^{\mathrm{d}}(\frac{1}{x})$ and $Q_1^{\mathrm{d}}(\frac{1}{x})$. But currently we cannot decouple them. The attempt to solve double Kreweras walks ends here.

10.2.4 Specializations

We can find some special values such that $Q(x,0)$ is solvable. If $Q(x,0)$ satisfies a linear equation with D-finite terms, we may solve it in the same way as those D-algebraic cases in Chapter 8. This implies that we should not multiply R_1, R_2 to get a quadratic function. Thus, the coefficients of $Q(x,0)$ and $Q(0,x)$ should have an $a \leftrightarrow b$ symmetry themselves. This implies $a = b$ and $Q(x,0) = Q(0,x)$. Now let us return to (10.2.20). We only have $Q_0^{\mathrm{d}}(\frac{1}{x})$ as an $[x^<]$

unknown series and $Q(x,0)$ as an $[x^>]$ unknown series. The equation then reads

$$\begin{aligned} Q_0^{\mathrm{d}}\left(\frac{1}{x}\right) &= -\frac{x^2 Q(x,0)\left(atxY_1+at-aY_1+Y_1\right)\left(atxY_1+atY_1-a+1\right)}{at(x+1)Y_1\left(atxY_1+at-ax+x\right)} \\ &+\frac{xY_0\left(2atx^2Y_0^2+2atx^2Y_0-atY_0-at-axY_0+xY_0\right)}{ct(x+1)\left(xY_0^2-1\right)\left(atY_0+at-axY_0+xY_0\right)} \\ &+(a^2ct^2xY_0+a^2ct^2x-2a^2ctx^2Y_0^2-a^2ctx^2Y_0 \\ &+a^2ctY_0-a^2ct+a^2cxY_0-2a^2tx^2Y_0^2-2a^2tx^2Y_0 \\ &+a^2tY_0+a^2t+a^2xY_0+4actx^2Y_0^2+3actx^2Y_0 \\ &-2actY_0-3acxY_0-axY_0+2cxY_0) \\ &\times\frac{xY_0}{act(x+1)\left(xY_0^2-1\right)\left(atY_0+at-axY_0+xY_0\right)}Q(0,0). \end{aligned} \tag{10.2.43}$$

We can solve this equation by the same discussion as that in Section 8.3. By some simplification, (10.2.43) is written as

$$P_{x,0}Q(x,0)+P_{d,\frac{1}{x}}Q_0^{\mathrm{d}}\left(\frac{1}{x}\right)+P_{00}+P_c=0, \tag{10.2.44}$$

where $P_{d,\frac{1}{x}}=1$ and $P_{x,0}$ can be canonically factored as

$$\begin{aligned} P_{x,0} = &\frac{1}{2a^2t^2+a^2tx^2-2a^2tx+a^2x^2-atx^2+2atx+at-2ax^2-ax+x^2+x} \\ &\times\frac{1}{(t^2x^4-4t^2x^3-6t^2x^2-4t^2x+t^2-2tx^3-2tx+x^2)}\times\frac{e^{L_+}e^{L_0}e^{L_-}}{(1+x)}. \end{aligned} \tag{10.2.45}$$

The factors in the denominator have 7 roots in all, and two of these cannot be expanded as formal series of t. We denote them as X_4, X_6. Then the solution $Q(0,0)$ is written as

$$Q(0,0)=-\frac{T_c}{T_{x,0}+T_{0,0}+T_{d,\frac{1}{x}}}, \tag{10.2.46}$$

where

$$T_{x,0}=[x^0]\frac{(x-X_4)(x-X_6)P_{x,0}}{e^{L_-}} \tag{10.2.47}$$

$$T_{0,0}=[x^0]\frac{(x-X_4)(x-X_6)P_{0,0}}{e^{L_-}} \tag{10.2.48}$$

$$T_{d,\frac{1}{x}}=[x^0]\frac{(x-X_4)(x-X_6)P_{d,\frac{1}{x}}}{e^{L_-}} \tag{10.2.49}$$

$$T_c=[x^0]\frac{(x-X_4)(x-X_6)P_c}{e^{L_-}} \tag{10.2.50}$$

Since $T_{x,0}, T_{0,0}, T_{d,\frac{1}{x}}$ are D-algebraic over t, $Q(0,0)$ is D-algebraic over t and $Q(x,y), Q(x,0), Q(0,y)$ are D-algebraic functions. $Q(0,0)$ is D-finite when L is either a rational function or a rational function times $\sqrt{\Delta}$. This implies $a=b=1$. In this case, $\det\mathbf{M}=0$ and the generating functions $Q(0,0), Q(x,0), Q(0,y), Q(x,y)$ are algebraic.

Chapter 11

Analytic methods

Besides the kernel method, there are also other analytic methods and combinatorial methods which have been applied to quarter-plane walk problems. We introduce two analytic methods in this chapter. They can both be found in [33]. The methods are based on Insight 4 in Section 4.2. We treat the kernel as a Riemann surface, and try to find two meromorphic functions $Q(x,0)$ and $Q(0,y)$ defined on the Riemann surface. Later in Chapter 12, we will combine the solution of the kernel method and these two analytic methods to discuss the properties of the generating functions.

11.1 A probabilistic model

We follow the discussion in [33]. Lattice paths are now treated as a probabilistic model. The model is generally the same as that discussed in Chapter 6. We consider a lattice walk with allowed steps $\mathcal{S} \subseteq \{-1,0,1\} \times \{-1,0,1\}$. The difference is that a probability p_{ij} ($i,j \in \{-1,0,1\}$) is applied to each step. The variable t is omitted from $Q(x,y)$ (or equivalently, let t equal some fixed value). This means we do not keep track of the length of the walk. We may assume the walk is ergodic (this can be done by choosing a proper p_{ij}). Then, when the length of the path is large, the probability of walks ending at an arbitrary point (k,l) is an invariant measure (a stationary distribution). We denote $q_{k,l}$ as the stationary distribution of walks ending at point (k,l). The generating function of the stationary distribution can still be written in the same form as (6.1.3). Let

$$\begin{aligned}\pi(x,y) &= \sum_{k,l\geq 0} q_{k,l} x^k y^l \\ \pi(x) &= \sum_{i\geq 0} q_{i,0} x^i \\ \tilde{\pi}(y) &= \sum_{j\geq 0} q_{0,j} y^j,\end{aligned} \tag{11.1.1}$$

Here $\pi(x,y)$, $\pi(x)$, $\tilde{\pi}(y)$ refer to the generating functions of the stationary distributions of walks ending at arbitrary points, on the x-axis or the y-axis. We then have

$$K(x,y)\pi(x,y) = q(x,y)\pi(x) + \tilde{q}(x,y)\tilde{\pi}(y) + q_0(x,y)\pi_{00}, \tag{11.1.2}$$

Comparing (11.1.2) and (6.1.3), they are both linear function of unknowns. We can find that the unknown $\pi(x,y)$ corresponds to $Q(x,y)$ in (6.1.3). Similarly $\pi(x)$ corresponds to $Q(x,0)$ and $\widetilde{\pi}(y)$ corresponds to $Q(0,y)$. They are defined as formal series of x,y, analytic in the domain $\{(x,y)\in\mathbb{C}^2 : |x|<1 \text{ and } |y|<1\}$. Also, π_{00} corresponds to terms of $Q(0,0)$, which are constants here. The derivation of (11.1.2) is very similar to that of (3.2.4) (or (3.5.2)). In an equilibrium state, the stationary distribution is an invariant measure. So we have

$$q_{i,j} = \sum_{\substack{l,m\in\{-1,0,1\}\\ m\le i\\ l\le j}} p_{l,m} q_{i-m,j-l}. \tag{11.1.3}$$

Then taking the generating function of $q_{i,j}$ gives (11.1.2). The kernel $K(x,y)$ reads

$$K(x,y) = 1 - \sum_{i,j} p_{i,j} x^i y^j, \tag{11.1.4}$$

and

$$xyK(x,y) = a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y). \tag{11.1.5}$$

We wish to only consider non-singular models, defined as follows.

Definition 19. *A quarter-plane random walk model is called singular if the polynomial $xyK(x,y)$ is either reducible in $\mathbb{R}$ or of degree 1 in at least one of x,y.*

If we treat t as a general constant in $xyK(x,y)$ in our model, all 23 models discussed previously are non-singular. In the probabilistic model, with general p_{ij}, $xyK(x,y)$ is irreducible.[1].

By Insight 4 of the kernel[2], irreducible equations $xYK(x,Y)=0$ and $XyK(X,y)=0$ define two multi-valued functions $Y(x)$ (or $X(y)$), which we consider as two Riemann surfaces $(x,Y(x))$ and $(X(y),y)$. These two surfaces are tori with genus 1 and they are conformally equivalent. We can define a third Riemann surface $\mathbb{S}$,

$$x(s)y(s)K(x(s),y(s)) = 0, \tag{11.1.6}$$

with two homomorphisms

$$\begin{aligned} h_x &: \mathbb{S}\to\mathbb{C}_x, \qquad h_x(s) = x(s)\\ h_y &: \mathbb{S}\to\mathbb{C}_y, \qquad h_y(s) = y(s) \end{aligned} \tag{11.1.7}$$

which project $\mathbb{S}$ onto $\mathbb{C}_x$ and $\mathbb{C}_y$. Since $\pi(x)$ $(Q(x,0))$ is analytic inside $|x|<1$ and $\tilde{\pi}(y)$ $(Q(0,y))$ is analytic inside $|y|<1$, the analytic domain of $\pi(x(s)), \tilde{\pi}(y(s))$ on this surface $\mathbb{S}$ is $h_x^{-1}(\mathbb{D})\cap h_y^{-1}(\mathbb{D})$ ($\mathbb{D}$ refers to the inner domain of unit circle).

On the surface we have

$$q(x(s),y(s))\pi(x(s)) + \tilde{q}(x(s),y(s))\tilde{\pi}(y(s)) + q_0(x(s),y(s))\pi_{00} = 0. \tag{11.1.8}$$

The symmetry group of the model defined in Section 3.5 also plays an important role in the analytic approach. We defined these two involutions ϕ,ψ as

$$\psi(u(x)) = u(x), \qquad \forall u(x)\in\mathbb{C}(x) \tag{11.1.9}$$

$$\psi(Y(x)) = \frac{c(x)}{a(x)Y(x)} = -\frac{b(x)}{a(x)} - Y(x), \tag{11.1.10}$$

$$\phi(w(y)) = w(y), \qquad \forall w(y)\in\mathbb{C}(y) \tag{11.1.11}$$

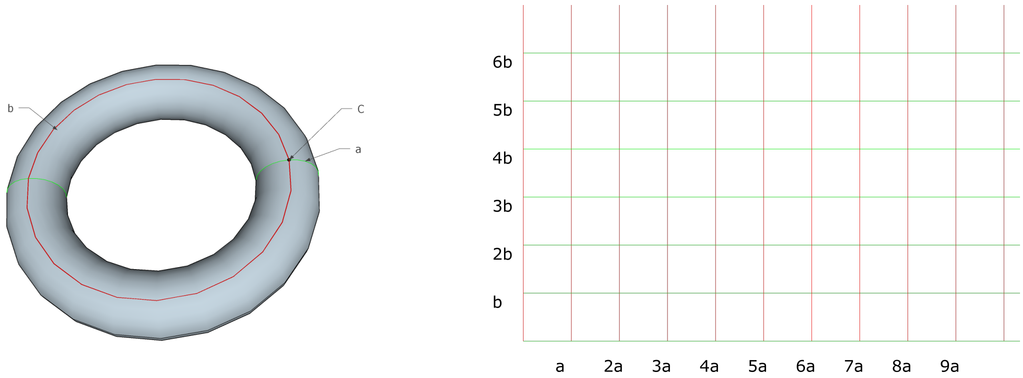
$$\phi(X(y)) = \frac{\tilde{c}(y)}{\tilde{a}(y)X(y)} = -\frac{\tilde{b}(y)}{\tilde{a}(y)} - X(y). \tag{11.1.12}$$

[1] With some special p_{ij}, the walk can be reducible. In that case, (11.1.14) has multiple roots.

[2] See Section 4.2

Since $xYK(x,Y) = 0$ and $XyK(X,y) = 0$ are both conformally equivalent to $X(s)Y(s)K(X(s),Y(s)) = 0$, we can regard ψ and ϕ as two automorphisms on the surface $(X(s),Y(s))$. These two automorphisms generate the symmetry group of the model.

To study functions on a Riemann surface, we lift them on to the universal covering $\mathbb{C}_\omega$. The universal covering of a torus is the complex plane where each parallelogram in the plane is a one-one mapping to the torus.



(a) Torus (b) Universal cover

Figure 11.1: a and b are two circles on the torus. They are mapped to lines on the complex plane parallel to the x and y axes. C is the intersection of a, b.

Any meromorphic function on the Riemann surface is a doubly-periodic elliptic function on the universal covering. Let us denote the two periods as ω_1, ω_2. By discussion in Section 2.7, we may express x, y by the Weierstrass $\wp$-functions as follows. We complete the square of the equation $xyK(x,y) = 0$ as

$$4a(x)^2y^2 + 4a(x)b(x)y + b(x)^2 = b(x)^2 - 4a(x)c(x), \tag{11.1.13}$$

where we define the discriminant as

$$D(x) = b(x)^2 - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \tag{11.1.14}$$

where the roots of the discriminant are x_1, x_2, x_3, x_4, which satisfy

$$\begin{aligned} -1 \le x_1 < x_2 \le 1 < x_3 < x_4, &\quad \text{if } d_4 > 0 \\ x_4 < -1 \le x_1 < x_2 < 1 \le x_3, &\quad \text{if } d_4 < 0 \\ x_4 = \infty, &\quad \text{if } d_4 = 0. \end{aligned} \tag{11.1.15}$$

Since the Weierstrass $\wp$-function [4] satisfies

$$\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega)^2 - g_3, \tag{11.1.16}$$

if $d_4 = 0$, we can write (11.1.13) as

$$\begin{aligned} 4a(x)^2y^2 + 4a(x)b(x)y + b(x)^2 &= \wp'(\omega)^2 \\ b(x)^2 - 4a(x)c(x) &= 4\wp(\omega)^3 - g_2\wp(\omega)^2 - g_3. \end{aligned} \tag{11.1.17}$$

The LHS of (11.1.16) is equal to the LHS of (11.1.13) and the RHS of (11.1.16) is equal to the RHS of (11.1.13). We directly express x, y in terms of $\wp(\omega)$ and $\wp'(\omega)$.

If $d_4 \neq 0$, let $u = \frac{1}{x-x_4}$. This moves the largest root x_4 to ∞. Then we multiply both sides of (11.1.13) by $\frac{1}{u^4}$. The LHS of (11.1.13) is still a square function and the RHS of (11.1.13) now is a cubic function of u. Let

$$(4a(x)^2y^2 + 4a(x)b(x)y + b(x)^2)\left(\frac{1}{x-x_4}\right)^4 = \wp'(\omega)^2$$
$$(b(x)^2 - 4a(x)c(x))\left(\frac{1}{x-x_4}\right)^4 = \frac{d_4(x-x_1)(x-x_2)(x-x_3)}{(x-x_4)^3} = 4\wp(\omega)^3 - g_2\wp(\omega)^2 - g_3. \tag{11.1.18}$$

We can express $\wp(\omega)$ in terms of u then in terms of x. The final expression is given by the following theorem.

Theorem 20 ([33, Chapter 3.3]).
Let $D(x)$ and d_4, d_3, d_2, d_1, d_0 be defined by (11.1.14),

- *If $d_4 \neq 0$, then $D'(x_4) > 0$,*

$$x(\omega) = x_4 + \frac{D'(x_4)}{\wp(\omega) - \frac{1}{6}D''(x_4)}$$

and

$$2a(x(\omega))y(\omega) + b(x(\omega)) = \frac{D'(x_4)\wp'(\omega)}{2(\wp(\omega) - \frac{1}{6}D''(x_4))^2}. \tag{11.1.19}$$

- *If $d_4 = 0$, then,*

$$x(\omega) = \frac{\wp(\omega) - \frac{d_2}{3}}{d_3}$$

and

$$2a(x(\omega))y(\omega) + b(x(\omega)) = -\frac{\wp'(\omega)}{2d_3}. \tag{11.1.20}$$

See [33] for a detailed proof. Here $\wp(\omega)$ refers to $\wp(\omega; \omega_1, \omega_2)$. If not mentioned, we will always abbreviate $\wp(\omega; \omega_1, \omega_2)$ as $\wp(\omega)$ for convenience in the following discussion.

By Theorem 20, we can calculate ω in terms of x, y. Since $\wp'(\omega) = \frac{d\wp(\omega)}{d\omega}$, by (11.1.17) and (11.1.18), we have

$$\int \frac{d\wp(\omega)}{2a(x(\omega))y(\omega) + b(x(\omega))} = \int d\omega \tag{11.1.21}$$

if $d_4 = 0$, and

$$\int \frac{(x-x_4)^2 d\wp(\omega)}{2a(x(\omega))y(\omega) + b(x(\omega))} = \int d\omega \tag{11.1.22}$$

if $d_4 \neq 0$. By (11.1.19), $\wp(\omega)$ can be written as a function of x. Thus, ω can be calculated as an integral of an algebraic function of x, t. It is a D-finite function of x, t.

The symmetry group of the model can also be lifted onto the universal cover. We still use the notation ϕ and ψ for the automorphisms on the universal cover. We further define $\delta = \psi\phi$. Since ϕ and ψ are involutions on the torus, $\phi^2 - 1$ and $\psi^2 - 1$ are both constants on the complex plane. A symmetric transformation on the universal cover of a torus is written as a composition of a reflection and a translation. Thus, ϕ, ψ and δ have the form

$$f(w) = \alpha w + \beta. \tag{11.1.23}$$

It has been proved [33, Chapter 3.1.2] that ϕ and ψ can be defined as

$$\begin{aligned}\psi(\omega) &= -\omega \\ \phi(\omega) &= -\omega + \omega_3 \\ \delta(\omega) &= \psi\phi(\omega) = \omega + \omega_3,\end{aligned} \tag{11.1.24}$$

where ω_2, ω_3 are real and ω_1 is purely imaginary. Now the problem becomes finding the meromorphic functions $\pi(x(\omega))$ and $\tilde{\pi}(y(\omega))$ on $K(x(\omega), y(\omega)) = 0$ which satisfy

$$0 = q\pi + \tilde{q}\tilde{\pi} + q_0\pi_{00} \tag{11.1.25}$$

and

$$\begin{aligned}\psi(\pi) &= \pi \\ \phi(\tilde{\pi}) &= \tilde{\pi},\end{aligned} \tag{11.1.26}$$

where we have abbreviated $\pi(x)$ as π and $\tilde{\pi}(y)$ as $\tilde{\pi}$.

In [33, Chapter 3.3], ω_1, ω_2, ω_3 are calculated to be

$$\begin{aligned}\omega_1 &= 2i\int_{x_1}^{x_2} \frac{dx}{\sqrt{D(x)}} \\ \omega_2 &= 2\int_{x_2}^{x_3} \frac{dx}{\sqrt{D(x)}} \\ \omega_3 &= 2i\int_{X(y_1)}^{x_1} \frac{dx}{\sqrt{D(x)}}.\end{aligned} \tag{11.1.27}$$

Until now, we have first expressed our problem on $\mathbb{C}_x$, then onto a Riemann surface and then on the universal covering. To take advantage of the Riemann surface, we consider the following lemma (which is **Lemma 2.3.4** in [33]),

Lemma 21. *Let M_y and M_x be the mean jump vector*

$$M_y = \sum_{i,j\in\{-1,0,1\}} jp_{ij}. \tag{11.1.28}$$

and

$$M_x = \sum_{i,j\in\{-1,0,1\}} ip_{ij}. \tag{11.1.29}$$

Let the random walk be non-singular and suppose $M_y \neq 0$. Let Γ be the unit circle $\{x : |x| = 1\}$. The algebraic function $Y(x)$ has two branches on Γ, denoted by $Y_0(x)$ and $Y_1(x)$, such that $Y_0(1) < Y_1(1)$.

1. *If $M_y > 0$, then $|Y_0(x)| < 1$ and $|Y_1(x)| \geq 1$ for $x \in \Gamma$, and the equality takes place only for $x = 1$, where $Y_1(1) = 1$. Moreover, $Y_0(x)$ (resp. $Y_1(x)$), $x \in \Gamma$, describes a real analytic curve, situated inside (resp. outside) the unit circle of y, for $x \neq 1$.*

2. *If $M_y < 0$, then $|Y_0(x)| \le 1$ for $x \in \Gamma$, and equality holds only for $x = 1$, where $Y_0(1) = 1$. Moreover, $|Y_1(x)| > 1$ for $x \in \Gamma$.*

If $M_x \neq 0$, similar properties hold for the algebraic function $X(y)$, the two branches of which are denoted respectively by $X_0(y)$ and $X_1(y)$.

We may lift the unit circle onto the Riemann surface $K(x(s), Y(x(s))) = 0$. Since the torus $K(x(s), Y(x(s))) = 0$ can be viewed as two glued tubes parameterized by (x, Y_0) and (x, Y_1), we further define

$$\begin{aligned} \Gamma_0 &= \{|x(s)| = 1\} \cap \{|y(s)| \le 1\} \\ \Gamma_1 &= \{|x(s)| = 1\} \cap \{|y(s)| \ge 1\} \\ \widetilde{\Gamma_0} &= \{|x(s)| \le 1\} \cap \{|y(s)| = 1\} \\ \widetilde{\Gamma_1} &= \{|x(s)| \ge 1\} \cap \{|y(s)| = 1\}, \end{aligned} \tag{11.1.30}$$

where Γ_0 (Γ_1) refers to the unit circle of x on the $(x, Y_0(x))$ $((x, Y_1(x)))$ part of the torus by Lemma 21. Similarly $\widetilde{\Gamma_0}$ ($\widetilde{\Gamma_1}$) refers to the unit circle of y on the $(X_0(y), y)$ $((X_1(y), y))$ part of the torus.

From Lemma 21, we have two important results.

1. $\Gamma_0, \Gamma_1, \widetilde{\Gamma_0}, \widetilde{\Gamma_1}$ are in the same homotopy class. It is easy to see Γ_0 and Γ_1 are in the same homotopy class because of the construction of the Riemann surface. If Γ_0 is in a different homotopy class with $\tilde{\Gamma}_0$, it will intersect both $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$. However, in Lemma 21, $|x| = 1$ and $|y| = 1$ only have one intersection.

2. By definition of $\pi(x)$ and $\tilde{\pi}(y)$, the boundary of the convergent area of both functions is inside $\Gamma_0 \cap \widetilde{\Gamma_0}$. By Lemma 21, the boundary of Γ_0 and $\widetilde{\Gamma_0}$ only intersect at point $x = y = 1$. This implies most of the other points are analytic on the boundary. This provides a way to analytically continue $\pi(x, y), \pi(x), \tilde{\pi}(y)$ to the domain $h_x^{-1}(\mathbb{D}) \cup h_y^{-1}(\mathbb{D})$.

These two inferences broaden the analytic domains of $Q(x, 0)$ and $Q(0, y)$. To get a further result, we recall the automorphisms ϕ, ψ. **Theorem 3.2.1** in [33] gives the following theorem,

Theorem 22. *π and $\tilde{\pi}$ can be continued as meromorphic functions to the whole Riemann surface.*

Proof. By the previous discussion, the convergent domain of π and $\tilde{\pi}$ is $h_x^{-1}(\mathbb{D}) \cap h_y^{-1}(\mathbb{D})$ on the Riemann surface. Now consider the homomorphism ψ. It permutes the two roots Y_0 and Y_1. And we have $\psi(y) = \frac{c(x)}{a(x)y}$ and $\psi(x) = x$. We can see that $\psi(y)$ maps $y \in \mathbb{D}$ to $y \in C_{\mathbb{D}}$. We apply ψ to (11.1.8) and get an equation of $\tilde{\pi}(\psi(y))$ and $\pi(x)$. This means $\tilde{\pi}$ is also analytic in $h_x^{-1}(\mathbb{D}) \cap h_y^{-1}(C_{\mathbb{D}})$ ($C_{\mathbb{D}}$ refers to the complement of $\mathbb{D}$) since all other terms including $\pi(x)$ are analytic in the equation. By the same discussion on ϕ, we can analytically continue π to the domain $h_x^{-1}(C_{\mathbb{D}}) \cap h_y^{-1}(\mathbb{D})$. And by the same discussion for δ, π and $\tilde{\pi}$ are analytic on the whole Riemann surface except some singularities on the boundary of $h_x^{-1}(\mathbb{D})$ and $h_y^{-1}(\mathbb{D})$. This finishes the proof. □

Now we can see the advantage of Insight 4. It broadens the meromorphic area to the whole surface such that any element in the symmetry group can be used on the functional equation.

We may lift $\pi, \tilde{\pi}$ onto the universal covering, and apply ϕ to (11.1.25). By linear combination, we eliminate $\tilde{\pi}$ (this is because it does not change under ϕ) and finally have

$$\delta(\pi) - f\pi = q_c \tag{11.1.31}$$

where $\delta = \psi\phi$, $f = \frac{q\phi(\tilde{q})}{\tilde{q}\phi(q)}$ and q_c are the functions independent of π. We may do this transformation multiple times,

$$\begin{aligned} \delta(\pi) - f\pi &= q_c \\ \delta^2(\pi) - \delta(f\pi) &= \delta(q_c) \\ &\vdots \\ \delta^i(\pi) - \delta^{i-1}(f\pi) &= \delta^{i-1}(q_c) \\ &\vdots \end{aligned} \tag{11.1.32}$$

When the group is finite and has n elements, we have $\delta^n = 1$. Restricting to finite groups, let

$$N(f) = \prod_{i=0}^{n-1} \delta^i(f) \tag{11.1.33}$$

and

$$Tr(f) = \sum_{i=0}^{n-1} \delta^i(f). \tag{11.1.34}$$

We can easily have a rational solution when $N(f) \neq 1$:

$$\rho = \frac{\sum_{i=0}^{n-1} \delta^i(q_c) \prod_{k=i+1}^{n-1} \delta^k(f)}{1 - N(f)}. \tag{11.1.35}$$

Theorems 4.8.1 and **4.8.3** in [33] give the final solution.

Theorem 23. *If $N(f) = 1$, then the general solution of (11.1.26) has the form*

$$\pi = w_1 + w_2 + \frac{w}{r}, \tag{11.1.36}$$

where

$$w_1 = \frac{1}{2n} \sum_{i=1}^{n-1} (i - n) \left[\prod_{j=i}^{n-1} \delta^j(f) \delta^{i-1}(\bar{q}_c) + \psi \left(\prod_{j=i}^{n-1} \delta^j(f) \delta^{i-1}(\bar{q}_c) \right) \right], \tag{11.1.37}$$

$$\bar{q}_c = q_c - \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{\delta^k(q_c)}{\prod_{i=1}^{k} \delta^i(f)} \right), \tag{11.1.38}$$

$$w_2 = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{\delta^k(q_c)}{\prod_{i=1}^{k} \delta^i(f)} \right) \tilde{\Phi}, \tag{11.1.39}$$

and $\tilde{\Phi}$ satisfies

$$\tilde{\Phi}\left(\omega + \frac{\omega_2}{2}\right) = \frac{\omega_1}{2\pi i} \zeta(\omega; \omega_1, \omega_3) - \frac{\omega}{\pi i} \zeta\left(\frac{\omega}{2}; \omega_1, \omega_3\right), \tag{11.1.40}$$

where $\zeta(\omega; \omega_1, \omega_3)$ is the Weierstrass ζ-function with periods ω_1 and ω_3 (see [4] for a definition).

The function w is an algebraic function of ω and satisfies $w = \phi(w) = \psi(w)$. The function r is an algebraic function defined by $f = \frac{r}{\delta(r)}$ and $\phi(r) = r$.

If $N(f) \neq 1$, then

$$\pi = \rho + \theta e^{\int (\frac{1}{n} Tr(\chi)\tilde{\Phi} + S - R) d\omega}, \tag{11.1.41}$$

where ρ is defined in (11.1.35), $\tilde{\Phi}$ is defined in (11.1.40), $\chi = \frac{1}{f}\frac{df}{d\omega}$ and

$$\begin{aligned} S &= \frac{1}{2n}\sum_{i=1}^{n-1}(i-n)\left[\prod_{j=i}^{n-1}\delta^{i-1}(\chi_2) - \psi\left(\prod_{j=i}^{n-1}\delta^{i-1}(\chi_2)\right)\right] \\ &= \frac{1}{n}\sum_{k=0}^{n-1} k\delta^k(\chi_2), \end{aligned} \tag{11.1.42}$$

where

$$\chi_2 = \chi - \frac{1}{n}Tr(\chi). \tag{11.1.43}$$

The function R is the solution of the homogeneous equations

$$\delta(R) = R \text{ and } \phi(R) = -R \tag{11.1.44}$$

and θ satisfies $\theta = \psi(\theta) = \phi(\theta)$.

Theorem 23 is the solution without t. We can extend the theorem to the situation when t is introduced. If we treat t as a constant and $t \leq \frac{1}{|S|}$ ($|S|$ is the cardinality of the step-set), the roots of the kernel with t satisfy Lemma 21. Further when $t \leq \frac{1}{|S|}$, the roots of the discriminant satisfy (11.1.15). By continuity, when $t \leq \frac{1}{|S|}$ we can always have different real roots.

By the property of $K(x,y)$, we know $Y_0(x)$ is a formal series of t without constant terms and $Y_1(x)$ contains $\frac{1}{t}$ in the expansion. When t is small enough, a series of t without constant terms is always inside the unit circle and a series with $\frac{1}{t}$ term is outside the unit circle[3]. Since ϕ and ψ permute the roots, by Lemma 21, for any point $(x, Y(x))$ where $Y(x)$ is a series of t without constant terms (inside the domain $\{|x| < 1\} \cap \{|y| < 1\}$) on the torus, ψ maps it to a point where $Y(x)$ is a series with $\frac{1}{t}$ terms (and hence it lies in $\{|x| < 1\} \cap \{|y| > 1\}$). Meanwhile ϕ maps $(X(y), y)$ to a point where $\phi(X(y))$ is a series with $\frac{1}{t}$ term. And δ maps $(x, Y(x))$ (or $(X(y), y)$) to a point where $X(y), Y(x)$ are both series with $\frac{1}{t}$ terms (in $\{|x| > 1\} \cap \{|y| > 1\}$). The analytic continuation and Theorem 22 still work. We further see that the convergent areas of (x,y) and $\delta(x,y)$ do not intersect each other. If x, y are formal series of t, then $\delta(x), \delta(y)$ contain $\frac{1}{t}$ in their expansions for $t \leq \frac{1}{k}$.

Thus, $Q(x,0)$ and $Q(0,y)$ can be analytically continued to the whole surface and Theorem 23 follows immediately. Notice that the extra condition for $Q(x,0)$ (π) to be a solution is that it is a formal series of x. Since an extra algebraic function w or (R,θ) appears in the solution, we can choose suitable functions to make this satisfied. By choosing the functions, we can get the explicit result of $Q(x,0)$.

Theorem 23 does not tell us anything about $Q(0,0)$, or any information about the point boundary terms of t. We still need to solve $Q(0,0)$ ourselves. Fortunately, (11.1.36) is an equation with kernel form. We can use the kernel method to solve it.

[3]Note that ϕ and ψ map the region inside the unit circle to the whole region outside the unit circle, since $\phi(y) = \frac{c(x)}{a(x)y}$, $\phi(x) = \frac{\tilde{c}(y)}{\tilde{a}(y)x}$, and $c(x), \tilde{c}(y)$ do not involve t.

By simple calculations, one will notice the condition $N(f) = 1$ implies that the full-orbit sum (5.4.1) in our calculation does not contain line boundary terms in the RHS. This is the case that the solution is D-finite over x. When $N(f) \neq 1$, which implies the full-orbit sum contains line boundary terms, we find the solution of these models is D-algebraic over x.

11.2 Riemann boundary value problem

Another analytic method for solving quarter-plane lattice walk problems is by techniques using singular integral equations. The idea is to reduce the problem to a Riemann boundary value problem (RBVP). One can refer to [33] for the basic introduction to RBVP. The probabilistic version of this solution was also proposed in [33]. Raschel [61] extended it into the counting problem.

Remark 7. *We are not going to use this method in Chapter 12. But since it is widely studied in the research of quarter-plane lattice walk problems, we give a brief introduction here.*

The quarter-plane lattice walk problem can be reduced to a problem called the Riemann-Carleman problem with a shift. Let L^+ be the domain to the left of the contour L. The standard Riemann-Carleman problem with a shift is described as follows,

Find a function Φ^+ holomorphic in some domain L^+, the limiting values of which on the closed contour L are continuous and satisfy the relation,

$$\Phi^+(\alpha(s)) = G(s)\Phi^+(s) + g(s), \qquad s \in L, \tag{11.2.1}$$

where

1. For any $s \in L$, g and G are Hölder continuous on L and $G(s) \neq 0$.
2. The function $\alpha(s)$ is an one-one mapping of $L \to L$ such that $\alpha(\alpha(s)) = s$ and α is Hölder continuous.

Note that a function f is said to be Hölder continuous [56] if for any two points s_1, s_2 on L, we can find positive constants A and μ, such that,

$$|f(s_1) - f(s_2)| \leq A|s_1 - s_2|^\mu. \tag{11.2.2}$$

Now, we reduce the lattice walk problem to a RBVP. In the following discussion, t is still treated as a small value such that Lemma 21 holds.

By Insight 2 in Section 4.2, the functional equation holds on the algebraic curve $K(X, y) = 0$ and we have

$$q(X_0, y)\pi(X_0) + \tilde{q}(X_0, y)\tilde{\pi}(y) + q_0(X_0, y)\pi_{00} = 0 \tag{11.2.3}$$
$$q(X_1, y)\pi(X_1) + \tilde{q}(X_1, y)\tilde{\pi}(y) + q_0(X_1, y)\pi_{00} = 0, \tag{11.2.4}$$

where

$$X_{1,0}(y) = \frac{-\tilde{b}(y) \pm \sqrt{\Delta}}{2\tilde{a}(y)} = \frac{-\tilde{b}(y) \pm \sqrt{d_4(y-y_1)(y-y_2)(y-y_3)(y-y_4)}}{2\tilde{a}(y)} \tag{11.2.5}$$

(assuming the discriminant has four roots y_1, y_2, y_3, y_4 and $d_4 > 0$). We are treating the kernel as a quadratic function of x with coefficients in $\mathbb{C}(y, t)$. Following the discussion in Lemma 21 and the discussion after Theorem 23, the roots of the discriminant obey (11.1.15) when t is small enough (just swap x and y). We may choose t small enough such that y_1, y_2 are inside the unit circle.

Now first consider the $\mathbb{C}_y \cup \infty$ plane. The complex plane is cut by the branch cuts $\overline{y_1y_2}$ and $\overline{y_3y_4}$. We define $\overrightarrow{y_1y_2}$ as the contour above $\overline{y_1y_2}$ in right direction and $\underrightarrow{y_1y_2}$ as the contour below $\overline{y_1y_2}$ in right direction, where $\overleftarrow{y_1y_2}$ and $\underleftarrow{y_1y_2}$ refer to the contours in left direction. Notice that $X_0(y)$ maps a point $y \in \mathbb{C}_y \cup \infty$ into a point in the $\mathbb{C}_x \cup \infty$. We consider the contour $X_0(s)$ in $\mathbb{C}_x \cup \infty$ plane where s is on the contour $\overrightarrow{y_1y_2}\underleftarrow{y_1y_2}$. The function $X_0(s)$ is continuous on $\overrightarrow{y_1y_2}$ and $\underleftarrow{y_1y_2}$. The values $X_0(y_1), X_0(y_2)$ are real and single-valued. So $X_0(s)$ is continuous on the contour $\overrightarrow{y_1y_2}\underleftarrow{y_1y_2}$. If s goes around $\overline{y_1y_2}$, $X_0(s)$ is a closed contour on $\mathbb{C}_x \cup \infty$. The contour is symmetric about the real axis and only intersects the real axis at two points since $\sqrt{\Delta}$ is pure imaginary on $\overline{y_1y_2}$ except the two end points. Further, if we denote $X_0(s)$ as the value of some point s above the branch cut, the value of the corresponding point s' below the branch cut will be $X_0(s') = X_1(s)$. Since $\phi(X_0) = X_1$, we have $\phi(X_0(s)) = X_0(s') = X_1(s)$.

Consider the value of (11.2.3) and (11.2.4) on $\overrightarrow{y_1y_2}\underleftarrow{y_1y_2}$. We have the following two equations

$$q(X_0(s), y)\pi(X_0(s)) + \tilde{q}(X_0(s), y)\tilde{\pi}(y(s)) + q_0(X_0(s), y)\pi_{00} = 0, \qquad s \in \overrightarrow{y_1y_2}\underleftarrow{y_1y_2} \tag{11.2.6}$$

$$q(X_1(s), y)\pi(X_1(s)) + \tilde{q}(X_1(s), y)\tilde{\pi}(y) + q_0(X_1(s), y)\pi_{00} = 0, \qquad s \in \overrightarrow{y_1y_2}\underleftarrow{y_1y_2}, \tag{11.2.7}$$

where $y = y(s)$. Since $y(s) = y(s')$, we omit the argument and only write y here. And by the definition of ϕ,

$$\begin{gathered} \phi(X_0(s)) = X_1(s) \\ \phi(y) = y. \end{gathered} \tag{11.2.8}$$

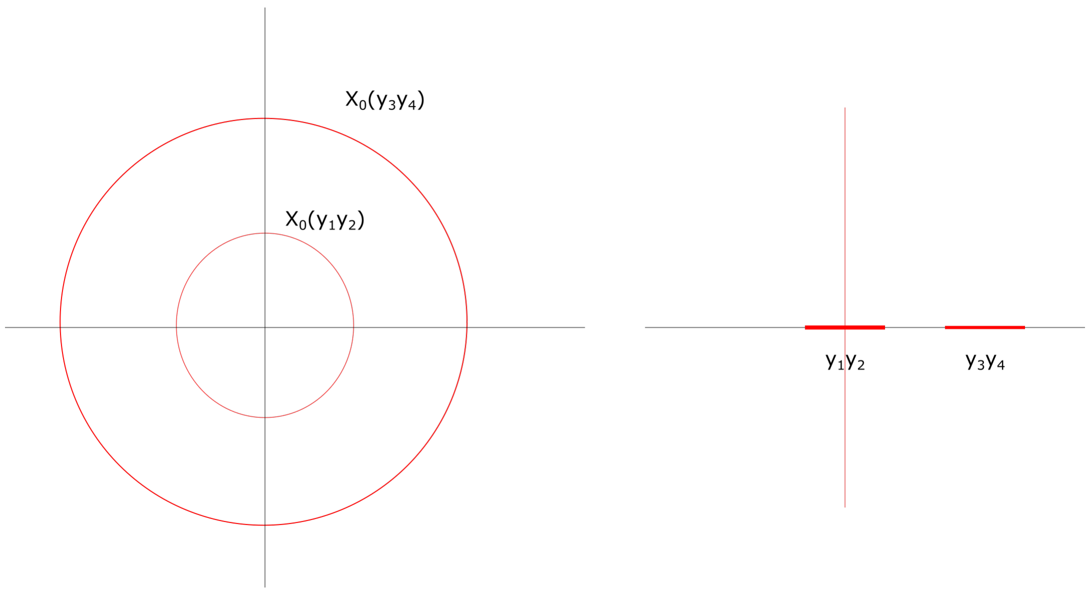


Figure 11.2: On the $\mathbb{C}_x$ plane, $X_0(y_1y_2)$ and $X_0(y_3y_4)$ are two circles. On the $\mathbb{C}_y$ plane, y_1y_2 and y_3y_4 are two segments on real axis.

Now (11.2.7) can be written as

$$q(\phi(X_0(s)), y)\pi(\phi(X_0(s))) + \tilde{q}(\phi(X_0(s)), y)\tilde{\pi}(y) + q_0(\phi(X_0(s)), y)\pi_{00} = 0. \tag{11.2.9}$$

Taking linear combinations, we can eliminate $\tilde{\pi}(y)$:

$$\pi(\phi(X_0(s))) - f(s)\pi(X_0(s)) = q_c(s), \qquad s \in \overrightarrow{y_1y_2}\overleftarrow{y_1y_2}. \tag{11.2.10}$$

Note that (11.2.10) looks like (11.2.1). However, we still need to guarantee $\pi(X_0(s))$ satisfies the holomorphic condition. The function π has been proved to be meromorphic on the Riemann surface in Theorem 22, so it is analytic on the $\mathbb{C}_x$ plane except for some singular points. Now we rewrite (11.2.6) as

$$q(x, Y_0(x))\pi(x) + \tilde{q}(x, Y_0(x))\tilde{\pi}(Y_0(x)) + q_0(X_0, Y_0(x))\pi_{00} = 0. \tag{11.2.11}$$

That is, we substitute $y = Y_0(x)$ into (11.2.6) and use the fact that $X_0(Y_0(x)) = x$ if we fix the branch of $Y_0(x)$.

We first prove $\tilde{\pi}$ is analytic inside the contour $X_0(\overrightarrow{y_1y_2})$. Consider the function $Y_0(x)$ on the contour $X_0(\overrightarrow{y_1y_2})$. It refers to the value on the line $\overrightarrow{y_1y_2}$ on the $\mathbb{C}_y$ plane. Thus $|Y_0(x)| < 1$ on the contour since we choose $\overrightarrow{y_1y_2}$ in the unit circle on the $\mathbb{C}_y$ plane. Then we consider the analyticity inside the circle $X_0(\overrightarrow{y_1y_2})$. Since on the $\mathbb{C}_y$ plane, $X_0(y)$ is analytic except $\overrightarrow{y_1y_2}$ and $\overrightarrow{y_3y_4}$, $Y_0(x)$ is analytic inside $X_0(\overrightarrow{y_1y_2})$ if $X_0(\overrightarrow{y_1y_2})$ is inside of $X_0(\overrightarrow{y_3y_3})$. (If $X_0(\overrightarrow{y_3y_4})$ is inside $X_0(\overrightarrow{y_1y_2})$, we let $y \to \psi(y)$ in (11.2.6) and change all the previous and following discussions from inside to outside $X_0(\overrightarrow{y_1y_2})$. All results still hold since $\pi(\psi(y))$ is analytic when $y \to \infty$.) So we need to prove that $X_0(\overrightarrow{y_1y_2})$ and $X_0(\overrightarrow{y_3y_4})$ do not cross.

Suppose these two circles intersect at x_p. First assume x_p is not real. By symmetry there must be another intersection at the conjugate point $\overline{x_p}$. Suppose x_p is in the upper half plane of $\mathbb{C}_x \cup \infty$, then

$$K(x_p, y) = \tilde{a}(y)x_p^2 + \tilde{b}(y)x_p + \tilde{c}(y) = 0 \tag{11.2.12}$$

has one root y_5 in $\overrightarrow{y_1y_2}$ and one root y_6 in $\overrightarrow{y_3y_4}$. The following two equations

$$\begin{aligned} \Re[\tilde{a}(y_{5,6})x_p^2 + \tilde{b}(y_{5,6})x_p + \tilde{c}(y_{5,6})] = 0 \\ \Im[\tilde{a}(y_{5,6})x_p^2 + \tilde{b}(y_{5,6})x_p + \tilde{c}(y_{5,6})] = 0 \end{aligned} \tag{11.2.13}$$

hold for y_5, y_6. If we write $x_p = u + iv$, then

$$\begin{aligned} \tilde{a}(y_{5,6})(u^2 - v^2) + \tilde{b}(y_{5,6})u + \tilde{c}(y_{5,6}) = 0 \\ 2\tilde{a}(y_{5,6})u + \tilde{b}(y_{5,6}) = 0. \end{aligned} \tag{11.2.14}$$

Notice that these two equations are both quadratic equations of y, so if they have two identical roots, one is a constant multiple of the other for general y. But we do not have $\tilde{c}(y)$ in the second equation. This means $\tilde{c}(y)$ can be expressed as a linear combination of $\tilde{a}(y)$ and $\tilde{b}(y)$. Thus $\tilde{a}(y), \tilde{b}(y), \tilde{c}(y)$ are linearly dependent with real coefficients. The curve $X_0(y) = \frac{-\tilde{b}(y) - \sqrt{\tilde{b}(y)^2 - 4\tilde{c}(y)\tilde{a}(y)}}{2\tilde{a}(y)}$ can be expressed by a function of $\frac{\tilde{b}(y)}{\tilde{a}(y)}$. Note also that $-\frac{\tilde{b}(y)}{\tilde{a}(y)} = X_0(y) + X_1(y)$, and $\Re(X_0(y)) = -\frac{\tilde{b}(y)}{2\tilde{a}(y)}$.

Now we choose a real value k such that $\Re(X_0(y)) = -\frac{\tilde{b}(y)}{2\tilde{a}(y)} = k$. We can solve this equation which has one or two roots. Notice that we are considering the intersecting case. When $k = \Re(x_p)$, we have two roots. Thus for general k, $-\frac{\tilde{b}(y)}{2\tilde{a}(y)} = k$ should have two roots y_a, y_b. If $(k, 0)$ is inside the contour $X_0(\overrightarrow{y_1y_2})$, it always has a root y_a such that $\Re(X_0(y_a)) = k$. Furthermore, this is an equation with real coefficients. When one root is real, the other is real. We always have another root y_b such that $y_b \in \overrightarrow{y_3y_4}$. But in fact as we saw earlier, $X_0(y)$ is determined only by $-\frac{\tilde{b}(y)}{\tilde{a}(y)} = 2k$ which is the same for y_a and y_b, so $X_0(y_a) = X_0(y_b)$. Hence the two circles intersect at a point whose real part is k. But k was arbitrary, so we must conclude that the two circles actually coincide.

Now instead assume x_p is real. Notice that $x_p = X_0(y_p) = \frac{-\tilde{b}(y_p)-\sqrt{\Delta(y_p)}}{2\tilde{a}(y_p)}$ and y_p is real on $\overrightarrow{y_1y_2}, \overrightarrow{y_3y_4}$. We immediately have $\Delta(y_p) = 0$, so we must have $y_p = y_i$ for some $i \in \{1, 2, 3, 4\}$. Suppose at x_p, the two roots are y_1 and y_3, we have

$$\begin{aligned} x_p &= -\frac{\tilde{b}(y_1)}{2\tilde{a}(y_1)} = -\frac{\tilde{b}(y_3)}{2\tilde{a}(y_3)} \\ \tilde{b}(y_1)^2 &- 4\tilde{a}(y_1)\tilde{c}(y_1) = 0 \\ \tilde{b}(y_3)^2 &- 4\tilde{a}(y_3)\tilde{c}(y_3) = 0. \end{aligned} \tag{11.2.15}$$

The first equation shows that y_1, y_3 are two roots of the equation $2x_p\tilde{a}(y) = \tilde{b}(y)$. The second and third equation can then be written as

$$x_p^2\tilde{a}(y_1) = \tilde{c}(y_1) \tag{11.2.16}$$

$$x_p^2\tilde{a}(y_3) = \tilde{c}(y_3). \tag{11.2.17}$$

Thus, y_1, y_3 are two roots of $x_p^2\tilde{a}(y) = \tilde{c}(y)$. Note thate $2x_p\tilde{a}(y) = \tilde{b}(y)$ and $x_p^2\tilde{a}(y) = \tilde{c}(y)$ are both linear or quadratic functions of y. They have identical roots if and only if they are exactly the same. We again have that $\tilde{c}(y)$ is a linear combination of $\tilde{a}(y)$ and $\tilde{b}(y)$. By the same discussion as the previous case, we conclude these two circles coincide.

We return to the question of analyticity. Since $Y_0(x)$ is analytic inside $X_0(\overrightarrow{y_1y_2})$, by the maximum modulus principle, all $Y_0(x)$ inside $X_0(\overrightarrow{y_1y_2})$ are less than 1. Therefore, $\tilde{\pi}(y)$ is holomorphic inside the contour $X_0(\overrightarrow{y_1y_2})$. Then in (11.2.11), the only singularities of $\pi(x)$ come from the singularities of $\tilde{q}(x, Y_0(x))$ and the zeros of $q(x, Y_0(x))$. To make it holomorphic, suppose we have r_i as singularities of π, and let

$$\pi_2 = \pi \prod (x - r_i). \tag{11.2.18}$$

Then we have

$$\pi_2(\phi(s)) - f_2(s)\pi_2(s) = q_{2c}(s), \qquad (x(s), y(s)) \in (X_0(y_1y_2), y_1y_2). \tag{11.2.19}$$

Now we find an equation on the complex plane $\mathbb{C}_x$, satisfying (11.2.1) on the contour $X_0(\overrightarrow{y_1y_2})$, and holomorphic in $X_0(\overrightarrow{y_1y_2})^+$. We can check that $f_2(s)$, $q_{2c}(s)$ are Hölder continuous on the contour $(X_0(\overrightarrow{y_1y_2}), \overrightarrow{y_1y_2})$ for lattice walk problems, since they are just polynomials of x, y. Also $\phi(\phi(s)) = s$. Thus, we have reduced the lattice walk problem to a RBVP. When $d_4 < 0$ we choose $\overrightarrow{y_2y_3}$ and $\overrightarrow{y_4y_1}$ to be the branch cutting lines. When the discriminant has three roots, we choose $\overrightarrow{y_1y_2}$ and $\overrightarrow{y_3\infty}$ to be the branch cutting lines. All discussions will be the same as previously discussed.

The solution follows **Proposition 11.6.2** in [33]:

Theorem 24. *Given the functional equation (6.1.3) with $a = b = c = 1$, for $x \in L^+(t)$, we have*

$$A(x, 0)Q(x, 0, t) - A(0, 0)Q(0, 0, t) = \frac{1}{2\pi i t}\int_L zY_0(z, t)\frac{\frac{d}{dz}w(z, t)}{w(z, t) - w(x, t)}dz, \tag{11.2.20}$$

where $w(z, t)$ is the solution of

$$w(\delta(z), t) - w(z, t) = 0, \qquad t \in L \tag{11.2.21}$$

and $Q(0, 0, t)$ can be calculated by the roots of $A(x, 0)$. The function $w(z, t)$ is called the conformal gluing function.

Remark 8. *Theorem 24 is valid for non-interacting cases or probabilistic models. For walks with interactions, we need to first change $A(x, 0)$ to $\frac{1}{a}(a - 1 - taA(x, Y_0))$. Then, in order to satisfy the condition that $\Phi^+(s)$ is holomorphic in L^+, we need to multiply the singularities into the integral as we did in (11.2.18). This may make $\frac{1}{a}(a - 1 - taA(x, Y_0))$ a bit different from the original $\frac{1}{a}(a - 1 - taA(x, Y_0))$ in the functional equation.*

Chapter 12

Algebraic, D-finite, D-algebraic, and hyper-transcendental properties of the solution

In previous chapters, we focused on how to find the solutions (generating functions) of quarter-plane lattice walks with boundary conditions. For some of the models, we currently cannot solve $Q(0,0)$ explicitly. Are we really unable to solve them by the kernel method or it is just that we have not found the right way? Besides, with interactions, some of the solutions are algebraic, some of them are D-finite and some of them are D-algebraic. The kernel method provides a good way to calculate them and a good expression for further study. But can we detect these properties before explicitly solving the models?

We appeal to other methods for help. In Chapter 11, we introduced two powerful analytic methods. Theorem 23 gives solutions to all walks associated with finite groups and Theorem 24 is a unified approach to all models, even for walks with infinite groups or for singular walks. With the help of Chapter 11 we are now able to detect the algebraic, D-finite, D-algebraic properties of $Q(x,0)$, $Q(0,y)$ and $Q(x,y)$ as functions of x,y without explicitly solving them. Actually Kurkova and Raschel have already discussed some of the properties via RBVP in [61, 48]. Because of the probabilistic background, these discussions are mostly about the properties over x and y.

When we begin to consider t, things become mysterious. The solutions in Chapter 11 all contain integrals, which make $Q(0,0)$ hard to solve. However, for physicists, $Q(0,0)$ is more important. For example, if we are counting the configurations of a polymer growth, we are interested in the surface tension under which the polymer will have an absorbed phase (that is the polymer tends to stick to the surface). If we model the polymer growth as a quarter-plane lattice walk, the problem becomes determining the interacting force which causes the walk to stay at the origin. This relies on the asymptotic behavior of $Q(0,0)$. This is the motivation for finding the explicit result of $Q(0,0)$. If $Q(0,0)$ is hard to solve, we at least want to find its singularities. This also means we want to find the homogeneous part of the linear differential equation it satisfies when it is D-finite, or the algebraic equation when it is D-algebraic.

In this section, we are going to discuss the properties of generating functions $Q(x,0)$, $Q(0,y)$, $Q(x,y)$ over x,y of our quarter-plane lattice walk models via the solution form of (11.1.36). Recall that

$$\pi = w_1 + w_2 + \frac{w}{r} \tag{12.0.1}$$

where

$$w_1 = \frac{1}{2n}\sum_{i=1}^{n-1}(i-n)\left[\prod_{j=i}^{n-1}\delta^j(f)\delta^{i-1}(\bar{q}_c) + \psi\left(\prod_{j=i}^{n-1}\delta^j(f)\delta^{i-1}(\bar{q}_c)\right)\right], \tag{12.0.2}$$

$$\bar{q}_c = q_c - \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{\delta^k(q_c)}{\prod_{i=1}^{k}\delta^i(f)}\right), \tag{12.0.3}$$

$$w_2 = \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{\delta^k(q_c)}{\prod_{i=1}^{k}\delta^i(f)}\right)\tilde{\Phi}, \tag{12.0.4}$$

and $\tilde{\Phi}$ satisfies

$$\tilde{\Phi}\left(\omega + \frac{\omega_2}{2}\right) = \frac{\omega_1}{2\pi i}\zeta(\omega;\omega_1,\omega_3) - \frac{\omega}{\pi i}\zeta\left(\frac{\omega}{2};\omega_1,\omega_3\right), \tag{12.0.5}$$

where $\zeta(\omega;\omega_1,\omega_3)$ is the Weierstrass ζ-function with periods ω_1 and ω_3 (see [4] for a definition).

The function w is an algebraic function of ω and satisfies $w = \phi(w) = \psi(w)$. The function r is an algebraic function defined by $f = \frac{r}{\delta(r)}$ and $\phi(r) = r$.

Combining the result of Theorem 23 and the kernel method, we are going to extend the results for properties over x, y to properties over the three variables x, y, t. The discussions and theorems in this section work for all models whose solutions can be written in the form (11.1.36). So we do not specify it is a model with or without interactions.

12.1 The criteria for $Q(x,0)$, $Q(0,y)$ and $Q(x,y)$

The following theorems are valid for quarter-plane lattice walks with unit step length following the definition in Chapter 8. By the discussion after Theorem 23, the solutions of the probabilistic model discussed in Section 11.1 can also be applied to our interacting quarter-plane lattice walk models. We use the solutions in Theorem 23 to prove the criteria. To match the solution of Theorem 23, we use most of the notations of the probabilistic model in Section 11.1. One can go to Section 11.1 for the definition of the universal cover $\mathbb{C}_\omega$, transformation δ, multiplication trace $N(f)$, periods ω_1, ω_2 and functions $w, w_1, w_2, \tilde{\Phi}$ etc. But we still write $Q(x,0)$ for the generating functions of walks ending on the x-axis instead of π and $Q(0,y)$ for the generating functions of walks ending on the y-axis instead of $\tilde{\pi}$ to show that we are considering a counting problem and not a probability problem. We require t to be small enough so that we can use the properties of the roots in Lemma 21. The definitions and properties of elliptic functions are introduced in Chapter 2. Recall from Section 3.5, for each model, we can find two involution ϕ and ψ such that the kernel remains the same under these two involutions and these two involutions generate the symmetry group of the model.

The proof of the criteria for $Q(x,0)$, $Q(0,y)$ and $Q(0,0)$ is based on the following lemma (**Lemma 4.73** in [33])

Lemma 25. *Consider a meromorphic function on the universal covering $\mathbb{C}_\omega$ of (11.1.8). It has periods ω_1, ω_2, where ω_1 is pure imaginary and ω_2 is real. Let δ be as defined in (11.1.24). That is $\delta(\omega) = \delta(\omega + \omega_3)$ where ω_3 is real. Then any solution of the equation*

$$w = \delta(w) \tag{12.1.1}$$

is algebraic over x if the symmetry group of the model is a finite group.

Proof. (12.1.1) implies $w(\omega) = w(\omega+\omega_3)$ by (11.1.24). And w is an elliptic function with periods ω_1, ω_3, where ω_1 is pure imaginary and ω_3 is real. Any elliptic function with periods ω_1, ω_3 can be written as a rational function of the Weierstrass functions $\wp(\omega;\omega_1,\omega_3), \wp'(\omega;\omega_1\omega_3)$.

When the group is finite, $\delta^n\wp(\omega) = \wp(\omega+n\omega_3) = \wp(\omega+m\omega_2)$ for some integer m relatively prime to n. So $n\omega_3 = 0 \bmod \omega_2$. Then $\omega_3 = \frac{m}{n}\omega_2$. $\omega_1, \omega_2, \omega_3$ can be calculated by (11.1.27). The value of m depends on the group, which is calculated explicitly in [61]. However, since we are proving the properties of the generating functions, it is enough to know m and n are integers. We can express $\wp(\omega;\omega_1,\frac{m}{n}\omega_2)$ by an algebraic function of $\wp(\omega), \wp'(\omega)$[1] (see (12.2.13) or [58]). And by the uniformization (11.1.19), we know the functions $\wp(\omega)$ and $\wp'(\omega)$ are rational functions of x, y. Then since y is an algebraic function of x, we have w algebraic over x. □

Remark 9. *By Lemma 25, we immediately know $\frac{w}{r}$ in (11.1.36) is algebraic over x. w_1 is algebraic by definition. Thus, the problem left for us now is the non-elliptic terms. Further, we can see that when the walk is associated with an infinite group, ω is not algebraic over x. Otherwise, we can always find $\frac{m}{n}$ such that $\omega_3 = \frac{m}{n}\omega_2$ and this implies the order of the group is finite.*

Theorem 26. *Let $N(f)$ be as defined in (11.1.33). Consider a quarter-plane walk whose symmetry group is a finite group.*

1. *If $N(f) = 1$, $Q(x,0)$, $Q(0,y)$ and $Q(x,y)$ are D-finite over x, y.*
2. *If $N(f) = 1$ and the full-orbit sum (5.4.1) equals 0, $Q(x,0)$, $Q(0,y)$ and $Q(x,y)$ are algebraic over x, y.*
3. *If $N(f) \neq 1$, then $Q(x,0)$ $Q(0,y)$ and $Q(x,y)$ are D-algebraic over x, y.*

Proof. We only prove the theorem for $Q(x,0)$. The result for $Q(0,y)$ can be proved similarly and $Q(x,y)$ follows by the functional equation. Apply the solution (11.1.36) and (11.1.41). π in these two equations can be written as $Q(x(\omega),0)$.

First consider the case $N(f) = 1$. One can also find the proof of this case in [34] (**Corollary 2.3**). Notice that except w_2, all terms are algebraic function of ω in (11.1.36). By Theorem 20 they are algebraic over x.

Now consider w_2. The non-algebraic term is $\tilde{\Phi}$. $\tilde{\Phi}$ is a linear function of $\zeta(\omega,\omega_1,\omega_3)$[2] and $\zeta(\omega/2,\omega_1,\omega_3)$. The derivative of a ζ-function is a $\wp$-function, and $\wp(\omega,\omega_1,\omega_3)$ is an elliptic function with periods ω_1, ω_3. By Lemma 25, $\wp(\omega,\omega_1,\omega_3)$ is algebraic over x. Another term in the derivative is $\wp(\frac{\omega}{2},\omega_1,\omega_3)$. By the addition theorem (12.2.21), $\wp(\frac{\omega}{2},\omega_1,\omega_3)$ is an algebraic function of $\wp(\omega,\omega_1,\omega_3)$ and thus an algebraic function of $\wp(\omega)$. Thus the derivative of $\tilde{\Phi}$ is algebraic. $\tilde{\Phi}$ is D-finite. Further, the ζ-function is not an elliptic function and thus $\tilde{\Phi}$ is not algebraic. We conclude that $Q(x,0)$ is D-finite over x. If $\frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{\delta^k(q_c)}{\prod_{i=1}^{k}\delta^i(f)}\right) = 0$, the coefficient of the $\tilde{\Phi}$ term is 0 and $Q(x,0)$ is algebraic.

If $N(f) \neq 1$, we apply (11.1.41). Rearrange the equation as

$$\frac{Q(x,0)-\rho}{\theta} = e^{\int(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)d\omega}. \tag{12.1.2}$$

First take the logarithm of (12.1.2) and then take derivatives with respect to x. The RHS then becomes $(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)\frac{d\omega}{dx}$. As we discussed above, all terms in $(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)$ are

[1] $\wp(\omega)$ refers to $\wp(\omega;\omega_1,\omega_2)$.

[2] See Chapter 2 for a definition of ζ function.

algebraic except $\tilde{\Phi}$. $\tilde{\Phi}$ is D-finite. $\frac{d\omega}{dx}$ is algebraic since x can be written as an algebraic function of $\wp(\omega)$ by Theorem 20. We thus have that $\frac{d\log\frac{Q(x,0)-\rho}{\theta}}{dx}$ is D-finite over x. The LHS is

$$\left(\frac{Q'(x,0)+\rho'}{Q(x,0)+\rho}-\frac{\theta'}{\theta}\right). \tag{12.1.3}$$

This implies that $Q(x,0)$ is D-algebraic over x. □

Theorem 26 works for all models associated with finite groups. For **Models 5-16** and **Model 18** without interactions, $N(f)=1$. With interactions, $N(f)\neq 1$. This explains why they become D-algebraic with interactions.

12.2 The criteria for $Q(0,0)$

The algebraic, D-finite, D-algebraic property of $Q(x,0)$ over x has been widely discussed in many works (for example [34, 48]). However, $Q(x,0)$ as a function of x,t has not been considered. The crucial variable is $Q(0,0)$, the generating function of walks ending at the origin. If we know the algebraic, D-finite, D-algebraic property of $Q(0,0)$ over t, then we know the property of $Q(x,0)$. Earlier we used the kernel method to explicitly solve lattice walk models and discuss the properties. The results are listed in Table 8.1. Unfortunately, we still have some unsolved cases and we cannot determine the property of $Q(0,0)$ over t.

We try to find some way to determine the property of generating functions over its variables before explicitly solving it. This may prove our explicit results, and may provide a way to judge the unsolved cases. The idea is based on transcendental extensions. In the kernel method, we substitute the roots of a quadratic factor into the numerator of the equation. This can be treated as working in a field extension of the numerator. By **Proposition 5.3** in [49],

Proposition 3. *Given a field $\mathbb{K}$ and elements $(\mathbf{x})=(x_1,\ldots,x_n)$ in some extension field, assume that there exist n polynomials f_i with coefficients in $\mathbb{K}$ such that:*

1. $f_i(\mathbf{x})=0$, *and*
2. $\det|\partial f_i/\partial x_j|\neq 0$

Then $\mathbb{K}(\mathbf{x})$ is separably algebraic[3] *over $\mathbb{K}$.*

We can determined the properties of $Q(0,0)$ over t by the coefficients in (11.1.36).

To understand how Proposition 3 works for lattice walk problems, we take reverse Kreweras walks as an example. We may treat the unknowns $Q(0,0),Q_{1,0},Q_{0,1}$ as $x_i,\ldots x_n$ in Proposition 3. Recall (8.2.33) and (8.2.34). We substitute roots of $P_{x,0},P_0^{\mathrm{d}}$ into these two equations and get equations independent of $Q(x,0)$. These equations are the $f_i(\mathbf{x})=0$ in Proposition 3. If we can find enough linearly independent equations, then the second condition $\det|\partial f_i/\partial x_j|\neq 0$ is satisfied. We can then solve the unknowns $Q(0,0),Q_{1,0},Q_{0,1}$ and the solution is separably algebraic over $\mathbb{K}$. Proposition 3 can also be adapted to the D-finite or D-algebraic case. We add transcendental terms T_1,T_2 (which appear as coefficients in the equation, see Section 8.1) into $\mathbb{K}$. Then $\mathbb{K}$ can be written as $\mathbb{R}(t,T_1,T_2)$. Then our solution is separably algebraic over $\mathbb{R}(t,T_1,T_2)$ and is thus D-algebraic.

[3] This means the minimal polynomial of each x_i can be factored into linear terms in the extension field.

In [19], Bousquet-Mélou and Jehanne give a **general strategy** for proving and solving certain functional equations. The aim is to find solutions $F(u)$ and $F_1, F_2 \dots F_k$ (functions of t) satisfying some certain polynomial equation,

$$P(F(u), F_1, F_2 \dots F_k, t, u) = 0, \tag{12.2.1}$$

where $F(u)$ is a function of u, t and $F_1, F_2 \dots F_k$ are functions of t. We may

1. Differentiate the equation with respect to u

$$F'(u)\frac{\partial P}{\partial x_0}(F(u), F_1, F_2 \dots F_k, t, u) + \frac{\partial P}{\partial x_{k+2}}(F(u), F_1, F_2 \dots F_k, t, u) = 0. \tag{12.2.2}$$

2. Find $U_i = U_i(t)$, $i = 1, 2 \dots k$ that satisfy

$$\frac{\partial P}{\partial x_0}(F(u), F_1, F_2 \dots F_k, t, u) = 0. \tag{12.2.3}$$

   Then

$$\frac{\partial P}{\partial x_{k+2}}(F(u), F_1, F_2 \dots F_k, t, u) = 0 \tag{12.2.4}$$

   automatically holds.

3. If we find k distinct U_i such that (12.2.3) holds, then we get $3k$ polynomial equations

$$\begin{aligned} P(F(U_i), F_1, F_2 \dots F_k, t, U_i) &= 0 \\ \frac{\partial P}{\partial x_0}(F(U_i), F_1, F_2 \dots F_k, t, U_i) &= 0 \\ \frac{\partial P}{\partial x_{k+2}}(F(U_i), F_1, F_2 \dots F_k, t, U_i) &= 0. \end{aligned} \tag{12.2.5}$$

If these $3k$ equations are independent, then these $3k$ equations characterize $3k$ unknowns $(U_i, F_i, F(U_i), i = 1, 2, \dots, k)$. We can solve F_i and $F(u)$ follows.

The key point is that we need k distinct roots otherwise the equation set (12.2.5) does not have full rank. An explicit theorem for the algebraic case with conditions was proved in **Theorem 3** in [19].

Theorem 27 ([19]). *Let $P(y_0, y_1, \dots y_k, t, v)$ be a polynomial in $k+3$ variables with coefficients in field $\mathbb{K}$. Consider a function*

$$F(u) = F_0(u) + tP(F(u), \Delta F(u), \Delta^{(2)}F(u), \dots \Delta^{(k)}F(u), t, u), \tag{12.2.6}$$

where $F_0(u)$ is a polynomial in u and

$$\Delta F(u) = \frac{F(u) - F(0)}{u} \tag{12.2.7}$$

and $\Delta^{(i)}$ is obtained by applying Δ i times

$$\Delta^{(i)}F(u) = \frac{F(u) - F(0) - uF' \dots - \frac{u^{i-1}}{(i-1)!}F^{(i-1)}}{u^i}. \tag{12.2.8}$$

Then the formal power series $F(u)$ is algebraic over $\mathbb{K}(u, t)$.

Proof. One can directly go to [19] to see the detailed proof. The basic idea is to construct a Vandermonde determinant so that the second condition of Proposition 3 is satisfied. By Proposition 3, we can prove all the solutions $F(u)$ and $\Delta^{(i)}F(u)$ are algebraic over $\mathbb{K}(u,t)$. The solutions of $\Delta^{(i)}F(u)$ provide equations for $F^{(i)}$ and these equations form another Vandermonde matrix. Thus we can solve $F^{(i)}$ and they are algebraic over $\mathbb{K}(u,t)$. □

Remark 10. *The form of* (12.2.6) *arises with Kreweras, reverse Kreweras and Gessel's walks without interactions. The equations of some D-finite and D-algebraic cases also have this form. Normally it is not hard to get an equation in the form of* (12.2.6) *if it is a model generated by the recurrence relation of one variable x (for example, directed lattice paths [2]). However, for the two-variable quarter-plane lattice walk models,* (12.2.6) *does not always work for an equation derived by the kernel method. For example, although we know reverse Kreweras walks are algebraic, the unknowns we have in* (8.2.33) *and* (8.2.34) *are $Q(0,0), Q_{1,0}, Q_{0,1}$ and these do not satisfy* (12.2.7).

The basic problem when applying the general strategy of Bousquet-Mélou and Proposition 3 is whether we can always find enough roots for (12.2.1). If we can, then we may assert that $Q(0,0)$ is D-algebraic over t. With the help of Proposition 3, Theorem 23 and the general strategy, we obtain the following theorem,

Theorem 28. *Let $N(f)$ be as defined in* (11.1.33).

1. *If $N(f) = 1$, $Q(0,0)$ is D-algebraic.*
2. *If $N(f) = 1$, and further the full orbit-sum*[4] *does not contain $Q(0,0)$, then $Q(0,0)$ is D-finite.*
3. *If $N(f) = 1$, and further the full orbit-sum equals 0, then $Q(0,0)$ is algebraic.*

We demonstrate how to apply Proposition 3 first in Theorem 28. That is, how to find the $f_i(\mathbf{x})$ in Proposition 3. Then later in Theorem 29, we will prove the property of the field $\mathbb{K}(\mathbf{x})$.

Proof. We still use the solution form (11.1.36). Now w, w_1, w_2 contain $Q(0,0)$ as an unknown. We want to apply Proposition 3 to show that the solution is separable algebraic over the field generated by the coefficients. The advantage of (11.1.36) is that it only has $Q(0,0)$ as an unknown function of t so we only need to find one equation to satisfy the first condition in Proposition 3. Now we apply the idea of the kernel method approach. (11.1.36) is in a kernel method form. In the kernel method approach, the LHS is $Q(x,0)$ $(\pi(x))$ and we substitute the roots of the denominator of the RHS into the numerator and let it equal 0. This implies that we need to find an x_0 such that,

- x_0 is a formal series of t.
- x_0 is a singularity of RHS of (11.1.36) (A root of the denominator of the RHS).

By the kernel method, if x_0 is a pth order singularity satisfying these two conditions, we have,

$$\lim_{x\to x_0} (x - x_0)^p \left(w_1 + w_2 + \frac{w}{r} \right) = 0. \tag{12.2.9}$$

The solution is algebraic over the field $\mathbb{K}$ generated by the coefficients in (11.1.36).

So what we need to do is to prove that this x_0 always exists. To do this, first notice $Q(x,0)$ is a meromorphic function on the torus $K(x(s), y(s)) = 0$[5]. It has at least one singularity,

[4] See (5.4.1) for the definition.

[5] See Chapter 11.

otherwise it would be a constant of x. So we at least have x_1 as a singularity. We further ask where these singularities come from. By the explicit solution form (11.1.36), there are four places that the singularities may appear:

1. First, $\prod_{i=1}^{k} \delta^i(f), k \in \{1, 2 \dots n\}$, which appears in denominator of w_1 and w_2. This term is unchanged by δ.

2. Second, $\tilde{\Phi}$.

3. Third, w.

4. Fourth, r.

The first, third and fourth terms are algebraic with coefficients in $\mathbb{C}(t)$. If they have singularities, their singularities are roots of some algebraic functions of x with coefficients in $\mathbb{C}(t)$. $\tilde{\Phi}$ is an integral of an algebraic function. The singularities of $\tilde{\Phi}$ are also algebraic roots. Thus x_1 is algebraic over t. We further need to prove we have a singularity that is a formal series of t.

Let us consider r first. Since r is defined as $f = r/\delta(r)$, the zeros of r are also the zeros of f. Then the singularities of the fourth case are contained in the first case. We discuss them together with the first case.

The first and third cases have the property $\delta(w) = w$. This implies that if x is a singularity of w, $\delta(x)$ is as well. By Theorem 6, Y_0 and Y_1, X_0 and X_1 have opposite valuations in t. By the discussion after Theorem 23, δ maps a point outside the disc $|x| < 1$ to a point inside the disc. Thus if the singularity x_1 has valuation -1 in t, then $\delta(x_1)$ is a formal power series of t. So if x_1 comes from the first or third cases, either x_1 is a formal series of t or $\delta(x_1)$ is.

For the second case, $\tilde{\Phi} = \frac{\omega_1}{2\pi i}\zeta(\omega;\omega_1,\omega_3) - \frac{\omega}{\pi i}\zeta(\frac{\omega}{2};\omega_1,\omega_3)$. The ζ function has the following property [4],

$$\zeta(\omega+\omega_i;\omega_1,\omega_3) = \zeta(\omega;\omega_1,\omega_3) + 2\zeta\left(\frac{\omega_i}{2};\omega_1,\omega_3\right), \qquad i \in \{1,3\}. \tag{12.2.10}$$

If x is a singularity of $\zeta(\omega;\omega_1,\omega_3)$, suppose it is the point $\omega_k = \wp^{-1}(x)$. By (12.2.10),

$$\zeta(\omega_k;\omega_1,\omega_3) = \zeta(\omega_k+\omega_3;\omega_1,\omega_3) - 2\zeta\left(\frac{\omega_3}{2};\omega_1,\omega_3\right). \tag{12.2.11}$$

If ω_k is the singularity of $\zeta(\omega_k;\omega_1,\omega_3)$, $\omega_k+\omega_3$ is the singularity of $\zeta(\omega_k+\omega_3;\omega_1,\omega_3)$. This implies $\delta(x(\omega)) = x(\omega+\omega_3)$ is also a singularity. Besides by the definition of $\zeta(\omega;\omega_1,\omega_3)$,

$$\zeta(\omega;\omega_1,\omega_3) = \frac{1}{\omega} + \sum_{m,n\neq 0}\left(\frac{1}{\omega - m\omega_1 - n\omega_3} + \frac{1}{m\omega_1+n\omega_3} + \frac{\omega}{(m\omega_1+n\omega_3)^2}\right). \tag{12.2.12}$$

Any singularity of $\zeta(\frac{\omega}{2};\omega_1,\omega_3)$ is also a singularity of $\zeta(\omega;\omega_1,\omega_3)$. The singularities of $\zeta(\omega;\omega_1,\omega_3)$ are actually the same as the singularities of $\wp(\omega;\omega_1,\omega_3)$. Thus, if x is a singularity of $\tilde{\Phi}$, $\delta(x)$ is also a singularity of $\tilde{\Phi}$. One of them must be formal series of t.

We have proved that if the singularity x_1 is not a formal series of t, we may choose $x_0 = \delta(x_1)$ to be the required singularity. Now we can use (12.2.9) to solve the equation. The solution is algebraic over the coefficients in the equation. We will later prove that all coefficients in (11.1.36) are D-finite over t in Theorem 29. Then we conclude $Q(0,0)$ is D-algebraic. Further, if the coefficients of $\tilde{\Phi}$ in w_2 do not contain $Q(0,0)$, then the coefficients of $Q(0,0)$ in (12.2.9) are polynomials and $Q(0,0)$ is D-finite by direct calculation. $\square$

With the help of Theorem 26 and Theorem 28, we can have a better understanding of Gessel's walk. By the full orbit-sum (10.1.16), we know that $N(f) = 1$. Thus, $Q(x, 0)$ is D-finite and $Q(0, 0)$ is D-algebraic. Further when $b = 1$, the full orbit-sum equals 0. In that case, $Q(x, 0)$ and $Q(0, 0)$ are algebraic. Here are the results of Theorem 26 and Theorem 28 for $N(f) = 1$ cases.

	Model	x, y	t	DF Condition	Alg Condition	Group
1	$\uparrow\leftarrow\downarrow\rightarrow$	DF	DAlg	$abc + ab - ac - bc = 0$ or $a = b = 1$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
2	$\nwarrow\swarrow\searrow\nearrow$	DF	DAlg	$c = ab$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
3	$\uparrow\nwarrow\swarrow\downarrow\searrow\nearrow$	DF	DAlg	$c = ab$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
4	$\uparrow\nwarrow\leftarrow\swarrow\downarrow\searrow\nearrow$	DF	DAlg	$c = ab$ and $1 - a - b - ab = 0$	–	4: $(\frac{1}{x}, y), (x, \frac{1}{y})$
17	$\uparrow\leftarrow\searrow$	DF	DAlg	$ab - ac - bc + abc = 0$	–	6: $(\frac{y}{x}, y), (x, \frac{x}{y})$
19	$\leftarrow\downarrow\nearrow$	DF	DAlg	$b + (-1 + b)c = 0$	$a = b$	6: $(\frac{1}{xy}, y), (x, \frac{1}{xy})$
20	$\uparrow\swarrow\rightarrow$	Alg	Alg	–	All	6: $(\frac{1}{xy}, y), (x, \frac{1}{xy})$
22	$\nwarrow\leftarrow\searrow\rightarrow$	DF	DAlg	$ab - ac - bc + abc = 0$ or $b = 1$	–	8: $(\frac{y}{x}, y), (x, \frac{x^2}{y})$
23	$\leftarrow\swarrow\rightarrow\nearrow$	DF	DAlg	–	$b = 1$	8: $(\frac{1}{xy}, y), (x, \frac{1}{x^2 y})$

Table 12.1: The DF condition and Alg Condition refer to the conditions under which $Q(0, 0)$ becomes D-finite or D-algebraic. There are some differences between this table and the tables in Chapter 8. Here, we can only get the $Q(0, 0)$ is D-algebraic for Kreweras walk (**Model 19**), but in Chapter 8, we solve the model and prove that $Q(0, 0)$ is actually algebraic. For Gessel's walk, we get the result that $Q(x, 0)$ is D-finite and $Q(0, 0)$ is D-algebraic, which is an improvement.

Theorem 29. *If we accept the solution of $Q(x, 0)$ in the form of (11.1.36) and further require $Q(x, 0)$ to be a formal series of t, then there exist w_1, w_2, w, r, such that the following three conditions are satisfied.*

1. *w_1, the coefficient of $\tilde{\Phi}$ in w_2, and r are algebraic over t.*
2. *$\tilde{\Phi}$ is D-finite over t.*
3. *w is D-finite over t.*

Note that we treat $Q(0, 0)$ as a constant coefficient here.

Proof. The first condition is satisfied by the definition of w_1, w_2 and r in the derivation of (11.1.36). When we derive (11.1.36), we ignored the condition that $Q(x, 0)$ must be a formal series of t. We further make $Q(x, 0)$ a formal series of t by the third term $\frac{w}{r}$ where w is an algebraic function with $w = \delta(w)$ (see below for further conditions on w). By definition, w_1, the coefficient of $\tilde{\Phi}$ in w_2 and r are rational functions of x, y, t. Since y is algebraic over x, t, these terms are algebraic over t.

In the second condition, $\tilde{\Phi}$ is expressed by $\zeta(\omega, \omega_1, \omega_3)$ and $\zeta(\frac{\omega}{2}, \omega_1, \omega_3)$. The derivatives of these are $\wp(\omega; \omega_1, \omega_3)$ and $\frac{1}{2}\wp(\frac{\omega}{2}; \omega_1, \omega_3)$. We are going to prove that $\wp(\omega; \omega_1, \omega_3)$ and $\wp(\frac{\omega}{2}; \omega_1, \omega_3)$ are algebraic over t and then $\zeta(\omega)$ will be D-finite.

We first consider $\wp(\omega, \omega_1, \omega_3)$. Since we are talking about lattice walks associated with a finite group, $(\psi\phi)^n = \delta^n = 1$. By Lemma 25, $\delta(\omega) = (\omega + \omega_3)$ and $\omega_3 = \frac{m}{n}\omega_2$ when the group is finite. We can write $\wp(\omega;\omega_1,\omega_3)$ as a rational function of $\wp(\omega;\omega_1,\omega_2)$. To do this, we first write $\wp(w;\omega_1,\frac{\omega_2}{n})$ as a function of $\wp(w)$. **Remark 16** in [47] gives an explicit expression

$$\wp\left(\omega;\omega_1,\frac{\omega_2}{n}\right)=\wp(\omega)+\sum_{h=1}^{n-1}\left[\wp\left(\omega+\frac{h\omega_2}{n}\right)-\wp\left(\frac{h\omega_2}{n}\right)\right], \tag{12.2.13}$$

where $\wp\left(\omega+\frac{h\omega_2}{n}\right)$ can be expressed by $\wp(\omega)$ and $\wp(\frac{h\omega_2}{n}),\wp'(\frac{h\omega_2}{n})$ via the addition formula (12.2.19) [4]. $\wp'(\frac{h\omega_2}{n})$ can further be written as an algebraic function of $\wp(\frac{h\omega_2}{n})$. So (12.2.13) expresses $\wp(\omega;\omega_1,\frac{\omega_2}{n})$ by an algebraic function of $\wp(\omega)$ and its special values $\wp(\frac{h\omega_2}{n})$.

We apply (12.2.13) in several ways. First, we can write $\wp(\omega;\omega_1,\omega_2)$ as a function of $\wp(\omega;\omega_1,m\omega_2)$ and $\wp(e_i;\omega_1,m\omega_2)$. We denote e_i,e_j,e_l for different special values (we do not specify e_i,e_j,e_l). Then we can write $\wp(\omega;\omega_1,\omega_3)=\wp(\omega;\omega_1,\frac{m}{n}\omega_2)$ as a function of $\wp(\omega;\omega_1,m\omega_2)$ and $\wp(e_j;\omega_1,m\omega_2)$. Thus, $\wp(\omega;\omega_1,\omega_3)$ is an algebraic function of $\wp(\omega;\omega_1,m\omega_2)$ and $\wp(e_l;\omega_1,m\omega_2)$. Since we are talking about t, we still need to prove all $\wp(e_l;\omega_1,m\omega_2)$ are algebraic over t.

Next, we try to express $\wp(e_i;\omega_1,m\omega_2)$ as functions of $\wp(e_j;\omega_1,\omega_2)$. To achieve this, we first consider $\wp(e_i;m\omega_1,m\omega_2)$ By the standard definition of $\wp(\omega;\omega_1,\omega_2)$,

$$\wp(\omega)=\frac{1}{\omega^2}+\sum_{(p,q)\neq(0,0)}\left[\frac{1}{(\omega-p\omega_1-q\omega_2)^2}-\frac{1}{(p\omega_1+q\omega_2)^2}\right]. \tag{12.2.14}$$

We have

$$\wp(s\omega)=\frac{1}{s^2\omega^2}+\sum_{(p,q)\neq(0,0)}\left[\frac{1}{(s\omega-p\omega_1-q\omega_2)^2}-\frac{1}{(p\omega_1+q\omega_2)^2}\right]=\frac{1}{s^2}\wp\left(\omega;\frac{\omega_1}{s},\frac{\omega_2}{s}\right). \tag{12.2.15}$$

Thus, for any e_i, $\wp(e_i,\omega_1,\omega_2)=m^2\wp(me_i;m\omega_1,m\omega_2)$. For the relation between $\wp(e_i;\omega_1,m\omega_2)$ and $r^2\wp(e_j;m\omega_1,m\omega_2)$ we again use (12.2.13). This time we use the corresponding $\frac{\omega_1}{n}$ form:

$$\wp(\omega;\frac{\omega_1}{n},\omega_2)=\wp(\omega)+\sum_{h=1}^{n-1}\left[\wp\left(\omega+\frac{h\omega_1}{n}\right)-\wp\left(\frac{h\omega_1}{n}\right)\right]. \tag{12.2.16}$$

Thus, we can express $\wp(\omega;\omega_1,m\omega_2)$ as function of $\wp(\omega;m\omega_1,m\omega_2)$ and $\wp(e_i;m\omega_1,m\omega_2)$. We substitute e_j in to (12.2.16), and have that $\wp(e_i;\omega_1,m\omega_2)$ is an algebraic function of $\wp(e_l;m\omega_1,m\omega_2)$. Again by (12.2.15), we know $\wp(e_l;m\omega_1,m\omega_2)=\frac{1}{m^2}\wp(\frac{e_l}{m};\omega_1,\omega_2)$. We finally change all the coefficients into values of $\wp$ functions with periods ω_1,ω_2. So our problem becomes whether these specific values of $\wp(\omega)$ are algebraic or not.

Notice that e_i is written as $\alpha\omega_1+\beta\omega_2$ where α and β are rational numbers. We prove that any value with the form $\wp(\alpha\omega_1+\beta\omega_2)$ is algebraic over t. First, by the property of Weierstrass $\wp$-function, we know $\wp(\omega)$ satisfies the equation

$$\wp'(\omega)=4\wp(\omega)^3-g_2\wp(\omega)-g_3. \tag{12.2.17}$$

And the roots of

$$4\wp(\omega)^3-g_2\wp(\omega)-g_3=0 \tag{12.2.18}$$

are $\wp(\frac{1}{2}\omega_1)$, $\wp(\frac{1}{2}\omega_2)$, $\wp(\frac{1}{2}(\omega_1+\omega_2))$. By the uniformization relations (11.1.19) and (11.1.20), we know (12.2.18) corresponds to the discriminant of kernel $K(x,y)$, thus g_2 and g_3 are algebraic over t. The roots are also algebraic. By the addition theorem of $\wp$-functions [4],

$$\wp(u+v)=\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2-\wp(u)-\wp(v), \tag{12.2.19}$$

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(u+v) & -\wp'(u+v) \end{vmatrix} = 0, \tag{12.2.20}$$

and

$$\begin{aligned}
\wp(2\omega) &= -2\wp(\omega) + \left[\frac{\wp''(\omega)}{\wp'(\omega)}\right]^2 \\
\wp\left(\frac{\omega}{2}\right) &= \wp(\omega) + \left(\wp(\omega) - \wp\left(\frac{1}{2}\omega_1\right)\right)^{\frac{1}{2}} \left(\wp(\omega) - \wp\left(\frac{1}{2}\omega_2\right)\right)^{\frac{1}{2}} \\
&\quad + \left(\wp(\omega) - \wp\left(\frac{1}{2}\omega_1\right)\right)^{\frac{1}{2}} \left(\wp(\omega) - \wp\left(\frac{1}{2}(\omega_1+\omega_2)\right)\right)^{\frac{1}{2}} \\
&\quad + \left(\wp(\omega) - \wp\left(\frac{1}{2}\omega_2\right)\right)^{\frac{1}{2}} \left(\wp(\omega) - \wp\left(\frac{1}{2}(\omega_1+\omega_2)\right)\right)^{\frac{1}{2}}
\end{aligned} \tag{12.2.21}$$

We can express $\wp(\alpha\omega_1 + \beta\omega_2)$ as an algebraic function of $\wp(\frac{\omega_1}{2})$, $\wp(\frac{\omega_1}{2})$, $\wp(\frac{1}{2}(\omega_1+\omega_2))$ and their derivatives. Thus, they are algebraic over t.

Now, we have proved $\wp(\omega;\omega_1,\omega_3)$ can be expressed as an algebraic function of $\wp(\omega)$ and $\wp(\alpha\omega_1+\beta\omega_2)$. And we have proved $\wp(\alpha\omega_1+\beta\omega_2)$ are algebraic functions of t. Thus $\wp(\omega;\omega_1,\omega_3)$ is algebraic over x, t. This means $\zeta(\omega,\omega_1,\omega_3)$ is D-finite over x, t and further has algebraic singularities. A similar proof can be applied to $\zeta(\frac{\omega}{2};\omega_1,\omega_3)$. Thus, the second condition is proved.

Now comes the final term w. This is an arbitrary function satisfying

$$\begin{aligned} \delta(w) &= w \\ \psi(w) &= w. \end{aligned} \tag{12.2.22}$$

By the definition of ψ and δ, the first equation means w is a rational function of $\wp(\omega;\omega_1,\omega_3)$ and $\wp'(\omega;\omega_1,\omega_3)$. The coefficients are arbitrary functions of t. We have proved $\wp(\omega;\omega_1,\omega_3)$ is algebraic over x, t. Thus, w is an algebraic function over x. Without any further requirements, w could be any kind of function of t since the coefficients can be chosen arbitrarily. We can choose coefficients that are algebraic over t and make w algebraic. However, since we require π (that is, $Q(x,0)$) to be a formal series of x and t in some small disc around $t = 0$, these coefficients cannot be chosen arbitrarily.

To find a suitable w, we expand the RHS of (11.1.36) as a series of t in some suitable annulus. Since w_1 and w_2 have singularities of both formal series of t and series with $\frac{1}{t}$, we can separate w_1, w_2 into $[x^<]$ and $[x^\geq]$ parts using partial fractions (similar to the idea of Theorem 6). Since both w_1 and w_2 have $[x^<]$ parts if we expand them as formal series of t, the coefficients in $\frac{w}{r}$ should be chosen such that $\frac{w}{r}$ cancels the $[x^<](w_1+w_2)$.

Let us write $\frac{w}{r}$ as $[x^\geq]\frac{w}{r} + [x^<]\frac{w}{r}$ in the same annulus. To satisfy the condition that $Q(x,0)$ is a formal series of x, $[x^<]\frac{w}{r} = -[x^<](w_1+w_2)$. We have proved that (w_1+w_2) is D-finite over t and their singularities are both algebraic over t. So $[x^<]\frac{w}{r}$ also has this property. In $[x^<]\frac{w}{r}$, both w and r provide singularities. We have proved that r is algebraic in x, t and it provides algebraic singularities. So w must also provide algebraic singularities for $[x^<]\frac{w}{r}$. We have $\delta(w) = w$. Since δ maps a singularity which is a series in t with valuation 1 to a singularity of series in t with valuation -1 (see the discussion after Theorem 23), the singularities of $\delta([x^\geq]\frac{w}{r})$ are also algebraic over t. Since the coefficients in $[x^<](w_1+w_2)$ are D-finite, we can choose suitable coefficients in w independent of the singularities and make w D-finite over t. □

Remark 11. *Theorem 28 says when the full orbit sum equals 0, $Q(0,0)$ is algebraic. But we do not reject the possibility that $Q(0,0)$ may be algebraic when the full orbit-sum does not equal 0. Our result in Theorem 29 is that w_1 is algebraic and w_2, w are D-finite. Following the general strategy of the kernel method, if we substitute the root x_0 in the numerator of (11.1.36), the non-algebraic terms may be cancelled in the calculation. This may lead to the case of Kreweras walks, where $Q(0,0)$ is algebraic but the full orbit-sum is not 0 and the generating function $Q(x,0)$ is D-finite.*

Following the same idea in Theorem 29, we can also prove that ρ, S and $Tr(\chi)$ in (11.1.41) are algebraic over t. We know that R has algebraic roots. To make $Q(x,0)$ a formal series of x, t, we still want the $[x^<]e^{\int(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)d\omega}$ to cancel the $[x^<]$ part of ρ. Since $[x^<]\rho$ is D-finite with algebraic singularities over t, $[x^<]e^{\int(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)d\omega}$ is also D-finite with algebraic singularities over t. The coefficient $[x^i]e^{\int(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)d\omega}$ can be calculated by first expanding $\frac{1}{n}Tr(\chi)\tilde{\Phi} + S - R$ into a series and then taking the exponent. The final expression depends on the singularities of $\frac{1}{n}Tr(\chi)\tilde{\Phi} + S - R$. Since all singularities are algebraic roots, the final expression can only be a D-finite function of x, t and the coefficients in the integral. These coefficients should also be D-finite since $[x^<]e^{\int(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)d\omega}$ itself is D-finite. $Tr(\chi)$ and S are defined as D-finite functions. This implies R also has D-finite coefficients otherwise $[x^{-i}]e^{\int(\frac{1}{n}Tr(\chi)\tilde{\Phi}+S-R)d\omega}$ cannot cancel $[x^{-i}]\rho$. But unfortunately, we cannot apply Proposition 3 since (11.1.41) is no longer a polynomial. We can still use the general strategy to solve it. The equation may look like

$$P(Q(0,0),t) + [x^>]e^{H(Q(0,0),t)} = 0, \tag{12.2.23}$$

where P, H are two D-finite functions. Clearly $Q(0,0)$ may not be algebraic over the coefficients. It may still be D-algebraic over t, but we cannot find the D-finite but not algebraic terms and cannot find the differential equation. However, in Chapter 8, for **Models 5** to **16** and **Model 18**, we directly find that $Q(0,0)$ is related to T_1, T_{00}, T_2, T_3 (see definitions in Section 8.3). We can definitely prove that they are D-algebraic and find the differential equation by the kernel method.

12.3 An extension of the general strategy

The solution of $Q(x,0)$ in the form (11.1.36) is powerful. With the help of (11.1.36), we can prove the D-algebraic, D-finite, algebraic properties of $Q(x,y)$, $Q(x,0)$ and $Q(0,0)$. The disadvantage of this solution form is that, we cannot directly find the D-finite terms in the solution. These D-finite but not algebraic terms provide singularities that describe the asymptotic behavior. In this section, we go back to results got by the kernel method approach in Chapter 8. The advantage of the kernel method is we can find these terms directly. We can find some equations of $Q(0,0)$ where each coefficient is known. Nevertheless, we sometimes get more than one unknown in these equations and the equations may not be solvable. Here, we face this problem.

Theorem 27 was based on the fact that we obtained (12.2.5) from one polynomial equation (12.2.1). However, when solving Kreweras and reverse Kreweras walks, we met a similar but slightly different situation. Observe that an irreducible equation of the form

$$\Lambda_F \Lambda_{FG} F(x) = \Lambda_{FG}\Lambda_G \sqrt{\Delta} G\left(\frac{1}{x}\right) + \sqrt{\Delta}A(x,\mathbf{U}) + B(x,\mathbf{U}) \tag{12.3.1}$$

arose (namely, (8.2.20) and (8.2.66)). Here,

(i) $F(x)$ and $G\left(\frac{1}{x}\right)$ are series in t whose coefficients are polynomials in x and $\frac{1}{x}$ respectively;

(ii) $\mathbf{U} = U_1, U_2, \ldots, U_k$. They are unknowns independent of x. $A(x, \mathbf{U})$ and $B(x, \mathbf{U})$ are linear combinations $U_1, \ldots, U_k$. For reverse Kreweras walks, the coefficients are in $\mathbb{R}(t, x)$. For Kreweras walks, the coefficients can be D-finite over x and t;

(iii) $\Lambda_F = \lambda_1 \ldots \lambda_\alpha$, $\Lambda_{FG} = \lambda_{\alpha+1} \ldots \lambda_\beta$, and $\Lambda_G = \lambda_{\beta+1} \ldots \lambda_\gamma$, where each λ_i is an irreducible quadratic function with coefficients in $\mathbb{C}(t)$, and each has one root which is a series of t and one root which is a series with a $\frac{1}{t}$ term. This property holds since the equation should be invariant under $\psi : x \to \frac{1}{x}$. Thus for each quadratic factors, ψ permutes the roots. If one is a series of t, the other should be a series with $\frac{1}{t}$.

(iv) Δ is a Laurent polynomial in $\mathbb{C}[t, x, \frac{1}{x}]$, which becomes a polynomial in t, x after multiplying by some power of x to clear the $\frac{1}{x}$ factors. This polynomial is either x times an irreducible cubic polynomial, or an irreducible quartic polynomial, or a product of two irreducible quadratic polynomials in x with different roots. It has two roots which are formal series of t without constant terms (one of which may be 0) and two roots which are series with a $\frac{1}{t}$ term. We denote these roots as X_1, X_2, X_3, X_4. The canonical factorization of $\sqrt{\Delta}$ can be written as $\sqrt{\Delta_0}\sqrt{\Delta_-}\sqrt{\Delta_+}$, where $\sqrt{\Delta_0}$, $\sqrt{\Delta_-}$, $\sqrt{\Delta_+}$ are defined as in Lemma 7.

For both models we then obtained two kernel-like equations by taking either the positive or negative part of (12.3.1) with respect to x. For reverse Kreweras this led to exactly three linearly independent equations in U_i. And since we have three unknown series, the kernel method gave the solution. For Kreweras we obtained four independent equations. We initially had four unknowns, but a fifth boundary term emerged when taking the positive and negative parts. It was only because we already knew the (algebraic) solution to one of the U_i that we were able to solve the system. This phenomenon also appears in many D-algebraic models.

By the canonical factorization (Lemma 7), we may divide (12.3.1) by $\sqrt{\Delta_+}$ and extract the non-negative and negative parts of each term with respect to x. This gives two equations

$$\Lambda_F \Lambda_{FG} \frac{F(x)}{\sqrt{\Delta_+}} = G_+(x) + A_+(x) + \frac{B(x, \ldots)}{\sqrt{\Delta_+}} \tag{12.3.2}$$

$$0 = \Lambda_{FG}\Lambda_G \sqrt{\Delta_0\Delta_-} G\left(\frac{1}{x}\right) - G_+(x) + \sqrt{\Delta_0\Delta_-} A(x, \mathbf{U}) - A_+(x, \mathbf{U}), \tag{12.3.3}$$

where

- $G_+(x)$ is the $[x^{\geq}]$ of $\Lambda_{FG}\Lambda_G\sqrt{\Delta_0\Delta_-}G(\frac{1}{x})$ – this can be expressed as a linear combination of boundary terms of $G(\frac{1}{x})$ with polynomial coefficients; and
- $A_+(x)$ is the $[x^{\geq}]$ of $\sqrt{\Delta_0\Delta_-}A(x, \ldots)$ – this is a linear combination of the unknowns U_i with polynomial coefficients.

Note that $G_+(x)$ may contain new unknowns. We put these unknowns together with the U_i. Assuming $A(x, \ldots)$ and $B(x, \ldots)$ are polynomials in x, t (for more general cases, see Remark 12), it will be useful to write

$$\begin{aligned} A(x, \mathbf{U}) &= (1, x, \ldots, x^m)\mathbf{M}_A(U_1, \ldots, U_k)^\mathsf{T} \\ B(x, \mathbf{U}) &= (1, x, \ldots, x^m)\mathbf{M}_B(U_1, \ldots, U_k)^\mathsf{T}, \end{aligned} \tag{12.3.4}$$

where k is the number of total unknowns after separation and $\mathbf{M}_A$ and $\mathbf{M}_B$ are matrices which are independent of x.

We may conclude the difference of the case discussed above in Section 12.3 with the condition in Theorem 27.

- We get the equations of (12.2.3) from two related equations (12.3.2) and (12.3.3), which are the positive and negative parts of (12.3.1).

- The unknowns $Q(0,0), Q_{1,0}, Q_{0,1}$ are not in the form of (12.2.8). The coefficients of $F^{(j)}, j \in \{0, 1, \dots i-1\}$ in (12.2.8) have degree $u^{-(i-j+1)}$ in u. We may construct a Vandermonde determinant from the coefficients of these $F^{(j)}$ in the equations of $\Delta^{(i)}F(u)$ and it is not 0. In our case, the coefficients of $Q_{1,0}$ and $Q_{0,1}$ have the same lowest degree in x since they come from taking $[x^{>}]$ terms of $Q(x,0)$ and $Q(0,x)$. By the same construction as (12.2.7), the determinant is not Vandermonde and it may be 0.

We want to extend Theorem 27 to our cases. Similar to Theorem 27, we give some conditions on the form of (12.3.1), (12.3.2) and (12.3.3).

Theorem 30. *Suppose equations (12.3.1), (12.3.2) and (12.3.3) satisfy conditions we discussed above, and in addition*

- *The number of unknowns U_i in (12.3.2) and (12.3.3) is k and the number of quadratic factors is γ.*

- *There is no common vector in the right null-spaces of $\mathbf{M}_A$ and $\mathbf{M}_B$.*

- *In (12.3.4), $(1, x, \dots, x^m)\mathbf{M}_A$ does not contain λ_i for $i \in [1, \beta]$ as a factor and similarly $(1, x, \dots, x^m)\mathbf{M}_B$ does not contain λ_j for $j \in [\alpha+1, \gamma]$ as a factor. This condition means we* cannot *write*

$$B(x, \dots) - \Lambda_F \Lambda_{FG} F(x) = \Lambda \left(P(x)F(x) - \sum_i R_i(x)U_i \right), \tag{12.3.5}$$

where Λ is a product of some λ_j, $j \in [1, \beta]$, and $P(x)$ and $R_i(x)$ are polynomials. Similarly, we cannot write

$$\Lambda_{FG}\Lambda_G\sqrt{\Delta}G\left(\frac{1}{x}\right) + \sqrt{\Delta}A(x, \mathbf{U}) = \sqrt{\Delta}\Lambda' \left(P'(x)G\left(\frac{1}{x}\right) - \sum_i R'_i(x)U_i \right), \tag{12.3.6}$$

where Λ' is a product of some λ_j, $j \in [\alpha+1, \gamma]$, and $P'(x)$ and $R'_i(x)$ are polynomials. For problems where this condition fails and (12.3.5) or (12.3.6) is true, we can rewrite $(P(x)F(x) - \sum_i R_i(x)U_i)$ or $\left(P'(x)G\left(\frac{1}{x}\right) - \sum_i R'_i(x)U_i\right)$ as $F_2(x)$ or $G_2(\frac{1}{x})$ and then consider the new equation.

- *Each root of λ_j, $j \in [1, \gamma]$, or a non-zero root of a factor $f(x)$ of Δ (over $\mathbb{C}(t)$) cannot be expressed by a rational function of roots of other factors. Mathematically speaking,*

$$x_j^{\pm} \notin \mathbb{C}(x_1^{\pm}, x_2^{\pm}, \dots, x_{j-1}^{\pm}, x_{j+1}^{\pm}, \dots, x_{\gamma}^{\pm}, X_1, X_2, X_3, X_4) \tag{12.3.7}$$

and

$$X_j \notin \mathbb{C}(x_1^{\pm}, x_2^{\pm}, \dots, x_{\gamma}^{\pm}, \mathcal{X}), \tag{12.3.8}$$

where $\mathcal{X}$ is the set of X_i, $i \in [1, 4]$, such that $f(X_i) \neq 0$.

Then if $k > \gamma$, the number of linearly independent equations obtained from (12.3.1) is at least γ. When $k \leq \gamma$, any set of k equations obtained is linearly independent, and thus we can solve $U_1, \dots, U_k$ and the solutions are algebraic.

Proof. Denote the roots of the quadratic factor λ_i by $x_i^{\pm}$, so that x_i^+ and $1/x_i^-$ are power series in t. Substituting x_i^+ into (12.3.2) for $i \in [1, \beta]$ gives

$$0 = G_+(x_i^+) + A_+(x_i^+, \mathbf{U}) + \frac{B(x_i^+, \mathbf{U})}{\sqrt{\Delta_{+i}}} \tag{12.3.9}$$

and similarly substituting x_i^- into (12.3.3) for $i \in [\alpha + 1, \gamma]$ gives

$$0 = -G_+(x_i^-) + \sqrt{\Delta_0 \Delta_{-i}} A(x_i^-, \mathbf{U}) - A_+(x_i^-, \mathbf{U}). \tag{12.3.10}$$

Here we have denoted $\Delta_{+i} = \Delta_+(x_i^+)$ and $\Delta_{-i} = \Delta_-(x_i^-)$. If $\alpha < \beta$ (which means there are common factors in the coefficients of $F(x)$ and $G(\frac{1}{x})$), then both roots of λ_j can be used if $j \in [\alpha + 1, \beta]$. We may choose either of them. For simplicity of the proof, we assume we choose the x^+ roots.

Now, let us choose γ distinct equations from distinct factors. Put all these together as a matrix and write them as $\mathbf{C}(U_1, \ldots, U_k)^\mathsf{T} = \mathbf{D}$, where

$$\mathbf{C} = \begin{pmatrix} a_{1,1} + b_{1,1}/\sqrt{\Delta_{+1}} & \cdots & a_{1,k} + b_{1,k}/\sqrt{\Delta_{+1}} \\ & \vdots & \\ a_{\beta,1} + b_{\beta,1}/\sqrt{\Delta_{+\beta}} & \cdots & a_{\beta,k} + b_{\beta,k}/\sqrt{\Delta_{+\beta}} \\ a_{\beta+1,1} + b_{\beta+1,1}\sqrt{\Delta_{-(\beta+1)}} & \cdots & a_{\beta+1,k} + b_{\beta+1,k}\sqrt{\Delta_{-(\beta+1)}} \\ & \vdots & \\ a_{\gamma,1} + b_{\gamma,1}\sqrt{\Delta_{-\gamma}} & \cdots & a_{\gamma,1} + b_{\gamma,k}\sqrt{\Delta_{-\gamma}} \end{pmatrix}. \tag{12.3.11}$$

We may first assume $k \geq \gamma$. The coefficient $a_{i,j} + b_{i,j}/\sqrt{\Delta_{\pm i}}$ is the coefficient of U_j in the ith equation (note that for readability, $\sqrt{\Delta_0}$ has been absorbed into $b_{i,j}$).

We want to prove that the rank of $\mathbf{C}$ is γ. Let $\mathbf{C}'$ be a $\gamma \times \gamma$ submatrix of $\mathbf{C}$. We rewrite the line and column number as $1, 2, \ldots, \gamma$ and let S_γ be the set of permutations of $\{1, 2, \ldots, \gamma\}$.

$$\det(\mathbf{C}') = \sum_{\sigma \in S_\gamma} \operatorname{sgn}(\sigma) \prod_{i=1}^{\gamma} \mathbf{C}_{i,\sigma_i} \tag{12.3.12}$$

$$= \sum_{\sigma \in S_\gamma} \operatorname{sgn}(\sigma) \prod_{i=1}^{\beta} \left(a_{i,\sigma_i} + b_{i,\sigma_i}/\sqrt{\Delta_{+i}}\right) \prod_{j=\beta+1}^{\gamma} \left(a_{j,\sigma_j} + b_{j,\sigma_j}\sqrt{\Delta_{-j}}\right). \tag{12.3.13}$$

We now rewrite this sum by splitting it into those terms which contain a factor of $\sqrt{\Delta_{+i}}$ or $\sqrt{\Delta_{-j}}$ and those which do not. Let $\mathcal{P}$ be the set of all ordered pairs $(\mathcal{Q}, \mathcal{R})$ where $\mathcal{Q} \subseteq \{1, \ldots, \beta\}$, $\mathcal{R} \subseteq \{\beta + 1, \ldots, \gamma\}$ and $\mathcal{Q} \cup \mathcal{R}$ is not empty. For $\sigma \in S_\gamma$ and $(\mathcal{Q}, \mathcal{R}) \in \mathcal{P}$, define

$$C_{\sigma,\mathcal{Q},\mathcal{R}} = \prod_{q' \in \mathcal{Q}'} a_{q',\sigma_{q'}} \prod_{r' \in \mathcal{R}'} a_{r',\sigma_{r'}} \prod_{q \in \mathcal{Q}} b_{q,\sigma_q} \prod_{r \in \mathcal{R}} b_{r,\sigma_r}, \tag{12.3.14}$$

where $\mathcal{Q}' = \{1, \ldots, \beta\} \setminus \mathcal{Q}$ and $\mathcal{R}' = \{\beta + 1, \ldots, \gamma\} \setminus \mathcal{R}$, and

$$J_{\mathcal{Q},\mathcal{R}} = \frac{\prod_{r \in \mathcal{R}} \sqrt{\Delta_{-r}}}{\prod_{q \in \mathcal{Q}} \sqrt{\Delta_{+q}}}. \tag{12.3.15}$$

Then

$$\det(\mathbf{C}') = \sum_{\sigma \in S_\gamma} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{\gamma} a_{i,\sigma_i} + \sum_{(\mathcal{Q},\mathcal{R}) \in \mathcal{P}} C_{\sigma,\mathcal{Q},\mathcal{R}} J_{\mathcal{Q},\mathcal{R}}\right) \tag{12.3.16}$$

$$= \sum_{\sigma \in S_\gamma} \operatorname{sgn}(\sigma) \prod_{i=1}^{\gamma} a_{i,\sigma_i} + \sum_{(\mathcal{Q},\mathcal{R}) \in \mathcal{P}} C_{\mathcal{Q},\mathcal{R}} J_{\mathcal{Q},\mathcal{R}}, \tag{12.3.17}$$

where

$$C_{\mathcal{Q},\mathcal{R}} = \sum_{\sigma \in S_\gamma} \operatorname{sgn}(\sigma) C_{\sigma,\mathcal{Q},\mathcal{R}}. \tag{12.3.18}$$

Now, we prove that (12.3.17) is not 0.

First, recall from (5.2.18) and (5.2.19) that Δ_- is a product of factors $\sqrt{1 - X_i/x}$ where X_i is a non-zero root of Δ which converges when $t \to 0$, while Δ_+ is a product of factors $\sqrt{1 - x/X_i}$ where X_i diverges as $t \to 0$. By our fourth condition, X_i/x_j cannot be reduced. Furthermore, since we require that each x_j comes from a different factor λ_j, no reduction can occur between any pair of $\sqrt{\Delta_{-r}}$ and $\sqrt{\Delta_{+q}}$ in $J_{\mathcal{Q},\mathcal{R}}$.

For different $J_{\mathcal{Q},\mathcal{R}}$, they contain different factors $\frac{\prod_{r \in \mathcal{R}} \sqrt{\Delta_{-r}}}{\prod_{q \in \mathcal{Q}} \sqrt{\Delta_{+q}}}$. So all $J_{\mathcal{Q},\mathcal{R}}$ are linearly independent from each other over $\mathbb{C}(x,t)$. This means unless all $C_{\mathcal{Q},\mathcal{R}} = 0$, the second part of (12.3.17) does not belong to $\mathbb{R}(t, x_1, x_2, \ldots, x_\gamma, X_1, X_2, X_3)$, since we always have $\sqrt{\Delta_\pm}$ in it. Then it cannot be cancelled by the first term of (12.3.17). So $|\mathbf{C}'| \neq 0$.

It remains then to show that at least one $C_{\mathcal{Q},\mathcal{R}} \neq 0$. Consider the case $\mathcal{Q} = \{1, \ldots, \beta\}$ and $\mathcal{R} = \{\beta+1, \ldots, \gamma\}$. Then $C_{\mathcal{Q},\mathcal{R}} = \det(\mathbf{C}'')$ where

$$\mathbf{C}'' = \begin{pmatrix} b_{1,1} & \cdots & b_{1,\gamma} \\ & \vdots & \\ b_{\beta,1} & \cdots & b_{\beta,\gamma} \\ b_{\beta+1,1} & \cdots & b_{\beta+1,\gamma} \\ & \vdots & \\ b_{\gamma,1} & \cdots & b_{\gamma,\gamma} \end{pmatrix}. \tag{12.3.19}$$

From (12.3.9), (12.3.10) and (12.3.4), it follows that

$$\mathbf{C}'' = \begin{pmatrix} (1, x_1^+, \ldots, (x_1^+)^m)\mathbf{M}'_A \\ \vdots \\ (1, x_\beta^+, \ldots, (x_\beta^+)^m)\mathbf{M}'_A \\ (1, x_{\beta+1}^-, \ldots, (x_{\beta+1}^-)^m)\mathbf{M}'_B \\ \vdots \\ (1, x_\gamma^-, \ldots, (x_\gamma^-)^m)\mathbf{M}'_B \end{pmatrix} \tag{12.3.20}$$

where $\mathbf{M}'_A$ (resp. $\mathbf{M}'_B$) is the matrix formed by the chosen γ columns of $\mathbf{M}_A$ (resp. $\mathbf{M}_B$).

Notice that by the third condition, each row $(1, x_i^\pm, \ldots, (x_i^\pm)^m)\mathbf{M}'_{A \text{ or } B}$ cannot be the $\mathbf{0}$ vector itself. Otherwise, there is some $i \in [1, \beta]$ such that x_i is a root of $(1, x, \ldots, x^m)\mathbf{M}_A = 0$, or some $i \in [\beta+1, \gamma]$ such that x_i is a root of $(1, x, \ldots, x^m)\mathbf{M}_B = 0$. This contradicts our third condition. Thus all rows of $\mathbf{C}''$ are not $\mathbf{0}$.

Now suppose $\det(\mathbf{C}'') = 0$, then there exists a non-zero vector $\mathbf{v}$ such that $\mathbf{C}''\mathbf{v} = \mathbf{0}$. This implies

$$\begin{aligned} (1, x_i^+, \ldots, (x_i^+)^m)\mathbf{M}'_A\mathbf{v} &= 0, \\ (1, x_j^-, \ldots, (x_j^-)^m)\mathbf{M}'_B\mathbf{v} &= 0. \end{aligned} \tag{12.3.21}$$

Since x_i and x_j are from different field extensions, $\mathbf{v}$ cannot cancel them simultaneously. Thus, $\mathbf{M}'_A\mathbf{v} = \mathbf{0}$ and $\mathbf{M}'_B\mathbf{v} = \mathbf{0}$. Then $(\mathbf{v}, 0 \ldots 0)$ is in the common right nullspace of $\mathbf{M}_A, \mathbf{M}_B$. This is a contradiction of our second condition. Thus, $C_{\mathcal{Q},\mathcal{R}} = \det \mathbf{C}'' \neq 0$. We have found γ linearly independent equations. Notice that in the proof, we take the x^+ roots of λ_i, $i \in [\alpha+1, \beta]$. The

x^- roots of λ_i, $i \in [\alpha+1, \beta]$ may provide more linear independent equations. So the number of linearly independent equations is at least γ.

For the $k < \gamma$ case, we change the matrix from $\gamma \times \gamma$ to $k \times k$. Then we can get the result by a similar proof. $\square$

Remark 12. *Theorem 30 works for reverse Kreweras walks. For reverse Kreweras walks, we have rational $A(x)$ and $B(x)$. We can easily change them to polynomials and apply Theorem 30. We may also extend the proof to the cases that $A(x, \mathbf{U})$ and $B(x, \mathbf{U})$ are D-finite. In this case, $A(x, \mathbf{U})$ and $B(x, \mathbf{U})$ are written as*

$$\begin{aligned} A(x, \mathbf{U}) &= (1, x, \dots)\mathbf{M}_A(U_1, \dots, U_k)^{\mathsf{T}} \\ B(x, \mathbf{U}) &= (1, x, \dots)\mathbf{M}_B(U_1, \dots, U_k)^{\mathsf{T}} \end{aligned} \tag{12.3.22}$$

The proof will be exactly the same except $\mathbf{M}_A$ and $\mathbf{M}_B$ are $\infty \times k$ matrices in this case. For ***Models 5–16****, we have $e^{L_-(x,t)}e^{L_0(t)}e^{L_+(x,t)}$ in the coefficients of $G(\frac{1}{x})$ instead of Δ. In these cases, we can see $\frac{\Pi_{r\in\mathcal{R}} e^{L_-(x_r,t)}}{\Pi_{q\in\mathcal{Q}} e^{L_+(x_q,t)}}$ are all linear independent and our conclusion still holds.*

Chapter 13

Conclusion and perspective

13.1 Summary

In the first part of our work, we have worked through the solutions of 23 quarter-plane lattice walk models with interactions. These are the models associated with finite groups, and we solved them by the kernel method. For all these models except double Kreweras walks, we found linear or polynomial equations of x that the generating functions $Q(x,0)$ satisfy. The coefficients of these equations contain boundary terms $Q_{i,j}$. This implies the generating function $Q(x,0)$ is algebraic over the coefficients in the equation. For all except double Kreweras walks and Gessel's walks, we found the explicit results of all generating functions $Q(x,y), Q(x,0), Q(0,y), Q(0,0)$. For Gessel's walks, we obtained a polynomial equation of $Q(x,0)$ and point boundary terms $Q_{i,j}$. For double Kreweras walks, we found a relation between $Q(x,0)$ and $Q(0,x)$. We also found the specializations that turn the generating functions from D-algebraic to D-finite or D-finite to algebraic for all models.

The second part of our work is to discover the properties of the generating functions in a general view. We have introduced two analytic methods. One is based on homomorphisms on Riemann surfaces and it can be applied to all 23 models associated with finite groups. The other is based on Riemann boundary value problems and it is a unified way to solve all models. We proved the properties of all the 23 models via the solution of the analytic methods.

13.2 Why walks with interactions?

It is natural to assign weights to each step and make lattice walks a probabilistic model like that in Section 11.1. However, boundary interactions are different from these kinds of probabilities. Our discussion is mostly about quarter-plane lattice walks with interactions. In Chapter 1, we proposed that the reason for studying lattice walks with interactions is that they biject to polymer models in physics. In this section, we give a reason from a mathematical point of view.

Recall Theorem 8, for step set $\mathcal{S} \subseteq \{-1,0,1\} \times \{-1,0,1\}$,

$$K(x,y)Q(x,y) = \frac{1}{c} + \frac{1}{a}(a-1-taA(x,y))Q(x,0) + \frac{1}{b}(b-1-tbB(x,y))Q(0,y) + \left(\frac{1}{abc}(ac+bc-ab-abc) + tG(x,y)\right)Q(0,0) \quad (13.2.1)$$

where $K(x,y) = 1 - tS(x,y)$ and $S(x,y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j$.

When $a = b = c = 1$, this equation becomes

$$xyK(x,y)Q(x,y) = xy - tA_{-1}(x)xQ(x,0) - tB_{-1}(y)yQ(0,y) + tG(x,y)xyQ(0,0) \quad (13.2.2)$$

In [61], this equation is denoted as

$$K'(x,y)Q(x,y) = xy + K'(x,0)Q(x,0) + K'(0,y)Q(0,y) - K'(0,0)Q(0,0) \tag{13.2.3}$$

where $K'(x,y) = xyK(x,y)$. We can see (13.2.3) is a linear equation of $K'(x,y)Q(x,y)$ and its boundary terms $K'(x,0)Q(x,0)$ and $K'(0,y)Q(0,y)$. Since the coefficient of $Q(x,0)$ does not involve y, the functional equation reveals some linear properties. The transformations act on each term separately. If we assign probabilities to each step direction then $K'(x,y)$ changes but $K'(x,0)$ is still a function of x and $K'(0,y)$ is still a function of y. The functional equation is still a linear function of a function of x and a function of y. However, boundary interactions break this symmetry. Many ideas and methods which work for unweighted models can be directly extended to the probabilistic ones, while they do not always work for interacting models. That is why interacting walks need extra study.

13.3 Why the kernel method?

We have used the kernel method to solve the models and used the analytic methods to prove some properties. One can also use analytic methods to solve the models, since Theorem 23 already gives the solution form with an arbitrary algebraic function w. By Theorem 29, we can compare the singularities of each term in the solution form (11.1.36) and find a suitable w such that the solution is a formal series. In [61, 33], the conformal gluing function $w(z,t)$ in the Riemann boundary value problem approach was solved explicitly and all models have results in a uniform way.

It seems that the kernel method is not that effective compared with the analytic methods, but it has its own advantages. We can easily get the linear differential equation of D-finite functions for D-finite models, which appears to be an open problem in [61]. This is because we have a good expression for $Q(x,0)$. Comparing with the integral of $\tilde{\Phi}$ and arbitrary function w in (11.1.36) or CGF w in (11.2.20), we can see that it is the terms $[x^>]\sqrt{\Delta}$ that makes $Q(x,0)$ D-finite in the solution via the kernel method. For example, for **Model 3**, $Q(x,0)$ is solved by taking the $[x^>]$ of (8.1.50), where S_1 and S_2 are known functions. They are both related to $[x^>]\sqrt{\Delta}$. If we just consider x, $Q(x,0)$ is a linear function of S_1, S_2 with polynomial coefficients in x. If we know the linear differential equation of $[x^>]\sqrt{\Delta}$, we can calculate the linear differential equation of $Q(x,0)$.

Equation (7.1.10) tells us that if we know the linear differential equation of $\sqrt{\Delta}$, then we know the linear differential equation of $[x^>]\sqrt{\Delta}$ and the linear differential equation of $Q(x,0)$ follows directly. The D-algebraic cases can be discussed similarly as well since they are rational functions of D-finite terms. For some of the models, $Q(0,0)$ is also written as a rational function of D-finite terms and we can get the differential equations directly. However, an easy approach to the differential equation of t for general $Q(0,0)$ is still unclear. For some models, we must solve $Q(0,0)$ by solving a linear equation set, but the calculation is too complicated. There is another method based on the idea of decoupling functions, discussed in [61]. That method is also good at finding the differential equations for an algebraic and D-algebraic $Q(x,0)$ over x. Currently, that method has only been shown to work for non-interacting or step-weighted walks.

Another reason for our focus on the kernel method comes from future study in higher dimensions, for example, 3D walks in the octant. The lack of knowledge about higher-dimensional manifolds or 2-dimensional manifolds with genus larger than 1 has restricted the progress on general lattice walk models via analytic methods. However, the kernel method provides another route. In [11], Bostan et. al. apply the kernel method to walks in the octant and solve models associated with finite groups without interactions. Although the kernel method is only useful for those walks with specific forms of generating functions, the solutions do not rely on knowledge about higher-dimensional manifolds. This may give a practical way to solve certain models in higher dimensions.

13.4 Perspectives

There are still many open questions. For example,

1. Higher-dimensional lattice walks. For physics, two-dimensional lattice walks only model single-interaction polymers (ie. polymers with surface tension on the wall and inner stress inside the polymer itself). To go to multiple interaction cases, we must map it to lattice walks in higher dimensions. This will increase the number of variables in the kernel and the symmetry group will become a polyhedral group. From the Riemann surface perspective, the functions are now on a higher dimensional manifold. Some specific walks have already been solved for non-interacting cases in [11]. However, with interactions or for general cases, the problems are more complicated.

2. Properties of walks with infinite groups. It has been proved in [29] that except for 9 special walks, the generating series of all non-singular walks with infinite groups are x and y-hypertranscendental[1]. The 9 special walks are D-algebraic. While for properties over the variable t, everything is still unclear.

3. Singularity analysis for solved models. First consider x. For those walks associated with finite groups, the singularities are very clear via the kernel method, since the solutions are rational functions of terms with known singularities. For the walks associated with infinite groups, it is an open question in [61]. Considering t: for those walks associated with finite groups, most models are solved with explicit solutions, and the singularities can be analyzed easily. There are already complete results for all non-interacting cases in [68]. However, for Kreweras and reverse Kreweras, since we solved the linear equation sets in the last step, the explicit results contain complicated determinants and we probably need some other techniques to deal with them.

4. Some extension of Bousquet-Mélou and Jehanne's theorem [19]. Many lattice walk problems can be converted to solving an equation in the form of (12.3.1). From an analytic point of view, (12.3.1) is in the form of (11.2.1), the Riemann-Carleman problem with a shift. We want to extend Bousquet-Mélou and Jehanne's theorem to this equation. Although we made some extension in Section 12.3, it is not a satisfactory result. The extra conditions currently only suit Kreweras and reverse Kreweras walks. So, an improvement of Bousquet-Mélou and Jehanne's theorem is our objective.

Quarter-plane lattice walk models have been studied for more than 50 years. Actually, among all the methods of solving quarter-plane lattice walk problems, the RBVP[2] appears first in [32, 53]. This is then followed by the homomorphism on Riemann surfaces [36]. The first time that the kernel method was applied to lattice walk problem was after the year 2000 [22]. Besides what we have discussed in this work, some new ideas, such as Tutte's invariants [61] and differential Galois theory [29] have also been introduced into this topic. The developments of quarter-plane lattice walk models are not confined to probabilistic or physical models, but

[1] See definitions in Chapter 2.
[2] Riemann boundary value problem approach

extend to the mathematical ideas and methods. The perspectives may be much wider than we suppose.

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