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Title:
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Date:
2025-01-01

Citation:
Zhang, F., Chen, P. & Wu, X. (2025). Optimal Benefit Distribution of a Tontine-like Annuity Fund with Age-Structured Models. Risks, 13 (1), pp.4-4. https://doi.org/10.3390/risks13010004.

risks

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Optimal Benefit Distribution of a Tontine-like Annuity Fund with Age-Structured Models

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Abstract: This paper introduces a tontine-like annuity fund designed to provide lifelong income to its participants. Initially, each member contributes a lump-sum payment into a trust fund as a joining premium. Participants then receive benefits over time, based on their survival. As members pass away, their share of payouts is redistributed among the survivors, resulting in increased payouts for those remaining. Differing from traditional tontines, which assume a uniform mortality risk, this fund accommodates participants of various ages and allows new members to join during its operation. To accommodate these features, the authors utilize age-structured models (ASMs) to determine fair premiums for new entrants and to analyze the dynamics of benefit distribution. The core objective of this paper is to develop a pension model using ASMs, recognizing its significant potential for adaptation and expansion. The primary mathematical approach employed is the Maximum Principle from optimal control theory, which helps in deriving explicit solutions for the optimal subsidy strategy. Through numerical examples and detailed illustrations, the paper demonstrates that participants who remain in the cohort longer receive greater subsidies. Additionally, the study finds that adverse shocks lead to a smaller population and thus fewer subsidies. Conversely, starting with a larger initial cohort population tends to increase the overall population, resulting in more subsidies. However, higher costs associated with subsidies lead to their reduction. Our analysis reveals the complex interplay of factors influencing the sustainability and effectiveness of the proposed annuity model.

Keywords: age-structured models; tontine; benefit distribution; maximum principle

Academic Editors: Tak Kuen Ken Siu and Hailiang Yang

Received: 24 November 2024
Revised: 22 December 2024
Accepted: 25 December 2024
Published: 30 December 2024

Citation: Zhang, Fan, Ping Chen, and Xueyuan Wu. 2025. Optimal Benefit Distribution of a Tontine-like Annuity Fund with Age-Structured Models. *Risks* 13: 4. https://doi.org/10.3390/risks13010004

1. Introduction

With aging populations and rising life expectancy, conventional Defined Benefit (DB) pensions in many countries have become significant fiscal burdens for central governments (see Cerami (2011)). To sustain their national pension systems, numerous governments have resorted to measures such as raising consumption taxes and providing financial subsidies, making effective pension reform a pressing issue. Earlier attempts to address these challenges included the issuance of longevity bonds; however, these efforts failed due to a lack of sufficient investor interest. More recently, the tontine pension model, characterized by fixed total payments, has gained renewed attention in both political and academic discussions.

Tontine presents an alternative approach to mitigating longevity risk by pooling investments from a group of members and distributing annual dividends proportionally among the surviving participants. Originating in the 1650s and attributed to Lorenzo de Tonti, this concept benefits members as they age by effectively sharing longevity risk within the group (see Weinert and Gründl (2021)). Tontine offers a potential solution to address pension reform challenges in aging populations. The earliest tontine insurance products were introduced in France in 1689 to help settle public sector debts and soon gained popularity as a financing mechanism in European royal courts. By the early 1900s, tontines dominated the American insurance market, involving about 9 million members. However, due to inadequate regulation and scandals, tontine products were banned in 1906 in both the UK and the US, where they became associated with greed and corruption. Due to the global trend of population aging, the economic burden of pensions in some countries, such as Japan, the UK, and China, has become exceptionally severe. The concept of tontines, which allows participants to share the longevity risk among themselves, has great potential to guide pension reforms in these countries.

Given the historical popularity of tontines and their potential to ease the financial strain on the current pension system, academia now starts to study this type of pension insurance. Tontine literature employs various stochastic processes[1] to model participant populations, which can be categorized into three main types. The first category includes single-cohort binomial models, which use a binomial framework for a homogeneous group established at inception, without allowing new members to join. Milevsky and Salisbury (2015) introduced the concept of "shares" in tontine, establishing the foundational model, while Chen et al. (2019) proposed "tonuity", a financial product offering tontine-like payoffs until a set age, followed by deferred annuity payments. The second category features single-cohort continuous models with an infinite pool, leveraging continuous population frameworks. Stamos (2008) pioneered such a model using a Poisson process, which was later refined by Bernhardt and Donnelly (2019, 2021) through stochastic optimization using the Hamilton–Jacobi–Bellman (HJB) equation. The third category encompasses mixed-cohort binomial models, where multiple cohorts are modeled within a binomial structure. Milevsky and Salisbury (2016) introduced the foundational model for mixed cohorts, while Chen et al. (2021) and Chen and Rach (2023) extended this framework by incorporating optimization techniques and modeling an integer number of cohorts, advancing the analytical depth of tontine studies.

In this paper, we begin with a single-cohort model inspired by the framework in Stamos (2008), removing the component of stochastic population. By allowing the population size to approach infinity, authors simplify the benefit distribution into a deterministic process. Building on the concept of "shares" introduced by Milevsky and Salisbury (2015), authors construct the model based on the underlying assets associated with these shares. Furthermore, authors incorporate age-structured models (ASMs) for population dynamics, introducing a continuous age dimension. The age-structured model (ASM) is a differential equation that describes the dynamics of population sizes across different age groups over time. This approach allows individuals of varying ages to join the pension fund either at its inception or at a later stage.

ASM is a well-developed mathematical model with a long history. ASMs originate from the Kermack–McKendrick Theory, a framework designed to predict the temporal evolution of case counts and distributions in infectious disease outbreaks. Building on the early work of Ross and Hudson (1917), Kermack and McKendrick (1927) introduced the Susceptible–Infected–Recovered (SIR) model, which has become a cornerstone of infectious disease modeling. This paper adopts the ASM mathematical framework embedded within the SIR model. Extensions of the Maximum Principle optimization theory to ASMs were developed by Feichtinger et al. (2003, 2006). The Maximum Principle in ASM is a set of computational methods used to determine the optimal control strategy. In the latter part of this paper, we closely follow this deterministic framework to formulate our optimization problem. Moreover, since the control variables in Feichtinger et al. (2003, 2006) are treated as external, authors similarly model our control variables as external subsidies to individual pension accounts.

Our proposed pension fund features a mixed-cohort structure and dynamic membership. It accommodates individuals of varying ages at inception by combining separately constructed age-structured individual pension accounts and a population model into a unified asset model. Unlike Chen et al. (2021) and Chen and Rach (2023), which rely on integer-based cohort models, our approach employs a continuous age dimension, integrating utility over the continuous age range rather than aggregating it over discrete cohorts as in Chen and Rach (2023). Additionally, the fund allows new members to join at various ages after inception using the ASM framework to determine fair premiums for late entrants using individual pension accounts. The dynamics of the fund's assets across time and age are shaped by investment returns, dividend rates, and contributions from new members, enabling a flexible and precise management approach.

Traditionally, ASM models have predominantly addressed interconversion among multiple compartments in fields such as epidemiology. The main contribution of this paper is the construction of a pension model based on ASM. A key innovation is the construction of a model featuring two ASM components: individual benefit distribution and population size. Our model is based on certain simplified assumptions, including age-dependent mortality rates (assumed to be time-independent), uniform initial conditions for individual pension accounts at age 65, and a zero inflation rate. Future research can expand on this work by incorporating more complex pension fund designs, facilitating independent advancements in both individual pension account modeling and population-level analyses.

The structure of this paper is as follows. Section 2 introduces the traditional binomial tontine model and derives a single-cohort tontine-like fund from it. Building on this foundation, an age-structured tontine-like fund is developed using the ASM framework. Section 3 extends this age-structured model by incorporating shocks and controls, optimizing overall benefits through the Maximum Principle method. Sections 4 and 5 present the numerical results and concluding remarks, respectively. Finally, Appendix A provides an outline of a standard mathematical approach for solving the ASMs.

2. Formulation of Tontine-like Annuity Fund

Stamos (2008) introduced the "perfect pool" assumption, which assumes that the annuity fund maintains an infinite number of members at any given time. While this assumption may appear reasonable initially, particularly with a large initial membership, it becomes increasingly unrealistic over time as the population size of the annuity fund inevitably decreases. To address this limitation, this paper allows for new members to join the fund at any time t, with their premiums determined at the time of entry. This approach ensures that the number of living members in the fund remains sufficiently large. In Section 2.1, authors outline the operation of a single-cohort tontine-like fund. In Section 2.2, authors extend this framework to develop a multi-cohort tontine-like fund using the ASM methodology.

2.1. Single-Cohort Tontine-like Fund

In this section, we explore the structure of the traditional single cohort tontine fund. Assuming a large population size, we analyze the distribution of benefits and the pricing for new members who join the fund while it is operational. Lastly, we consider how the single cohort model can be extended to the ASM domain.

2.1.1. Single-Cohort Tontine Fund

We consider a traditional homogeneous-cohort tontine. We assume that there are n individuals, with the same age a_0 and hence the same mortality risk, forming a tontine at time t_0. Within the rest of this paper, we assume $t_0 = 0$. Without loss of generality, we assume each member can only buy one share of the product, then the tontine has a total of n shares at time t_0. During the operation of the tontine, the number of shares remains constant. As long as there is at least one member still alive, the rate of the benefit of each share, denoted by $c(t)$, is to be paid. According to the rules of tontine, the overall benefit $n \times c(t)$ is shared equally among all living members at time $t > t_0$. According to Milevsky and Salisbury (2015, 2016), the benefit of each living member at time t is given by

$$p(t) = \begin{cases} \frac{nc(t)}{N(t)}, & N(t) > 0, \\ 0, & N(t) = 0, \end{cases}$$

where $N(t)$ denotes the number of alive members in the tontine pool at time t and is a random variable at any given time t.[2] The benefit rate can also be written as $p(t) = nc(t)/N(t)I_d(N(t) > 0)$, where $I_d(.)$ is an indicator function. We assume the remaining lifetimes of the tontine members are independent of each other. Based on this assumption, the model simplifies the population analysis by ignoring the mortality correlations between participants. Thus, $N(t)$ follows a binomial (Bin) distribution Bin $(n,\ {}_tp_{a_0})$, where ${}_tp_{a_0}$ denotes the probability that an individual aged a_0 survives to age $a_0 + t$. This setting is also adopted to facilitate the derivation of the age-structured tontine in the subsequent sections. However, there are tontine that are not homogeneous; see Forman and Sabin (2016) and Denuit et al. (2022) for details.

2.1.2. Key Assumptions and Derivations

Following Stamos (2008), we explore the limiting scenario of a conventional tontine with a substantial initial member count. According to the law of large numbers, when $n \to \infty$,

$$\frac{N(t)}{n} \xrightarrow{a.s.} {}_t\, p_{a_0},$$

where $\xrightarrow{a.s.}$ denotes almost surely convergence. The use of the law of large numbers is based on the assumption that the number of participants is large; therefore, in practice, a sufficiently large number of initial pool is essential. According to McKeever (2009), a tontine product issued in Britain in 1757 failed to launch due to insufficient participation.

Moreover, we let τ denote the limiting age for human beings. We have ${}_{\tau-a_0}p_{a_0} = 0$. For any $t \in (0, \tau - a_0)$, we have $0 < {}_tp_{a_0} < 1$; therefore,

$$\lim_{n\to\infty} \Pr(N(t) > 0 | t < \tau - a_0) = \lim_{n\to\infty} 1 - (1 - {}_tp_{a_0})^n = 1. \tag{1}$$

That is,

$$I_d(N(t) > 0) \xrightarrow{p} 1,$$

where $\xrightarrow{p}$ denotes the converge in probability. Hence, for $t < \tau - a_0$, we have

$$p(t) = \frac{nc(t)}{N(t)} I_d(N(t) > 0) \xrightarrow{p} \frac{c(t)}{{}_tp_{a_0}},$$

where the limiting case has a continuous form of benefits over time. Although taking this limit neglects a portion of the stochasticity, this mathematical approach lays the groundwork for the subsequent derivation of the age-structured tontine-like fund.

We denote the continuous benefit of a living member at time t using $b(t)$ within the rest of this paper, i.e.,

$$b(t) = \frac{c(t)}{{}_t p_{a_0}}. \tag{2}$$

We let $\mu(a)$ denote the force of mortality of an individual at age a; then, the associated survival probability is

$${}_t p_{a_0} = \exp\left(-\int_{a_0}^{a_0+t} \mu(s)ds\right). \tag{3}$$

For simplicity, we assume that the force of mortality depends on age only and does not change over time in this paper.

Next, we investigate the sources of growth in $b(t)$. For $t > 0$, we have

$$\begin{aligned} \frac{d}{dt}b(t) &= \frac{d}{dt}\left(c(t)\exp\left(\int_{a_0}^{a_0+t}\mu(s)ds\right)\right) \\ &= \frac{d}{dt}c(t)\exp\left(\int_{a_0}^{a_0+t}\mu(s)ds\right) + c(t)\mu(a_0+t)\exp\left(\int_{a_0}^{a_0+t}\mu(s)ds\right) \\ &= \frac{b(t)}{c(t)}\frac{d}{dt}c(t) + \mu(a_0+t)b(t). \end{aligned} \tag{4}$$

Following Dickson et al. (2019), we use reserve to refer to the wealth of each share; then, we denote $h(t)$ as the balance of the reserve of a share at time t. The initial balance $h(0)$ is just the amount contributed for each share by members initially. Then, $c(t)$ can be considered as the dividend of each share's reserve account at time t. For the convenience of mathematical derivations, we assume a constant dividend rate, denoted by $m > 0$, within the rest of this section. Then, $c(t) = mh(t)$.

2.1.3. Dynamics of Reserves and Wealth

We let $W(t)$ denote the total wealth of the fund at time t. Then, $W(0) = nh(0)$ and $W(t) = nh(t)$. In this paper, we assume the tontine fund invests exclusively in bonds with a rate of return, $r \geq 0$. Thus,

$$dW(t) = (r-m)W(t)dt,$$

and hence

$$W(t) = W(0)e^{(r-m)t}.$$

Here, we assume $m > r$, such that the total wealth of the fund can be reduced over time. Further, we have

$$h(t) = \frac{1}{n}W(0)e^{(r-m)t} = h(0)e^{(r-m)t}, \tag{5}$$

$$c(t) = mh(0)e^{(r-m)t}, \tag{6}$$

and

$$dc(t) = mdh(t) = (r-m)c(t)dt. \tag{7}$$

Plugging (7) into (4) offers

$$\frac{d}{dt}b(t) = (r-m)b(t) + \mu(a_0+t)b(t), \tag{8}$$

which implies that the benefit growth rate per surviving member is $r + \mu(a_0+t)$. This uncovers the dual origins of a tontine member's benefit augmentation: the primary source is the investment income and the secondary source is the passing of fellow members.

Remark 1. *Stamos (2008) also obtained similar results to the above ones, but their method of proof was based on the expectation of the Poisson process, while our proof is a direct application of the law of large numbers.*

Similar to one share's reserve account, we let $h_L(t)$ denote a living member's reserve account and assume that each living member's benefit is payable using $h_L(t)$. Then we obtain

$$b(t) = mh_L(t). \tag{9}$$

From (2), (8), and $c(t) = mh(t)$, we have

$$h_L(t) = \frac{h(t)}{{}_t p_{a_0}}$$

and

$$\begin{aligned} \frac{d}{dt}h_L(t) &= \frac{1}{m}\frac{d}{dt}b(t) = \frac{1}{m}(r - m + \mu(a_0 + t))b(t) \\ &= (r - m + \mu(a_0 + t))h_L(t). \end{aligned} \tag{10}$$

The economic implication of (10) is that members of a tontine obtain the right of inheritability of reserve balances due to the death of other members.

2.1.4. Age-Structured Fund Extensions

The reserve of a living member, $h_L(t)$, as defined above, aligns conceptually with the approaches in existing literature (see Milevsky and Salisbury (2015), and Bernhardt and Donnelly (2019)) but adopts a distinct calculation method. In those works, reserves are modeled as discounted expectations of future benefits. In contrast, we treat pension products as financial instruments, calculating the present reserve based on past contributions, payouts, and investment returns. This $h_L(t)$ can also be used for pricing purposes regarding the intermediate entrants into the pension plan. This concept plays a significant role in the derivation of the dynamic equation for the overall wealth in ASM pension schemes. Additionally, we consider the reserve for each share $h(t)$, which is crucial in the subsequent construction of ASM pension schemes.

Next, we examine the intermediate joiners' premiums. We assume that at time t, an individual aged $a_0 + t$ joins the fund. The fair joining premium they need to pay should be $h_L(t)$. We use $*$ to distinguish variables before and after this new member joins. The total wealth after this member joins becomes

$$W^*(t) = N(t) \cdot h_L(t) + h_L(t)$$

and the reserve for each share becomes

$$h_L^*(t) = \frac{W^*(t)}{N^*(t)} = \frac{N(t) \cdot h_L(t) + h_L(t)}{N(t) + 1} = h_L(t), \tag{11}$$

where $N^*(t) = N(t) + 1$. That is to say, the wealth per living member after the joining of the new member remains $h_L(t)$. Similarly, we can show that

$$b^*(t) = mh_L^*(t) = mh_L(t) = b(t). \tag{12}$$

From (11) and (12), we conclude that if the contribution amount of an intermediate joining member equals the current reserve balance per existing member, the additional membership does not alter the reserve balance per member. As a result, the dynamics of $b(t)$ and $c(t)$ remain unaffected. In this single-cohort model, individuals joining at time

t must belong to the cohort of age $a_0 + t$ at time t. However, in the subsequent multi-cohort model, individuals have the flexibility to join their respective cohorts at various ages.

By allowing new members to join during the operation of the pension fund, we ensure that the membership pool remains sufficiently large, enabling the law of large numbers to apply. Essentially, our single-cohort tontine-like fund functions as a pooled annuity fund (see Stamos (2008), Bernhardt and Donnelly (2019) and Dagpunar (2021)). When the membership pool is relatively small, the traditional binomial tontine model offers better description than our approach. However, our model is more suitable for larger membership pools. The following age-structured tontine-like fund is similarly designed to operate effectively with a large membership pool.

2.2. Age-Structured Tontine-like Fund

In practice, a natural time for individuals to join a pooled annuity fund is upon retirement, which occurs at varying time points. Consequently, it is practical to include members of different ages within the same annuity pool. In this context, the previously introduced single-cohort model becomes impractical, necessitating the incorporation of an age dimension into the tontine framework, both in theory and in application. The assumption of a large population in the single-cohort model aligns with the large-population assumption of the ASM population model, making our approach more appropriate for addressing large-scale pension systems, such as national pension schemes.

Several studies, including Sabin (2010), Donnelly et al. (2014), and Milevsky and Salisbury (2016), have already explored similar concerns. Their frameworks typically consist of a discrete number of cohorts, each modeled using a binomial framework. However, since our single-cohort model is derived by taking the limit of an initial population to create a continuous representation, our multi-cohort extension adopts an ASM framework with both a continuous age dimension and a continuous time dimension. Additionally, we draw inspiration from Feichtinger et al. (2003, 2006), who introduced an optimal production problem involving controlled machine numbers. Following their approach, we formulate an optimal pension fund problem within our framework. The age-structured tontine model proposed here is particularly suited for scenarios involving large membership pools.

We now proceed with our proposed age-structured tontine-like fund. At time $t_0 = 0$, the fund's inception, initial members are grouped into different cohorts based on their joining age, with each cohort following the single-cohort model discussed in Section 2.1. Individuals aged 65 and above may join the fund at any time $t \geq 0$. We let $y(t, a)$, where $a \geq 65$ and $t \geq 0$, denote the size of the cohort aged a at time t within the fund, and let $q(t, a)$ represent the number of newly joined members aged a at time t. The following boundary conditions apply:

- when $t = 0$, $g(a) \triangleq y(0, a)$ is the cohort size of members aged a at the inception of the fund;
- when $a = 65$, $q(t) \triangleq y(t, 65)$ is the number of members who join the fund aged 65 at time t;
- $g(a) > 0$ and $q(t) > 0$.

At the fund's inception, each member is required to pay a one-time lump-sum joining fee, entitling them to future dividends. Under the single-cohort model, this lump-sum payment is represented by $h(0)$. Since the fund comprises multiple cohorts at inception, we use $v(a)$ to denote the initial contribution amount for a member aged a, where $v(a)$ is assumed to be a continuously differentiable function.

In Section 2.1, we show that as long as the intermediate joining fee at time $t > 0$ equals $h_L(t)$, the wealth balance and benefit rate per living member within a cohort remain unchanged when additional members join. Following this principle, the fair joining fee for a member aged $a > 65$ at time $t > 0$ is defined as the current reserve balance per member in cohort (t, a) within the fund. Assuming sufficiently large initial cohort sizes and normal fund operations, it is reasonable to expect that the size of cohort (t, a) remains non-zero. For members joining the fund at age 65 and time $t > 0$, their joining fee is treated as the initial fee for cohort $(t, 65)$.

In our multi-cohort model, the payment required from individuals joining at an intermediate time t is determined by the reserve $h_L(t, a)$, which depends on both time t and age a. The boundary conditions for $h_L(t, a)$ at $t = 0$ and $a = 65$ are defined as follows:

$$h_L(0, a) = v(a), \quad h_L(t, 65) = v(65).$$

These conditions specify that the initial reserve h_L at time $t = 0$ for age a equals $v(a)$, and the reserve for individuals aged 65 at any time t equals $v(65)$. The detailed calculation of $h_L(t, a)$ is provided later.

For illustrative purposes, we divide all cohorts into three categories based on their establishment times:

Cohorts established at the fund's inception: These cohorts consist of individuals whose initial ages at the fund's inception are greater than 65. The age–time relationship for any individual in this category satisfies the condition $a > t + 65$.

Cohorts established during the fund's operation: These cohorts are formed after the fund's inception, and all founding members are exactly 65 years old at the time of establishment. The age–time relationship for any individual in this category satisfies the condition $65 \leq a < t + 65$.

The founding cohort: This category includes only a single cohort established at $t_0 = 0$, comprising members aged exactly 65 at the fund's inception. The age–time relationship for any individual in this cohort satisfies $a = t + 65$, and the initial population size of this cohort is denoted by $g(65)$.

This classification provides a structured understanding of how cohorts are defined and evolve within the multi-cohort framework.

We consider an example illustrated in Figure 1. We assume that an individual aged 69 at time $t = 7$ wishes to join the pension plan immediately. This individual belongs to the cohort established at $t = 3$, where the initial ages of all members were 65. Consequently, the current reserve value for each living member in this cohort, $h_L(7, 69)$, represents the price this person needs to pay. Now, we consider another example where an individual joins the fund at $t = 2$ with age 69. In this case, the individual corresponds to the cohort established at the inception of the fund, where the founding members were all 67 years old. Here, the relevant price is determined by the current reserve value for this cohort, $h_L(2, 69)$.

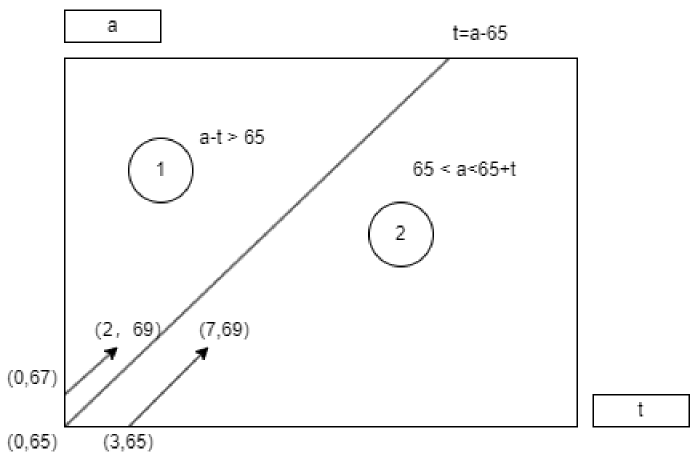

Figure 1. Age-structured tontine-like fund.

Having outlined how the age-structured tontine-like fund is constructed, we now introduce the dynamic equation governing the cohort sizes within the age-structured population system.

Theorem 1. *The dynamic equation of $y(t,a)$ is given by*

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})y(t,a)=-\mu(a)y(t,a)+q(t,a), \tag{13}$$

with boundary conditions $y(0,a)=g(a)$ and $y(t,65)=q(t)$. The closed-form solutions of $y(t,a)$ are

$$y(t,a)=\begin{cases} g(a-t)\,{}_tp_{a-t}+\int_0^t q(v,a-t+v)\,{}_{t-v}p_{a-t+v}dv, & a-t\geq 65, \\ q(t-a+65)\,{}_{a-65}p_{65}+\int_{t-a+65}^t q(v,a-t+v)\,{}_{t-v}p_{a-t+v}dv, & 65\leq a<65+t. \end{cases}$$

Proof. Following the initial assumption in this section, i.e., only individuals aged 65 or above can join the age-structured tontine fund, we now determine the various cohort sizes in the fund. We let $Y(t,a)$ denote the total number of members in the fund aged between 65 and a (inclusive) at time t. Then, we have, for $t\geq 0, a\geq 65$,

$$Y(t,a)=\int_{65}^a y(t,s)ds,$$

where $y(t,a)$ satisfies the following partial differential equation (see Inaba (2017), p. 85 in chap. 2):

$$\frac{\partial y(t,a)}{\partial t}+\frac{\partial y(t,a)}{\partial a}=-\mu(a)y(t,a)+q(t,a).$$

A simple observation is that $q(0,a)=g(a)$. As previously assumed, τ is the limiting age of human beings, so we have, for $a\in(65,\tau)$, $\int_{65}^{\tau}\mu(a)da=+\infty$. Following Inaba (2017), the closed-form solutions of $y(t,a)$ is

$$y(t,a)=\begin{cases} g(a-t)\,{}_tp_{a-t}+\int_0^t q(v,a-t+v)\,{}_{t-v}p_{a-t+v}dv, & a-t\geq 65, \\ q(t-a+65)\,{}_{a-65}p_{65}+\int_{t-a+65}^t q(v,a-t+v)\,{}_{t-v}p_{a-t+v}dv, & 65\leq a<65+t. \end{cases}$$

This ends the proof. □

Remark 2. *This is the basic population model of ASM. The derivation of the PDE (13) can be found in Li and Brauer (2008), and detailed solutions to this PDE are provided in (A1). For an intuitive understanding, refer to Figure 1. This model captures the population dynamics within the pension system, where each cohort is established at the boundary and subsequently evolves, experiencing deaths $-\mu(a)y(t,a)$ and new entrants $q(t,a)$ along its progression. Unlike Chen and Rach (2023), which models an integer number of cohorts, our approach employs continuous age dynamics. This distinction allows for a more accurate representation of age progression, aligning better with real-world demographic changes. In reality, individuals of the same age often do not share the same birthdays, and their mortality forces can vary. By using a continuous age variable a, this model is able to account for such variations effectively.*

Next, we discuss the benefits and balances under this extended fund scheme. For any cohort at time t, identified by its age a at t, we define the continuous benefit per share and per living member as $c(t,a)$ and $b(t,a)$, respectively. Based on the results of the single-cohort model (2), the benefit per living member at each time-age point takes the form

$$b(t,a) = \begin{cases} \frac{c(t,a)}{{}_tp_{a-t}}, & a \geq t+65, \\ \frac{c(t,a)}{{}_{a-65}p_{65}}, & 65 \leq a < t+65, \end{cases} \tag{14}$$

where ${}_tp_{a-t}$ and ${}_{a-65}p_{65}$ offer the probability of a founding member, under different categories, surviving to the current time age point. Similar to the single-cohort case, we assume that the benefits are paid according to a constant dividend rate, $m > 0$, i.e., $b(t,a) = mh_L(t,a)$, and the whole fund is be invested in bonds with a rate of return $r \geq 0$. Then, we have

Theorem 2. *The benefit function, $b(t,a)$, of the age-structured tonine-like pension fund satisfies the following dynamic equation:*

$$(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial a})b(t,a) = [r - m + \mu(a)]\frac{c(t,a)}{{}_tp_{a-t}} = [r - m + \mu(a)]b(t,a), \tag{15}$$

with boundary conditions

$$b(0,a) = mv(a),$$
$$b(t,65) = mv(65).$$

Proof. We first consider living members aged a at time $t \geq 0$ in the tontine pool, where $a \geq t + 65$. Their initial age at time 0 is then $a_0 := a - t$. Clearly, a_0 can be used to identify cohorts among the living members. Following (14), we have

$$b(t,a) = \frac{c(t,a)}{{}_tp_{a_0}}.$$

From (6), we have

$$c(t,a) = mv(a-t)e^{(r-m)t},$$

which has the following partial derivatives:

$$\frac{\partial}{\partial t}c(t,a) = -mv'(a-t)e^{(r-m)t} + (r-m)c(t,a),$$
$$\frac{\partial}{\partial a}c(t,a) = mv'(a-t)e^{(r-m)t},$$

where $v'()$ is the first derivative of $v()$. Then, we have

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)c(t,a) = (r-m)c(t,a).$$

The deterministic assumption in the single-cohort model enables this simple result in the multi-cohort model. From (3), we can obtain

$$\frac{\partial}{\partial a}\frac{1}{{}_tp_{a-t}} = \frac{\mu(a) - \mu(a-t)}{{}_tp_{a-t}},$$
$$\frac{\partial}{\partial t}\frac{1}{{}_tp_{a-t}} = \frac{\mu(a-t)}{{}_tp_{a-t}}.$$

As a result, we have

$$\begin{aligned}\frac{\partial}{\partial t}b(t,a) &= \frac{1}{{}_tp_{a-t}}\Big[-mv'(a-t)e^{(r-m)t}+(r-m)c(t,a)+\mu(a-t)c(t,a)\Big],\\ \frac{\partial}{\partial a}b(t,a) &= \frac{1}{{}_tp_{a-t}}\Big\{mv'(a-t)e^{(r-m)t}+[\mu(a)-\mu(a-t)]c(t,a)\Big\},\end{aligned}$$

and then

$$\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\Big)b(t,a)=[r-m+\mu(a)]\frac{c(t,a)}{{}_tp_{a-t}}=[r-m+\mu(a)]b(t,a).$$

This is the dynamic equation of the continuous benefit density $b(t,a)$. Boundary conditions are given by

$$\begin{aligned}b(0,a)&=c(0,a)=mv(a),\\ b(t,65)&=c(t,65)=mv(65).\end{aligned}$$

The proof for the case $t+65>a$ is similar, so we omit it here. This ends the proof. □

Remark 3. *We observed that* (8)*, the single-cohort result, is a simplified case of* (15)*. In the multi-cohort context, our result reveals that the growth rate of individual benefit functions for different ages at the same time is related to the force of mortality across various ages.*

Theorem 1 derives the dynamic equation for the benefits. Given that our designed pension fund distributes dividends according to a fixed proportion m, we can also calculate the dynamic equation for the individual fund assets at each age-time point. The initial point for an individual's asset is the amount $v(\cdot)$ they contributed upon joining.

Lemma 1. *The dynamic equation of a living member's reserve account, $h_L(t,a)$, is given by*

$$\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\Big)h_L(t,a)=[r-m+\mu(a)]h_L(t,a), \tag{16}$$

with boundary conditions $h_L(0,a)=v(a)$ and $h_L(t,65)=v(65)$.

Proof. The proof is trivial since $b(t,a)=mh_L(t,a)$. □

Remark 4. *It is straightforward to determine that when $\frac{\partial}{\partial a}=0$, Equation* (16) *reduces to the single-cohort scenario in* (10)*.*

We define the total fund wealth at each time age point (t,a) as the product of $y(t,a)$ and the associated account balance per living member aged a at time t, i.e.,

$$W(t,a)=y(t,a)h_L(t,a), \tag{17}$$

with boundary conditions $W(0,a)=g(a)v(a)$ and $W(t,65)=q(t)h_L(t,65)$. We have the following result:

Theorem 3. *The dynamic equation of $W(t,a)$ is*

$$\frac{\partial W(t,a)}{\partial t}+\frac{\partial W(t,a)}{\partial a}=(r-m)W(t,a)+q(t,a)h_L(t,a). \tag{18}$$

Proof. According to (17), the partial derivatives of $W(t,a)$ are

$$\frac{\partial W(t,a)}{\partial t} = \frac{\partial y(t,a)}{\partial t} h_L(t,a) + \frac{\partial h_L(t,a)}{\partial t} y(t,a),$$

$$\frac{\partial W(t,a)}{\partial a} = \frac{\partial y(t,a)}{\partial a} h_L(t,a) + \frac{\partial h_L(t,a)}{\partial a} y(t,a).$$

Then, we have

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big) W(t,a) &= \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big) y(t,a) \cdot h_L(t,a) + y(t,a) \cdot \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big) h_L(t,a), \\ &= \big[-\mu(a) y(t,a) + q(t,a)\big] h_L(t,a) + y(t,a)\big\{[r - m + \mu(a)] h_L(t,a)\big\} \\ &= (r-m) W(t,a) + q(t,a) h_L(t,a), \end{aligned}$$

where $q(t,a)h_L(t,a)$ is the additional wealth brought in by those who joined the fund at time t at age a. □

We let $W(t)$ denote the total wealth of the age-structured tontine-like fund at time t; then, we can see that $W(t) = \int_{65}^{\tau} W(t,a)da$. In particular, $W(0)$ is the total initial wealth of the fund.

Remark 5. *One can see from (18) that there are two origins of wealth growth in our tontine-like fund at any given time t: the investment income at time t and the added wealth contributed by newcomers who join the fund at time t.*

Equations (17) and (18) form the foundation for the formulation of the optimal control problem in the next section. Future research can build upon this foundational age-structured pension model by introducing modifications to address new objectives and scenarios. For instance, the model could be expanded to address variations in mortality rates influenced by regional, gender, or other socio-economic factors. Specifically, we could define $y_1(t,a)$ and $y_2(t,a)$ to represent male and female populations, respectively, with $\mu_1(t,a)$ and $\mu_2(t,a)$ denoting the corresponding instantaneous mortality rates for each gender. Given the fundamental equation $W(t,a) = y(t,a) \times hL(t,a)$, it would also be necessary to formulate $hL_1(t,a)$ and $hL_2(t,a)$ to accommodate these distinctions.

In the next section, we incorporate control variables and random shocks to develop our optimal control problem.

3. An Optimal Control of the Age-Structured Tontine-like Fund

Section 2 constructs an age-structured tontine-like fund that allows individuals to join at various age at intermediate times. In this section, we study an optimal subsidy problem from the fund operator's view. In addition, we introduce shocks to the model to represent the random fluctuations in mortality rates that are caused by natural disasters, new advances in medical science, and so on. In the following, we define the mortality shocks as a random variable, allowing us to follow the optimization procedure in Feichtinger et al. (2003, 2006).

Following Lin and Cox (2005) and Chen et al. (2019), we introduce the mortality shock via a random variable ϵ, which modifies the survival function. We let φ denote the realization of ϵ and p^* represent the perturbed survival probability; then, we have

$$
\begin{aligned}
{}_tp^*_{a_0} &= {}_tp_{a_0}^{1-\epsilon} = \exp\Big(-\int_{a_0}^{a_0+t}(1-\epsilon)\mu(s)ds\Big),\\
E[{}_tp^*_{a_0}] &= \int_{-\infty}^{1} {}_tp_{a_0}^{1-\varphi} f_\epsilon(\varphi)d\varphi,
\end{aligned}
$$

where $f_\epsilon(\varphi)$ denotes the probability density function of ϵ and the supporting domain of φ is $(-\infty, 1)$. When community faces natural disasters or infectious diseases, these random events can reduce survival probabilities, such that ${}_tp^*_{a_0} <_t p_{a_0}$. We denote the extent of this decrease with the random variable $\epsilon = \varphi$, resulting in ${}_tp_{a_0}^{1-\varphi} <_t p_{a_0}$, implying that $\varphi < 0$. Conversely, the introduction of a new drug by a company constitutes a random event that could increase survival probabilities, leading to ${}_tp^*a_0 >_t p_{a_0}$. In this scenario, ${}_tp_{a_0}^{1-\varphi} >_t p_{a_0}$, indicating $\varphi \in (0, 1)$. Therefore, when no random events occur, we have $\varphi = 0$. In the following, we use superscript $*$ to denote the updated version under the random shock ϵ of any relevant function we defined before. As a result, the new benefit of a living member aged a at time t takes an updated form

$$
b^*(t,a) = \frac{c(t,a)}{{}_tp^*_{a_0}} = \frac{c(t,a)}{{}_tp^*_{a-t}}.
$$

To obtain an updated partial differential equation for $b^*(t,a)$, we need to determine $\frac{\partial}{\partial t}\frac{1}{{}_tp^*_{a-t}}$ and $\frac{\partial}{\partial a}\frac{1}{{}_tp^*_{a-t}}$. We have

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{{}_tp^*_{a-t}} &= \frac{\partial}{\partial t}\left(\frac{1}{{}_tp_{a-t}}\right)^{1-\epsilon} = (1-\epsilon)\left(\frac{1}{{}_tp_{a-t}}\right)^{-\epsilon}\frac{\partial}{\partial t}\frac{1}{{}_tp_{a-t}}\\
&= (1-\epsilon)\left(\frac{1}{{}_tp_{a-t}}\right)^{1-\epsilon}\mu(a-t) = \frac{(1-\epsilon)\mu(a-t)}{{}_tp^*_{a-t}},\\
\frac{\partial}{\partial a}\frac{1}{{}_tp^*_{a-t}} &= \frac{\partial}{\partial a}\left(\frac{1}{{}_tp_{a-t}}\right)^{1-\epsilon} = (1-\epsilon)\left(\frac{1}{{}_tp_{a-t}}\right)^{-\epsilon}\frac{\partial}{\partial a}\frac{1}{{}_tp_{a-t}}\\
&= (1-\epsilon)\left(\frac{1}{{}_tp_{a-t}}\right)^{1-\epsilon}[\mu(a)-\mu(a-t)] = \frac{(1-\epsilon)[\mu(a)-\mu(a-t)]}{{}_tp^*_{a-t}}.
\end{aligned}
$$

Following the derivations of (15), we can obtain

$$
\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big)b^*(t,a) = (1-\epsilon)[r-m+\mu(a)]\frac{c(t,a)}{{}_tp^*_{a-t}} = (1-\epsilon)[r-m+\mu(a)]b^*(t,a). \tag{19}
$$

Since $b^*(t,a) = mh^*_L(t,a)$, we can obtain the following partial differential equation of $h^*_L(t,a)$,

$$
\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big)h^*_L(t,a) = (1-\epsilon)[r-m+\mu(a)]h^*_L(t,a). \tag{20}
$$

Further, the population size under shock, $y^*(t,a)$, satisfies the following partial differential equation

$$
\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big)y^*(t,a) = -(1-\epsilon)\mu(a)y^*(t,a) + q(t,a) \tag{21}
$$

with boundary conditions $y^*(0,a) = g(a)$ and $y^*(t,65) = q(t)$.

Based on (20), we introduce a control variable $I(t,a) > 0$ on the right-hand side, where $I(t,a)$ represents any subsidy applied to the pension fund. Additionally, we let $I_0(t)$ denote the subsidy specifically allocated to new entrants joining the pension fund at the age of 65. This subsidy could come from a company or a governmental body. For example, the government might subsidize a portion of the joining fee to incentivize more individuals to participate in the tontine fund, given that the tontine-like system transfers longevity risk to fund members and alleviates the government's financial burden.

We now formulate an optimization problem over a finite time horizon $[0, T]$, where $T = \tau - a_0$. The objective is to maximize the total benefits received by all surviving members, net of the subsidy costs.

Our age-structured model with the control and the random shock becomes

$$\begin{aligned}\frac{\partial h_L^*(t,a)}{\partial t} + \frac{\partial h_L^*(t,a)}{\partial a} &= (1-\epsilon)(r - m + \mu(a))h_L^*(t,a) + I(t,a), \\ h_L^*(0,a) &= v(a), \\ h_L^*(t,65) &= v(65) + I_0(t).\end{aligned} \tag{22}$$

Following Feichtinger et al. (2006), we consider the total benefits received by all surviving members, net of the subsidy costs. We let $y^{\varphi}(t,a) := y^*(t,a)|\epsilon = \varphi$ and $h_L^{\varphi}(t,a) := h_L^*(t,a)|\epsilon = \varphi$. The objective function is given by

$$\begin{aligned}&\max_{I(t,a),I_0(t)} E_\epsilon\Bigg[\int_0^T e^{-rt}\int_{65}^{\tau} y^*(t,a)\Big(mh_L^*(t,a) - C(a)I(t,a) - \frac{D(a)}{2}I^2(t,a)\Big)dadt \\ &\qquad - \int_0^T e^{-rt}q(t)\Big(C_0 I_0(t) + \frac{D_0}{2}{I_0}^2(t)\Big)dt\Bigg] \\ &= \max_{I(t,a),I_0(t)} \int_0^T e^{-rt}\int_{65}^{\tau}\int_{-\infty}^{1} y^{\varphi}(t,a)\Big(mh_L^{\varphi}(t,a) - C(a)I(t,a) - \frac{D(a)}{2}I^2(t,a)\Big)f_\epsilon(\varphi)d\varphi dadt \\ &\qquad - \int_0^T e^{-rt}q(t)\Big(C_0 I_0(t) + \frac{D_0}{2}{I_0}^2(t)\Big)dt,\end{aligned} \tag{23}$$

where $C(a)I(t,a)$ and $C_0 I_0(t)$ denote the cost of subsidies with pre-determined constants $1 < C(a), C_0 < 2$, and $\frac{D(a)}{2}I^2(t,a)$ and $\frac{D_0}{2}{I_0}^2(t)$ represent the operational costs with pre-determined constants $0 < D(a), D_0 < 1$.

The objective function outlined above aligns closely with the multi-cohort framework of our proposed age-structured tontine model, enabling a continuous evaluation of the overall benefits for members across all ages and time points. This approach extends existing optimal control studies in pension literature, which are either single-cohort based (e.g., Bernhardt and Donnelly (2019)) or multi-cohort based but limited to an integer number of cohorts (e.g., Milevsky and Salisbury (2016) and Chen and Rach (2023)).

In solving the proposed optimal control problem, we closely follow the theoretical results presented in Feichtinger et al. (2003, 2006), with particular emphasis on the maximum principle method detailed in the latter paper. For the detailed mathematical derivations of ASM optimization theory, refer to Feichtinger et al. (2003). It is important to note that the results provide necessary optimality conditions.

We calculate the Hamiltonian function of our problem as

$$\begin{aligned}H &= \int_{-\infty}^{1} y^{\varphi}(t,a)\Big(mh_L^{\varphi}(t,a) - C(a)I(t,a) - \frac{D(a)}{2}I^2(t,a)\Big)f_\epsilon(\varphi)d\varphi \\ &\quad + \lambda(t,a)\int_{-\infty}^{1}\Big((1-\varphi)(r - m + \mu(a))h_L^{\varphi}(t,a) + I(t,a)\Big)f_\epsilon(\varphi)d\varphi,\end{aligned}$$

where $\lambda(t,a)$ is Lagrange multiplier and the boundary Hamiltonian function is

$$H_0 = -q(t)C_0 I_0(t) - q(t)\frac{D_0}{2} {I_0}^2(t) + \lambda(t,65)\Big(q(t)v(65) + I_0(t)\Big).$$

The first-order conditions for the optimality are

$$\frac{\partial H}{\partial I} = 0 \quad \text{and} \quad \frac{\partial H_0}{\partial I_0(t)} = 0,$$

which are equivalent to

$$C(a)E_\epsilon[y^*(t,a)] + D(a)I(t,a)E_\epsilon[y^*(t,a)] + \lambda(t,a) = 0$$

and

$$-q(t)C_0 - q(t)D_0 I_0(t) + \lambda(t,65) = 0,$$

where $E_\epsilon[y^*(t,a)] = \int_{-\infty}^{1} y^\varphi(t,a) f_\epsilon(\varphi) d\varphi$. Thus, the necessary optimality conditions are

$$I(t,a) = \frac{\lambda(t,a) - C(a)E_\epsilon[y^*(t,a)]}{D(a)E_\epsilon[y^*(t,a)]}, \tag{24}$$

and

$$I_0(t) = \frac{\lambda(t,65) - q(t)C_0}{q(t)D_0}. \tag{25}$$

Further, the dynamic equation of the Lagrange multiplier $\lambda(t,a)$ with the shock is given by

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big)\lambda(t,a) &= r\lambda(t,a) - \frac{\partial H}{\partial h_L^\varphi} \\
&= \int_{-\infty}^{1} \Big(\lambda(t,a)(\varphi r + (1-\varphi)(m-\mu(a))) - m y^\varphi(t,a)\Big) f_\epsilon(\varphi) d\varphi \\
&= \lambda(t,a)[rE(\epsilon) + (1-E(\epsilon))(m-\mu(a))] - mE_\epsilon[y^*(t,a)],
\end{aligned} \tag{26}$$

with boundary conditions

$$\lambda(t,\tau) = 0 \quad \text{and} \quad \lambda(T,a) = 0. \tag{27}$$

The method of characteristics (see (A1)) is adopted to solve this partial differential Equation (26) under conditions (27). As a result, we have

$$\lambda(t,a) = \begin{cases} \int_t^T mE_\epsilon[y^*(v,a-t+v)] e^{-\int_t^v [rE(\epsilon) + (1-E(\epsilon))(m-\mu(a-t+s))]ds} dv, & 65 \le a < 65+t, \\ \int_a^\tau mE_\epsilon[y^*(t-a+v,v)] e^{-\int_a^v [rE(\epsilon) + (1-E(\epsilon))(m-\mu(s))]ds} dv, & a-t \ge 65. \end{cases}$$

The result can be further simplified to

$$\lambda(t,a) = \begin{cases} \int_t^T \frac{m}{{}_{v-t}p_a^{1-E(\epsilon)}} e^{-(v-t)[rE(\epsilon) + m(1-E(\epsilon))]} E_\epsilon[y^*(v,a-t+v)] dv, & 65 \le a < 65+t, \\ \int_a^\tau \frac{m}{{}_{v-a}p_a^{1-E(\epsilon)}} e^{-(v-a)[rE(\epsilon) + m(1-E(\epsilon))]} E_\epsilon[y^*(t-a+v,v)] dv, & a-t \ge 65. \end{cases}$$

Due to (24) and (25), we obtain a closed-form solution of $I(t,a)$ as follows:

$$I(t,a) = \begin{cases} \dfrac{\int_t^T \frac{m}{\frac{1-E(\epsilon)}{{}_{v-t}p_a}} e^{-(v-t)[rE(\epsilon)+m(1-E(\epsilon))]} E_\epsilon[y^*(v,a-t+v)]dv - C(a)E_\epsilon[y^*(t,a)]}{D(a)E_\epsilon[y^*(t,a)]}, & 65 \le a < 65+t, \\ \dfrac{\int_a^\tau \frac{m}{\frac{1-E(\epsilon)}{{}_{v-a}p_a}} e^{-(v-a)[rE(\epsilon)+m(1-E(\epsilon))]} E_\epsilon[y^*(t-a+v,v)]dv - C(a)E_\epsilon[y^*(t,a)]}{D(a)E_\epsilon[y^*(t,a)]}, & a-t \ge 65. \end{cases} \tag{28}$$

Also, we have

$$I_0(t) = \frac{\int_t^T \frac{m}{\frac{1-E(\epsilon)}{{}_{v-t}p_{65}}} e^{-(v-t)[rE(\epsilon)+m(1-E(\epsilon))]} E_\epsilon[y^*(v, 65-t+v)]dv - q(t)C_0}{q(t)D_0}. \tag{29}$$

Traditional tontine models predominantly featured a single cohort and were unable to accommodate populations with multiple age groups. Looking ahead, it is beneficial to examine how specific decisions impact the parameters $y(t,a)$ and $h_l(t,a)$. By employing simulations to generate surface graphs of $y(t,a)$, $h_l(t,a)$, and $W(t,a)$, we can visually assess the effects of these decisions. Additionally, our model can be expanded to account for economic shocks, such as recessions. For instance, the rate r can be adapted to $r(t)$, decreasing in response to a recession occurring at time t. Alternatively, $r(t)$ can be modeled as a stochastic process to reflect economic variability.

Remark 6. *The mortality shock, ϵ, considered in the above optimization problem, is modeled as a random variable independent of time. The method outlined in Feichtinger et al. (2006) remains applicable in this context. By applying the existing ASM optimization theory to pension-related problems, this paper contributes to advancing the current pension literature.*

4. Numerical Analysis

In this section, we illustrate the above optimal control results via some numerical examples. We show that the setting of external control variables matters in the resultant position of each cohort in the final optimal pension structure.

4.1. Baseline Model

The key assumptions in our numerical analysis are summarised in Table 1. Most of these assumptions are commonly used in the current literature. In addition, we assume that the mortality shock ϵ follows a truncated $N(-0.0035, 0.0814^2)$ on $(-\infty, 1)$.

Table 1. Parameter values and functional specifications to be used for the numerical analysis.

Factor	Value	Factor	Value
$\mu(a)$	$0.1e^{0.1(a-88.72)}$	$C(a)$	$0.013(80+0.5a)$
$g(a)$	$110-a$	$D(a)$	$0.2e^{0.01a}$
$v(a)$	$100+a$	r	2%
$q(t,a)$	$0.05(65+t-0.5a)$	m	4%
$q(t)$	$45+0.05t$	τ	105

A brief justifications of our key assumptions is given below.

- $g(a) = 110 - a$: when this tontine-like fund was established, the initial population size was such that there were fewer people in the older age groups.
- $q(t,a) = 0.05(65 + t - 0.5a)$: throughout the operation of the fund, it may attract a growing interest from new participants at the same age, i.e., $q(t,a)$ increases when t increases. On the other hand, the joining fee increases when the age of newcomers

increases, i.e., $q(t,a)$ being a decreasing function of a. Without loss of generality, we set the number of mid-entry participants as $q(t,a) = (65 + t - 0.5a)/20$.

- $v(a) = 100 + a$: as older individuals have higher mortality rates, the asset growth rate for older members within this fund is higher. Hence, there is a greater inclination for older people to invest more money in this fund. Therefore, we assume that older individuals need to pay higher joining fees initially.
- $C(a) = 0.013(80 + 0.5a)$ and $D(a) = 0.2e^{0.01a}$: we believe that allocating resources to older members entails elevated costs and operational outlays.

We first estimate $E_\epsilon[y^*(t,a)]$. Substituting the above factors into (21) yields

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)y^*(t,a) = -0.1(1-\epsilon)e^{0.1a-8.872}y^*(t,a) + 0.05(65 + t - 0.5a)$$

with boundary conditions $y^*(0,a) = 110 - a$ and $y^*(t,65) = 45 + 0.05t$. Based on the key idea shown in Figure 1, i.e., $\Delta t \equiv \Delta a$, $y^*(t,a)$ can be recursively approximated for a given step size h, i.e.,

$$y^*(t+h, a+h) \approx y^*(t,a) - 0.1h(1-\epsilon)e^{0.1a-8.872}y^*(t,a) + 0.05h(65 + t - 0.5a). \tag{30}$$

For $0 \le t \le 40$, $65 \le a \le \tau$, we construct a set of grids $\{t_i = ih, a_j = 65 + jh\}$, $i = 0,1,\ldots,i_{max}$, $j = 0,1,\ldots,j_{max}$, where $i_{max} = \lfloor \frac{40}{h} \rfloor$ and $j_{max} = \lfloor \frac{\tau-65}{h} \rfloor$. We let $y^*_{ij} := y^*(t_i, a_j)$; then, (30) provides, for $i = 0,\ldots,i_{max}-1$, $j = 0,\ldots,j_{max}-1$,

$$y^*_{i+1,j+1} \approx \left[1 - 0.1h(1-\epsilon)e^{0.1(65+jh)-8.872}\right]y^*_{ij} + 0.05h(65 + ih - 0.5(65 + jh)) \tag{31}$$

with $y^*_{0j} = 45 - jh$ and $y^*_{i0} = 45 + 0.05ih$.

To estimate $E_\epsilon[y^*_{ij}]$, following the distributional assumption of ϵ, we generate 1000 realizations of ϵ, denoted by $\varphi_1, \ldots, \varphi_{1000}$. Then, we have

$$E_\epsilon[y^*(t_i,a_j)] \approx \frac{1}{1000}\sum_{n=1}^{1000} y^{\varphi_n}(t_i,a_j),$$

where $y^{\varphi_n}(t_i,a_j)$ satisfies (31) with ϵ replaced by φ_n. The following figures of $E_\epsilon[y^*(t,a)]$ are generated for $h = 0.5$.

Figure 2 illustrates the cohort size of the age-structured pension fund at any time–age point (t,a), where $t \in [0,40]$ and $a \in [65,105]$, under the assumed mortality shock. Additionally, Figure 3 depicts the size changes over time for selected cohorts with initial ages $a_0 = 70, 75, \ldots, 100$, while Figure 4 shows the size changes for the cohort with $a_0 = 65$ that commenced at different times $t = 0, 5, \ldots, 35$. From these figures, it is evident that older cohorts, such as those with $a_0 \ge 80$ formed at $t = 0$, do not grow in size over time. In contrast, younger cohorts formed at the fund's inception and cohorts established after $t = 0$ grow in size up to a certain point before starting to decline. One possible explanation for this difference is that individuals aged 80 and above generally have lower life expectancy, making pension investment less attractive to them. Conversely, younger individuals are more likely to join the pension fund during its operation to secure financial protection for their retirement.

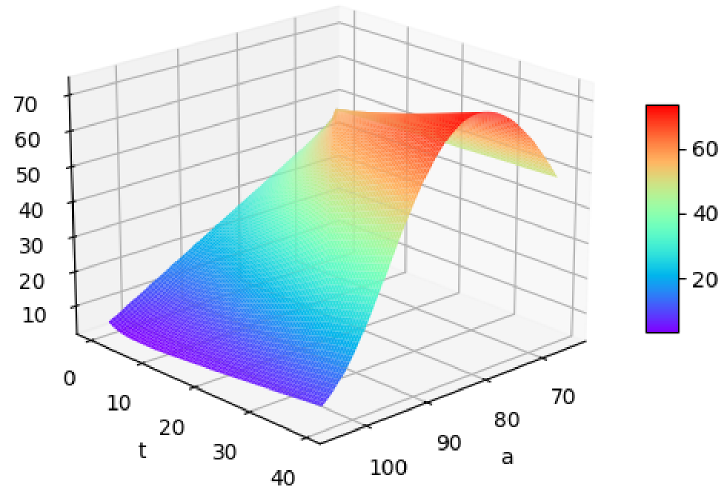

Figure 2. $E_\epsilon(y^*(t,a))$ with $t \le 40$ and $a \in [65, 105]$.

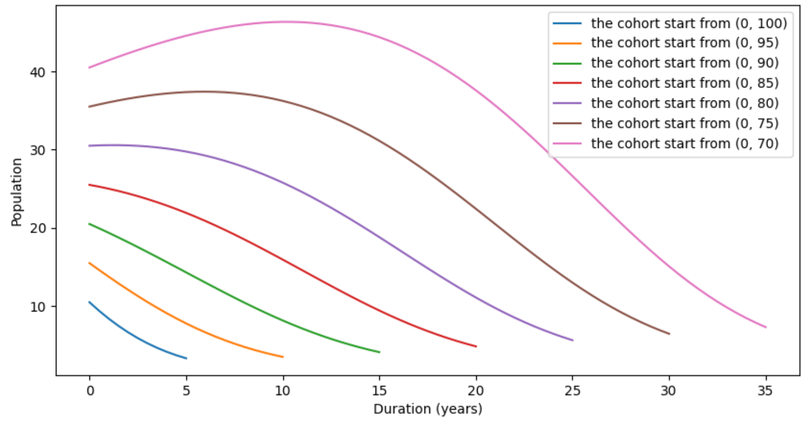

Figure 3. The cohort that exists at the beginning of the fund.

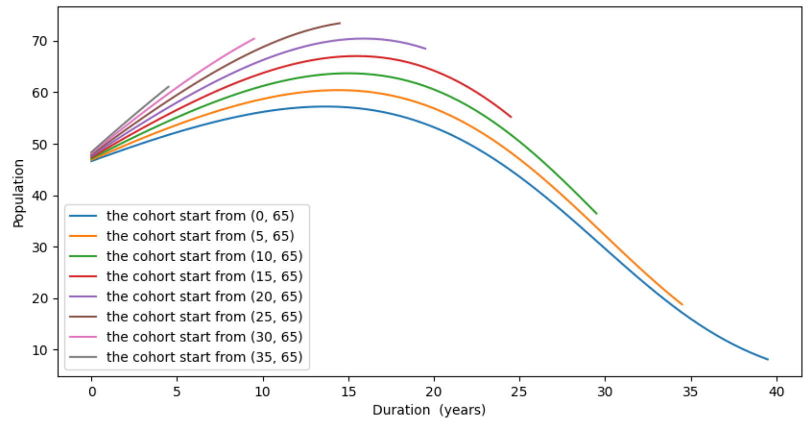

Figure 4. The cohort established during the operation of the fund.

In Figure 3, we illustrate the population dynamics of cohorts formed at the inception of the fund, selecting 5-year intervals for analysis. Initially, the cohort beginning at (0, 70) experiences an increase in population size due to new memberships; however, it later diminishes in size. In contrast, the cohort starting from (0, 100) consistently shows a decline in population size, despite new members joining, due to the high instantaneous mortality rate at this age. Furthermore, we calculate the ending time age point by adding the duration to the starting point. For instance, the cohort originating from (0, 70) with a duration of 35 years concludes at the time age point (35, 105). Similarly, the cohort from (0, 85), lasting

20 years, also ends at (35, 105). Notably, despite the varying starting ages, all cohorts terminate at age 105, the upper limit set by our population model.

In Figure 4, we examine cohorts formed during the fund's operation at 5-year intervals, focusing on their population changes. All selected cohorts have a starting age of 65, initially showing an increase in population size. However, variations in their ending ages lead to different final population sizes. For instance, the cohort beginning at (35, 65) concludes at (40, 70), maintaining a relatively high population size. Conversely, the cohort that starts at (0, 65) ends at (40, 105), resulting in a smaller population size by the end. The ending time age points for these cohorts vary between (40, 70) and (40, 105), as the fund's termination time is set to 40 years.

Based on the assumptions we made at the beginning of this section, from (28) and (29), we determine that

$$I(t,a) = \begin{cases} \dfrac{\int_t^{40} \frac{0.04}{{}_{v-t}p_a^{1.0035}} e^{-0.04(v-t)} E_\epsilon[y^*(v,a-t+v)]dv - 0.1e^{0.1a-8.872} E_\epsilon[y^*(t,a)]}{0.2e^{0.01a} E_\epsilon[y^*(t,a)]}, & 65 \le a < 65+t, \\ \dfrac{\int_a^{105} \frac{0.04}{{}_{v-a}p_a^{1.0035}} e^{-0.04(v-a)} E_\epsilon[y^*(t-a+v,v)]dv - 0.1e^{0.1a-8.872} E_\epsilon[y^*(t,a)]}{0.2e^{0.01a} E_\epsilon[y^*(t,a)]}, & a-t \ge 65, \end{cases}$$

and

$$I_0(t) = \frac{\int_t^{40} \frac{0.04}{{}_{v-t}p_{65}^{1.0035}} e^{-0.04(v-t)} E_\epsilon[y^*(v, 65-t+v)]dv - (45+0.05t)0.2e^{0.65}}{(45+0.05t)0.2e^{0.65}}.$$

Figure 5 shows the values of $I(t,a)$ within the specified range. It demonstrates that, to maximize the overall benefit, the optimal control strategy involves allocating more subsidies to cohorts that are expected to remain in the tontine-like fund for a longer duration. For example, this includes cohorts with smaller a_0 at time 0 and cohorts with $a_0 = 65$ formed at earlier times.

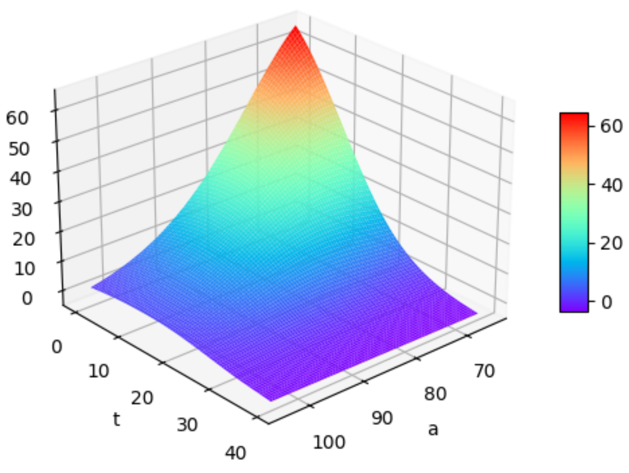

Figure 5. $I(t,a)$ with $t \le 40$ and $a \in [65, 105]$.

Similarly, from (22), we have

$$\frac{\partial h_L^*(t,a)}{\partial t} + \frac{\partial h_L^*(t,a)}{\partial a} = (1-\epsilon)\Big(0.1e^{0.1a-8.872} - 0.02\Big)h_L^*(t,a) + I(t,a),$$
$$h_L^*(0,a) = 100 + a, \qquad h_L^*(t,65) = 165 + I_0(t).$$

Using the same approach as for estimating $y^*(t,a)$, along with the previously obtained results for $I(t,a)$ and $I_0(t)$, we can estimate the values of $b^*(t,a) = 0.04h_L^*(t,a)$. Subsequently, the values of $E_\epsilon(c(t,a)) = E_\epsilon(b^*(t,a)_t p^*_{a-t})$ can be calculated. These results are summarized in Figure 6.

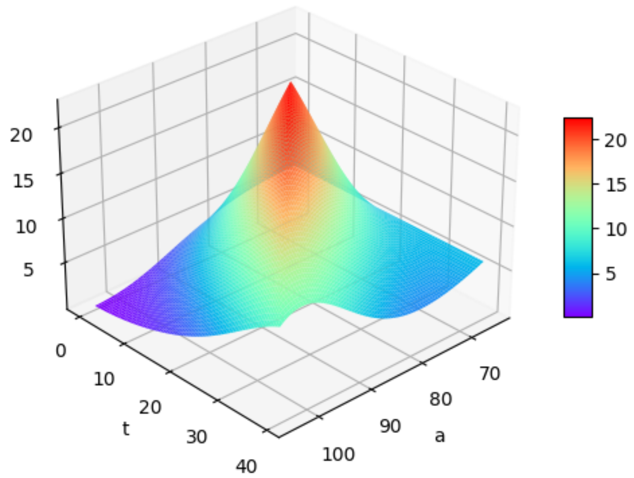

Figure 6. $E_\epsilon[c(t,a)]$ with $t \leq 40$ and $a \in [65, 105]$.

Figure 6 demonstrates that while this tontine-like fund is expected to provide higher benefits for the elderly, this is primarily true for those who have remained in the fund for longer duration—namely the youngest cohorts at the fund's inception. In contrast, older cohorts at the fund's inception should not expect to receive significant benefits under the proposed objective of our optimal control problem. The younger cohorts are prioritized, receiving more subsidies and benefits, as their longer membership duration makes them more critical to maximizing the overall benefit of the fund.

It is worth noting that the benefits for cohorts formed after time 0, i.e., $c(t,a)$ with $65 \leq a < 65 + t$, are not fully visualized in Figure 6 due to the time constraint ($t \leq 40$) imposed in our numerical analysis. If this constraint were lifted, these cohorts would also enjoy higher benefits and subsidies, as their extended membership duration in the pension fund would make them significant contributors to the overall objectives of the scheme.

4.2. The Impacts of Model Parameters

This section studies the impacts of mortality shocks, cohort sizes and subsidies on the optimal control strategies.

4.2.1. The Impacts of Mortality Shocks

In this subsection, we modify only the parameter for mortality shocks, ϵ, adjusting it from a normal distribution of $\mathrm{N}(-0.0035, 0.0814^2)$ to $\mathrm{N}(-0.03, 0.4^2)$.

As shown in Section 3, an increase in the magnitude of negative ϵ values represents a more severe epidemic, leading to an elevated instantaneous mortality rate. Consequently, as shown in Table 2, the population size decreases compared to the baseline model. This reduction aligns with our expectations; the intensification of negative shocks correlates with a more severe epidemic, which in turn increases the mortality rate and reduces the population size.

Table 2. Selected points for $E_\epsilon(y^*(t,a))$ with more severe mortality shock.

t = 10				
a	70	80	90	100
$N(-0.0035, 0.0814^2)$	53.68	45.35	24.75	7.55
$N(-0.03, 0.4^2)$	53.47	44.65	23.76	6.90
t = 30				
a	70	80	90	100
$N(-0.0035, 0.0814^2)$	59.48	67.00	47.03	13.89
$N(-0.03, 0.4^2)$	59.27	65.82	44.58	12.46

We select the cohort with the longest duration for our analysis. As depicted in Figure 7, the cohort subjected to the altered shocks is significantly smaller, with the disparity increasing over time. This observation aligns with our expectations; more pronounced negative shocks elevate the instantaneous mortality rate, leading to a more substantial difference in the cumulative number of deaths over time.

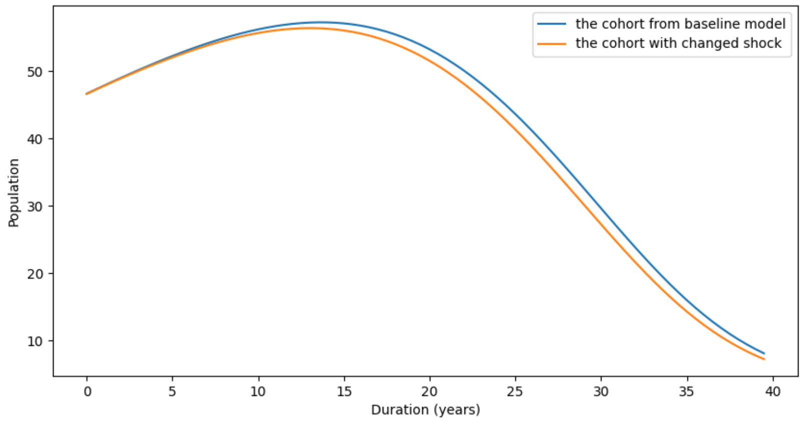

Figure 7. The cohort start from $(0, 65)$.

Table 3 shows that subsidies diminish when subjected to the altered shocks, a result that we anticipated. This decline can be attributed to the analytical formulation of $I(t,a)$, which includes $y(t,a)$ in the numerator. As the negative shocks increase the mortality rate, the population size $y(t,a)$ diminishes, subsequently reducing $I(t,a)$.

Table 3. Selected points for $I(t,a)$ with more severe mortality shock.

t = 10				
a	70	80	90	100
$N(-0.0035, 0.0814^2)$	38.81	29.01	14.90	5.31
$N(-0.03, 0.4^2)$	37.29	27.97	14.32	5.10
t = 30				
a	70	80	90	100
$N(-0.0035, 0.0814^2)$	0.16	0.70	0.93	0.16
$N(-0.03, 0.4^2)$	0.12	0.31	0.49	0.10

Furthermore, selected data points in Table 4 demonstrate a decrease in the expected $E_\epsilon[c(t,a)]$ under the changed shocks. This reduction is consistent with our expectations and logically follows from the decrease in subsidies $I(t,a)$, naturally leading to lower benefits.

Table 4. Selected points for $E_\epsilon[c(t,a)]$ with more severe mortality shock.

t = 10				
a	70	80	90	100
$\mathrm{N}(-0.0035, 0.0814^2)$	11.07	22.46	11.87	3.66
$\mathrm{N}(-0.03, 0.4^2)$	10.84	21.88	11.60	3.61
t = 30				
a	70	80	90	100
$\mathrm{N}(-0.0035, 0.0814^2)$	5.80	8.96	15.64	11.36
$\mathrm{N}(-0.03, 0.4^2)$	5.74	8.63	15.05	11.04

4.2.2. The Impacts of Cohort Sizes

In this subsection, we maintain all parameters except for the cohort size, adjusting $g(a) = 110 - a$ and $q(t) = 45 + 0.05t$ to $g(a) = 130 - 1.2a$ and $q(t) = 52 + 0.07t$. This alteration suggests a more active initial enrollment in the pension fund, leading to a larger starting cohort size.

As depicted in Table 5, the population size is marginally larger compared to the baseline model, a result that aligns with our expectations. This increase is directly attributable to the larger initial cohort size. Given that the instantaneous mortality rate remains constant, a larger initial number of participants naturally results in an increased population size over time.

Table 5. Selected points for $E_\epsilon(y^*(t,a))$ with larger initial cohort sizes.

t = 10				
a	70	80	90	100
$g(a) = 110 - a, q(t) = 45 + 0.05t$	53.68	45.35	24.75	7.55
$g(a) = 130 - 1.2a, q(t) = 52 + 0.07t$	58.85	50.03	26.77	7.84
t = 30				
a	70	80	90	100
$g(a) = 110 - a, q(t) = 45 + 0.05t$	59.48	67.00	47.03	13.89
$g(a) = 130 - 1.2a, q(t) = 52 + 0.07t$	65.03	71.18	49.06	14.23

For a more extended observation, we selected the cohort with the longest duration for comparison. Figure 8 illustrates that although the cohort with the adjusted size starts significantly larger, the difference diminishes over time. This pattern meets our expectations because, despite the increased initial numbers, the instantaneous mortality rate and mid-duration enrolments do not change, leading to a gradual narrowing of the gap as time progresses.

Table 6 illustrates that subsidies increase under the influence of a larger cohort size, which is consistent with expectations. This is because the analytical solution of $I(t,a)$ includes $y(t,a)$ in the numerator. A larger initial cohort size represents a larger population size, and a larger population size leads to greater subsidies $I(t,a)$.

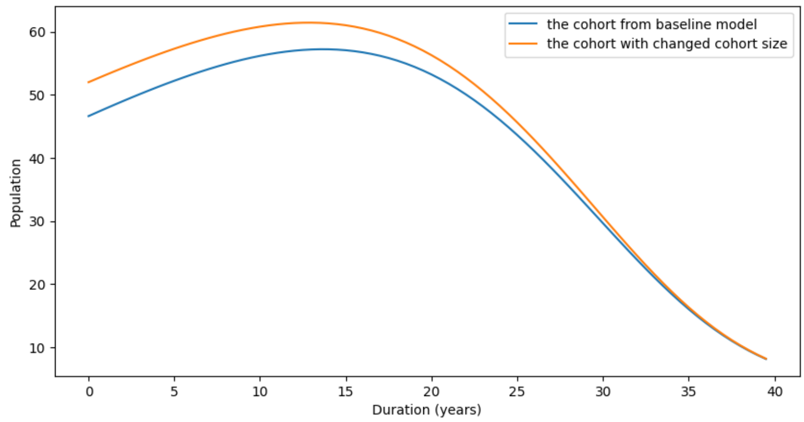

Figure 8. The cohort start from $(0, 65)$.

Table 6. Selected points for $I(t, a)$ with with larger initial cohort sizes.

	t = 10			
a	70	80	90	100
$g(a) = 110 - a, q(t) = 45 + 0.05t$	38.81	29.01	14.90	5.31
$g(a) = 130 - 1.2a, q(t) = 52 + 0.07t$	41.68	30.99	16.00	5.87
	t = 30			
a	70	80	90	100
$g(a) = 110 - a, q(t) = 45 + 0.05t$	0.16	0.70	0.93	0.16
$g(a) = 130 - 1.2a, q(t) = 52 + 0.07t$	0.18	0.84	1.05	0.17

By observing selected points in Table 7, we can see that the expected $E_\epsilon[c(t, a)]$ increases under changed cohort size, which is consistent with our expectations. This is because subsidies $I(t, a)$ increased, and naturally the benefits also increased.

Table 7. Selected points for $E_\epsilon[c(t, a)]$ with larger initial cohort sizes.

	t = 10			
a	70	80	90	100
$g(a) = 110 - a, q(t) = 45 + 0.05t$	11.07	22.46	11.87	3.66
$g(a) = 130 - 1.2a, q(t) = 52 + 0.07t$	11.51	23.82	12.49	3.80
	t = 30			
a	70	80	90	100
$g(a) = 110 - a, q(t) = 45 + 0.05t$	5.80	8.96	15.64	11.36
$g(a) = 130 - 1.2a, q(t) = 52 + 0.07t$	5.86	9.34	16.52	12.01

4.2.3. The Impacts of Subsidies

In this subsection, we focuse on modifying parameters $C(a)$ and $D(a)$ to assess their impact on subsidies while keeping other elements constant. The parameters were adjusted from $C(a) = 0.013(80 + 0.5a)$ and $D(a) = 0.2e^{0.01a}$ to $C(a) = 0.1(90 + 0.3a)$ and $D(a) = 0.3e^{0.02a}$, reflecting an increase in both the cost of subsidies and operating costs, which suggests lower operational efficiency of the fund.

Since $C(a)$ and $D(a)$ do not depend on the population model $y(t, a)$, there is no need to update the visual representations related to $y(t, a)$. Table 8 shows that the subsidies $I(t, a)$ decrease as a result of these higher costs, aligning with our expectations. An increase in subsidy costs implies reduced efficiency, leading to a decreased likelihood of government support for these subsidies.

Table 8. Selected points for $I(t,a)$ with higher subsidy costs.

t = 10				
a	70	80	90	100
$C(a) = 0.013(80 + 0.5a)$, $D(a) = 0.2e^{0.01a}$	38.81	29.01	14.90	5.31
$C(a) = 0.1(90 + 0.3a)$, $D(a) = 0.3e^{0.02a}$	33.38	24.83	11.69	2.98
t = 30				
a	70	80	90	100
$C(a) = 0.013(80 + 0.5a)$, $D(a) = 0.2e^{0.01a}$	0.16	0.70	0.93	0.16
$C(a) = 0.1(90 + 0.3a)$, $D(a) = 0.3e^{0.02a}$	0.04	0.14	0.23	0.06

Furthermore, as evidenced by data points in Table 9, the expected $E_\epsilon[c(t,a)]$ also decreases in scenarios with adjusted subsidies parameters. This decline is directly linked to the reduction in subsidies $I(t,a)$, naturally leading to diminished benefits.

Table 9. Selected points for $E_\epsilon[c(t,a)]$ with higher subsidy costs.

t = 10				
a	70	80	90	100
$C(a) = 0.013(80 + 0.5a)$, $D(a) = 0.2e^{0.01a}$	11.07	22.46	11.87	3.66
$C(a) = 0.1(90 + 0.3a)$, $D(a) = 0.3e^{0.02a}$	10.09	20.33	10.75	3.34
t = 30				
a	70	80	90	100
$C(a) = 0.013(80 + 0.5a)$, $D(a) = 0.2e^{0.01a}$	5.80	8.96	15.64	11.36
$C(a) = 0.1(90 + 0.3a)$, $D(a) = 0.3e^{0.02a}$	4.82	6.83	13.33	10.08

5. Conclusions

The main contribution of this paper is the construction of a multi-cohort tontine-like pension fund with a continuous age dimension, allowing individuals to join the fund at any time during its operation. This approach ensures a consistently large pool of participants, made possible by the ASM framework, which effectively models population dynamics through the inclusion of new entrants using $q(t,a)$. This work is the first attempt to construct pension models based on ASM. For future research, various extensions to this model can be explored. For instance, the population function $y(t,a)$ could be adapted to include gender distinctions, and the individual pension account balance $h_L(t,a)$ would be adjusted accordingly. Additionally, we incorporate a random mortality shock and an external control variable on individual accounts into the basic model. By applying the optimization methods developed in Feichtinger et al. (2003, 2006), we derive the analytical solutions. Our numerical analysis reveals that the optimal control results $I(t,a)$ prioritize cohorts expected to remain in the fund for longer periods.

Based on the numerical results, the optimal subsidy strategy $I(t,a)$ appears to favor cohorts with longer durations, resulting in these cohorts receiving significantly higher benefits $c(t,a)$ compared to those with shorter durations. We modify the corresponding parameters from three perspectives: mortality shock, cohort sizes, and subsidies, and create tables and graphs to observe the impacts. We find that more negative mortality shocks lead to higher instantaneous mortality rates, which in turn result in smaller populations, fewer subsidies and benefits. This indicates that major events like pandemics can impact pensions, and governments and fund managers need to be prepared in advance. We also find that a larger initial cohort size leads to a larger population size, resulting in more subsidies and benefits. For participants, this means they can choose to join cohorts with larger numbers. Additionally, we discover that higher costs of subsidies lead to fewer subsidies and benefits. For governments and managers, improving administrative efficiency and reducing the costs of subsidies can indeed improve the overall economic situation of pensions.

This tontine-like model has some limitations that are worth addressing and expanding upon. For example, in our model, $\mu(a)$ is independent of time. However, with the advancement of medical technology over time, we can extend $\mu(a)$ to $\mu(t,a)$ in the future. Another point is that we assume the fund only invests in bonds, whereas in reality, pension funds typically invest in both stocks and bonds. Therefore, in the future, the individual pension account $h_L(t,a)$ could be structured to invest in stocks, incorporating stochastic price processes typically used in such investments.

Author Contributions: Conceptualization, F.Z. and P.C.; methodology, F.Z.; software, F.Z.; validation, F.Z., P.C. and X.W.; formal analysis, X.W.; investigation, X.W.; resources, P.C.; writing—original draft preparation, F.Z.; writing—review and editing, P.C. and X.W.; visualization, F.Z.; supervision, P.C. and X.W.; project administration, P.C.; funding acquisition, P.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by China Scholarship Council—University of Melbourne PhD Scholarship grant number 20201012052748.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Appendix A. The Characteristic Line Method for ASMs

We set an ASM $y(t,a)$ given by

$$\frac{\partial y(t,a)}{\partial t} + \frac{\partial y(t,a)}{\partial a} = u(t,a)y(t,a) + f(t,a), \quad \text{(A1)}$$

with boundary condition

$$y(0,a) = y_0(a);$$
$$y(t,0) = v(t).$$

Method of characteristic:

If $a \geq t$, the start time age point of $y(t,a)$ is $y(0,a-t)$. We set $a = t + c$. (A1) becomes

$$\frac{dy(t,t+c)}{dt} = u(t,t+c)y(t,t+c) + f(t,t+c).$$

Using the one-dimensional ordinary differential nonhomogeneous equation formula, we can derive

$$y(t,t+c) = y(0,c)exp(\int_0^t u(x,x+c)dx) + \int_0^t f(s,c+s)exp(\int_s^t u(x,x+c)dx)ds,$$

Replacing c with $a - t$, we obtain

$$y(t,a) = y(0,a-t)exp(\int_0^t u(x,x+a-t)dx) + \int_0^t f(s,a-t+s)exp(\int_s^t u(x,x+a-t)dx)ds.$$

If $t > a$, the starting time-age point of $y(t,a)$ is $y(t-a,0)$. We set $t = a+c$. (A1) becomes

$$\frac{dy(a+c,a)}{da} = u(a+c,a)y(a+c,a) + f(a+c,a).$$

Using the one-dimensional ordinary differential nonhomogeneous equation formula, we can derive

$$y(a+c,a) = y(c,0)exp(\int_0^a u(x+c,x)dx) + \int_0^a f(s+c,s)exp(\int_s^a u(x+c,x)dx)ds.$$

Replacing c by $t-a$, we obtain

$$y(t,a) = y(t-a,0)exp(\int_0^a u(x+t-a,x)dx) + \int_0^a f(s+t-a,s)exp(\int_s^a u(x+t-a,x)dx)ds.$$

Therefore, the closed-form solutions of $y(t,a)$ is

$$y(t,a) = \begin{cases} y_0(a-t)e^{\int_0^t u(x,x+a-t)dx} + \int_0^t f(s,a-t+s)e^{\int_s^t u(x,x+a-t)dx}ds, & a-t \geq 0, \\ v(t-a)e^{\int_0^a u(x+t-a,x)dx} + \int_0^a f(s+t-a,s)e^{\int_s^a u(x+t-a,x)dx}ds, & 0 \leq a < t. \end{cases}$$

Notes

1. Stochastic processes are a commonly used mathematical model to describe random risks.
2. $N(t) = 0$ represents the scenario where all individuals in the pool have deceased. Naturally, in this case, the fund manager no longer needs to make payments for this pool.

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