Author/s:
Gunna, Ajeeth Reddy

Title:
Combinatorics of Symmetric Functions through Lattice Models

Date:
2024-01

Persistent Link:
https://hdl.handle.net/11343/345207

# Combinatorics of Symmetric Functions through Lattice Models

**Ajeeth Gunna**

**ORCID: 0000-0001-7918-0531**

A THESIS SUBMITTED IN TOTAL FULFILLMENT OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE

DEPARTMENT OF MATHEMATICS AND STATISTICS

**THE UNIVERSITY OF MELBOURNE**

January 10, 2024

అంకితం

గున్న విజయ లక్ష్మి
గున్న నిహంత్

# ABSTRACT

Symmetric functions emerge in many fields of mathematics, serving as characters in representation theory, polynomial representatives in the cohomology rings of varieties in algebraic geometry, and generating series in enumerative combinatorics, among others. Beyond these connections, the ring of symmetric functions itself contains a rich combinatorial theory that propels their systematic study. This PhD thesis explores the application of exactly solvable lattice models to symmetric functions in four distinct papers.

**Paper 1: Vertex models of canonical Grothendieck polynomials and their duals**

We study exactly solvable lattice models associated to canonical Grothendieck polynomials and their duals. We derive inversion relations and Cauchy identities.

**Paper 2: Crystals and integrable systems for edge labeled tableaux**

We define an integrable five vertex model whose partition function is the generating function $E^{\lambda}$ of edge labeled tableau of shape $\lambda$. Using this, we prove a Cauchy-type identity. We give a crystal structure on edge labeled tableau to give a positive Schur expansion of $E^{\lambda}$.

**Paper 3: Littlewood–Richardson Coefficients of spin Hall–Littlewood Functions**

We provide a combinatorial formula for the Littlewood–Richardson (LR) coefficients of spin Hall–Littlewood functions, and factorial versions of them. This is achieved by representing these functions and the LR coefficients as the partition function of a lattice model and applying the underlying Yang-Baxter equation. Our combinatorial expression is in terms of generalised honeycombs; the latter were introduced by Knutson and Tao for ordinary LR coefficients and applied to the computation of Hall polynomials by Zinn–Justin.

**Paper 4: Shuffle algebras, Lattice paths and Macdonald functions**

We consider partition functions on the $N \times N$ square lattice with the local Boltzmann weights given by the $R$-matrix of the $U_t(\widehat{sl}(n+1|m))$ quantum algebra. We identify boundary states such that the square lattice can be viewed on a conic surface. The partition function $Z_N$ on this lattice computes the weighted sum over all possible closed coloured lattice paths with $n+m$ different colours: $n$ "bosonic" colours and $m$ "fermionic" colours. Each bosonic (fermionic) path of colour $i$ contributes a factor of $z_i$ ($w_i$) to the weight of the configuration. We show the following:

(i.) $Z_N$ is a symmetric function in the spectral parameters $x_1 \dots x_N$ and generates basis elements of the commutative trigonometric Feigin–Odesskii shuffle algebra. The generating function of $Z_N$ admits a shuffle-exponential formula analogous to the Macdonald Cauchy kernel.

(ii.) $Z_N$ is a symmetric function in two alphabets $(z_1 \dots z_n)$ and $(w_1 \dots w_m)$. When $x_1 \dots x_N$ are set to be equal to the box content of a skew Young diagram $\mu/\nu$ with $N$ boxes the partition function $Z_N$ reproduces the skew Macdonald function $P_{\mu/\nu}[w-z]$.

# DECLARATION OF AUTHORSHIP

This is to certify that

i. the thesis comprises only my original work towards the Doctorate of Philosophy except where indicated in the preface,

ii. due acknowledgment has been made in the text to all other material used; and

iii. the thesis is fewer than 100 000 limit in length, exclusive of tables, maps, bibliographies and appendices as approved by the Research Higher Degrees Committee.

Ajeeth Gunna.

# PREFACE

- A. Gunna and P. Zinn-Justin. Vertex models for Canonical Grothendieck polynomials and their duals. Algebraic Combinatorics, Volume 6 (2023) no. 1, pp. 109-163. doi: 10.5802/alco.235. `arXiv:2009.13172`
- A. Gunna and T. Scrimshaw. "Crystals and integrable systems for edge labeled tableaux", Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Banglore), Séminaire Lotharingien de Combinatoire 86B (2022), Article # 61, 12pp.
- A. Gunna, M. Wheeler, and P. Zinn–Justin. Littlewood–Richardson coefficients for spin Hall–Littlewood functions. Preprint, 2023.
- A. Garbali and A. Gunna. Shuffle Algebras, Lattice paths and Macdonald functions. Preprint, `arXiv:2312.06138` ,2023.
- I acknowledge the funding I received in the form of the Melbourne Research Scholarship and Science Abroad Travel Scholarship.

# ACKNOWLEDGMENTS

I am sincerely thankful to Paul Zinn-Justin for agreeing to supervise me. His consistent encouragement and the freedom he gave me to explore my interests have been invaluable. His demand for elegance in one's work has been a continuous source of inspiration.

I express gratitude to Michael Wheeler for investing his valuable time in teaching me and for his unwavering commitment to inspiring young mathematicians. His perseverance and insistence on clarity have left a profound and lasting impact on my journey.

I am thankful to Sasha Garbali for introducing me to Shuffle algebras. His generosity with his time and his knowledge is unparalleled. I would like to thank him for making my time in Melbourne memorable.

I am grateful to Sasha and Michael for providing me with financial support for the last 6 months for my Ph.D. without which I would have had a difficult time.

I thank Travis Scrimshaw for not only agreeing to collaborate on a paper but also for providing guidance beyond the confines of Mathematics.

I am thankful to have the support of Jules Lamers, who has consistently checked in on me and shown interest in my work.

I must acknowledge my teachers from my schooling years for their numerous selfless acts during my formative years – Mr Subramaniam, Mrs Brunda, Mr Ravi Chandra, and Mr Kishore.

I thank the anonymous referees for their careful reading and the comments which have significantly improved this thesis.

## Contents

# INTRODUCTION

**Symmetric polynomials.** We refer to a polynomial in $n$ variables that remains unchanged under the permutation of its variables as a symmetric polynomial. These polynomials play a crucial role in various areas of mathematics, they appear as characters in representation theory, as polynomial representatives of varieties in their cohomology ring, as generating series in enumerative combinatorics etc,.

Let $\Lambda$ be the ring of symmetric polynomials. Among many interesting questions regarding symmetric polynomials, we consider two questions. Suppose that $(F_\lambda)$ (indexed by partitions $\lambda$) forms a basis for $\Lambda$. Is there a *Cauchy identity* involving $F_\lambda$ which takes the following form:

$$\sum_{\lambda} F_\lambda(x_1, \dots, x_n) G_\lambda(y_1, \dots, y_n) = \prod_{1 \le i,j \le n} \frac{1}{1 - x_i y_j},$$

where $G_\lambda$ is a family of symmetric polynomials dual to $(F_\lambda)$.

Secondly, since $(F_\lambda)$ form a basis, is there a combinatorial formula for the coefficients $c^{\nu}_{\lambda,\mu}$ in the following expansion?

$$F_\lambda(x_1, \dots, x_n) F_\mu(x_1, \dots, x_n) = \sum_{\nu} c^{\nu}_{\lambda,\mu} F_\nu(x_1, \dots, x_n).$$

The coefficients $c^{\nu}_{\lambda,\mu}$ are referred to as *Littlewood–Richardson coefficients* (LR). These coefficients have various interpretations. For example, the LR coefficients of Schur polynomials appear in the representation theory of $GL(n)$; they also appear in the cohomology of the Grassmannian. Meanwhile, the *Hall polynomials*, which are the LR coefficients of Hall–Littlewood polynomials, count short exact sequences of finite abelian $p$-groups.

In this thesis, we explore Cauchy identities and LR coefficients of many interesting families of symmetric polynomials using exactly solvable lattice models.

**Exactly solvable lattice models.** In the past three decades, exactly solvable lattice models have been extensively used to study symmetric polynomials. They have also seen broad applications in recent years in various areas, with recent applications to the K-theory of the Grassmannian [22, 15, 16], alternating sign matrices [18], Whittaker functions [6, 7], probability theory [3, 4], and symmetric functions [1, 5].

In a nutshell, symmetric polynomials are expressed as partition functions of a lattice model. A lattice model is built from the so-called $R$ matrices of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n)$. By leveraging the underlying Yang–Baxter equation, one can derive many identities, such as branching formulae, Cauchy identities, Littlewood identities, etc. The lattice model formulation of symmetric polynomials provides conceptually simpler proofs.

While the $R$-matrices corresponding to Quantum algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$ are enough to give a lattice model formulation for a symmetric polynomial, one can not compute LR coefficients using them. In [36], Zinn-Justin computed LR coefficients of Schur polynomials using lattice models. The key observation there is that we need to consider a lattice model made up of the $R$ matrices of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_3)$ to compute LR coefficients. This approach was later used to compute the LR coefficients of Grothendieck polynomials [30], Hall–Littlewood polynomials [38], Schubert polynomials [17] etc.

**Vertex models for Grothendieck polynomials and their duals.** Symmetric (or stable) Grothendieck polynomials ($G_\lambda$), a $K$-theoretic deformation of Schur polynomials, were first introduced by Fomin and Kirillov in [13]. They are non-homogeneous polynomials where the terms of lowest total degree is Schur polynomial. They are obtained as the limit of double Grothendieck polynomials, which were introduced by Lascoux and Schützenberger as representatives for the structure sheaves of Schubert varieties in a flag variety [20]. Building on the work of [13], Buch studied various combinatorial properties of symmetric Grothendieck polynomials, including tableau definitions and a combinatorial interpretation of their LR coefficients [8].

Dual Grothendieck polynomials ($g_\lambda$) were introduced by Lam and Pylyavskyy in [19]. These polynomials are defined as a generating series of reverse plane partitions and are dual to symmetric Grothendieck polynomials under the Hall inner product. Since their introduction, there has been a significant amount of research on these polynomials, including work by Yeliussizov on their properties, such as a determinantal formula, a connection to last passage percolation, and a Cauchy identity [31, 32, 33, 34, 35].

In [13], Fomin and Kirillov first discovered the connection between Grothendieck polynomials and quantum integrability. However, it was not until later that Grothendieck polynomials were reformulated in the context of exactly solvable lattice models [37], where the connection to quantum integrability became more explicit. In recent years, there has been a significant amount of research on exploring the various applications of this connection [21, 22, 17, 9]. Similarly, there has been some recent attention on the use of lattice models to study dual Grothendieck polynomials ($g_\lambda$) [23].

In the first paper, we study two types of lattice models for both $G_\lambda$ and $g_\lambda$. These models appear to be new. We derive Cauchy identities, Littlewood identities and inversion relations.

**Crystals and integrable systems for edge labeled tableaux.** Factorial Schur polynomials ($s_\lambda(x|a)$) are a generalisation of Schur polynomials and serve as polynomial representatives of Schubert varieties in the equivariant cohomology ring of the Grassmannian [**?**]. Thomas and Yong [27] provided a solution to the LR coefficients of $s_\lambda(x|a)$ in terms of *edge-labeled tableaux* (ELT).

Schur polynomials are defined as a generating series of semistandard Young tableaux. Analogously, in the second paper, we inquire about the properties of the generating series of ELT. By introducing *Edge Schur functions* ($E^\lambda(x|a)$) as a generating series of ELT, we establish that they are symmetric functions and prove that they satisfy a Cauchy identity with factorial Schur polynomials.

Another natural question in the study of $E^\lambda(x|a)$ involves their expansion in terms of Schur polynomials. We discover the existence of a $\mathcal{U}_q(\mathfrak{sl}_n)$-crystal structure on edge-labeled tableaux by breaking the tableau into diagonals. An immediate consequence is a positive Schur expansion of $E^\lambda(x|a)$ through the counting of the highest weight elements.

**Littlewood–Richardson coefficients of spin Hall–Littlewood functions.** Hall–Littlewood polynomials are symmetric polynomials that interpolate between monomial symmetric polynomials and Schur polynomials. They have been studied as partition functions of a lattice model [29, 28]. The LR coefficients of Hall–Littlewood polynomials are referred to as *Hall polynomials*. In [38], Zinn-Justin provided a formula for Hall polynomials in terms of honeycombs, which were previously introduced as an LR rule for Schur polynomials.

In [2], Borodin introduced *spin Hall–Littlewood functions* as a partition function of a lattice model. Borodin proved many interesting properties of these functions, including the Cauchy identity, Pieri rule, branching formulae, etc. These functions are a generalisation of Hall–Littlewood polynomials.

In the third paper, we derive a combinatorial formula for the LR coefficients of spin Hall–Littlewood functions. This is achieved by considering a hexagonal $\mathcal{U}_q(\widehat{\mathfrak{sl}}_3)$ lattice model. We prove that the partition function of that lattice is equal to the product of two spin Hall–Littlewood functions. We show that, through repeated application of the Yang–Baxter equation, the partition function of the same lattice is a combinatorial object multiplied by a spin Hall–Littlewood function. Thereby generalising the honeycomb formula for Hall polynomials by Zinn-Justin.

In the same paper, we introduce another generalisation of Hall–Littlewood polynomials called *factorial Hall–Littlewood polynomials*. These polynomials involve two sets of variables and the polynomials are symmetric in only one set of variables. We derive a combinatorial formula for their LR coefficients as well.

**Shuffle algebras, lattice paths and Macdonald functions.** The trigonometric Feigin–Odesskii shuffle algebra, denoted as $\mathcal{A}$ [26, 12, 10, 24] is an algebra of symmetric rational functions with a non-commutative multiplication. Within this algebra, there exists a commutative subalgebra [10], denoted as $\mathcal{A}^\circ$. Recently, a connection between the Feigin–Odesskii shuffle algebra $\mathcal{A}^\circ$ and lattice partition functions associated with the vertex model of $\mathcal{U}_t(\widehat{\mathfrak{sl}}_n)$ has been established [25, 14].

In the fourth paper, following [14], we investigate domain-wall boundary partition functions of vertex models, with a special relationship between the spectral parameters. These partition functions are symmetric rational functions and satisfy the wheel conditions and hence they are elements of $\mathcal{A}^\circ$. We also realize the shuffle product of these elements as lattice partition functions.

As an application of this result, we compute vertex model partition functions on the cone with the Boltzmann weights determined by the $R$-matrix of the Quantum super algebra $\mathcal{U}_t(\widehat{\mathfrak{sl}}(n+1|m))$ algebra. The generating function of these partition functions is given by the mixed Cauchy kernel [11], an object similar to the Cauchy kernel in the Macdonald theory.

As a consequence of this, we get a lattice model formulation for skew Macdonald polynomials. Our model for Macdonald polynomials is fundamentally different to the ones that exist in the literature.

## REFERENCES

[1] A. Aggarwal, A. Borodin, and M. Wheeler. Colored fermionic vertex models and symmetric functions. Preprint, `arXiv:2101.01605`, 2021.

[2] A. Borodin. On a family of symmetric rational functions. *Adv. Math.*, 306:973–1018, 2017.

[3] A. Borodin, A. Bufetov, and M. Wheeler. Between the stochastic six vertex model and Hall–Littlewood processes. *J. Combin. Theory Ser. A*, 2020. To appear.

[4] A. Borodin and M. Wheeler. Coloured stochastic vertex models and their spectral theory. Preprint, `arXiv:1808.01866`, 2018.

[5] A. Borodin and M. Wheeler. Nonsymmetric Macdonald polynomials via integrable vertex models. *Trans. Amer. Math. Soc.*, 375(12):8353–8397, 2022.

[6] B. Brubaker, V. Buciumas, D. Bump, and H. Gustafsson. Colored vertex models and Iwahori Whittaker functions. Preprint, `arXiv:1906.04140`, 2019.

[7] B. Brubaker, V. Buciumas, D. Bump, and H. Gustafsson. Metaplectic Iwahori Whittaker functions and supersymmetric lattice models. Preprint, `arXiv:2012.15778`, 2021.

[8] A. Buch. A Littlewood-Richardson rule for the $K$-theory of Grassmannians. *Acta Math.*, 189(1):37–78, 2002.

[9] V. Buciumas and T. Scrimshaw. Double Grothendieck polynomials and colored lattice models, 2020.

[10] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, and S. Yanagida. A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials. *J. Math. Phys.*, 50(9):095215, 2009. `arXiv:0904.2291`.

[11] B. Feigin, A. Hoshino, J. Shibahara, J. Shiraishi, and S. Yanagida. Kernel function and quantum algebras. Preprint, `arXiv:1002.2485`, 2010.

[12] B. Feigin and A. Tsymbaliuk. Equivariant $K$-theory of Hilbert schemes via shuffle algebra. *Kyoto J. Math.*, 51(4):831–854, 2011. `arXiv:0904.1679`.

[13] S. Fomin and A. Kirillov. Yang–Baxter equation, symmetric functions and Grothendieck polynomials, 1993.

[14] A. Garbali and P. Zinn-Justin. Shuffle algebras, lattice paths and the commuting scheme. *Contemp. Math. Special Issue: Hypergeometry, Integrability and Lie Theory*, 780, 2022. `arXiv:2110.07155`.

[15] V. Gorbounov and C. Korff. Quantum integrability and generalised quantum Schubert calculus. *Adv. Math.*, 313:282–356, 2017.

[16] A. Gunna and P. Zinn-Justin. Vertex models for canonical Grothendieck polynomials and their duals. *Algebr. Comb.*, 6(1):109–162, 2023.

[17] A. Knutson and P. Zinn-Justin. Schubert puzzles and integrability I: invariant trilinear forms, 2017.

[18] G. Kuperberg. Another proof of the alternating-sign matrix conjecture. *Internat. Math. Res. Notices*, (3):139–150, 1996.

[19] T. Lam and P. Pylyavskyy. Combinatorial Hopf Algebras and K-Homology of Grassmanians. *International Mathematics Research Notices*, 2007(9):rnm125, 2007.

[20] Alain Lascoux and Marcel-Paul Schützenberger. Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(11):629–633, 1982.

[21] K. Motegi and K. Sakai. K-theoretic boson-fermion correspondence and melting crystals. *Journal of Physics A: Mathematical and Theoretical*, 47, 11 2013.

[22] K. Motegi and K. Sakai. Vertex models, TASEP and Grothendieck polynomials. *J. Phys. A*, 46(35):355201, 26, 2013.

[23] K. Motegi and T. Scrimshaw. Refined dual grothendieck polynomials, integrability, and the schur measure, 2020.

[24] A. Neguţ. The shuffle algebra revisited. *Int. Math. Res. Not.*, 2014(22):6242–6275, 2014. `arXiv:1209.3349`.

[25] A. Neguţ. A tale of two shuffle algebras. Preprint, `arXiv:1908.08395`, 2019.

[26] O. Schiffmann and E. Vasserot. The elliptic Hall algebra and the $K$-theory of the Hilbert scheme of $\mathbb{A}^2$. *Duke Math. J.*, 162(2):279–366, 2013. `arXiv:0905.2555`.

[27] H. Thomas and A. Yong. Equivariant Schubert calculus and jeu de taquin. *Ann. Inst. Fourier (Grenoble)*, 68(1):275–318, 2018.

[28] M. Wheeler and P. Zinn-Justin. Refined Cauchy/Littlewood identities and six-vertex model partition functions: III. deformed bosons. *Adv. Math.*, 299:543–600, 2016.

[29] M. Wheeler and P. Zinn-Justin. Hall polynomials, inverse Kostka polynomials and puzzles. *Journal of Combinatorial Theory, Series A*, 159:107–163, 2018.

[30] M. Wheeler and P. Zinn-Justin. Littlewood-Richardson coefficients for Grothendieck polynomials from integrability. *J. Reine Angew. Math.*, 757:159–195, 2019.

[31] D. Yeliussizov. Duality and deformations of stable Grothendieck polynomials. *Journal of Algebraic Combinatorics*, 45(1):295–344, 2017.

[32] D. Yeliussizov. Symmetric Grothendieck polynomials, skew Cauchy identities, and dual filtered Young graphs. *J. Comb. Theory, Ser. A*, 161:453–485, 2019.

[33] D. Yeliussizov. Dual Grothendieck polynomials via last-passage percolation. *Comptes Rendus. Mathématique*, 358(4):497–503, 2020.

[34] D. Yeliussizov. Enumeration of plane partitions by descents. *Journal of Combinatorial Theory, Series A*, 178:105367, 2021.

[35] D. Yeliussizov. Random plane partitions and corner distributions. *Algebraic Combinatorics*, 4(4):599–617, 2021.

[36] P. Zinn-Justin. Littlewood–Richardson coefficients and integrable tilings. *Electron. J. Combin.*, 16(1):Research Paper 12, 33, 2009.

[37] P. Zinn-Justin. *Six-vertex, loop and tiling models: integrability and combinatorics.* Lambert Academic Publishing, 2009. Habilitation thesis.

[38] P. Zinn-Justin. Honeycombs for Hall polynomials. *Electron. J. Combin.*, 27:P2.23, 2019.

# PAPER I: VERTEX MODELS FOR CANONICAL GROTHENDIECK POLYNOMIALS AND THEIR DUALS

*Algebraic Combinatorics*
Volume 6, issue 1 (2023), p. 109–162
`https://doi.org/10.5802/alco.235`

# Vertex models for Canonical Grothendieck polynomials and their duals

Ajeeth Gunna & Paul Zinn-Justin

Abstract We study exactly solvable lattice models associated to canonical Grothendieck polynomials and their duals. We derive inversion relations and Cauchy identities.

## 1. Introduction

Grothendieck polynomials were introduced by Lascoux and Schutzenberger in [10] as representatives of $K$-theoretic Schubert classes in flag varieties. Their connection to quantum integrability was noticed as early as [3], though it took some time to reformulate Grothendieck polynomials in the context of exactly solvable lattice models [19], where quantum integrability is most explicit. Recently, a large literature has developed around these ideas [2, 7, 13, 14, 15]. In this work we focus on *symmetric* Grothendieck polynomials (also called stable Grothendieck polynomials), i.e. the ones that are related to the $K$-theory of Grassmannians [1], though we expect many of our ideas to be applicable to more general (partial) flag varieties. We also consider their duals, in the sense of product/coproduct duality.(1) We propose some new formulations of both Grothendieck and dual Grothendieck in terms of certain "bosonic" exactly solvable lattice models.

Let $\Lambda$ be the ring of symmetric functions. Even though the elements of $\Lambda$ are not polynomials, by abuse of language we shall refer to them as polynomials, identifying a symmetric function $F$ with the corresponding symmetric polynomial $F(x_1,\ldots,x_n)$. Schur polynomials $s_\lambda$ (where $\lambda$ runs over all partitions) form an orthonormal basis of $\Lambda$ under the Hall inner product. The involution map $\omega$, which sends $e_k$ (*elementary symmetric polynomials*) to $h_k$ (*complete homogeneous symmetric polynomials*), maps $s_\lambda$ to $s_{\lambda'}$ where $\lambda'$ is the transpose of $\lambda$. Let $\tilde{\Lambda}$ be the completion of $\Lambda$, which is obtained by allowing infinite linear combinations of $s_\lambda$.

Grothendieck polynomials $G_\lambda$ are non homogeneous symmetric polynomials; with the appropriate choice of variables, $G_\lambda = s_\lambda +$ higher order terms. When the number

---

*Manuscript received 18th December 2020, revised 2nd March 2022, accepted 7th March 2022.*

Keywords. Grothendieck polynomials, Exactly solvable lattice models.

(1) These should not be confused with the dual Grothendieck polynomials that are e.g. considered in [15]. The latter are dual w.r.t. the natural scalar product of $K$-theory. In contrast, ours are dual w.r.t. the Hall inner product.

ISSN: 2589-5486 `http://algebraic-combinatorics.org/`

of variables grows, their degree grows, so they must be considered as elements of $\tilde{\Lambda}$. The structure constants $c^{\nu}_{\lambda,\mu}$ defined by

$$G_\lambda G_\mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} G_\nu,$$

satisfy $c^{\nu'}_{\lambda',\mu'} = c^{\nu}_{\lambda,\mu}$ [4, Ex. 9.20]. However the image of $G_\lambda$ under $\omega$ is *not* $G_{\lambda'}$. This implies that the family of polynomials $\omega(G_{\lambda'})$ has the same structure constants as Grothendieck polynomials. We shall not be dealing with structure constants in this paper, reserving them for subsequent work [5] and only mention them as motivation for what follows.

In [8], Lam and Pylyavskyy defined *dual Grothendieck polynomials* ($g_\lambda$) as certain generating functions of *reverse plane partitions*. These polynomials are dual to $G_\lambda$ under the Hall inner product, and are of the form $g_\lambda = s_\lambda +$ lower order terms. Similarly to Grothendieck polynomials, the image of $g_\lambda$ under the involution map is not $g_{\lambda'}$.

In [16], Yeliussizov introduced a two parameter version of Grothendieck polynomials and their dual, which he called *canonical Grothendieck polynomials* and *dual stable canonical Grothendieck polynomials*. For more detailed combinatorial properties and definitions we refer the reader to [16]. Canonical Grothendieck polynomials and their dual satisfy the following relations:

$$\omega(G^{(\alpha,\beta)}_{\lambda}) = G^{(\beta,\alpha)}_{\lambda'} \qquad \omega(g^{(\alpha,\beta)}_{\lambda}) = g^{(\beta,\alpha)}_{\lambda'}.$$

In this paper, we shall study two types of vertex models, based on the way partitions are encoded, for both $G^{(\alpha,\beta)}_{\lambda}$ and $g^{(\alpha,\beta)}_{\lambda}$. We call a vertex model a *row model* (resp. *column model*) when the partitions are encoded by row (resp. column) multiplicities. Section 2 is devoted to the former, Section 3 to the latter. All the models studied in this paper appear to be new (see also [14]).

We then introduce (Section 4) *generalised Grothendieck polynomials* which are obtained by attaching additional variables to the vertical lines of the underlying lattice model. Along the process, we recover the *generalised dual Grothendieck polynomials* defined by Yeliussizov [18].

In Section 5, we show that the transfer matrices of these lattice models satisfy remarkable *inversion relations*. These show a deep connection between row and column lattice models, thus embodying the involution $\omega$ at the level of transfer matrices. This should be reminiscent of similar relations satisfied by the usual free fermionic *vertex operators* related to Schur functions (see e.g. [20], or [19] and references therein); indeed, our transfer matrices can be thought of as deformations of these vertex operators.

Finally, in Section 6, we show how "quantum integrability" in the form of RLL relations immediately implies the *Cauchy* identities

$$(1) \qquad \sum_{\lambda} G^{(-\alpha,-\beta)}_{\lambda}(x_1,x_2,\dots,x_m) g^{(\alpha,\beta)}_{\lambda}(y_1,y_2,\dots,y_n) = \prod_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} \frac{1}{1-x_i y_j}$$

$$(2) \qquad \sum_{\lambda} G^{(-\beta,-\alpha)}_{\lambda'}(x_1,x_2,\dots,x_m) g^{(\alpha,\beta)}_{\lambda}(y_1,y_2,\dots,y_n) = \prod_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} (1+x_i y_j)$$

for (generalised) Grothendieck polynomials and their duals. By specializing $\alpha = 0$ and $\beta = 1$, we recover the *Cauchy identity* for Grothendieck polynomials and its dual [9, 17]. Throughout this paper, $\alpha$ and $\beta$ are constants and our results hold for all values, with some intricacies for $\beta = 0$.

The appendix contains proofs of the RLL relations.

## 2. Row Vertex Models

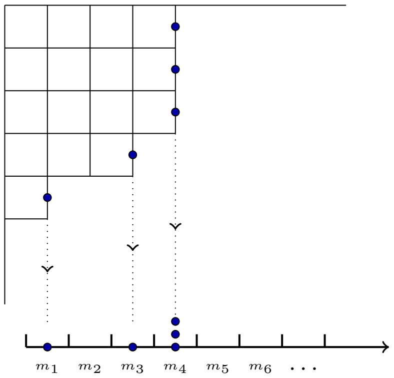
2.1. Definition of Physical space. Let $V^r$ be an infinite dimensional vector space with basis indexed by collections of nonnegative integers $(m_i)_{i\in\mathbb{Z}_{>0}}$ such that only a finite number of $m_i$s are nonzero; we view it as a subspace of $\bigotimes_{i=1}^{\infty} V_i$ where each $V_i = \operatorname{Span}(|0\rangle, |1\rangle, \dots)$ has a basis indexed by a single nonnegative integer:

$$V^r = \operatorname{Span}\{|m_1\rangle \otimes |m_2\rangle \otimes |m_3\rangle \cdots\} \qquad m_i \geqslant 0,\ i \geqslant 1. \tag{3}$$

We shall identify partitions with basis elements of $V^r$. Given a partition $\lambda$, which we view as a Young diagram, let $|\lambda\rangle$ be the basis vector with integers

$$m_i(\lambda) = \text{number of rows of size } i \text{ of } \lambda$$

(hence the superscript $r$). For example, we identify the partition $\lambda = (4,4,4,3,1)$ with the basis element $|1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |3\rangle \otimes |0\rangle \dots$ of $V^r$:



All the vertex models studied in this paper follow a general template. In order to not repeat ourselves, we shall study this model in detail and then skip the general arguments in other models.

### 2.2. Row vertex model for canonical Grothendieck polynomials.

2.2.1. *Conventions.* We use the standard diagrammatic formalism to interpret lattice models in terms of linear operators. We briefly review it here, and fix conventions.

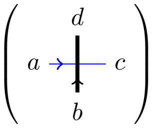
All our lattice models are defined on some domain of the plane which consists of edges and vertices of valency 4. Edges traverse vertices to form lines, which are given a certain orientation: in all that follows, the domain is a (rectangular) region of the square lattice, so that lines can be either horizontal (also called "auxiliary" lines), in which case they are oriented left to right, or vertical (also called "physical" lines), in which case they are oriented bottom to top.

To each line is associated a vector space, and juxtaposition of lines corresponds to tensor product (the order of the factors is the order of the incoming external lines). These vector spaces come equipped with a basis labelled by the various states that edges of the lattice model carry. In our case, vertical lines are numbered $1, 2, \dots$ from left to right, and vertical edges carry a nonnegative integer, so that to vertical line numbered $i$ we assign the vector space $V_i$ (and collectively they form the "physical space" $V^r$). Horizontal edges can carry either labels $0, 1$, in which case we call the horizontal line fermionic and assign to it a space $F \cong \mathbb{C}^2$ (possibly adding a subscript to distinguish the various horizontal lines), or it can carry a nonnegative integer (bosonic line), in which case we call the corresponding vector space $W$. Graphically, when the auxiliary line is fermionic, we draw thin lines. When they are bosonic, we draw thick lines.

Finally, an important convention is that we transpose all linear operators in order to facilitate reading expressions from left to right; this means that if incoming lines at a vertex form $A \otimes B$ and outgoing lines form $C \otimes D$, then to the vertex is associated a linear operator from $C \otimes D$ to $A \otimes B$. We hope that this does not cause any confusion.

2.2.2. *Definition of the $L$ matrix.* In this subsection, the auxiliary line is fermionic. To every vertex we assign a (Boltzmann) weight that depends on the local configuration (i.e. states of the edges) around it. The weights are given as follows:

(4) $w_x$

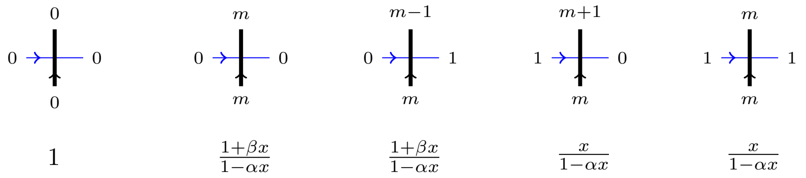


$$\equiv w_x(a,b;c,d) = \delta_{a+b,c+d} \begin{cases} \frac{x}{1-\alpha x} & \text{when } a=1, \\ \frac{1+\beta x}{1-\alpha x} & \text{when } a=0 \text{ and } b \neq 0, \\ 1 & a,b,c,d=0, \end{cases}$$

where $a,c \in \{0,1\},$ and $b,d \in \mathbb{Z}_{\geqslant 0}$.

Let us now represent the vertices graphically with their Boltzmann weights written below them.

(5)



The corresponding linear operator is the so-called $L$ matrix; it acts on $F_i \otimes V_j$, where $F_i = \operatorname{span}\{|0\rangle, |1\rangle\}$. Let us first define annihilation and creation operators, $\phi_j$ and $\phi_j^\dagger$, acting on the $j^{th}$ factor $V_j$ of $V^r$:

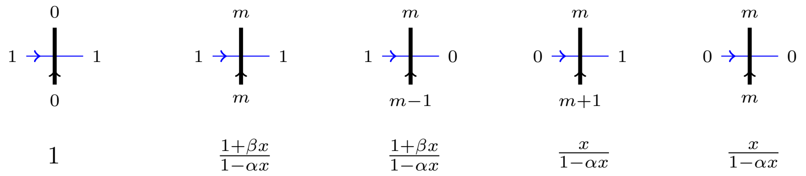
$$\phi_j |m\rangle = |m-1\rangle \qquad \phi_j^\dagger |m\rangle = |m+1\rangle$$
$$\phi_j |0\rangle = |0\rangle .$$

Then

$$L_{i,j}(x) = \frac{1}{1-\alpha x} \begin{pmatrix} \delta_{0,m}(1-\alpha x) + (1-\delta_{0m})(1+\beta x) & (1+\beta x)\phi_j \\ x\phi_j^\dagger & x \end{pmatrix}. \tag{6}$$

We shall now define dual $L$ matrices, $L^*$. We obtain $L^*$ by flipping the vertices upside down and replacing $0's$ with $1's$ and vice versa on the horizontal edges.

(7)

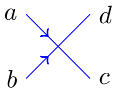


Then define $L^*$ acting on $F_i \otimes V_j$ as follows:

$$L^*_{i,j}(x) = \frac{1}{1-\alpha x} \begin{pmatrix} x & x\phi_j^\dagger \\ (1+\beta x)\phi_j & \delta_{0,m}(1-\alpha x) + (1-\delta_{0m})(1+\beta x) \end{pmatrix}. \tag{8}$$

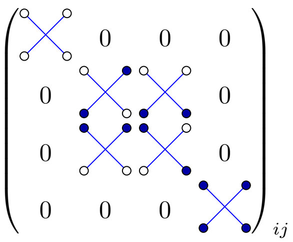
2.2.3. *$R$-matrix and Yang–Baxter relations.* Consider the vector spaces $F_i, F_j$ where $i < j$. Then we define a $R$-matrix which acts linearly on $F_i \otimes F_j$ as follows,

$$R_{i,j}(x_i,x_j) : |a\rangle \otimes |b\rangle \mapsto \sum_{c,d \text{ where } a+b=c+d} R^{a,d}_{b,c}(x_i,x_j)\, |c\rangle \otimes |d\rangle .$$

Graphically, we represent the entry $R^{a,d}_{b,c}$ as . We now give the $R$ matrix that underpins the integrability of the vertex model presented above. For convenience, let us represent $|0\rangle$ and $|1\rangle$ of $F$ as empty or occupied:

(9)

$R_{i,j}(x,y) =$



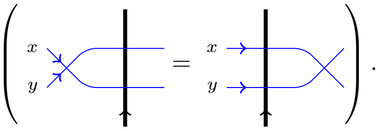
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1+\beta x)}{(1+\beta y)}\frac{y}{x} & 0 \\ 0 & 1 & 1-\frac{(1+\beta x)}{(1+\beta y)}\frac{y}{x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \in \mathrm{End}(F_i \otimes F_j).$$

One recognizes this as the $R$-matrix of the five-vertex model [6] with spectral parameter $\frac{x}{1-\beta x}$. It can be obtained as a limit of the $R$ matrix of the stochastic six-vertex model where the quantum parameter is sent to 0.(2)

Together with the $L_{i,n}$ and $L_{j,n}$ matrices, $R_{ij}$ satisfies the RLL relation in End $(F_i \otimes F_j \otimes V_n)$:

(10)

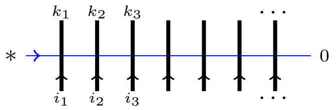
$$R_{ij}(x,y)L_{i,n}(x)L_{j,n}(y) = L_{j,n}(y)L_{i,n}(x)R_{ij}(x,y)$$



We skip the proof of the above equation; it is best checked by computer with symbolic calculation software.

2.2.4. *Transfer matrices.* We shall now build a vertex model based on the $L$-matrix above. It is convenient to depict a single row of the model as in the following picture:

$$w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\}) =$$



where the $*$ on the left means that we are summing over all possible states. Even though we are considering an infinitely large row of vertices, the weight is uniquely defined. To see this, fix the labels on the top and bottom. Since there are only finitely many non zero labels on top and bottom, sufficiently far to the right the horizontal labels are constant, and we choose them to be 0s. Graphically, we show this by assigning 0 to the horizontal edge on the far right. Then, when the bottom and top labels are fixed, there is a unique configuration because of the local conservation around every vertex.

We now define the corresponding *transfer matrix* $T$ which acts linearly on $V^r$ as follows,

(11) $$T(x) : |i_1\rangle \otimes |i_2\rangle \otimes \cdots \mapsto \sum_{k_1,k_2\dots \geqslant 0} w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})\, |k_1\rangle \otimes |k_2\rangle \otimes \cdots.$$

---

(2) The $L$-matrices (6) and (8) can presumably be obtained as a similar limit of the $R$-matrix of $U_q(\widehat{\mathfrak{sl}(2)})$ where the vertical space is a suitable Verma module representation [11].

One can rewrite it in terms of the $L$-matrix as

$$T(x) = \lim_{n\to\infty} \langle *|\, L_{01}(x)L_{02}(x)\dots L_{0n}(x)\, |0\rangle \tag{12}$$

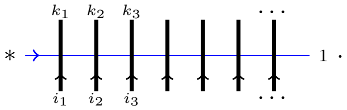
where the vector space attached to the horizontal line is labelled 0 by the subscript, whereas the vertical lines are labelled $1, 2, \dots$. Here $|0\rangle$ is the basis vector of the horizontal space, whereas $\langle *|$ is the sum of basis vectors of the dual of the horizontal space. The limit is entry-wise and is well-defined because of the aforementioned uniqueness of the configuration.

Similarly, we can define the dual transfer matrices $T^*$:

$$T^*(x) : |i_1\rangle \otimes |i_2\rangle \otimes \cdots \mapsto \sum_{k_1,k_2,\dots \geqslant 0} w^*(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})\, |k_1\rangle \otimes |k_2\rangle \otimes \cdots \tag{13}$$

where the right boundary is fixed to be 1:

$$w^*(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\}) =$$

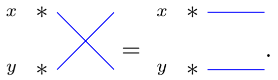


Equivalently,

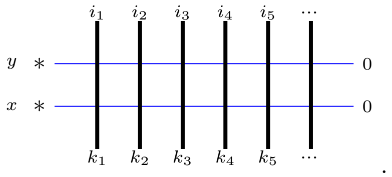
$$T^*(x) = \lim_{n\to\infty} \langle *|\, L^*_{01}(x)L_{02}(x)\dots L^*_{0n}(x)\, |1\rangle\,. \tag{14}$$

Throughout this paper we use the same conventions to define transfer matrices.

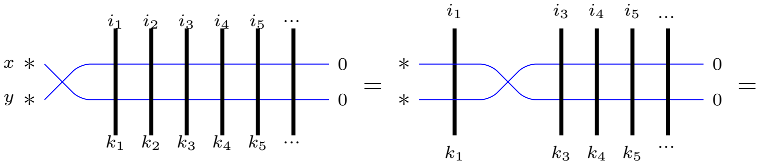
2.2.5. *Commutation relation of the transfer matrices.* Observe that the sum of the entries in a column of the $R$ matrix is always 1. This means that the state which is the sum of all possible states is an eigenvector of the $R$ matrix with eigenvalue 1. This property can reinterpreted as the fact that the partition function of a single vertex with fixed boundaries on the right and free boundary on the left is always 1:

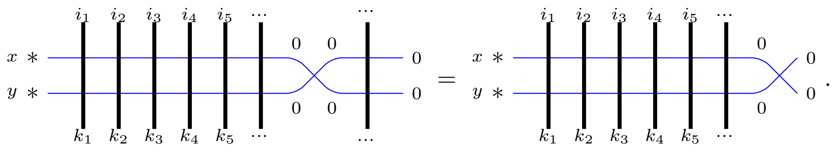


Consider the product of two transfer matrices, $T(x)$ and $T(y)$. Graphically, taking the product amounts to stacking the two row to row transfer matrices one upon the other. Observe that the boundary on the left is free and for sufficiently large $n$, the boundary on the right is fixed. Recall that an edge with $*$ is a free boundary. Thus $T(x)T(y)$ is

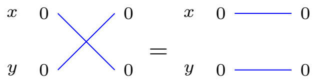


Now multiply $T(x)T(y)$ on the left by $R(x, y)$, and apply the RLL relation finitely many times:

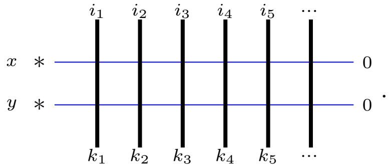




Sufficiently far to the right, we are left with a cross where all edges are labelled 0:



and the corresponding entry of the $R$ matrix is 1. Thus we get $T(y)T(x)$:



2.2.6. *Canonical Grothendieck polynomials.* Given that the transfer matrices commute, the polynomials defined using them are invariant under permutation of the variables. It is also easy to see that $T(0) = 1$, so that these polynomials satisfy the stability property which makes them an element of $\tilde{\Lambda}$. We now prove that the polynomials defined using $T$ are *canonical Grothendieck polynomials.*

Before we prove it, let us recall the branching formula for $G_\lambda^{(\alpha,\beta)}$ from [16, Proposition 8.8]. For a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, denote $\bar{\lambda} = (\lambda_2, \lambda_3, \dots)$.

We have

$$G_\lambda^{(\alpha,\beta)}(x_1, \dots, x_n, x_{n+1}) = \sum_{\lambda/\mu \text{ hor. strip}} G_\mu^{(\alpha,\beta)}(x_1, \dots, x_n) G_{\lambda /\!\!/ \mu}^{(\alpha,\beta)}(x_{n+1}), \tag{15}$$

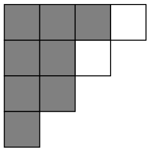
and

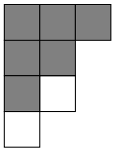
$$G_{\lambda /\!\!/ \mu}^{(\alpha,\beta)}(x) = \left(\frac{x}{1-\alpha x}\right)^{|\lambda/\mu|} \left(\frac{1+\beta x}{1-\alpha x}\right)^{r(\mu/\bar{\lambda})}, \tag{16}$$

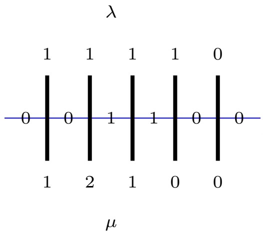
where $r(\lambda/\mu)$ is the number of non zero rows of $\lambda/\mu$. We shall take this branching formula as the definition of $G_\lambda^{(\alpha,\beta)}$.

Remark 2.1. In order to dispel any confusion, we point out that $G_{\lambda /\!\!/ \mu}(x_1, \dots, x_n)$ polynomials are not the same as skew Grothendieck polynomials $G_{\lambda/\mu}$. For a simple counter example, observe that for any partition $\lambda$, we have $G_{\lambda /\!\!/ \lambda}^{(\alpha,\beta)}(x) = \left(\frac{1+\beta x}{1-\alpha x}\right)^{r(\lambda/\bar{\lambda})} \neq G_{\lambda/\lambda}^{(\alpha,\beta)}(x) = 1.$

Let us now look at an example to understand $r(\mu/\bar{\lambda})$. Consider the partitions $\lambda = (4,3,2,1)$ and $\mu = (3,2,2,1)$. Then $\bar{\lambda} = (3,2,1)$ and $r(\mu/\bar{\lambda}) = 2$:

$\lambda/\mu =$ 

$\mu/\bar{\lambda} =$ 



.

We can alternatively formulate $r(\mu/\bar{\lambda})$ as the number of removable boxes of $\mu$ that do not lie in the same column as any box of $\lambda/\mu$.

As a consequence of recording partitions with row multiplicities, every vertex with a non zero label on the bottom edge corresponds to a removable box of $\mu$. If a box is added to the $i^{th}$ column of $\mu$, then the removable box corresponding to that vertex at site $i$ will be in the same column as the new box. So $r(\mu/\bar{\lambda})$ is precisely the number of vertices with zero label on the left edge and a non zero label on the bottom edge.

Theorem 2.2. *The canonical Grothendieck polynomials $G_\lambda^{(\alpha,\beta)}(x)$ are given by*

$$G_\lambda^{(\alpha,\beta)}(x_1,\dots,x_n) = \langle 0|\, T(x_1)\dots T(x_n)\,|\lambda\rangle \tag{17}$$

$$G_\lambda^{(\alpha,\beta)}(x_1,\dots,x_n) = \langle \lambda|\, T^*(x_n)\dots T^*(x_1)\,|0\rangle \tag{18}$$

*where $|\lambda\rangle = \bigotimes_{i=1}^{\infty} |m_i(\lambda)\rangle$, and similarly for the dual state $\langle\lambda|$.*

*Proof.* We shall prove (17), and (18) follows immediately as a consequence of the way we defined the $L^*$ matrix. Fix $\lambda$, then we can just consider the finite transfer matrix of size $\lambda_1$. Inserting a complete set of states before the final transfer matrix, we have the following branching formula

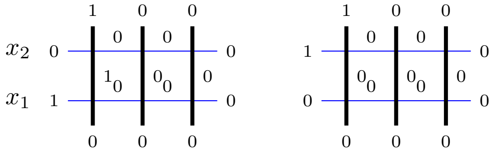
$$G_\lambda^{(\alpha,\beta)}(x_1,\dots,x_n,x) = \sum_\mu \langle 0|\, T(x_1)\dots T(x_n)\,|\mu\rangle \langle\mu|\, T(x)\,|\lambda\rangle\,.$$

On comparing the branching formula for $G_\lambda^{(\alpha,\beta)}$ (eq. (15)), it is enough to show $G_{\lambda/\!\!/\mu}^{(\alpha,\beta)}(x) = \langle\mu|\, T(x)\,|\lambda\rangle$. Recall that for a horizontal strip $\lambda/\mu$, we have

$$G_{\lambda/\!\!/\mu}^{(\alpha,\beta)}(x) = \left(\frac{x}{1-\alpha x}\right)^{|\lambda/\mu|} \left(\frac{1+\beta x}{1-\alpha x}\right)^{r(\mu/\bar{\lambda})}.$$

Based on the vertices that we used to define $T$, one easily observes that $\langle\mu|\, T(x)\,|\lambda\rangle \neq 0$ if and only if $\lambda/\mu$ is a horizontal strip. The label 1 on the left edge at site $i$ amounts to adding a box in the $i^{th}$ column from the left. For every such vertex, we get a factor of $\frac{x}{1-\alpha x}$. From our previous analysis, we see that $r(\mu/\bar{\lambda})$ is exactly the number of vertices with the label 0 on the left edge and a non zero label on the bottom edge, where each such vertex has a weight of $\frac{1+\beta x}{1-\alpha x}$. ☐

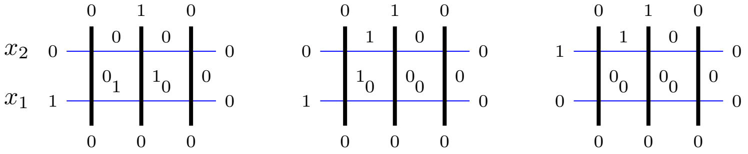
Example 2.3. For partition $\lambda = (1,0)$ we have the following two possible configurations on the left, each with a unique configuration on the interior.



Therefore,

$$G_\lambda^{(\alpha,\beta)}(x_1,x_2) = \left(\frac{x_1}{1-\alpha x_1}\right)\left(\frac{1+\beta x_2}{1-\alpha x_2}\right) + \left(\frac{x_2}{1-\alpha x_2}\right).$$

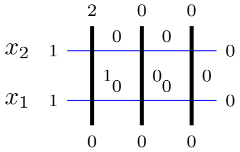
Example 2.4. For partition $\lambda = (2,0)$, we have the following configurations.



$$G_\lambda^{(\alpha,\beta)}(x_1,x_2) =$$

$$\left(\frac{x_1}{1-\alpha x_1}\right)^2\left(\frac{1+\beta x_2}{1-\alpha x_2}\right) + \left(\frac{x_1}{1-\alpha x_1}\right)\left(\frac{x_2}{1-\alpha x_2}\right)\left(\frac{1+\beta x_2}{1-\alpha x_2}\right) + \left(\frac{x_2}{1-\alpha x_2}\right)^2$$

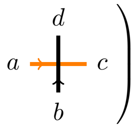
Example 2.5. For the partition $\lambda = (1,1)$, there is a unique configuration, where the overall weight is the polynomial $G_{(1,1)}^{(\alpha,\beta)}(x_1,x_2)$.



$$G_\lambda^{(\alpha,\beta)}(x_1,x_2) = \left(\frac{x_1}{1-\alpha x_1}\right)\left(\frac{x_2}{1-\alpha x_2}\right).$$

2.3. Row vertex model for dual canonical Grothendieck polynomials. In this section, we consider a similar vertex model as the one introduced in Section 2.2, but with a bosonic auxiliary line. This means that that we shall associate an infinite dimensional vector space to the values a horizontal line can carry. The Boltzmann weights of the vertices are the following:

(19)



$$w_x\left(\begin{array}{ccc} & d & \\ a & & c \\ & b & \end{array}\right) \equiv w_x(a,b;c,d) = \delta_{a+b,c+d}\begin{cases}(\alpha+\beta)^{a-d-1}(x+\alpha)\beta^d & a > d,\\ \beta^{a-1}x & 0 < a \leqslant d,\\ 1 & a = 0,\end{cases}$$

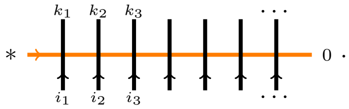
where $a,b,c,d \in \mathbb{Z}_{\geqslant 0}$.

Let $W = \mathrm{Span}\{|j\rangle\}_{j\in\mathbb{Z}_{\geqslant 0}}$ be an infinite dimensional vector space, and for $1 \leqslant i \leqslant n$, let $W_i$ be a copy of $W$. Then we define an $l$ matrix which acts linearly on $W_i \otimes V_j$ as follows:

$$l_{i,j}\left(x_i\right) : |a\rangle \otimes |b\rangle \mapsto \sum_{c,d \text{ where } a+b=c+d} w_{x_i}\Big(a,b;c,d\Big)\,|c\rangle \otimes |d\rangle\,. \tag{20}$$

Let $w(\{i_1,i_2,\ldots\};\{k_1,k_2,\ldots\})$ be the weight of single row of vertices.

$$w(\{i_1,i_2,\ldots\};\{k_1,k_2,\ldots\}) =$$



We define the transfer matrix $t$ which acts linearly on $V^r$ as follows:

$$t(x) : |i_1\rangle \otimes |i_2\rangle \otimes \cdots \mapsto \sum_{k_1,k_2\ldots \geqslant 0} w(\{i_1,i_2,\ldots\};\{k_1,k_2,\ldots\})\,|k_1\rangle \otimes |k_2\rangle \otimes \cdots. \tag{21}$$

As the horizontal lines are bosonic, we represent the $r$-matrix as a cross of thick lines (

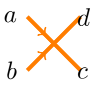
). Consider the vector spaces $W_i, W_j$ where $i<j$. Define an $r$-matrix which acts linearly on $W_i \otimes W_j$ as follows:

$$r_{i,j}(x_i,x_j) : |a\rangle \otimes |b\rangle \mapsto \sum_{c,d \text{ where } a+b=c+d} r^{a,d}_{b,c}(x_i,x_j)\,|c\rangle \otimes |d\rangle\,. \tag{22}$$

where the entries of $r$-matrix here are the following:

(23)

$$r^{a,d}_{b,c}(x,y) =$$

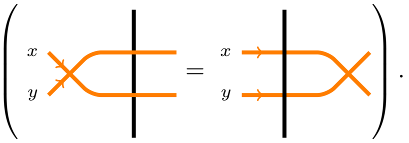


$$= \delta_{a+b,c+d} \begin{cases} 0 & b>c \\ 1 & b=c=0 \\ \dfrac{y}{x} & b=c>0 \\ \left(1-\dfrac{y}{x}\right)\left(1-\dfrac{y}{\beta}\right)^{a-d-1} & b=0, c\neq 0 \\ \left(1-\dfrac{y}{x}\right)\left(1-\dfrac{y}{\beta}\right)^{a-d-1}\left(\dfrac{y}{\beta}\right) & b>0, b<c. \end{cases}$$

Together with matrices $l_{i,n}$ and $l_{j,n}$, $r_{ij}$ satisfies the RLL relation in End $(W_i \otimes W_j \otimes V_n)$ (see Appendix A.2):

(24)

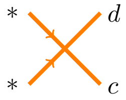
$$r_{ij}(x,y)l_{i,n}(x)l_{j,n}(y) = l_{j,n}(y)l_{i,n}(x)r_{ij}(x,y)$$



Remark 2.6. Observe that the $r$ matrix is not defined at $\beta=0$. However, the weights (19) for the $l$-matrix force all horizontal labels to be 0 or 1 as $\beta$ is sent to zero, and such entries of the $r$-matrix are well defined. We shall also study a different model for $g^{(\alpha,\beta)}_{\lambda}$ polynomials when $\beta=0$ in Section 3.4.

2.3.1. *Eigenvector of the $r$-matrix.* We proceed as in the previous section, showing the state which is sum of all the possible states is an eigenvector of the $r$-matrix. We show this by computing the partition function of a single vertex with fixed right boundary and free left boundary:

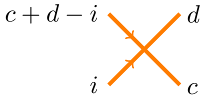
$$\mathcal{Z}(c,d) =$$



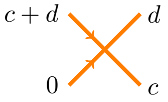
(where $c$, $d$ are non negative integers) is constant and equal to 1.

We compute:

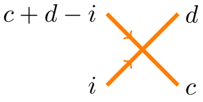
$$\mathcal{Z}(c,d) = \sum_{i=0}^{c}$$



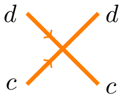
$$=$$



$$+\sum_{i=1}^{c-1}$$



$$+$$



$$\begin{aligned}
&= \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-1} + \sum_{i=1}^{c-1}\left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-i-1}\left(\frac{y}{\beta}\right) + \frac{y}{x}\\
&= \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-1} + \frac{y}{\beta}\left(\frac{1-\frac{y}{x}}{1-\frac{y}{\beta}}\right)\sum_{i=1}^{c-1}\left(1-\frac{y}{\beta}\right)^{i} + \frac{y}{x}\\
&= \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-1} + \frac{y}{\beta}\left(\frac{1-\frac{y}{x}}{1-\frac{y}{\beta}}\right)\frac{\left(1-\frac{y}{\beta}\right)\left(1-\left(1-\frac{y}{\beta}\right)^{c-1}\right)}{1-\left(1-\frac{y}{\beta}\right)} + \frac{y}{x}\\
&= \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-1} + \left(1-\frac{y}{x}\right)\left(1-\left(1-\frac{y}{\beta}\right)^{c-1}\right) + \frac{y}{x}\\
&= \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-1} + \left(1-\frac{y}{x}\right) - \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-1} + \frac{y}{x}\\
&= 1.
\end{aligned}$$

By repeating the same argument as in Section 2.2.5, we get the commutation relation of the transfer matrices,

$$t(x)t(y) = t(y)t(x).$$

Therefore, the polynomials defined using $t$ are invariant under permutation of variables.

### 2.3.2. *Canonical dual Grothendieck polynomials.*

In order to formulate the branching formula for $g_\lambda^{(\alpha,\beta)}$, we need to establish some statistics on partitions.

For a skew-partition $\lambda/\mu$, define

$$\begin{aligned}
r(\lambda/\mu) &= \text{number of non zero rows,}\\
c(\lambda/\mu) &= \text{number of non zero columns,}\\
b(\lambda/\mu) &= \text{number of connected components.}
\end{aligned}$$

Let us now recall the branching formula of $g_\lambda^{(\alpha,\beta)}$ from [16, Theorem 8.6]. For $\lambda,\mu$, we have

$$g_\lambda^{(\alpha,\beta)}(x_1,\dots,x_n,x_{n+1}) = \sum_{\mu\subseteq\lambda} g_\mu^{(\alpha,\beta)}(x_1,\dots,x_n) g_{\lambda/\mu}^{(\alpha,\beta)}(x_{n+1}), \tag{25}$$

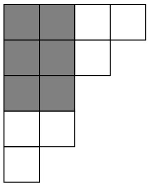
where

$$g_{\lambda/\mu}^{(\alpha,\beta)}(x) = \beta^{r(\lambda/\mu)-b(\lambda/\mu)}(\alpha+\beta)^{|\lambda/\mu|-r(\lambda/\mu)-c(\lambda/\mu)+b(\lambda/\mu)}x^{b(\lambda/\mu)}(\alpha+x)^{c(\lambda/\mu)-b(\lambda/\mu)} \tag{26}$$

whenever $\mu\subseteq\lambda$ and 0 otherwise.

We shall use this branching formula as the definition of $g_\lambda^{(\alpha,\beta)}$. Let us compute some examples to understand the above statistics.

$\lambda/\mu$=(4,3,2,2,1)/(2,2,2)

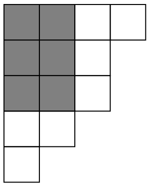


$r(\lambda/\mu)=4$

$c(\lambda/\mu)=4$

$b(\lambda/\mu)=2$

$\lambda/\mu$=(4,3,3,2,1)/(2,2,2)

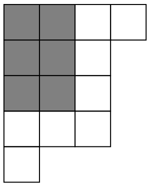


$r(\lambda/\mu)=5$

$c(\lambda/\mu)=4$

$b(\lambda/\mu)=2$

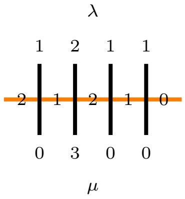
$\lambda/\mu$=(4,3,3,3,1)/(2,2,2)

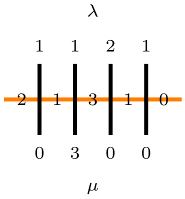


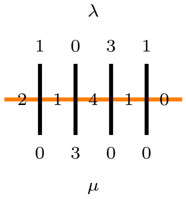
$r(\lambda/\mu)=5$

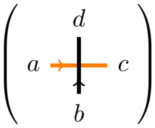
$c(\lambda/\mu)=4$

$b(\lambda/\mu)=1$

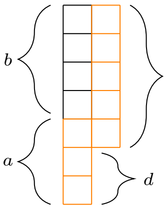




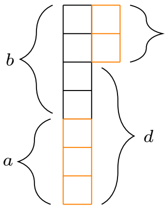


Let us unpack the information contained at a vertex. Consider a vertex  at site $i$. The label on the left edge $a$ corresponds to adding $a$ boxes to the $i^{th}$ column of $\mu$. The label $d$ is the number rows of $\lambda$ with size $i$. We want to understand number of row of size $i$ in $\lambda/\mu$. There are three types of vertices, $b<c$, $b>d$, and $b=c$. Let us look at the labels on the $i^{th}$ column of $\lambda/\mu$ in terms of the Young diagrams.

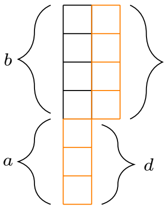
case: $b<c$



case: $b>c$



case: $b=c$



From the above pictures, it is evident that the number of non zero rows of size $i$ in $\lambda/\mu$ is $\min(a, d)$. Also observe that, in the last two cases, the skew diagram is disjoint. Therefore, the number of connected components is the number of vertices where $b > c$ or $b = c$. Finally, the $i^{th}$ column of $\lambda/\mu$ is non zero if and only if some boxes are added to it, i.e. when $a \neq 0$.

Theorem 2.7. *The dual canonical Grothendieck polynomials $g_\lambda^{(\alpha,\beta)}(x)$ are given by*

$$g_\lambda^{(\alpha,\beta)}(x_1, \ldots, x_n) = \langle 0| \, t(x_1) \ldots t(x_n) \, |\lambda\rangle \tag{27}$$

*where $|\lambda\rangle = \bigotimes_{i=1}^{\infty} |m_i(\lambda)\rangle$.*

*Proof.* On comparing with the branching formula (25), it enough to show that for $\mu \subseteq \lambda$,

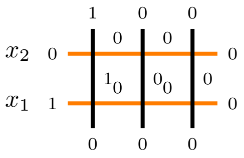
$$g_{\lambda/\mu}^{(\alpha,\beta)}(x) = \langle \mu| \, t(x) \, |\lambda\rangle \, .$$

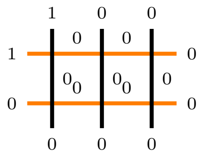
Recall that for $\mu \subseteq \lambda$, we have

$$g_{\lambda/\mu}^{(\alpha,\beta)}(x) = \\ \beta^{r(\lambda/\mu)-b(\lambda/\mu)}(\alpha+\beta)^{\lambda/\mu - r(\lambda/\mu)-c(\lambda/\mu)+b(\lambda/\mu)} x^{b(\lambda/\mu)}(\alpha+x)^{c(\lambda/\mu)-b(\lambda/\mu)}.$$

Let us study the exponent of $\beta$ in $\langle \mu| \, t(x) \, |\lambda\rangle$. Recall that the number of rows of size $i$ in $\lambda/\mu$ is $\min(a, d)$. The connected components are recorded by vertices where $b \geqslant c$. Therefore, by assigning the weight $\beta^d$ when $b < c$ and $\beta^{a-1}$ whenever $b \geqslant c$, we get the exponent of $\beta$ in the overall weight as $r(\lambda/\mu) - b(\lambda/\mu)$, which is precisely the exponent of $\beta$ in $g_\lambda^{(\alpha,\beta)}(x)$. Similarly, by doing the same for the other factors, one recovers the Boltzmann weights. ☐

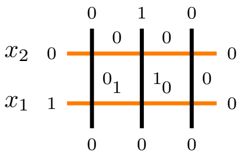
Example 2.8. For the partition $\lambda = (1, 0)$ we have the following two configurations corresponding to

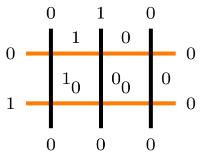


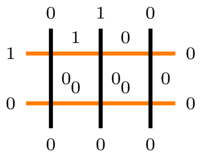


$$g_\lambda^{(\alpha,\beta)}(x_1, x_2) = x_1 + x_2.$$

Example 2.9. For the partition $\lambda = (2, 0)$, we have

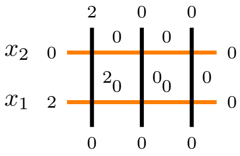


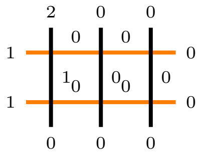


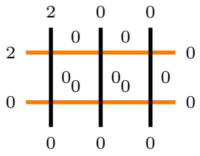


$$g_\lambda^{(\alpha,\beta)}(x_1, x_2) = (x_1+\alpha)x_1 + x_1x_2 + (x_2+\alpha)x_2 = x_1^2 + x_1x_2 + x_2^2 + \alpha(x_1+x_2).$$

Example 2.10. For the partition $\lambda = (1, 1)$, we have

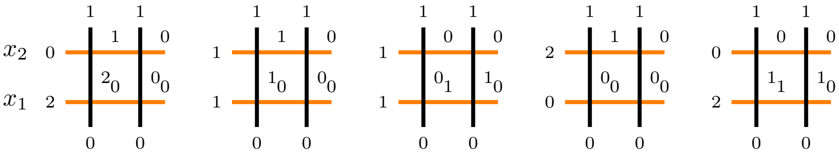






$$g_\lambda^{(\alpha,\beta)}(x_1,x_2) = \beta x_1 + x_1x_2 + \beta x_2 = x_1x_2 + \beta(x_1+x_2).$$

Example 2.11. For the partition $\lambda = (2,1)$, we have



$$g_\lambda^{(\alpha,\beta)}(x_1,x_2) = \beta x_1x_2 + x_1x_2^2 + (x_1+\alpha)x_1x_2 + (x_2+\alpha)\beta x_2 + (x_1+\alpha)\beta x_1.$$

## 3. Column Vertex Models

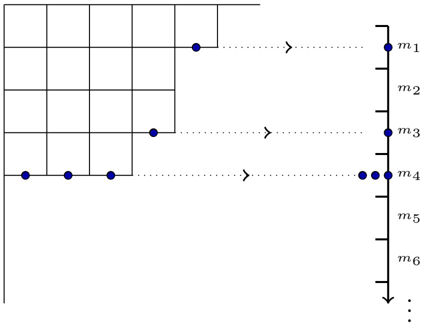
3.1. Definition of Physical space. Recall that we identify partitions with basis elements of $V^r$ by recording row multiplicities. In this section, for the physical space, we use the same vector space ($V^r$) that we used in the earlier section. We shall denote it by $V^c$. Even though $V^c$ and $V^r$ are identical, we distinguish them by the way we identify the partitions with the basis elements.

$$V^c = \operatorname{Span}\{|m_1^c\rangle \otimes |m_2^c\rangle \otimes |m_3^c\rangle \cdots\} \qquad m_i^c \geqslant 0,\ i \geqslant 1. \tag{28}$$

Given a partition, which we view as Young diagram, let $|\lambda^c\rangle$ be the basis vector with integers

$$m_i^c(\lambda) = \text{number of columns of size } i \text{ of } \lambda.$$

For example, we identify the partition $\lambda = (5,4,4,3)$ with the basis element $|1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |3\rangle \otimes |0\rangle \dots$ of $V^c$:



It is useful to note that $m_i^c(\lambda) = m_i^r(\lambda^{'})$.

### 3.2. Column vertex model for canonical Grothendieck polynomials.

3.2.1. *Definition of $\widetilde{L}$-matrix and $\widetilde{R}$ matrix.* The main difference of the model considered in this section from the row model of $G_\lambda^{(\alpha,\beta)}$ is that the horizontal line can now carry any non negative integer. For every vertex, we assign the Boltzmann weights in the following way:

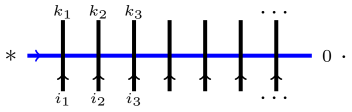
$$w_x\left(\begin{array}{ccc} & d & \\ a & \rightarrow\!\!\!+\!\!\!\uparrow & c \\ & b & \end{array}\right) \equiv w_x(a,b;c,d) = \delta_{a+b,c+d}\begin{cases}\left(\frac{x}{1-\alpha x}\right)^a & b=c\\ \left(\frac{x}{1-\alpha x}\right)^a\left(\frac{1+\beta x}{1-\alpha x}\right) & b>c\\ 0 & b<c\end{cases} \tag{29}$$

where $a, b, c, d$ are non negative integers. Let $W = \text{Span}\{|j\rangle\}_{j\in\mathbb{Z}_{\geqslant 0}}$ be an infinite dimensional vector space, and for $1 \leqslant i \leqslant n$, let $W_i$ be a copy of $W$. Let $V_j \cong W$ be another copy. Then we define a $\widetilde{L}$ matrix which acts linearly on $W_i \otimes V_j$ as follows:

$$\widetilde{L}_{i,j}\left(x_i\right) : |a\rangle \otimes |b\rangle \mapsto \sum_{c,d \text{ where } a+b=c+d} w_{x_i}\Big(a,b;c,d\Big)\,|c\rangle \otimes |d\rangle\,. \tag{30}$$

Let $w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})$ be the weight of single row of vertices:

$$w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\}) =$$



.

We now define the transfer matrix $\widetilde{T}$ which acts linearly on $V^c$ as follows,

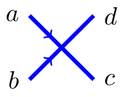
$$\widetilde{T}(x) : |i_1\rangle \otimes |i_2\rangle \otimes \cdots \mapsto \sum_{k_1,k_2\dots\geqslant 0} w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})\,|k_1\rangle \otimes |k_2\rangle \otimes \cdots. \tag{31}$$

Consider the vector spaces $W_i, W_j$ where $i < j$. Then we define an $R$-matrix which acts linearly on $W_i \otimes W_j$ as follows,

$$\widetilde{R}_{i,j}(x_i, x_j) : |a\rangle \otimes |b\rangle \mapsto \sum_{c,d \text{ where } a+b=c+d} \widetilde{R}^{ad}_{bc}(x_i, x_j)\,|c\rangle \otimes |d\rangle\,, \tag{32}$$

where the entries are:

$$\widetilde{R}^{a\,d}_{b\,c}\,(x,y) =$$

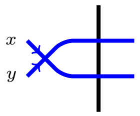


$$= \left(\frac{\dfrac{x}{1-\alpha x}}{\dfrac{y}{1-\alpha y}}\right)^a \begin{cases} 0 & \text{when } b<c \\ 1 & \text{when } b=c \\ \dfrac{1}{1-\alpha x} - \dfrac{x}{(1-\alpha x)y} & \text{otherwise.} \end{cases}$$

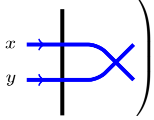
Together with $\widetilde{L}_{i,n}$ and $\widetilde{L}_{j,n}$ matrices, $\widetilde{R}_{ij}$ satisfies the RLL relation in End $(W_i \otimes W_j \otimes V_n)$ (see Appendix A.1):

(33)

$$\widetilde{R}_{ij}(x,y)\widetilde{L}_{i,n}(x)\widetilde{L}_{j,n}(y) = \widetilde{L}_{j,n}(y)\widetilde{L}_{i,n}(x)\widetilde{R}_{ij}(x,y) \Bigg($$



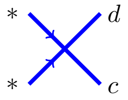
$$=$$



$$\Bigg).$$

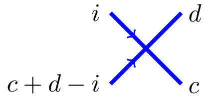
3.2.2. *Eigenvector of the $\tilde{R}$ matrix.* As in the previous section, we show that the partition function with single vertex and fixed right boundary and free boundary condition on the left

$$\mathcal{Z}(c,d) =$$



(where $c, d$ are nonnegative integers) is constant and equal to 1. We compute:

$$\mathcal{Z}(c,d) = \sum_{i=0}^{d}$$



$$\begin{aligned}
&= \left(\frac{1}{1-\alpha x} - \frac{x}{(1-\alpha x)y}\right) \sum_{i=0}^{d-1} \left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{i} + \left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{d} \\
&= \left(\frac{y-x}{y(1-\alpha x)}\right) \left(\frac{1-\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{d}}{1-\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)}\right) + \left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{d} \\
&= 1.
\end{aligned}$$

By repeating the same argument as in Section 2.2.5, we get the commutation relation of the transfer matrices:

$$\widetilde{T}(x)\widetilde{T}(y) = \widetilde{T}(y)\widetilde{T}(x).$$

Therefore, the polynomials defined using $t$ are invariant under permutation of variables.

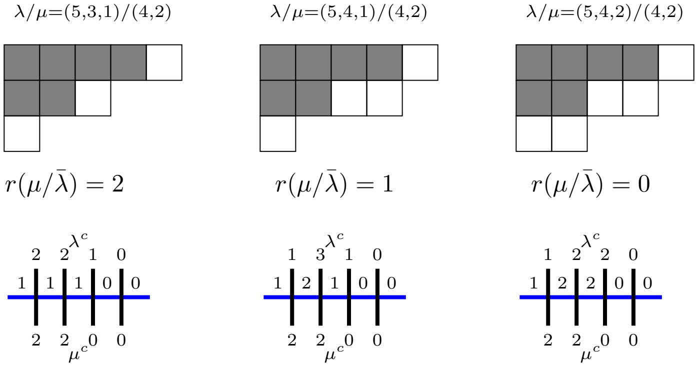
3.2.3. *Canonical Grothendieck polynomials.* Given that the transfer matrices commute, the polynomials defined using $\widetilde{T}$ are invariant under permutation of variables. We now prove that the polynomials defined using $T$ are *canonical Grothendieck polynomials*.

Theorem 3.1. *The canonical Grothendieck polynomials* $G_\lambda^{(\alpha,\beta)}(x)$ *are given by*

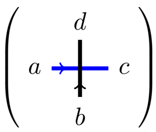
$$G_\lambda^{(\alpha,\beta)}(x_1,\dots,x_n) = \langle 0|\, \widetilde{T}(x_1)\cdots\widetilde{T}(x_n)\, |\lambda^c\rangle \tag{34}$$

*where* $|\lambda^c\rangle = \bigotimes_{i=1}^{\infty} |m_i^c(\lambda)\rangle$.

Example 3.2. Let us observe some examples to understand $r(\mu/\bar{\lambda})$.



*Proof.* Let us now understand the local configuration of vertices of this model. Consider a vertex



at site $i$. The label $a$ corresponds to adding $a$ boxes to $i^{th}$ row of $\mu$. By recording the left nodes, we get $\lambda/\mu$. Then, in-order to get a horizontal strip, the number of boxes that can be added to $i^{th}$ row should be at most $b$.

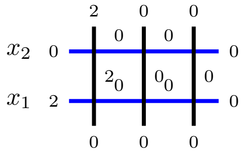
When $c < b$, we have a removable box in $i^{th}$ row that is not in the same column with any box of $\lambda/\mu$. Therefore, $r(\mu/\bar{\lambda})$ is precisely the number of vertices where $c < b$.

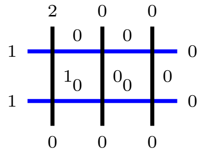
Following the reasoning in Theorem 2.2, it is enough to show that $\langle u^c| \widetilde{T}(x) |\lambda^c\rangle = G^{(\alpha,\beta)}_{\lambda /\!\!/ \mu}$ for a horizontal strip $\lambda/\mu$. Recall that for a horizontal strip, we have

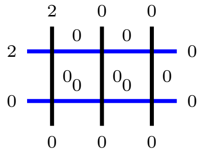
$$G^{(\alpha,\beta)}_{\lambda /\!\!/ \mu}(x) = \left(\frac{x}{1-\alpha x}\right)^{|\lambda/\mu|} \left(\frac{1+\beta x}{1-\alpha x}\right)^{r(\mu/\bar{\lambda})}.$$

Observe that $\langle u^c| \widetilde{T}(x) |\lambda^c\rangle \neq 0$ if and only if $\lambda/\mu$ is a horizontal strip. From the above analysis and the way Boltzmann weights are defined, the proof is now immediate. ☐

Example 3.3. For the partition $\lambda = (2, 0)$, we have

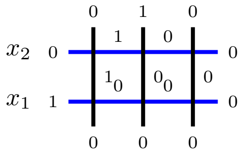






$$G^{(\alpha,\beta)}_{\lambda}(x_1, x_2) =$$
$$\left(\frac{x_1}{1-\alpha x_1}\right)^2 \left(\frac{1+\beta x_2}{1-\alpha x_2}\right) + \left(\frac{x_1}{1-\alpha x_1}\right)\left(\frac{x_2}{1-\alpha x_2}\right)\left(\frac{1+\beta x_2}{1-\alpha x_2}\right) + \left(\frac{x_2}{1-\alpha x_2}\right)^2.$$

Example 3.4. For the partition $\lambda = (1, 1)$, we have



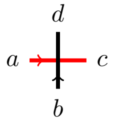
$$G^{(\alpha,\beta)}_{\lambda}(x_1, x_2) = \left(\frac{x_1}{1-\alpha x_1}\right)\left(\frac{x_2}{1-\alpha x_2}\right).$$

## 3.3. Column vertex model dual canonical Grothendieck polynomials.

### 3.3.1. *Definition of $\widetilde{l}$-matrix and $\widetilde{r}$-matrix.*

We consider the same vertex model as row model of $g^{(\alpha,\beta)}_{\lambda}$, but with different Boltzmann weights. For every vertex, we assign the Boltzmann weights in the following way:

(35)



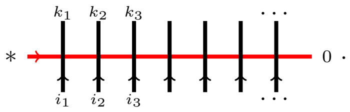
$$w_x\left(\quad\right) \equiv w_x(a, b; c, d) = \delta_{a+b,c+d} \begin{cases} (\alpha+\beta)^{a-d-1}\beta(x+\alpha)^d & 0 < a > d \\ x(x+\alpha)^{a-1} & 0 < a \leqslant d \\ 1 & a = 0, \end{cases}$$

where $a, b, c, d \in \mathbb{Z}_{\geqslant 0}$. Let $W = \mathrm{Span}\{|j\rangle\}_{j\in\mathbb{Z}_{\geqslant 0}}$ be an infinite dimensional vector space, and for $1 \leqslant i \leqslant n$, let $W_i$ be a copy of $W$. Let $V_j \cong W_i$ be a vector space. Then we define a $\widetilde{l}$-matrix which acts linearly on $W_i \otimes V_j$ as follows,

$$\widetilde{l}_{i,j}(x_i) : |a\rangle \otimes |b\rangle \mapsto \sum_{c,d \text{ where } a+b=c+d} w_{x_i}\Big(a, b; c, d\Big) |c\rangle \otimes |d\rangle . \tag{36}$$

As usual, let $w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})$ be the weight of single row of vertices.

$$w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\}) =$$



We now define the transfer matrix $t$ which acts linearly on $V^c$ as follows:

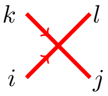
$$\tilde{t}(x) : |i_1\rangle \otimes |i_2\rangle \otimes \cdots \mapsto \sum_{k_1, k_2 \dots \geqslant 0} w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\}) \, |k_1\rangle \otimes |k_2\rangle \otimes \cdots. \tag{37}$$

Consider the vector spaces $W_i, W_j$ where $i < j$. Then we define a $\widetilde{r}$-matrix which acts linearly on $W_i \otimes W_j$ as follows:

$$\widetilde{r}_{i,j}(x_i, x_j) : |a\rangle \otimes |b\rangle \mapsto \sum_{c,d \text{ where } a+b=c+d} \widetilde{r}^{a,d}_{b,c}(x_i, x_j) \, |c\rangle \otimes |d\rangle \, . \tag{38}$$

where the entries are the following:

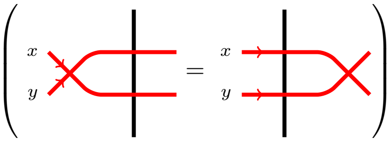
$$\widetilde{r}^{k,l}_{i,j}(x, y) = \tag{39}$$



$$= \begin{cases} 0 & i < j \\ 1 & k = l = 0 \\ \dfrac{x}{y}\left(\dfrac{y+\alpha}{x+\alpha}\right)^{1-k} & k = l > 0 \\ \left(1 - \dfrac{x}{y}\right) & k = 0 \\ \dfrac{x}{y}\left(\dfrac{y+\alpha}{x+\alpha} - 1\right)\left(\dfrac{y+\alpha}{x+\alpha}\right)^{-k} & k > 0. \end{cases}$$

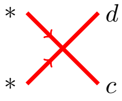
Together with matrices $\widetilde{l}_{i,n}$ and $\widetilde{l}_{j,n}$ , $\widetilde{r}_{ij}$ satisfies the RLL relation in $\mathrm{End}\,(W_i \otimes W_j \otimes V_n)$ (see Appendix A.3).

$$\widetilde{r}_{ij}(x, y)\widetilde{l}_{i,n}(x)\widetilde{l}_{j,n}(y) = \widetilde{l}_{j,n}(y)\widetilde{l}_{i,n}(x)\widetilde{r}_{ij}(x, y) \tag{40}$$



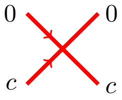
3.3.2. *Eigenvector of the $\widetilde{r}$ matrix.* We proceed as in previous sections, computing the partition function of a single vertex with fixed right boundary with the Boltzmann weights of $\widetilde{r}$ matrix. We claim that for any non-negative integers $c, d$,

$$\mathcal{Z}(c, d) =$$



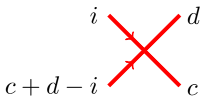
the partition function is constant and is equal to 1. Let us first consider the case where $d = 0$. Then there is a unique vertex as the bottom left entry should be greater than or equal to $c$ and also should satisfy the conservation. The weight of the unique configuration is 1.

$$\mathcal{Z}(c, 0) =$$

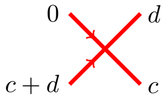


We now compute for the case where $d > 0$:

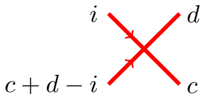
$$\mathcal{Z}(c,d) = \sum_{i=0}^{d}$$



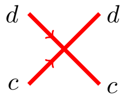
$$=$$



$$+\sum_{i=1}^{d-1}$$



$$+$$



$$\begin{aligned}
&= \left(1-\frac{x}{y}\right) + \frac{x}{y}\left(\frac{y+\alpha}{x+\alpha}-1\right)\sum_{i=1}^{d-1}\left(\frac{x+\alpha}{y+\alpha}\right)^{i} + \frac{x}{y}\left(\frac{x+\alpha}{y+\alpha}\right)^{d-1}\\
&= \left(1-\frac{x}{y}\right) + \frac{x}{y}\left(1-\left(\frac{x+\alpha}{y+\alpha}\right)^{d-1}\right) + \frac{x}{y}\left(\frac{x+\alpha}{y+\alpha}\right)^{d-1}\\
&= 1.
\end{aligned}$$

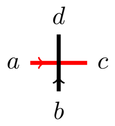
Using the argument in Section 2.2.5, we get the commutation relation of the transfer matrices,

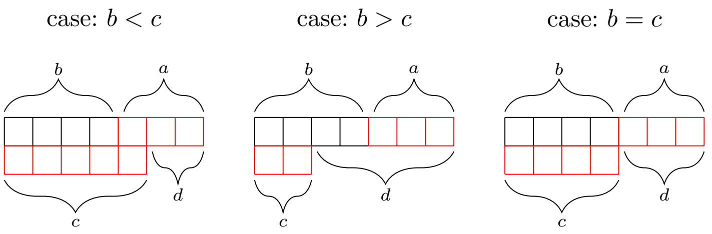
$$\widetilde{t}(x)\widetilde{t}(y) = \widetilde{t}(y)\widetilde{t}(x).$$

Therefore, the polynomials defined using $\widetilde{t}$ are invariant under permutation of variables.

3.3.3. *Dual canonical Grothendieck polynomials.* Recall that for a skew-partition $\lambda/\mu$, we have

$$\begin{aligned}
r(\lambda/\mu) &= \text{number of non zero rows},\\
c(\lambda/\mu) &= \text{number of non zero columns},\\
b(\lambda/\mu) &= \text{number of connected components}.
\end{aligned}$$

We shall unpack the information contained at a vertex like we did in the case of row model of $g_{\lambda}^{(\alpha,\beta)}$. Consider a vertex $\Bigg($  $\Bigg)$ at site $i$. The label on the left edge $a$, corresponds to adding $a$ boxes to the $i^{th}$ row of $\mu$. The label $d$, is the number columns of $\lambda$ with size $i$. We want to understand number of columns of size $i$ in $\lambda/\mu$. There are three types of vertices, $b < c$, $b > d$, and $b = c$. Let us look at the labels on $i^{th}$ row of $\lambda/\mu$ in-terms of the Young diagram.


case: $b < c$ case: $b > c$ case: $b = c$

It is evident from the pictures that the number of non zero columns of size $i$ in $\lambda/\mu$ is $\min(a,d)$. Also, observe that the vertices where $b \leqslant c$ detect the number of connected components. The number of non empty rows in $\lambda/\mu$ is equal to the number of vertices where $a \neq 0$.

Theorem 3.5. *The dual canonical Grothendieck polynomials* $g_\lambda^{(\alpha,\beta)}(x)$ *are given by*

$$g_\lambda^{(\alpha,\beta)}(x_1,\dots,x_n) = \langle 0|\, \tilde{t}(x_1)\dots\tilde{t}(x_n)\,|\lambda^c\rangle \tag{41}$$

*where* $|\lambda^c\rangle = \bigotimes_{i=1}^{\infty} |m_i^c(\lambda)\rangle$.

*Proof.* Following the reasoning as in Theorem 2.2, it enough to show that for $\mu \subseteq \lambda$,

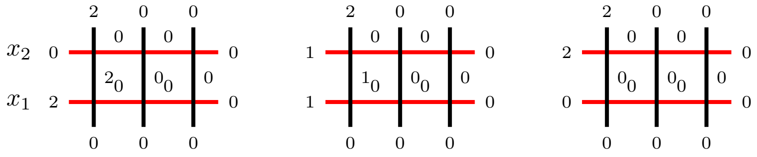
$$g_{\lambda/\mu}^{(\alpha,\beta)}(x) = \langle \mu^c|\, t(x)\,|\lambda^c\rangle\,.$$

Recall that for $\mu \subseteq \lambda$, we have

$$g_{\lambda/\mu}^{(\alpha,\beta)}(x) = \\ \beta^{r(\lambda/\mu)-b(\lambda/\mu)}(\alpha+\beta)^{\lambda/\mu-r(\lambda/\mu)-c(\lambda/\mu)+b(\lambda/\mu)}x^{b(\lambda/\mu)}(\alpha+x)^{c(\lambda/\mu)-b(\lambda/\mu)}.$$

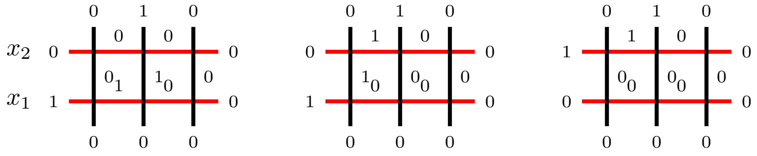
Let us deal the $\beta$ factor in in $\langle\mu|\,\tilde{t}(x)\,|\lambda\rangle$. Observe that the $\beta$ appears in a Boltzmann weight of a vertex only when $a \neq 0$ and $b < c$. From our previous analysis, we see that such vertices precisely count $r(\lambda/\mu) - b(\lambda/\mu)$. Similarly, we can check for all the other factors in $\langle\mu|\,\tilde{t}(x)\,|\lambda\rangle$. ☐

Example 3.6. For the partition $\lambda = (2,0)$, we have the following three configurations:



$$g_\lambda^{(\alpha,\beta)}(x_1,x_2) = x_1(x_1+\alpha) + x_1x_2 + x_2(x_2+\alpha).$$

Example 3.7. For the partition $\lambda = (1,1)$, we have



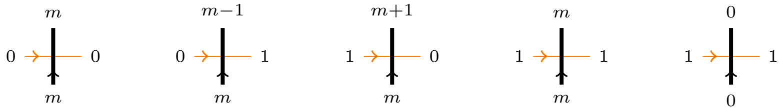
$$g_\lambda^{(\alpha,\beta)}(x_1,x_2) = \beta x_1 + x_1x_2 + \beta x_2.$$

### 3.4. Vertex model for $j$ polynomials.

3.4.1. *Definition of* $\mathfrak{l}$ *matrix.* In this subsection, the auxiliary line is fermionic. Let

$$\mathfrak{l}_{i,j}(x) = \begin{pmatrix} 1 & \phi_j \\ x\phi_j^\dagger & x+\delta_{0,m} \end{pmatrix} \tag{42}$$

be the $\mathfrak{l}$-matrix acting on $F_i \otimes V_j$. Below we represent the entries of $\mathfrak{l}$ graphically:
(43)



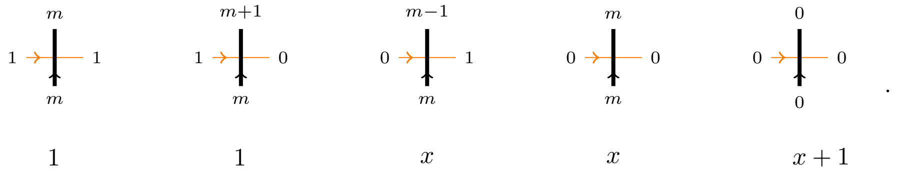
.

1 1 $x$ $x$ $x+1$

Similarly, we have the dual $\mathfrak{l}^*$ matrices,

$$\mathfrak{l}^*_{i,j}(x) = \begin{pmatrix} x + \delta_{0,m} & x\phi_j \\ \phi_j^\dagger & 1 \end{pmatrix} \tag{44}$$

(45)



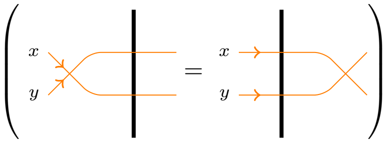
The R matrix that makes this model integrable is same as (9) with $\beta = 0$. When $\beta = 0$ we denote this matrix as $R(y/x)$.

$$R_{ij}(y/x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{y}{x} & 0 \\ 0 & 1 & 1 - \dfrac{y}{x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \text{End}(F_i \otimes F_j). \tag{46}$$

The $R$ matrix together with matrices $\mathfrak{l}_i$ and $\mathfrak{l}_j$, satisfies the RLL relation in End $(F_i \otimes F_j \otimes V_n)$:

(47)

$$R_{ij}(y/x)\mathfrak{l}_{i,n}(x)\mathfrak{l}_{j,n}(y) = \mathfrak{l}_{j,n}(y)\mathfrak{l}_{i,n}(x)R_{ij}(y/x)$$

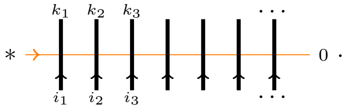


3.4.2. *Row-row transfer matrices.* We now define the transfer matrix $\mathfrak{t}$ which acts linearly on $V^r$ as follows:

$$\mathfrak{t}(x) : |i_1\rangle \otimes |i_2\rangle \otimes \cdots \mapsto \sum_{k_1,k_2\dots\geqslant 0} w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})\, |k_1\rangle \otimes |k_2\rangle \otimes \cdots, \tag{48}$$

where $w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})$ is the weight of the single of row of vertices.

$$w(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\}) =$$



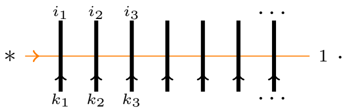
Remark 3.8. Observe that the transfer matrix $t$ from row model of $g_\lambda^{(1,0)}$ and $\mathfrak{t}$ are the same.

Similarly, we define the dual transfer matrix $\mathfrak{t}^*$ which acts linearly on $V^r$ as follows:

(49)

$$\mathfrak{t}^*(x) : |i_1\rangle \otimes |i_2\rangle \otimes \cdots \mapsto \sum_{k_1,k_2\dots\geqslant 0} w^*(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})\, |k_1\rangle \otimes |k_2\rangle \otimes \cdots,$$

where $w^*(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\})$ is the weight of the single row of vertices made of the vertices of $\mathfrak{l}^*$.

$w^*(\{i_1, i_2, \dots\}; \{k_1, k_2, \dots\}) =$ 

.

3.4.3. *j polynomials.* Recall that we denote dual Grothendieck polynomials by $g_\lambda$, which is the $\alpha = 0$ and $\beta = 1$ specialization of $g_\lambda^{(\alpha,\beta)}$. Then the $\omega(g_\lambda)$ polynomials are called *weak dual Grothendieck polynomials* and we shall denote them by $j_\lambda$:

$$j_\lambda = \omega(g_\lambda) = \omega(g_\lambda^{(0,1)}) = g_{\lambda'}^{(1,0)}.$$

When $\beta = 0$, the branching formula of $g_\lambda^{(\alpha,\beta)}$ reduces to the following [17]:

$$j_\lambda(x_1, \dots, x_n, x_{n+1}) = \sum_\mu j_{\lambda/\mu}(x_{n+1}) j_\mu(x_1, \dots, x_n),$$

where $j_{\lambda/\mu}(x)$ is defined as follows,

$$j_{\lambda/\mu}(x) = \begin{cases} x^{c(\lambda/\mu)}(1+x)^{|\lambda/\mu| - c(\lambda/\mu)} & \lambda/\mu \text{ vert. strip,} \\ 0 & \text{otherwise.} \end{cases}$$

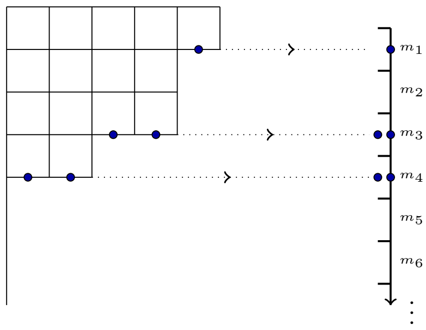
Theorem 3.9. *The dual weak Grothendieck polynomials $j_\lambda(x)$ are given by*

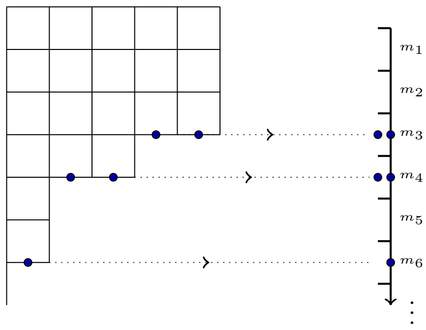
$$j_\lambda(x_1, \dots, x_n) = \langle 0 | \mathfrak{t}(x_1) \dots \mathfrak{t}(x_n) | \lambda^c \rangle \tag{50}$$

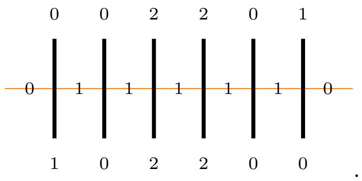
$$j_\lambda(x_1, \dots, x_n) = \langle \lambda^c | \mathfrak{t}^*(x_n) \dots \mathfrak{t}^*(x_1) | 0 \rangle \tag{51}$$

*where $|\lambda^c\rangle = \bigotimes_{i=1}^\infty |m_i^c(\lambda)\rangle$, and similarly for the dual state $\langle \lambda^c |$.*

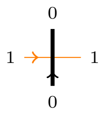
*Proof.* Before we prove (50), let us observe an example to understand the vertices. For $\mu = (5, 4, 4, 3)$ and $\lambda = (5, 5, 5, 3, 1, 1)$,







.

From the example above, observe that having 1 on the left horizontal edge at site $i$ amounts to adding a box in row $i$ of the Young diagram of $\mu$. The number of such vertices amounts to the number of boxes added. The vertex  at site $i$ can be read as adding a box in two successive rows in the same column. So every such vertex amounts to $|\lambda/\mu| - c(\lambda/\mu)$. Using similar reasoning as in Theorem 2.2 and with the preceding analysis, the proof is immediate. ☐

## 4. GENERALISED POLYNOMIALS

In this section, we shall generalise the polynomials by introducing additional variables which are attached to the vertical lines of the underlying lattice model. In order to do that, we need the R matrix that underpins the integrability of a lattice model to satisfy the so-called *difference property*. Usually the difference property refers to entries of the R matrix being invariant under translation of the spectral variables. In this paper, we say that an R matrix satisfies the difference property when the entries are invariant under scaling of the spectral parameters i.e. the non constant entries are polynomials in ratio of the spectral variables.

### 4.1. DIFFERENCE PROPERTY OF THE R MATRICES.

In this subsection, we study the difference property of the various R matrices studied in this paper.

4.1.1. R *matrices of Row models.* Consider the $R$ matrix ((9)) for canonical Grothendieck polynomials. Observe that when $\beta = 0$, it satisfies the difference property. It also satisfies the property for general $\alpha$ and $\beta$ if we consider the spectral variables to be $\dfrac{x}{1-\beta x}$ instead of $x$.

$$R(y/x) = R(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{y}{x} & 0 \\ 0 & 1 & 1-\dfrac{y}{x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{52}$$

In the case of $g_\lambda^{(\alpha,\beta)}$, the $r$-matrix does not satisfy the difference property. It is also not defined for $\beta = 0$. Hence, we studied a different model for the case where $\beta = 0$. Observe that the R matrix for the vertex model of $j_\lambda = g_{\lambda'}^{(1,0)}$ is the same as that in (52).

4.1.2. R *matrices of column models.* The $\widetilde{R}$ matrix of the column model of $G_\lambda^{(\alpha,\beta)}$,

$$\widetilde{R}^{a\,d}_{b\,c}(y,x) = \left(\frac{\dfrac{x}{1-\alpha x}}{\dfrac{y}{1-\alpha y}}\right)^a \begin{cases} 0 & \text{when } b < c, \\ 1 & \text{when } b = c, \\ \dfrac{1}{1-\alpha x} - \dfrac{x}{(1-\alpha x)y} & \text{when } b > c \end{cases}$$

satisfies the difference property when $\alpha = 0$. It also satisfies the difference property in general, when we consider the spectral variables to be $\frac{x}{1+\alpha x}$ and $\frac{y}{1+\alpha y}$ instead of $x$ and $y$.

The $\widetilde{r}$-matrix of the column model of $g_\lambda^{(\alpha,\beta)}$:

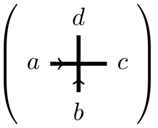
$$\widetilde{r}_{i,j}^{k,l}(x,y) = \begin{cases} 0 & i<j \\ 1 & k=l=0 \\ \dfrac{x}{y}\left(\dfrac{y+\alpha}{x+\alpha}\right)^{1-k} & k=l>0 \\ \left(1-\dfrac{x}{y}\right) & k=0 \\ \dfrac{x}{y}\left(\dfrac{y+\alpha}{x+\alpha}-1\right)\left(\dfrac{y+\alpha}{x+\alpha}\right)^{-k} & k>0 \end{cases} \tag{53}$$

satisfies the difference property when $\alpha = 0$.

4.2. Generalised polynomials. To summarize, in the case of row models we can only generalise $G_\lambda^{(\alpha,0)}$ and $g_\lambda^{(\alpha,0)}$. Similarly, in the case of column models we can generalise $G_\lambda^{(0,\beta)}$ and $g_\lambda^{(0,\beta)}$.

We generalise the polynomials by assigning variables to vertical lines. Let us assign the variable $z_i$ to the $i^{th}$ vertical line from the left. In any model, the weight of a vertex formed with the intersection of $i^{th}$ horizontal line and $j^{th}$ vertical line is defined as follows:

$$\widetilde{w}_{(x_i,z_j)}\left(\vphantom{\Big|}\right.$$



$$\left.\vphantom{\Big|}\right) = w_{\left(\frac{x_i}{z_j}\right)}(a,b;c,d).$$

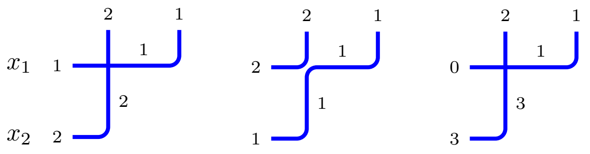
Let us name these polynomials.

(i) We call $G_\lambda^\alpha = G_\lambda^{(0,-\alpha)}$ *generalised Grothendieck polynomials.*
(ii) We call $g_\lambda^\alpha = g_\lambda^{(0,\alpha)}$ *generalised dual Grothendieck polynomials.*
(iii) We call $J_\lambda^\alpha = G_{\lambda'}^{(-\alpha,0)}$ *generalised weak Grothendieck polynomials.*
(iv) We call $j_\lambda^\alpha = g_{\lambda'}^{(\alpha,0)}$ *generalised weak dual Grothendieck polynomials.*

When $\alpha = 1$, we shall drop the superscript.

Example 4.1. For the partition $\lambda = (1)$, the *generalised Grothendieck polynomial* $G_\lambda(x_1, z_1)$ is $\dfrac{x_1}{z_1}$. For comparison, the *double Grothendieck polynomial* is $x_1+y_1(1-x_1)$ [12]. We observe that these two generalisations of Grothendieck polynomials are not the same.

Example 4.2. Let us look at a non trivial example. For the partition $\lambda = (3, 1)$ we have the following three configurations (edges with labels 0 suppressed):



$$G_{(3,1)}^{(0,-1)}(x_1,x_2) =$$
$$\left(\frac{x_1^2}{z_1z_2}\right)\left(1-\frac{x_1}{z_1}\right)\left(\frac{x_2}{z_1}\right)^2 + \left(\frac{x_1^3}{z_1^2z_2}\right)\left(\frac{x_2}{z_1}\right) + \left(1-\frac{x_1}{z_1}\right)\left(\frac{x_1}{z_2}\right)\left(\frac{x_2}{z_1}\right)^3.$$

Remark 4.3. Observe that by setting both $\alpha$ and $\beta$ to 0, we get a generalised version of *Schur polynomials. Generalised Schur polynomials* from the row model and the column model are not the same. Let us denote the generalised Schur from the row (resp. column) model by $s^r$ (resp. $s^c$).

Example 4.4. For the partition $\lambda = (3,1)$, we have

$$s^r_{(3,1)}(x_1,x_2;z_1,z_2,\dots) = \left(\frac{x_1^3}{z_1z_2z_3}\right)\left(\frac{x_2}{z_1}\right) + \left(\frac{x_1^2}{z_1z_2}\right)\left(\frac{x_2^2}{z_1z_3}\right) + \left(\frac{x_1}{z_1}\right)\left(\frac{x_2^3}{z_1z_2z_3}\right)$$

$$s^c_{(3,1)}(x_1,x_2;z_1,z_2,\dots) = \left(\frac{x_1^2}{z_1z_2}\right)\left(\frac{x_2}{z_1}\right)^2 + \left(\frac{x_1^3}{z_1^2z_2}\right)\left(\frac{x_2}{z_1}\right) + \left(\frac{x_1}{z_2}\right)\left(\frac{x_2}{z_1}\right)^3.$$

$s^r_\lambda$ is a monomial multiple of $s_\lambda$, where the monomial is obtained by recording the columns of the Young diagram. Similarly, in the case of $s^c_\lambda$ the monomial is obtained by recording the rows of the Young diagram.

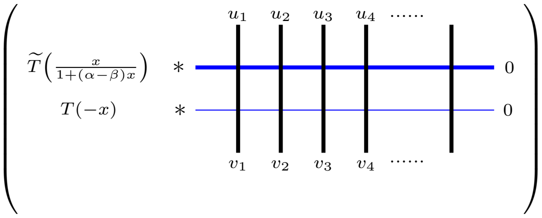
## 5. Duality between Column and Row models

In this section, we shall study a relation between the transfer matrix of the row and column model of $G^{(\alpha,\beta)}_\lambda$. Let us recall the necessary notation from various sections. The transfer matrices of the row model for $G^{(\alpha,\beta)}_\lambda$ are denoted by $T$, and those of the column model are denoted by $\widetilde{T}$.

Proposition 5.1 (Inversion relation). *The transfer matrices $\widetilde{T}$ and $T$ satisfy the following identity:*

(54)

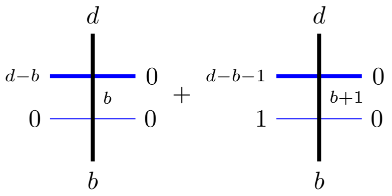
$$T(-x)\widetilde{T}\left(\frac{x}{1+(\alpha-\beta)x}\right) = 1$$



.

*Proof.* We shall prove the proposition for transfer matrices of size 1 and then apply induction to the size of the transfer matrices. Assign weights of $T(-x)$ for the vertex at the bottom and the weights of $\widetilde{T}\left(\frac{x}{1+(\alpha-\beta)x}\right)$ for the other vertex. Write $z$ for the vertical spectral parameter.

Observe that when $b = d$, there is a unique configuration with total weight 1. Now assume $b \neq d$



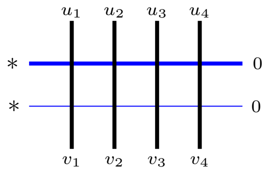
When $b > 0$,

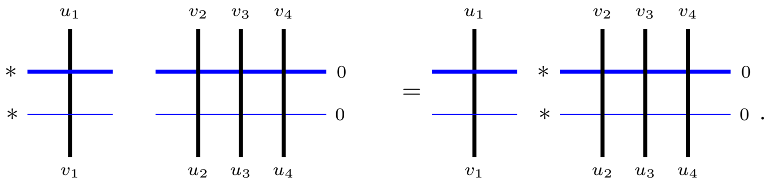
$$\begin{aligned}&= \left(\frac{1-\beta\left(\frac{x}{z}\right)}{1+\alpha\left(\frac{x}{z}\right)}\right)\left(\frac{\frac{x}{z}}{1-\beta\left(\frac{x}{z}\right)}\right)^{d-b}\left(\frac{1+\alpha\left(\frac{x}{z}\right)}{1-\beta\left(\frac{x}{z}\right)}\right)+\\ &\quad\left(\frac{-\frac{x}{z}}{1+\alpha\left(\frac{x}{z}\right)}\right)\left(\frac{\frac{x}{z}}{1-\beta\left(\frac{x}{z}\right)}\right)^{d-b-1}\left(\frac{1+\alpha\left(\frac{x}{z}\right)}{1-\beta\left(\frac{x}{z}\right)}\right)\\ &= 0.\end{aligned}$$

When $b = 0$,

$$\begin{aligned}&= \left(\frac{\frac{x}{z}}{1-\beta\left(\frac{x}{z}\right)}\right)^{d}+\left(\frac{-\frac{x}{z}}{1+\alpha\left(\frac{x}{z}\right)}\right)\left(\frac{\frac{x}{z}}{1-\beta\left(\frac{x}{z}\right)}\right)^{d-1}\left(\frac{1+\alpha\left(\frac{x}{z}\right)}{1-\beta\left(\frac{x}{z}\right)}\right)\\ &= 0.\end{aligned}$$

In order to apply the induction argument, we need to show that the transfer matrix of size $n+1$ can be written as a multiple of the transfer matrix of size $n$. Consider a transfer matrix of size $n+1$ where top and bottom labels are fixed.





Observe that when the left most boundary is fixed, then the contribution from the first site is fixed. Therefore, we can move the free boundary condition across the physical line at site 1. We then apply induction. ☐

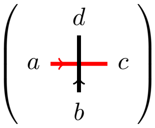
Define $\widetilde{T}^*(x) \in \mathrm{End}(V^c)$ as the adjoint of $\widetilde{T}(\mathrm{x})$:

$$\langle\lambda^c|\,\widetilde{T}^*(x)\,|\mu^c\rangle = \langle\mu^c|\,\widetilde{T}(x)\,|\lambda^c\rangle\,.$$

Then the inversion relation between $T^*$ and $\widetilde{T}^*$ follows from the definition of $\widetilde{T}^*$ and the inversion relation of $T$ and $\widetilde{T}$:

$$T^*(-x)\widetilde{T}^*\left(\frac{x}{1+(\alpha-\beta)x}\right) = 1. \tag{55}$$

In the case of $g_\lambda^{(\alpha,\beta)}$, such an inversion relation does not exist for general $\alpha$ and $\beta$. But there is an inversion relation in the case where $\alpha = 0$ and $\beta = 1$. Since we are concerned with $g_\lambda^{(0,1)}$, it is convenient to specialize the Boltzmann weights from the column model of $g_\lambda^{(\alpha,\beta)}$.

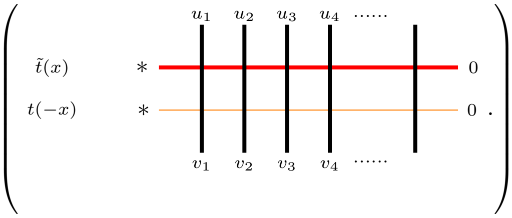


$$w_{(x,z)}\left(\quad\right) = w_{\left(\frac{x}{z}\right)}(a,b;c,d) = \left(\frac{x}{z}\right)^{\min(a,d)}.$$

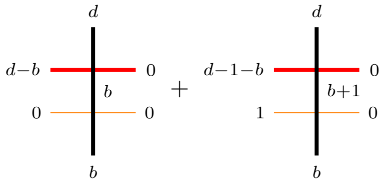
Finally, we recall that the transfer matrix $t$ of $g^{(1,0)}$ from the row model and the transfer matrix $\mathfrak{t}$ of $j$ polynomials are the same. Therefore, graphically we represent $t$ with fermionic auxiliary line.

Proposition 5.2. *The transfer matrices of $g_\lambda^{(1,0)}$ from the row model and transfer matrices of $g_\lambda^{(0,1)}$ from the column model satisfy the following relation:*

$$t(-x)\tilde{t}(x) = 1$$

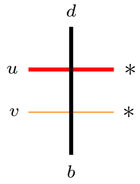


*Proof.* The proof is similar to Proposition 5.1. We shall prove the statement for transfer matrices of size 1. When $b = d$, there is a unique configuration with weight 1 and when $b \neq d$, we have two configurations which add up to 0.

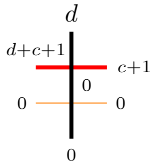


$$\left(\frac{x}{z}\right)^{d-b} + \left(\frac{-x}{z}\right)\left(\frac{x}{z}\right)^{d-1-b} = 0.$$

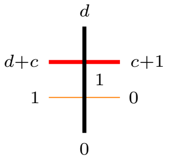
Assume $b > 0$, then for any fixed $v, u, b, d$ the Boltzmann weights are fixed irrespective of the right boundary.



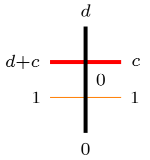
We can then simply slide the free boundary condition. Special care needs to be taken when $b = 0$. When $b = 0$, we have the following configurations:



$$= \left(\frac{x}{z}\right)^{d}$$



$$= \left(-\frac{x}{z}\right)^{d+1}$$



$$= \left(1 - \frac{x}{z}\right)\left(\frac{x}{z}\right)^{d}.$$

Observe that in the case of the first configuration, there is a unique configuration suggesting that the right boundary is not free. But we can get away with it by adding the weight of the first configuration with the weight of the second configuration. Then we obtain $\left(1 - \frac{x}{z}\right)\left(\frac{x}{z}\right)^{d}$ times the transfer matrix of size $n$. ☐

## 6. Cauchy identities

In this section, we shall prove Cauchy identities involving $G_\lambda^{(-\alpha,-\beta)}$ and $g_\lambda^{(\alpha,\beta)}$. We shall prove them by using the commutation relations between various combinations of the transfer matrices.

Proposition 6.1. *The transfer matrix* $T^*(x)$ *(eq. (13)) from the row model of* $G_\lambda^{(-\alpha,-\beta)}(x)$*, and the transfer matrix* $t(y)$*(eq. (21)) from the row model of* $g_\lambda^{(\alpha,\beta)}(y)$ *satisfy*

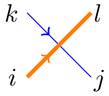
$$t(y)T^*(x) = \frac{1}{1-xy}T^*(x)t(y). \tag{56}$$

*Proof.* Firstly, note that the product of $T^*$ and $t$ is well defined only when we assume that the spectral variables satisfy $|x| < 1$ which ensures that the terms with unbounded degree are equal to 0.

The proof is similar to the way we proved that the transfer matrices commute. The $\mathfrak{R}(x,y)$ matrix

(57)

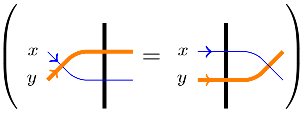
$$\mathfrak{R}(x,y) \in \operatorname{End}(F\otimes W), \quad \mathfrak{R}^{k,l}_{i,j} =$$



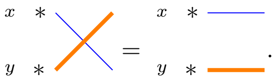
$$= \begin{cases} 1-xy & j=k=1, i=l=0 \\ xy & k=l=0, i=j=1 \\ 1-x\beta & k=1 \\ x\beta & k=0 \\ 1 & i=k=l=j=0 \end{cases}$$

where $k,j \in \{0,1\}$ and $i,l \in \mathbb{Z}_{\geqslant 0}$, together with the $L^*(x)$ matrix of $G_\lambda^{(-\alpha,-\beta)}(x)$ and the $l(y)$ matrix of $g_\lambda^{(\alpha,\beta)}(y)$ satisfies the RLL relation (see Appendix A.4):
(58)

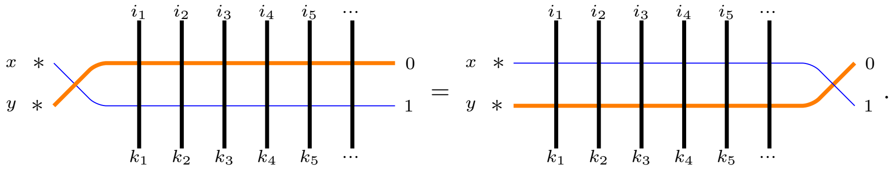
$$\mathfrak{R}(x,y)L^*(x)l(y) = l(y)L^*(x)\mathfrak{R}(x,y) \in \operatorname{End}(F\otimes W\otimes V)$$



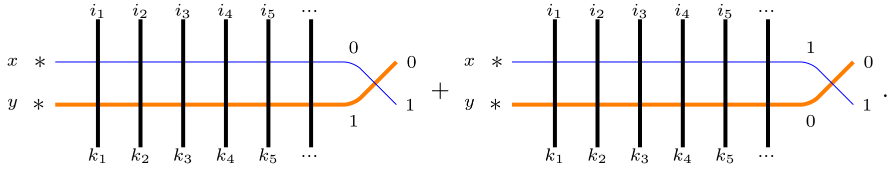
Observe that the sum of all possible states is an eigenvector $\mathfrak{R}$.



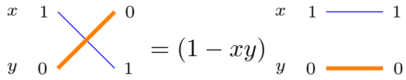
Then after multiplying the $\mathfrak{R}$-matrix to $T^*(x)t(y)$ and repeatedly applying the RLL relation we get the following equation:



Given below are the two possible configurations on the right hand side of the equation.



The partition function of the first configuration has terms of unbounded degree and hence is 0. Therefore, there is a unique configuration on the right hand side and the entry corresponding to the cross is $(1 - xy)$.



This implies the desired commutation relation:

$$T^*(x)t(y) = (1 - xy)t(y)T^*(x). \qquad \square$$

THEOREM 6.2. *Canonical Grothendieck polynomials and their duals satisfy the following Cauchy identity:*

$$\sum_{\lambda} G_{\lambda}^{(-\alpha,-\beta)}(x_1, x_2, \ldots, x_m) g_{\lambda}^{(\alpha,\beta)}(y_1, y_2, \ldots, y_n) = \prod_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} \frac{1}{1 - x_i y_j}. \tag{59}$$

*Proof.* Let

$$\mathcal{G}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = \langle 0| \, t(y_1)t(y_2) \cdot \cdots \cdot T^*(x_2)T^*(x_1) \, |0\rangle \, .$$

Then by Theorems 2.2 and 2.7 we get that

$$\mathcal{G}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = \sum_{\lambda} G_{\lambda}^{(-\alpha,-\beta)}(x_1, x_2, \ldots, x_m) g_{\lambda}^{(\alpha,\beta)}(y_1, y_2, \ldots, y_n).$$

By repeatedly applying the commutation relation of Proposition 6.1, we obtain

$$\begin{aligned} \mathcal{G}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) &= \prod_{\substack{1 \leqslant i \leqslant n, \\ 1 \leqslant j \leqslant m}} \frac{1}{1 - x_i y_j} \langle 0| \, T^*(x_1)T^*(x_2) \cdots t(y_2)t(y_1) \, |0\rangle \\ &= \prod_{\substack{1 \leqslant i \leqslant n, \\ 1 \leqslant j \leqslant m}} \frac{1}{1 - x_i y_j}. \qquad \square \end{aligned}$$

REMARK 6.3. Recall that $g_{\lambda}^{(\alpha,\beta)}$ and $G_{\lambda}^{(\alpha,\beta)}$ were defined using the branching formulae (see (15) and (25)). The Cauchy identity then implies that these two families are dual with respect to the Hall inner product.

We can derive a skew version of the identity if we choose a different vector and covector. Let

$$\mathcal{G}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = \langle \mu| \, t(y_1)t(y_2) \cdot \cdots \cdot T^*(x_2)T^*(x_1) \, |\lambda\rangle \, ,$$

then using the same reasoning as in the above theorem we get the following identity:

$$\begin{aligned} &\sum_{\nu} G_{\nu /\!\!/ \lambda}(x_1, x_2, \ldots, x_m) g_{\nu/\lambda}(y_1, y_2, \ldots, y_n) = \\ &\qquad \prod_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m} \frac{1}{1 - x_i y_j} \sum_{\nu} G_{\mu /\!\!/ \nu}(x_1, x_2, \ldots, x_m) g_{\lambda/\nu}(y_1, y_2, \ldots, y_n). \end{aligned}$$

We can do the same for all identities in this section but for simplicity we shall stick to the non-skew identities.

COROLLARY 6.4. *Generalised weak Grothendieck polynomials and their duals satisfy the following identity:*

(60)

$$\sum_{\lambda} J_{\lambda}^{\alpha}(x_1, \ldots, x_m; z_1, z_2, \ldots) j_{\lambda}^{\alpha}(y_1, \ldots, y_n; \frac{1}{z_1}, \frac{1}{z_2}, \ldots) = \prod_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} \frac{1}{1 - x_i y_j}.$$

*Proof.* Observe that when $\beta = 0$, the non constant entries of $\mathfrak{R}$ are $xy$ and $1 - xy$. If we introduce the inhomogeneities as $xz$ and $\frac{y}{z}$, then the $\mathfrak{R}$ matrix remains the same and thereby gives the same commutation relation. Then the proof of the identity is same as that of Theorem 6.2. ☐

We now prove the following Cauchy identity:

$$\sum_{\lambda} G_{\lambda'}^{(-\beta,-\alpha)}(x_1, x_2, \dots, x_m) g_{\lambda}^{(\alpha,\beta)}(y_1, y_2, \dots, y_n) = \prod_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} (1 + x_i y_j). \tag{61}$$

We can prove this identity by proving a commutation relation between $T^*$ and $\tilde{t}$. But we shall prove it using the inversion relation from (54).

Theorem 6.5. *Canonical Grothendieck polynomials and their duals satisfy the following Cauchy identity:*

$$\sum_{\lambda} G_{\lambda'}^{(-\beta,-\alpha)}(x_1, x_2, \dots, x_m) g_{\lambda}^{(\alpha,\beta)}(y_1, y_2, \dots, y_n) = \prod_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} (1 + x_i y_j).$$

*Proof.* By substituting $-x$ for $x$ in (56), we get the following relation.

$$t(y)T^*(-x) = \frac{1}{1+xy} T^*(-x)t(y).$$

By multiplying $\widetilde{T}^*\left(\frac{x}{1+x(\beta-\alpha)}\right)$ on both sides and applying the inversion relation (55), we get the following relation:

$$\widetilde{T}^*\left(\frac{x}{1+x(\beta-\alpha)}\right)t(y) = \frac{1}{1+xy} t(y)\widetilde{T}^*\left(\frac{x}{1+x(\beta-\alpha)}\right).$$

Let

$$\mathcal{G}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) = \\ \langle 0|\, t(y_1)t(y_2)\cdots \widetilde{T}^*\left(\frac{x_1}{1+(\beta-\alpha)x_1}\right)\cdots T^*\left(\frac{x_n}{1+(\beta-\alpha)x_n}\right)|0\rangle .$$

Then by Theorem 2.7 and the definition of $\widetilde{T}^*$ we obtain

$$\mathcal{G}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) = \sum_{\lambda} G_{\lambda'}^{(-\beta,-\alpha)}(x_1, \dots, x_n) g_{\lambda}^{(\alpha,\beta)}(y_1, \dots, y_m)$$

On the other hand by repeatedly applying the commutation relation we get that

$$\begin{aligned}&\mathcal{G}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)\\ &= \prod_{\substack{1\leqslant i\leqslant n,\\ 1\leqslant j\leqslant m}} (1 + x_i y_j)\, \langle 0|\, \widetilde{T}^*\left(\frac{x_1}{1+(\beta-\alpha)x_1}\right)\cdots T^*\left(\frac{x_n}{1+(\beta-\alpha)x_n}\right)t(y_1)t\cdots(y_m)\,|0\rangle\\ &= \prod_{\substack{1\leqslant i\leqslant n,\\ 1\leqslant j\leqslant m}} (1 + x_i y_j).\end{aligned}$$

☐

Corollary 6.6. *Generalised Grothendieck polynomials and generalised weak dual Grothendieck polynomials satisfy the following identity:*

$$\sum_{\lambda} G_{\lambda}(x_1, \dots, x_m; z_1, z_2, \dots) j_{\lambda}(y_1, \dots, y_n; \frac{1}{z_1}, \frac{1}{z_2}, \dots) = \prod_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} (1 + x_i y_j)\,. \tag{62}$$

*Proof.* Plug in $\beta = 0$ and $\alpha = 1$ in Theorem 6.5 and use the inhomogeneous transfer matrices. ☐

Theorem 6.7. *Generalised Grothendieck polynomials and their duals satisfy the following identity:*

(63)
$$\sum_\lambda G_\lambda(x_1,\ldots,x_m;z_1,\ldots,z_m)g_\lambda(y_1,\ldots,y_n;\frac{1}{z_1},\ldots,\frac{1}{z_m}) = \prod_{1\leqslant i\leqslant m,1\leqslant j\leqslant n}\frac{1}{1-x_iy_j}.$$

*Proof.* Recall the commutation relation from Theorem 6.5:

$$\widetilde{T}^*\left(\frac{x}{1+x(\beta-\alpha)}\right)t(y) = \frac{1}{1+xy}t(y)\widetilde{T}^*\left(\frac{x}{1+x(\beta-\alpha)}\right).$$

In order to apply the inversion relation among the transfer matrices of the dual Grothendieck polynomials, we need to specialize the above commutation relation with $\alpha = 1$ and $\beta = 0$. We then get the following relation:

$$\widetilde{T}^*\left(\frac{x}{1-x}\right)t(-y) = \frac{1}{1-xy}t(-y)\widetilde{T}^*\left(\frac{x}{1-x}\right).$$

We now multiply the above equation by $\tilde{t}(y)$ and apply the inversion relation (Proposition 5.2):

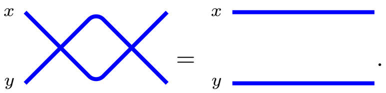
$$\tilde{t}(y)\widetilde{T}^*\left(\frac{x}{1-x}\right) = \frac{1}{1-xy}\widetilde{T}^*\left(\frac{x}{1-x}\right)\tilde{t}(y).$$

The result then follows immediately from the definition of $\widetilde{T}^*$, and Theorem 3.5. □

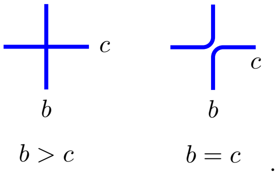
Proposition 6.8. *Generalised Grothendieck polynomials satisfy*

$$G_\lambda(z_1,\ldots,z_m;z_1,\ldots,z_m) = 1.$$

*Proof.* When $\alpha = 0$ and $\beta = -1$, $\widetilde{R}$ and $\widetilde{L}$ are same. The $\widetilde{R}$ matrix satisfies the unitary relation (for a proof refer to Appendix A.1.5):

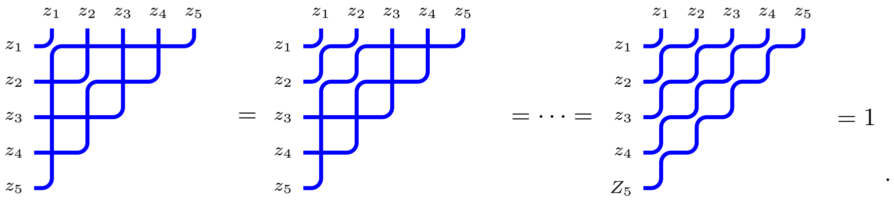


Recall that there are two types of vertices (*crossing* and *elbow*) in the column model for Grothendieck polynomials:



Consider the model with $m$ rows i.e. Grothendieck polynomial in $m$ variables. We know that the number of variables restricts us to partitions with highest column being less than or equal to $m$. So the only inhomogeneities are $z_1,\ldots,z_m$.

Recall that the Boltzmann weight of a crossing has a $\left(1-\frac{x}{z}\right)$ factor. So, when we set $x_i = z_i$, the contribution of a configuration to the partition function is non zero only when the vertices along the diagonal are entirely elbows.



Then we get the desired result by repeatedly applying the unitary relation. ☐

As a consequence of the above proposition, we recover an identity for *generalised dual Grothendieck polynomials*, which is proved by Yeliussizov in [18].

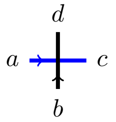
Corollary 6.9. *Dual Grothendieck polynomials satisfy the following identity:*

$$\sum_{l(\lambda)\leqslant m} g_\lambda(y_1,\dots,y_n;\frac{1}{z_1},\dots,\frac{1}{z_m}) = \prod_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} \frac{1}{1-z_i y_j}.$$

*Proof.* Set $x_i = z_i$ in (63). ☐

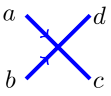
## Appendix A. RLL relations

A.1. RLL for the column model of $G_\lambda^{(\alpha,\beta)}$. For convenience, let us recall the Boltzmann weights of the column model of $G_\lambda^{(\alpha,\beta)}$ and the entries of the $\widetilde{R}$ matrix.



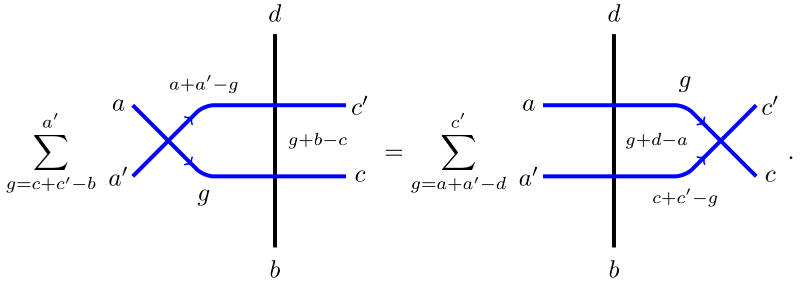
$$(64)\qquad w_x\left(\quad\right) \equiv w_x(a,b;c,d) = \delta_{a+b,c+d}\begin{cases} \left(\frac{x}{1-\alpha x}\right)^a & b=c \\ \left(\frac{x}{1-\alpha x}\right)^a\left(\frac{1+\beta x}{1-\alpha x}\right) & b>c \\ 0 & b<c \end{cases}$$

where $a,b,c,d$ are non negative integers.



$$\widetilde{R}^{a\,d}_{b\,c}(y,x) = \qquad = \left(\frac{\frac{x}{1-\alpha x}}{\frac{y}{1-\alpha y}}\right)^a \begin{cases} 0 & \text{when } b<c \\ 1 & \text{when } b=c \\ \dfrac{1}{1-\alpha x} - \dfrac{x}{(1-\alpha x)y} & \text{otherwise.} \end{cases}$$

Let us try to understand the range of $g$. First observe that whenever $a' < g$, the summation is 0 because of the $\widetilde{R}$-matrix. Based on the Boltzmann weights, the contribution of the top vertex is non zero if and only if $g+b-c \geqslant c'$. Therefore, $g$ on *LHS* can at most be $a'$, and it has to be at least $c'+c-b$. Similarly, on the *RHS* we have $a+a'-d \leqslant c'$.



A.1.1. *Assume $b > c$ and $d > a$.* Let us now compute the $LHS$.

$$\begin{aligned}LHS = &\left(\frac{y-x}{y(1-\alpha x)}\right)\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{a}\left(\frac{y}{1-\alpha y}\right)^{d}\left(\frac{x}{1-\alpha x}\right)^{c+c'-b}\left(\frac{1+\beta x}{1-\alpha x}\right)+\\ &\left(\frac{y-x}{y(1-\alpha x)}\right)\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{a}\left(\frac{y}{1-\alpha y}\right)^{a+a'}\left(\frac{1+\beta y}{1-\alpha y}\right)\left(\frac{1+\beta x}{1-\alpha x}\right)\\ &\left(\sum_{g=c+c'-b+1}^{a'-1}\left(\frac{x(1-\alpha y)}{x(1-\alpha x)}\right)^{g}\right)+\\ &\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{a}\left(\frac{x}{1-\alpha x}\right)^{a'}\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{y}{1-\alpha y}\right)^{a}\left(\frac{1+\beta y}{1-\alpha y}\right)\\ =&\left(\frac{1+\beta x}{1-\alpha x}\right)^{2}\left(\frac{x}{1-\alpha x}\right)^{a+c+c'-b}\left(\frac{y}{1-\alpha y}\right)^{d-a}.\end{aligned}$$

We compute the right hand side of the equation:

$$\begin{aligned}RHS = &\left(\frac{y-x}{y(1-\alpha x)}\right)\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{a+a'-d}\left(\frac{x}{1-\alpha x}\right)^{a}\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{y}{1-\alpha y}\right)^{a'}+\\ &\left(\sum_{g=a+a'-d+1}^{c'-1}\left(\frac{y-x}{y(1-\alpha x)}\right)\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{g}\left(\frac{x}{1-\alpha x}\right)^{a}\right.\\ &\left.\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{y}{1-\alpha y}\right)^{a'}\left(\frac{1+\beta y}{1-\beta y}\right)\right)+\\ &\left(\frac{x(1-\alpha y)}{y(1-\alpha x)}\right)^{c'}\left(\frac{x}{1-\alpha x}\right)^{a}\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{y}{1-\alpha y}\right)^{a'}\left(\frac{1+\beta y}{1-\beta y}\right)\\ =&\left(\frac{1+\beta x}{1-\alpha x}\right)^{2}\left(\frac{x}{1-\alpha x}\right)^{a+c+c'-b}\left(\frac{y}{1-\alpha y}\right)^{d-a}.\end{aligned}$$

A.1.2. *Assume $a < d$ and $b = c$.* From the computation of the previous case, we can get the $LHS$ by multiplying $\frac{1-\alpha x}{1+\beta x}$ to the $LHS$ of previous computation.

$$LHS = \left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{x}{1-\alpha x}\right)^{a+c'}\left(\frac{y}{1-\alpha y}\right)^{d-a}.$$

On the right hand side, there is only one case because of the global condition, $a + a' + b = c + c' + d$.

$$\begin{aligned}RHS = &\left(\frac{x}{1-\alpha x}\right)^{a+c'}\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{y}{1-\alpha y}\right)^{a'-c'}\\ = &\left(\frac{x}{1-\alpha x}\right)^{a+c'}\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{y}{1-\alpha y}\right)^{d-a}.\end{aligned}$$

A.1.3. *Assume $a = d$ and $b > c$.* $RHS$ of the present case is a $\frac{1-\alpha x}{1+\beta x}$ multiple of the $RHS$ of the Appendix A.1.1.

$$\begin{aligned} RHS = &\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{x}{1-\alpha x}\right)^{a+c+c'-b}\left(\frac{y}{1-\alpha y}\right)^{d-a} \\ &\left(\frac{1+\beta x}{1-\alpha x}\right)\left(\frac{x}{1-\alpha x}\right)^{a+c+c'-b}. \end{aligned}$$

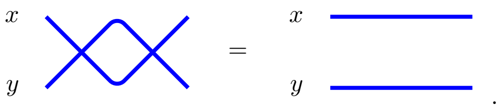
For the $LHS$, there is just one valid configuration:

$$\begin{aligned} LHS = &\left(\frac{x}{1-\alpha x}\right)^{a}\left(\frac{y}{1-\alpha y}\right)^{-a}\left(\frac{y}{1-\alpha y}\right)^{a}\left(\frac{x}{1-\alpha x}\right)^{a'}\left(\frac{1+\beta x}{1-\alpha x}\right) \\ = &\left(\frac{x}{1-\alpha x}\right)^{a+a'}\left(\frac{1+\beta x}{1-\alpha x}\right) \\ = &\left(\frac{x}{1-\alpha x}\right)^{a+c+c'-b}\left(\frac{1+\beta x}{1-\alpha x}\right). \end{aligned}$$

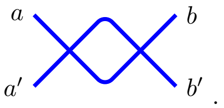
In the final step, we substitute $a' = c+c'-b$, which follows from the global condition.

A.1.4. *Assume $a = d$ and $b = c$.* Recall from Appendix A.1.3 that when $a = d$, there is a unique configuration on $LHS$. Similarly, recall from Appendix A.1.2 that there is a unique configuration on $RHS$ when $b = c$. We have already computed these two configurations and they are equal.

A.1.5. *Unitary relation for the $\widetilde{R}$ matrix.* The $\widetilde{R}$ satisfies the unitary relation:



Consider the following configuration:



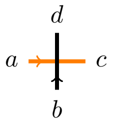
When $a = b$ and $a' = b'$, there is a unique configuration with weight 1. Now let us assume $a \neq b$ and $a' \neq b'$.

$$\begin{aligned} &= \left(\frac{x}{y}\right)^{a}\left(1-\frac{x}{y}\right)\left(\frac{y}{x}\right)^{a+a'-b'} + \sum_{g=b'+1}^{a'-1}\left(\frac{x}{y}\right)^{a}\left(1-\frac{x}{y}\right)\left(\frac{y}{x}\right)^{a+a'-g}\left(1-\frac{y}{x}\right)+ \\ &\quad \left(\frac{x}{y}\right)^{a}\left(\frac{y}{x}\right)^{a}\left(1-\frac{y}{x}\right) \\ &= \left(1-\frac{x}{y}\right)\left(\frac{y}{x}\right)^{a'-b'} + \left(\frac{y}{x}\right)^{a'-b'-1}\left(1-\frac{y}{x}\right)\left(1-\left(\frac{x}{y}\right)^{a'-b'-1}\right) + \left(1-\frac{y}{x}\right) \\ &= 0. \end{aligned}$$

A.2. RLL relation for row model of $g_\lambda^{(\alpha,\beta)}$. We recall the Boltzmann weights and $r$-matrix of the row model of $g_\lambda^{(\alpha,\beta)}$:

(65)

$w_x$

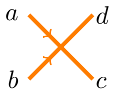


$$\equiv w_x(a,b;c,d) = \delta_{a+b,c+d}\begin{cases} (\alpha+\beta)^{a-d-1}(x+\alpha)\beta^d & a>d \\ \beta^{a-1}x & 0<a\leqslant d \\ 1 & a=0, \end{cases}$$

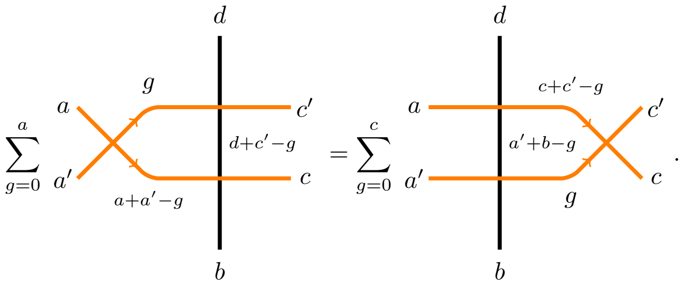
where $a,b,c,d \in \mathbb{Z}_{\geqslant 0}$.

The entries of the $r$-matrix are the following:

(66) $r_{b,c}^{a,d}(x,y) =$



$$= \begin{cases} 0 & b>c \\ 1 & b=c=0 \\ \dfrac{y}{x} & b=c>0 \\ \left(1-\dfrac{y}{x}\right)\left(1-\dfrac{y}{\beta}\right)^{a-d-1} & b=0 \\ \left(1-\dfrac{y}{x}\right)\left(1-\dfrac{y}{\beta}\right)^{a-d-1}\left(\dfrac{y}{\beta}\right) & b>0. \end{cases}$$



Before we start proving the relation, let us analyze the cases we need to consider. Firstly, from the $LHS$, the weight of bottom vertex is fixed based on whether $b \leqslant c$ or $b > c$. Similarly, on the $RHS$, the weight of the top vertex is fixed based on the relation between $a$ and $d$.

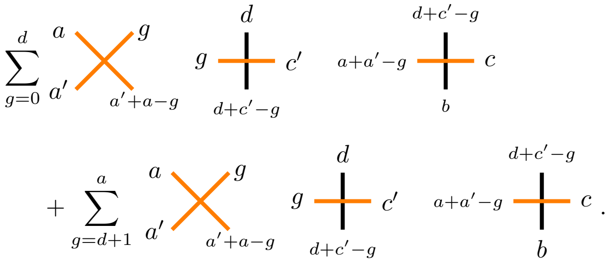
We now look at the cases that arise from considering the entries of the $r$ matrix. Firstly, the entries of the $r$ matrix on $LHS$ depends on whether $a' > 0$ or $a' = 0$. Similarly, the entry of $r$ matrix on $RHS$ depends on whether $c > 0$ or $c = 0$.

In total, there are sixteen cases to consider. We shall divide these cases into four categories based on the conditions on $b, c$ and $a, d$. Then for each such case, we shall consider four sub-cases by the conditions on $a'$ and $c$.

A.2.1. *Assume $b < c$ and $d < a$. Assume $a' > 0$ and $c > 0$..*

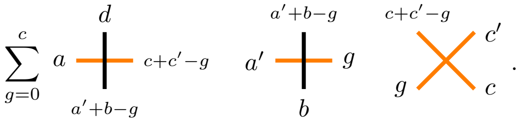
To ease up the computation, we break up the summation into two parts, $0 \leqslant g \leqslant d$ and then $d < g \leqslant a$.

Under the assumption that $b < c$, we have



$$\begin{aligned}
LHS = {} & \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{a-1}\left(\frac{y}{\beta}\right)(\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'}+ \\
& \sum_{g=1}^{d}\left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{a-g-1}\left(\frac{y}{\beta}\right)(\beta^{g-1}y)\left((\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'-g}\right)+ \\
& \sum_{g=d+1}^{a-1}\left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{a-g-1}\left(\frac{y}{\beta}\right)\left((\alpha+\beta)^{g-d-1}\beta^{d}(y+\alpha)\right) \\
& \left((\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'-g}\right) \\
& +\left(\frac{y}{x}\right)\left((\alpha+\beta)^{a-d-1}\beta^{d}(y+\alpha)\right)\left((\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'-a}\right) \\
= {} & (\alpha+\beta)^{c+a-b-d-2}\beta^{2d+c'-a}(x+\alpha)\left(y+\alpha\frac{y}{x}\right).
\end{aligned}$$

We compute the right hand side: Assume that $b < c$.



$$\begin{aligned}
\frac{RHS}{(\alpha+\beta)^{a-d-1}\beta^{d}(x+\alpha)} = {} & \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-1}\beta^{a'-1}y+ \\
& \sum_{g=1}^{b}\left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-g-1}\frac{y}{\beta}\beta^{a'-1}y+ \\
\sum_{g=b+1}^{c-1}\left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{c-g-1} & \frac{y}{\beta}\left((\alpha+\beta)^{g-b-1}(y+\alpha)\beta^{a'+b-g}\right)+ \\
& \frac{y}{x}\left((\alpha+\beta)^{c-b-1}(y+\alpha)\beta^{a'+b-c}\right) \\
= {} & \left(1-\frac{y}{x}\right)(\alpha+\beta)^{c-b-1}\beta^{a'+b-c}y+ \\
& \frac{y}{x}\left((\alpha+\beta)^{c-b-1}(y+\alpha)\beta^{a'+b-c}\right) \\
RHS = {} & (\alpha+\beta)^{a+c-d-b-2}\beta^{d+a'+b-c}\left(y+\alpha\frac{y}{x}\right)(x+\alpha) \\
= {} & (\alpha+\beta)^{a+c-d-b-2}\beta^{2d+c'-a}\left(y+\alpha\frac{y}{x}\right)(x+\alpha).
\end{aligned}$$

*Assume $a' = 0$ and $c > 0$.*

$$\begin{aligned}
LHS = {} & \Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-1}(\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'}+\\
& \sum_{g=1}^{d}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}\big(\beta^{g-1}y\big)\Big((\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'-g}\Big)+\\
& \sum_{g=d+1}^{a+b-c}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}\Big((\alpha+\beta)^{g-d-1}\beta^{d}(y+\alpha)\Big)\\
& \Big((\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'-g}\Big)\\
= {} & \Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{c-b-1}(\alpha+\beta)^{a-d-1}(x+\alpha)\beta^{d}.
\end{aligned}$$

We compute the $RHS$:

$$\begin{aligned}
RHS = {} & (\alpha+\beta)^{a-d-1}\beta^{d}(x+\alpha)\left(\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{c-1}+\right.\\
& \left.\left(\sum_{g=1}^{b}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{c-g-1}\frac{y}{\beta}\right)\right)\\
= {} & (\alpha+\beta)^{a-d-1}\beta^{d}(x+\alpha)\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{c-b-1}.
\end{aligned}$$

Since we assumed $b > c$, we do not need to consider the cases where $c = 0$.

A.2.2. *Assume $b \geqslant c$ and $d < a$. Assume $a' > 0$ and $c > 0$ .*

$$\begin{aligned}
LHS = {} & \Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-1}\Big(\frac{y}{\beta}\Big)x\beta^{a+a'-1}+\\
& \sum_{g=1}^{d}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}\Big(\frac{y}{\beta}\Big)\big(\beta^{g-1}y\big)x\beta^{a+a'-g-1}+\\
& \sum_{g=d+1}^{a-1}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}\Big(\frac{y}{\beta}\Big)\Big((\alpha+\beta)^{g-d-1}\beta^{d}(y+\alpha)\Big)x\beta^{a+a'-g-1}\\
& +\Big(\frac{y}{x}\Big)\Big((\alpha+\beta)^{a-d-1}\beta^{d}(y+\alpha)\Big)x\beta^{a'-1}\\
= {} & \Big(\frac{y}{\beta}\Big)(\alpha+\beta)^{a-d-1}\beta^{a'+d}(x+\alpha).
\end{aligned}$$

Since $a < d$ and $b \geqslant c$, the Boltzmann weights from the vertices are fixed. When we factor out the contribution from the overall sum, we are left with entries of the $r$-matrix with fixed right boundary. Then from the fact that sum of all the possible states is an eigenvector of the $r$ matrix, we get that the overall sum of $RHS$ is just the product of the fixed Boltzmann weights.

$$RHS = \big((\alpha+\beta)^{a-d-1}(x+\alpha)\beta^{d}\big)\Big(\beta^{a'-1}y\Big).$$

*Assume $a' = 0$.*

$$LHS = \Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-1}x\beta^{a-1}+$$

$$\begin{aligned}
&\sum_{g=1}^{d}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}(\beta^{g-1}y)x\beta^{a-g-1}+\\
&\sum_{g=d+1}^{a-1}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}\Big((\alpha+\beta)^{g-d-1}\beta^{d}(y+\alpha)\Big)x\beta^{a+a'-g-1}\\
&+\Big((\alpha+\beta)^{a-d-1}\beta^{d}(y+\alpha)\Big)\\
=\ &(\alpha+\beta)^{a-d-1}\beta^{d}(x+\alpha).
\end{aligned}$$

For $RHS$, we again use the fact that the sum of all the possible states is an eigenvector of the $r$-matrix to conclude that the $RHS$ is just the product the Boltzmann weights. Observe that the contribution from the bottom vertex is 1. So, the overall $RHS$ is just the Boltzmann weight of the top vertex.

$$RHS=(\alpha+\beta)^{a-d-1}(x+\alpha)\beta^{d}.$$

A.2.3. *Assume $a \leqslant d$ and $b < c$.* Since we are assuming $b < c$, its follows that $c > 0$. *Assume $a' > 0$.*

$$\begin{aligned}
LHS=\ &\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-1}\Big(\frac{y}{\beta}\Big)(\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'}+\\
&\sum_{g=1}^{a-1}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}\Big(\frac{y}{\beta}\Big)(\beta^{g-1}y)\Big((\alpha+\beta)^{b-c-1}(x+\alpha)\beta^{d+c'-g}\Big)+\\
&+\Big(\frac{y}{x}\Big)(\beta^{a-1}y)\Big((\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'-a}\Big)\\
=\ &(\alpha+\beta)^{c-b-1}(x+\alpha)\beta^{d+c'}\Big(\frac{y}{\beta}\Big).
\end{aligned}$$

Given that $b < c$, we can get the $RHS$ computation from Appendix A.2.1. The only difference being the Boltzmann weight corresponding to the top vertex.

$$\begin{aligned}
RHS=\ &x\beta^{a-1}(\alpha+\beta)^{c-b-1}\beta^{a'+b-c}\Big(\Big(1-\frac{y}{x}\Big)y+\frac{y}{x}(y+\alpha)\Big)\\
=\ &(\alpha+\beta)^{c-b-1}\beta^{a'+b-c+a}\Big(\frac{y}{\beta}\Big)(x+\alpha)\\
=\ &(\alpha+\beta)^{c-b-1}\beta^{d+c'}\Big(\frac{y}{\beta}\Big)(x+\alpha).
\end{aligned}$$

We do not need to consider the case where $a' = 0$ as the global condition forces $c'$ to negative.

$$c'=(a-d)+(b-c)<0.$$

A.2.4. *Assume $a \leqslant d$ and $b \geqslant c$. Assume $a' > 0$.*

$$\begin{aligned}
LHS=\ &\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-1}\Big(\frac{y}{\beta}\Big)\beta^{a+a'-1}x+\\
&\sum_{g=1}^{a-1}\Big(1-\frac{y}{x}\Big)\Big(1-\frac{y}{\beta}\Big)^{a-g-1}\Big(\frac{y}{\beta}\Big)\beta^{g-1}y\beta^{a+a'-g-1}x+\\
&\frac{y}{x}\beta^{a-1}y\beta^{a'-1}x\\
=\ &\Big(y\beta^{a'-1}\Big)\Big(x\beta^{a-1}\Big).
\end{aligned}$$

As a result of the assumptions, the Boltzmann weights are fixed. Using the fact that sum of all the possible states is an eigenvector, we get that $RHS$ is just the product of the two fixed Boltzmann weights.

$$RHS = \left(x\beta^{a-1}\right)\left(y\beta^{a'-1}\right).$$

*Assume* $a' = 0$.

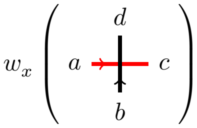
$$\begin{aligned} LHS = & \left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{a-1}\beta^{a-1}x+ \\ & \sum_{g=1}^{a-1}\left(1-\frac{y}{x}\right)\left(1-\frac{y}{\beta}\right)^{a-g-1}\beta^{g-1}y\beta^{a-g-1}x+\beta^{a-1}x \\ = & \ \beta^{a-1}x. \end{aligned}$$

On the $RHS$, it's just the Boltzmann weight of the top vertex.

$$RHS = \beta^{a-1}x.$$

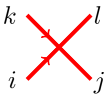
A.3. RLL FOR THE COLUMN MODEL OF $g_\lambda^{(\alpha,\beta)}$. We recall the Boltzmann weights, and the entries of $\tilde{r}$-matrix.

(67)

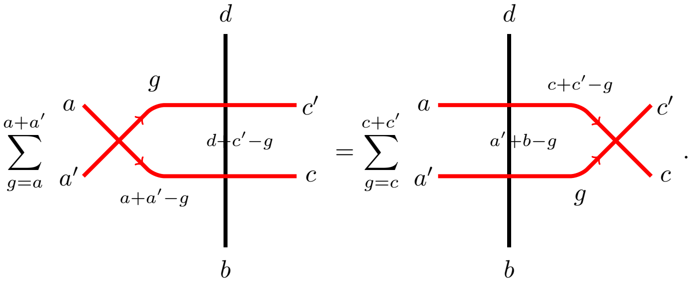


$$\equiv w_x(a,b;c,d) = \delta_{a+b,c+d}\begin{cases}\beta(\alpha+\beta)^{a-d-1}(x+\alpha)^d & 0<a>d\\ x(x+\alpha)^{a-1} & 0<a\leqslant d\\ 1 & a=0.\end{cases}$$

(68) $\widetilde{r}_{i,j}^{k,l}(x,y) =$



$$= \begin{cases} 0 & i<j\\ 1 & k=l=0\\ \dfrac{x}{y}\left(\dfrac{y+\alpha}{x+\alpha}\right)^{1-k} & k=l>0\\ \left(1-\dfrac{x}{y}\right) & k=0\\ \dfrac{x}{y}\left(\dfrac{y+\alpha}{x+\alpha}-1\right)\left(\dfrac{y+\alpha}{x+\alpha}\right)^{-k} & k>0.\end{cases}$$



Firstly, on the $LHS$ the Boltzmann weight of the bottom vertex is fixed based on the relation between $b$ and $c$. Similarly, from the $RHS$, we see that the weight of the top vertex is determined by the relation between $a$ and $d$. So, we need to assume certain relations between $b$ and $c$, and $a$ and $d$. Furthermore, observe that entries of the $\widetilde{r}$ matrix depends on whether the top left label is equal to or greater than 0.

Therefore, in each subsection we assume some combination of relations between $b$ and $c$, and $a$ and $d$, and a condition on $c'$.

A.3.1. *Assume $a > d$ and $b \geqslant c$.* As $a > d$, we get that $a > 0$. Similarly, from the bottom label of the top vertex on $LHS$, we get that $c' \neq 0$. Therefore, we only need to consider the case where $a > 0$ and $c' > 0$.

$$\begin{aligned}
LHS = {} & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}\beta(\alpha+\beta)^{a-d-1}(y+\alpha)^{d}x(x+\alpha)^{a'-1}+ \\
& \sum_{g=a+1}^{a+a'-1}\left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}\beta(\alpha+\beta)^{g-d-1}(y+\alpha)^{d}x(x+\alpha)^{a'+a-g-1} \\
& \left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}\beta(\alpha+\beta)^{a+a'-d-1}(y+\alpha)^{d} \\
= {} & \left(\frac{x}{y}\right)\beta(\alpha+\beta)^{a-d-1}(y+\alpha)^{d-a}x(x+\alpha)^{a'+a-1}\left(\frac{y-\beta}{x-\beta}\right)- \\
& \left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\beta(\alpha+\beta)^{a+a'-d-1}(y+\alpha)^{d-a+1}(x+\alpha)^{a-1}\left(\frac{\beta}{x-\beta}\right).
\end{aligned}$$

We compute the $RHS$. Observe that we need assume a condition on $c'$. First, let us assume that $c' > 0$.

Observe that the label $a' + b - g$ has to be positive. Based on our assumptions and the global condition, we conclude that

$$a' + b - (c + c') = d - a < 0.$$

Therefore, the range of $g$ is from $a$ to $a' + b$.

$$\begin{aligned}
\frac{RHS}{\beta(\alpha+\beta)^{a-d-1}(x+\alpha)^{d}} = {} & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-c'}y(y+\alpha)^{a'-1} \\
+ \sum_{g=c+1}^{b}\left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1+g-c'-c} & \left(\frac{y-x}{y+\alpha}\right)y(y+\alpha)^{a'-1} \\
+ \sum_{g=b+1}^{a'+b}\left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1+g-c'-c} & \left(\frac{y-x}{y+\alpha}\right)\beta(\alpha+\beta)^{g-b-1}(y+\alpha)^{a'+b-g} \\
RHS = {} & \left(\frac{x}{y}\right)\left(\frac{y-\beta}{x-\beta}\right)x(x+\alpha)^{a+a'-1}(y+\alpha)^{d-a}\beta(\alpha+\beta)^{a-d-1}- \\
& \left(\frac{x}{y}\right)(y-x)\left(\frac{\beta^{2}}{x-\beta}\right)(y+\alpha)^{d-a}(x+\alpha)^{a-1}(\alpha+\beta)^{a+a'-d-1}.
\end{aligned}$$

A.3.2. *Assume $a \leqslant d$ and $b \geqslant c$.* We now assume that $a > 0$ and $c' > 0$.

$$\begin{aligned}
LHS = {} & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}y(y+\alpha)^{a-1}x(x+\alpha)^{a'-1}+ \\
& \sum_{g=a+1}^{d}\left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}y(y+\alpha)^{g-1}x(x+\alpha)^{a'+a-g-1}+ \\
& \sum_{g=d+1}^{a+a'-1}\left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}\beta(\alpha+\beta)^{g-d-1}(y+\alpha)^{d}x(x+\alpha)^{a'+a-g-1}+
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}\beta(\alpha+\beta)^{a+a'-d-1}(y+\alpha)^{d}\\
=\ &\left(\frac{x^{3}}{y}\right)\left(\frac{y-\beta}{x-\beta}\right)(y+\alpha)^{d-a}(x+\alpha)^{2a+a'-d-2}-\\
&\left(\frac{x}{y}\right)\left(\frac{y-x}{x-\beta}\right)\beta(\alpha+\beta)^{a+a'-d-1}(y+\alpha)^{d-a}(x+\alpha)^{a-1}.
\end{aligned}$$

$$\begin{aligned}
\frac{RHS}{x(x+\alpha)^{a-1}} =\ &\left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-c'}y(y+\alpha)^{a'-1}+\\
&\sum_{g=c+1}^{b}\left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1+g-c'-c}\left(\frac{y-x}{y+\alpha}\right)y(y+\alpha)^{a'-1}+\\
&\sum_{g=b+1}^{c+c'-1}\left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1+g-c'-c}\left(\frac{y-x}{y+\alpha}\right)\beta(\alpha+\beta)^{g-b-1}(y+\alpha)^{a'+b-g}+\\
&\left(1-\frac{x}{y}\right)\beta(\alpha+\beta)^{c+c'-b-1}(y+\alpha)^{a'+b-c-c'}\\
RHS =\ &\left(\frac{x^{3}}{y}\right)\left(\frac{y-\beta}{x-\beta}\right)(y+\alpha)^{d-a}(x+\alpha)^{2a+a'-d-2}-\\
&\left(\frac{x}{y}\right)\left(\frac{y-x}{x-\beta}\right)\beta(\alpha+\beta)^{a+a'-d-1}(y+\alpha)^{d-a}(x+\alpha)^{a-1}.
\end{aligned}$$

*Assume* $a=0$ *and* $c'>0$.

$$\begin{aligned}
LHS =\ &x(x+\alpha)^{a'-1}+\sum_{g=1}^{d}\left(1-\frac{x}{y}\right)y(y+\alpha)^{g-1}x(x+\alpha)^{a'-g-1}+\\
&\sum_{g=d+1}^{a'-1}\left(1-\frac{x}{y}\right)\beta(\alpha+\beta)^{g-d-1}(y+\alpha)^{d}x(x+\alpha)^{a'-g-1}+\\
&\left(1-\frac{x}{y}\right)\beta(\alpha+\beta)^{a'-d-1}(y+\alpha)^{d}\\
=\ &x(x+\alpha)^{a'-d-1}(y+\alpha)^{d}\left(\frac{x}{y}\right)\left(\frac{y-\beta}{x-\beta}\right)-\\
&\left(1-\frac{x}{y}\right)\beta(y+\alpha)^{d}(\alpha+\beta)^{a'-d-1}\left(\frac{\beta}{x-\beta}\right).
\end{aligned}$$

Observe that, while computing $RHS$ in the case where $a\leqslant d$, it is only in the final step we multiply the Boltzmann weight of the top vertex. Here, the Boltzmann weight of the top vertex is 1. Therefore,

$$\begin{aligned}
RHS =\ &\left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1+b-c-c'}(y+\alpha)^{a'-1}\left(\frac{(y-\beta)x}{x-\beta}\right)-\\
&\beta(\alpha+\beta)^{c+c'-b-1}(y+\alpha)^{d}\left(\frac{\beta(y-x)}{y(x-\beta)}\right)\\
=\ &\left(\frac{x^{2}}{y}\right)\left(\frac{y-\beta}{x-\beta}\right)(y+\alpha)^{d}(x+\alpha)^{a'-d-1}-\\
&\left(1-\frac{x}{y}\right)\left(\frac{\beta}{x-\beta}\right)\beta(y+\alpha)^{d}(\alpha+\beta)^{a'-d-1}.
\end{aligned}$$

In the above computation we have assumed $a' > d$. We now consider the case where $a' \leqslant d$.

$$\begin{aligned} LHS = {} & x(x+\alpha)^{a'-1} + \sum_{g=1}^{a'-1} \left(1 - \frac{x}{y}\right) y(y+\alpha)^{g-1} x(x+\alpha)^{a'-g-1} + \\ & \left(1 - \frac{x}{y}\right) y(y+\alpha)^{a'-1} \\ = {} & y(y+\alpha)^{a'-1}. \end{aligned}$$

On the $RHS$, the weights of the vertices are fixed for all $g$. Since the right boundary of the cross is fixed and the fact that sum of all possible states is an eigenvector, we get that $RHS = y(y+\alpha)^{a'-1}$.

A.3.3. *Assume $a > d$ and $b < c$.* As $a > d$, we will have $a > 0$. Also $c' > 0$, otherwise we get contradiction on the range of $g$.

$$\begin{aligned} & LHS = \\ & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a} \beta(\alpha+\beta)^{a-d-1}(y+\alpha)^{d}\beta(\alpha+\beta)^{a'+a-d-c'-1}(x+\alpha)^{d+c'-a} + \\ & \sum_{g=a+1}^{d+c'} \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}\left(\frac{y-x}{y+\alpha}\right) \\ & \beta(\alpha+\beta)^{g-d-1}(y+\alpha)^{d}\beta(\alpha+\beta)^{a+a'-d-c'-1}(x+\alpha)^{d+c'-g} \\ = {} & \left(\frac{x}{y}\right)\beta^2(\alpha+\beta)^{a+c-b-d-2}(y+\alpha)^{d+1-a}(x+\alpha)^{d+c'-1} + \\ & \left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\beta^2(\alpha+\beta)^{a+b-c-d-1}(y+\alpha)^{d+1-a} \\ & (x+\alpha)^{d+c'-1}\left(\frac{1-\left(\dfrac{\alpha+\beta}{x+\alpha}\right)^{d+c'-a}}{x-\beta}\right). \end{aligned}$$

We compute the $RHS$:

$$\begin{aligned} & RHS = \\ & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-c'} \beta(\alpha+\beta)^{a-d-1}(x+\alpha)^{d}\beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{a'+b-c} + \\ & \sum_{g=c+1}^{a'+b} \left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1+g-c-c'} \left(\beta(\alpha+\beta)^{a-d-1}(x+\alpha)^{d}\right) \\ & \left(\beta(\alpha+\beta)^{g-b-1}(y+\alpha)^{a'+b-g}\right) \\ = & \left(\frac{x}{y}\right)\beta^2(\alpha+\beta)^{a+c-b-d-2}(y+\alpha)^{d-a+1}(x+\alpha)^{d+c'-1} + \\ & \left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\beta^2(\alpha+\beta)^{a+c-b-d-1}(y+\alpha)^{d-a+1}(x+\alpha)^{d+c'-1} \end{aligned}$$

$$\left(\frac{1-\left(\frac{\alpha+\beta}{x+\alpha}\right)^{d+c'-a}}{x-\beta}\right).$$

A.3.4. *Assume $a \leqslant d$ and $b < c$. Assume $a > 0$ and $c' > 0$.*

Recall that we have the global condition $a + a' + b = c + c' + d$. Observe that, because of the assumptions, the range of $g$ on the $LHS$ is $a$ to $d + c'$.

On the $RHS$, we have $a' + b - c - c' = d - a \geqslant 0$. Therefore, the range of $g$ is from $c$ to $c + c'$

$$\begin{aligned}
LHS = {} & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a} y(y+\alpha)^{a-1}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'-a} + \\
& \sum_{g=a+1}^{d}\left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a} y(y+\alpha)^{g-1}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'-g} + \\
& \sum_{g=d+1}^{d+c'}\left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a} \beta(\alpha+\beta)^{g-d-1} \\
& (y+\alpha)^{d}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'-g} \\
= {} & \left(\frac{x^2}{y}\right)\beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{d-a}(x+\alpha)^{a+c'-1}\left(\frac{y-\beta}{x-\beta}\right) - \\
& \left(\frac{x}{y}\right)\beta^2\left(\frac{y-x}{x-\beta}\right)(\alpha+\beta)^{c+c'-b-1}(y+\alpha)^{d-a}(x+\alpha)^{a-1}.
\end{aligned}$$

$$\begin{aligned}
\frac{RHS}{x(x+\alpha)^{a-1}} = {} & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-c'} \beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{a'+b-c} + \\
& \sum_{g=c+1}^{c+c'-1}\left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1+g-c'-c}\left(\frac{y-x}{y+\alpha}\right)\beta(\alpha+\beta)^{g-b-1}(y+\alpha)^{a'+b-g} + \\
& \left(1-\frac{x}{y}\right)\beta(\alpha+\beta)^{c+c'-b-1}(y+\alpha)^{a'+b-c-c'} \\
RHS = {} & \left(\frac{x^2}{y}\right)\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{a+c'-1}(y+\alpha)^{a'+b-c-c'}\left(\frac{y-\beta}{x-\beta}\right) - \\
& \left(\frac{x}{y}\right)\beta^2(\alpha+\beta)^{c+c'-b-1}(y+\alpha)^{a'+b-c-c'}(x+\alpha)^{a-1}\left(\frac{y-x}{x-\beta}\right).
\end{aligned}$$

*Assume $a = 0$ and $c' > 0$.*

$$\begin{aligned}
LHS = {} & \beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'} + \\
& \sum_{g=1}^{d}\left(1-\frac{x}{y}\right)y(y+\alpha)^{g-1}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'-g} + \\
& \sum_{g=d+1}^{d+c'}\left(1-\frac{x}{y}\right)\beta(\alpha+\beta)^{g-d-1}(y+\alpha)^{d}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'-g} \\
= {} & \left(\frac{x}{y}\right)\beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{d}(x+\alpha)^{c'}\left(\frac{y-\beta}{x-\beta}\right) -
\end{aligned}$$

$$\left(1-\frac{x}{y}\right)\left(\frac{\beta^2}{x-\beta}\right)(\alpha+\beta)^{c'+c-b-1}(y+\alpha)^d.$$

Observe when $a=0$, the only difference in the $RHS$ from the earlier case is the weight of the Boltzmann weight of the top vertex.

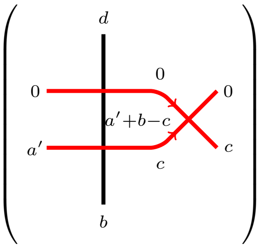
$$\begin{aligned} RHS = {} & \left(\frac{x}{y}\right)\beta(\alpha+\beta)^{c-b-1}(y+\alpha)^d(x+\alpha)^{c'}\left(\frac{y-\beta}{x-\beta}\right)- \\ & \left(1-\frac{x}{y}\right)\left(\frac{\beta^2}{x-\beta}\right)(\alpha+\beta)^{c+c'-b-1}(y+\alpha)^d. \end{aligned}$$

*Assume $a=0$ and $c'=0$.*

$$\begin{aligned} LHS = {} & \beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'}+ \\ & \sum_{g=1}^{d}\left(1-\frac{x}{y}\right)y(y+\alpha)^{g-1}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d-g}+ \\ = {} & \beta(\alpha+\beta)^{c-b-1}(y+\alpha)^d. \end{aligned}$$

On the $RHS$, there is a unique configuration.

$$\begin{aligned} RHS = {} & \beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{a'+b-c} \\ = {} & \beta(\alpha+\beta)^{c-b-1}(y+\alpha)^d \end{aligned}$$



.

*Assume $a>0$ and $c'=0$.*

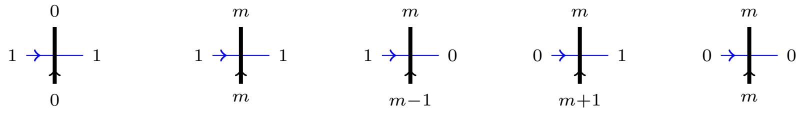
$$\begin{aligned} LHS = {} & \left(\frac{x}{y}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}y(y+\alpha)^{a-1}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'-a}+ \\ & \sum_{g=a+1}^{d}\left(\frac{x}{y}\right)\left(\frac{y-x}{y+\alpha}\right)\left(\frac{y+\alpha}{x+\alpha}\right)^{1-a}y(y+\alpha)^{g-1}\beta(\alpha+\beta)^{c-b-1}(x+\alpha)^{d+c'-g}+ \\ = {} & \beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{d-a}x(x+\alpha)^{a-1}. \end{aligned}$$

Just like in the previous case, we have a unique configuration.

$$\begin{aligned} RHS = {} & x(x+\alpha)^{a-1}\beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{a'+b-c} \\ = {} & x(x+\alpha)^{a-1}\beta(\alpha+\beta)^{c-b-1}(y+\alpha)^{d-a}. \end{aligned}$$

A.4. RLL for the Cauchy identity. We prove a relation between $L^*$, the dual $L$ matrix of row model of $G_\lambda^{(-\alpha,-\beta)}$, and $l$, from the row model of $g_\lambda^{(\alpha,\beta)}$.

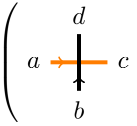
For convenience let us recall all the characters of the play:



| | | | | |
|---|---|---|---|---|
| $1$ | $\frac{1-\beta x}{1+\alpha x}$ | $\frac{1-\beta x}{1+\alpha x}$ | $\frac{x}{1+\alpha x}$ | $\frac{x}{1+\alpha x}$ |

(69)

$w_x$

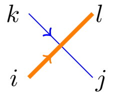


$$\equiv w_x(a,b;c,d) = \delta_{a+b,c+d} \begin{cases} (\alpha+\beta)^{a-d-1}(x+\alpha)\beta^d & a > d \\ \beta^{a-1}x & 0 < a \leqslant d \\ 1 & a = 0, \end{cases}$$

where $a, b, c, d \in \mathbb{Z}_{\geqslant 0}$.

The $\mathfrak{R}$ matrix $\in \mathrm{End}(F \otimes W)$:

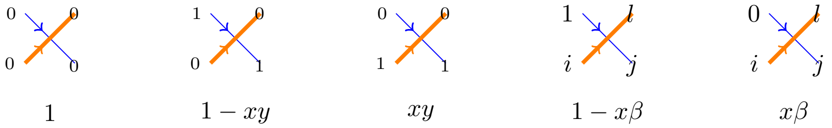
(70) $\mathfrak{R}_{i,j}^{k,l} =$



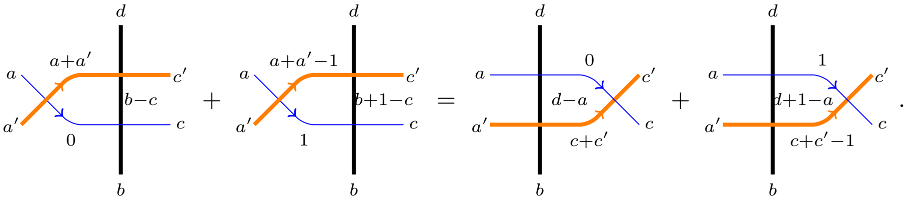
$$= \begin{cases} 1 - xy & j = k = 1, i = l = 0 \\ xy & k = l = 0, i = j = 1 \\ 1 - x\beta & k = 1 \\ x\beta & k = 0 \\ 1 & i = k = l = j = 0 \end{cases}$$

where $k, j \in \{0, 1\}$ and $i, l \in \mathbb{Z}_{\geqslant 0}$,

(71)



together with $L^*$ and $l$ satisfy the RLL relation. We shall prove the following equation:



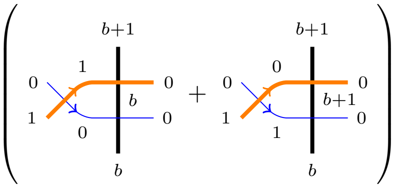
Based on the entries of the $R$-matrix, we shall assume certain condition on $a, a'$ and $c, c'$. We shall divide the conditions of $a, a'$ into three cases, $a + a' = 0$,$a + a' = 1$ and $a + a' > 1$. We do the same with the conditions on $c, c'$. Therefore, we shall have a total of 9 cases to consider.

A.4.1. *Assume* $a+a' = 0$ *and* $c+c' = 0$. Observe that, because of the global condition, $b$ is equal to $d$. So each side will have a unique configuration with identical weight.

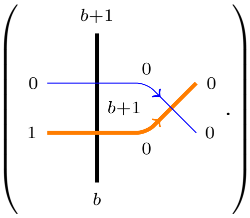
A.4.2. *Assume* $a + a' = 0$. When $a + a' = 0$, there is a unique configuration on $LHS$. On the RHS we use the fact that sum of all the possibles states is an eigenvector of the $\mathfrak{R}$ matrix to conclude that the weight of $RHS$ is just the product of the Boltzmann weights.

A.4.3. *Assume* $a = 0$ *and* $a' = 1$ *and* $c + c' = 0$.

$$\begin{aligned} LHS = {} & (x\beta)y\left(\frac{x}{1+\alpha x}\right) + (xy)\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = {} & \frac{xy}{1+\alpha x} \end{aligned}$$

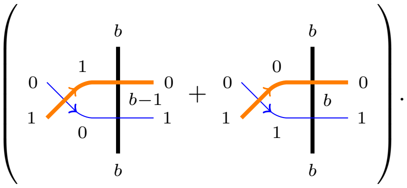


$$RHS = \left(\frac{x}{1+\alpha x}\right)(y)$$



A.4.4. *Assume $a = 0$ and $a' = 1$ and $c' = 0$ and $c = 1$.* When $b > 0$:

$$\begin{aligned} LHS = \ & (x\beta)y\left(\frac{x}{1+\alpha x}\right) + xy\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = \ & \frac{xy}{1+\alpha x} \end{aligned}$$

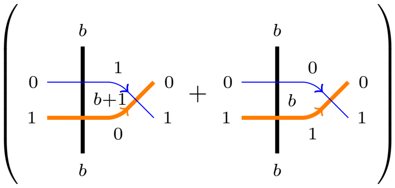


When $b = 0$, only the second configuration from the left is a valid configuration.

$$LHS = \ xy$$

When $b > 0$, we have:

$$RHS =$$



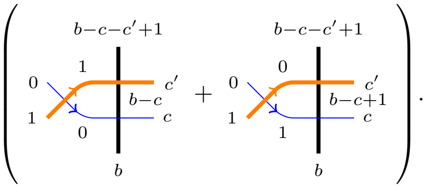
$$\begin{aligned} &= \left(\frac{x}{1+\alpha x}\right)y(1-xy) + \left(\frac{x}{1+\alpha x}\right)y(xy) \\ &= \frac{xy}{1+\alpha x}. \end{aligned}$$

When $b = 0$,

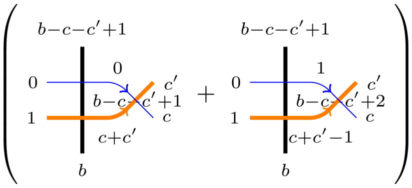
$$\begin{aligned} RHS &= \left(\frac{x}{1+\alpha x}\right)y(1-xy) + \frac{x}{1+\alpha x}(y+\alpha)(xy) \\ &= \left(\frac{xy}{1+\alpha x}\right)(1-xy+xy+\alpha x) \\ &= xy. \end{aligned}$$

A.4.5. *Assume $a = 0$ and $a' = 1$ and $c + c' \geqslant 1$ and $c' \neq 0$.*

$$\begin{aligned} LHS = \ & (x\beta)y\left(\frac{x}{1+\alpha x}\right) + xy\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = \ & \frac{xy}{1+\alpha x} \end{aligned}$$



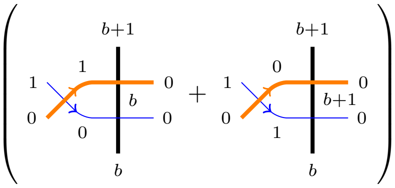
$$RHS =$$



$$= \left(\frac{x}{1+\alpha x}\right)y(x\beta) + \left(\frac{x}{1+\alpha x}\right)y(1-x\beta)$$
$$= \frac{xy}{1+\alpha x}.$$

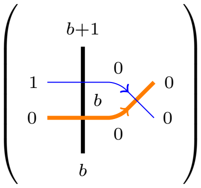
A.4.6. *Assume $a = 1$ and $a' = 0$ and $c + c' = 0$.*

$$LHS =$$



$$= (1-x\beta)y\left(\frac{x}{1+\alpha x}\right) + (1-xy)\left(\frac{1-\beta x}{1+\alpha x}\right)$$
$$= \frac{1-\beta x}{1+\alpha x}.$$

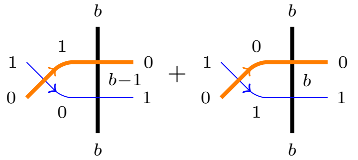
$$RHS = \left(\frac{1-\beta x}{1+\alpha x}\right)$$



.

A.4.7. *Assume $a = 1$ and $a' = 0$ and $c = 1$ and $c' = 0$.* When $b > 0$, we have

$$LHS =$$



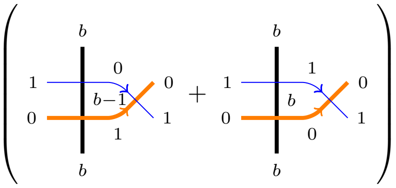
$$= (1-x\beta)y\left(\frac{x}{1+\alpha x}\right) + (1-xy)\left(\frac{1-\beta x}{1+\alpha x}\right)$$
$$= \frac{1-\beta x}{1+\alpha x}.$$

When $b = 0$, only the second configuration from the left is valid.

$$LHS = (1-xy).$$

When $b > 0$, we have

$$RHS = (xy)\left(\frac{1-\beta x}{1+\alpha x}\right) + (1-xy)\left(\frac{1-\beta x}{1+\alpha x}\right)$$
$$= \frac{1-\beta x}{1+\alpha x}$$



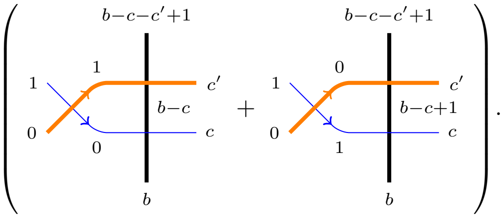
.

When $b = 0$, only the second configuration is valid.

$$RHS = 1 - xy.$$

A.4.8. *Assume $a = 1$ and $a' = 0$ and $c + c' \geqslant 1$ and $c' \neq 0$.* When $b - c - c' + 1 \geqslant 1$, we have

$$\begin{aligned} LHS = {} & (1-x\beta)y\left(\frac{x}{1+\alpha x}\right) + \\ & (1-xy)\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = {} & \frac{1-\beta x}{1+\alpha x} \end{aligned}$$

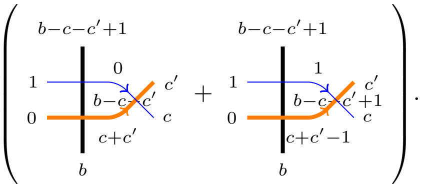


When $b - c - c' + 1 = 0$, we have

$$\begin{aligned} LHS = {} & (1-x\beta)(y+\alpha)\left(\frac{x}{1-\alpha x}\right) + (1-xy)\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = {} & (1-x\beta). \end{aligned}$$

When $b - c - c' + 1 \geqslant 1$, we have

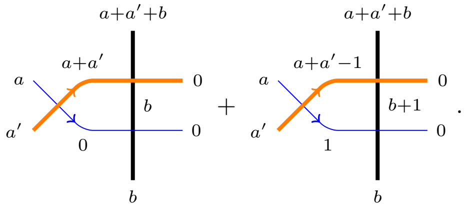
$$\begin{aligned} RHS = {} & (x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right) + (1-x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = {} & \left(\frac{1-\beta x}{1+\alpha x}\right) \end{aligned}$$



When $b - c - c' + 1 = 0$, only the second configuration survives.

$$RHS = (1 - x\beta).$$

A.4.9. *Assume $a + a' > 1$ and $c + c' = 0$.*



When $a = 0$,

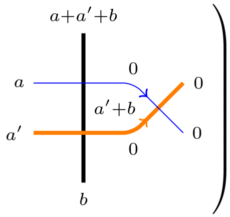
$$\begin{aligned} LHS = {} & (x\beta)y\beta^{a'-1}\left(\frac{x}{1+\alpha x}\right) + (x\beta)y\beta^{a'-2}\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = {} & (xy)\beta^{a'}\left(\frac{x}{1+\alpha x}\right) + \frac{xy\beta^{a'-1}}{1+\alpha x} - (xy)\beta^{a'}\left(\frac{x}{1+\alpha x}\right) \\ = {} & \frac{xy\beta^{a'-1}}{1+\alpha x}. \end{aligned}$$

When $a = 1$,

$$\begin{aligned} LHS = \;& (1 - x\beta)y\beta^{a'}\left(\frac{x}{1+\alpha x}\right) + (1 - x\beta)y\beta^{a'-1}\left(\frac{1-\beta x}{1+\alpha x}\right) \\ = \;& (1 - x\beta)y\beta^{a'-1}\left(\frac{x\beta}{1+\alpha x} + \frac{1-\beta x}{1+\alpha x}\right) \\ = \;& (1 - x\beta)y\left(\frac{\beta^{a'-1}}{1+\alpha x}\right). \end{aligned}$$

When $a = 0$,

$$RHS = \left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1}$$

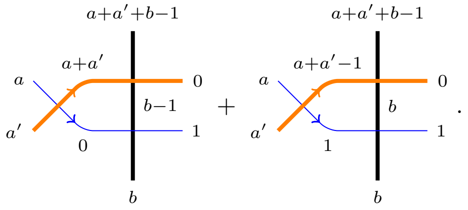


.

When $a = 1$,

$$RHS = \left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1}.$$

A.4.10. *Assume* $a + a' > 1$ *and* $c = 1$ *and* $c' = 0$.



When $a = 0$ and $b > 0$

$$\begin{aligned} LHS = \;& (x\beta)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1} + (x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-2} \\ = \;& \frac{xy\beta^{a'-1}}{1+\alpha x}. \end{aligned}$$

When $a = 0$ and $b = 0$

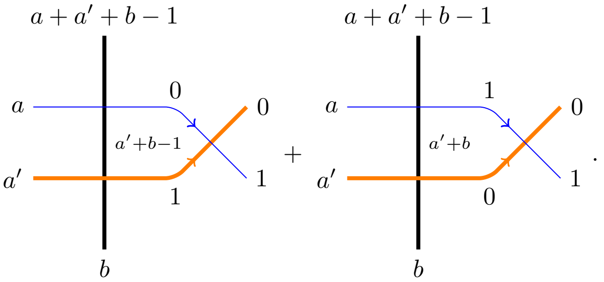
$$\begin{aligned} LHS = \;& x\beta(y\beta^{a'-2}) \\ = \;& xy\beta^{a'-1}. \end{aligned}$$

When $a = 1$ and $b > 0$

$$\begin{aligned} LHS = \;& (1 - x\beta)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'} + (1 - x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1} \\ = \;& \left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1}. \end{aligned}$$

When $a = 1$ and $b = 0$

$$LHS = (1 - x\beta)y\beta^{a'-1}.$$



When $a=0$ and $b>0$

$$\begin{aligned} RHS = {}& (xy)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1} + (1-xy)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1} \\ = {}& \left(\frac{xy\beta^{a'-1}}{1+\alpha x}\right). \end{aligned}$$

When $a=0$ and $b=0$

$$\begin{aligned} RHS = {}& (xy)\left(\frac{x}{1+\alpha x}\right)(y+\alpha)\beta^{a'-1} + (1-xy)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1} \\ = {}& xy\beta^{a'-1}. \end{aligned}$$

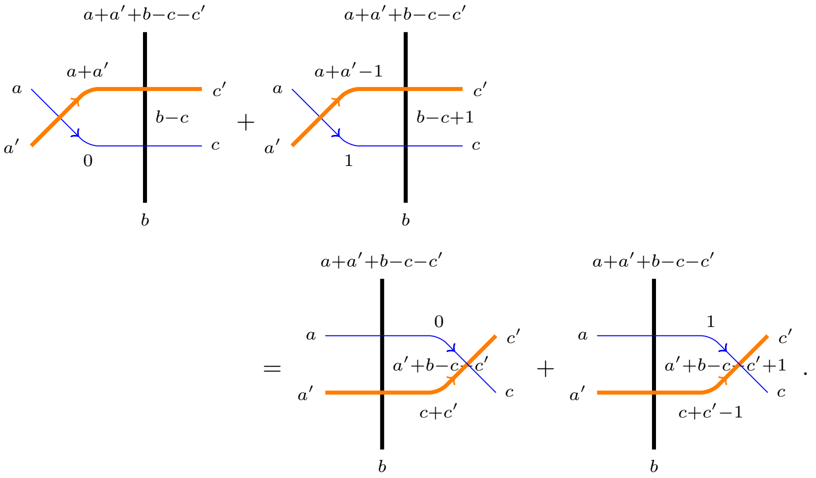
When $a=1$ and $b>0$

$$\begin{aligned} RHS = {}& (xy)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1} + (1-xy)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1} \\ = {}& \left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1}. \end{aligned}$$

When $a=1$ and $b=0$

$$\begin{aligned} RHS = {}& (xy)\left(\frac{1-\beta x}{1+\alpha x}\right)(y+\alpha)\beta^{a'-1} + (1-xy)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1} \\ = {}& (1-\beta x)y\beta^{a'-1}. \end{aligned}$$

A.4.11. *Assume* $a+a'>1$ *and* $c+c'\geqslant 1$ *and* $c'\neq 0$.



When $a = 0$ and $b - c - c' \geqslant 0$

$$LHS = (x\beta)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1} + (x\beta)\left(\frac{1-\beta x}{1+\alpha}\right)y\beta^{a'-2},$$

$$RHS = (x\beta)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1} + (1-x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1}.$$

When $a = 1$ and $b - c - c' \geqslant 0$,

$$LHS = (1-x\beta)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'} + (1-x\beta)\left(\frac{1-\beta x}{1+\alpha}\right)y\beta^{a'-1},$$

$$RHS = (x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1} + (1-x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1}.$$

When $a = 0$ and $b - c - c' = -1$,

$$LHS =(x\beta)\left(\frac{x}{1+\alpha x}\right)(y+\alpha)\beta^{a'+b-c-c'} + (x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(y\beta^{a'-2}),$$

$$RHS =(x\beta)\left(\frac{x}{1+\alpha x}\right)(y+\alpha)\beta^{a'+b-c-c'} + (1-\beta x)\left(\frac{x}{1+\alpha x}\right)y\beta^{a'-1}.$$

When $a = 1$ and $b - c - c' = -1$,

$$LHS =(1-x\beta)\left(\frac{x}{1+\alpha x}\right)(y+\alpha)\beta^{a'} + (1-x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(y\beta^{a'-1}),$$

$$RHS =(x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(y+\alpha)\beta^{a'-1} + (1-\beta x)\left(\frac{1-\beta x}{1+\alpha x}\right)y\beta^{a'-1}.$$

When $a = 0$ and $a' - 1 > a' + b - c - c'$,

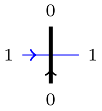
$$\begin{aligned}LHS =&(x\beta)\left(\frac{x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-1}(y+\alpha)\beta^{a'+b-c-c'}+\\&(x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-2}(y+\alpha)\beta^{a'+b-c-c'},\end{aligned}$$

$$\begin{aligned}RHS =&(x\beta)\left(\frac{x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-1}(y+\alpha)\beta^{a'+b-c-c'}+\\&(1-x\beta)\left(\frac{x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-2}(y+\alpha)\beta^{a'+b-c-c'+1}.\end{aligned}$$

When $a = 1$ and $b - c - c' < -1$,

$$\begin{aligned}LHS =&(1-x\beta)\left(\frac{x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-1}(y+\alpha)\beta^{1+a'+b-c-c'}+\\&(1-x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-2}(y+\alpha)\beta^{a'+b-c-c'+1},\end{aligned}$$

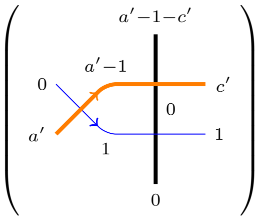
$$\begin{aligned}RHS =&(x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-1}(y+\alpha)\beta^{a'+b-c-c'}+\\&(1-x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(\alpha+\beta)^{c'+c-b-2}(y+\alpha)\beta^{a'+b-c-c'+1}.\end{aligned}$$

In all the cases, we have assumed that the vertex  does not appear. Let us now study the conditions on the nodes where such a vertex can occur.

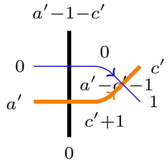
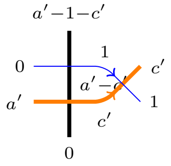
Observe that it can appear in the second configuration of $LHS$ when $b = 0$ and $c = 1$. Similarly, it can appear in the second configuration of $RHS$ when $a+a'+b-c-c' = 0$ and $a'+b-c-c'+1 = 0$, which reduces to the conditions $a = 1$ and $a'+b-c-c' = -1$.

When $a = 0$, $b = 0$ and $c = 1$,

$$LHS = (x\beta)(\alpha+\beta)^{c'-1}(y+\alpha)\beta^{a'-1-c'}$$



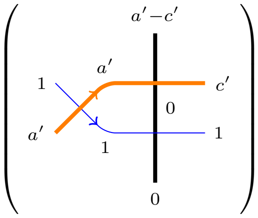
.

$RHS =$  $+$ 

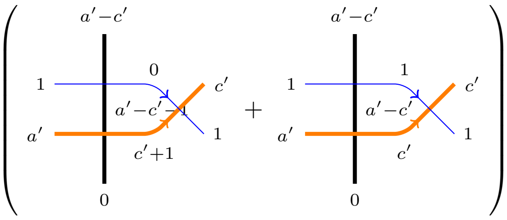
$$\begin{aligned}
&=(x\beta)\left(\frac{x}{1+\alpha x}\right)(\alpha+\beta)^{c'}(y+\alpha)\beta^{a'-c'-1}\\
&\quad+(1-x\beta)\left(\frac{x}{1+\alpha x}\right)(\alpha+\beta)^{c'-1}(y+\alpha)\beta^{a'-c'}\\
&=x(\alpha+\beta)^{c'-1}(y+\alpha)\beta^{a'-c'}.
\end{aligned}$$

When $a = 1$ and $b = 0$ and $c = 1$,

$$LHS = (1-x\beta)(\alpha+\beta)^{c'-1}(y+\alpha)\beta^{a'-c'}$$



.

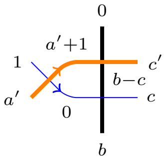


.

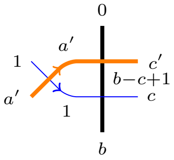
$$\begin{aligned}RHS =&(x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(\alpha+\beta)^{c'}(y+\alpha)\beta^{a'-c'-1}+\\&(1-\beta x)\left(\frac{1-\beta x}{1+\alpha x}\right)(\alpha+\beta)^{c'-1}(y+\alpha)\beta^{a'-c'}\\=&(1-\beta x)(\alpha+\beta)^{c'-1}(y+\alpha)\beta^{a'-c'}.\end{aligned}$$

When $a = 1$ and $a' + b - c - c' = -1$,

$LHS =$

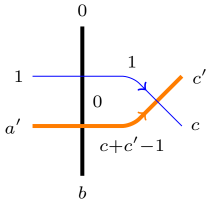


$+$



$$\begin{aligned}&= (1-x\beta)\left(\frac{x}{1+\alpha x}\right)(\alpha+\beta)^{a'}(y+\alpha)+\\&(1-x\beta)\left(\frac{1-\beta x}{1+\alpha x}\right)(\alpha+\beta)^{a'-1}(y+\alpha)\\&=(1-x\beta)(y+\alpha)(\alpha+\beta)^{a'-1}.\end{aligned}$$

$RHS = (1-x\beta)(\alpha+\beta)^{a'-1}(y+\alpha)$ $\Bigg($



$\Bigg).$

When we combine both the conditions, $b = 0$ and $c = 1$, and $a = 1$ and $a+b-c-c' = -1$, we get $a = 1$ and $a' = c'$ for which we have already computed $LHS$ and $RHS$.

*Acknowledgements.* The authors would like to thank J. Lamers for a careful reading of the manuscript, and T. Scrimshaw for discussions. The authors would also like to thank the two anonymous reviewers whose comments/suggestions helped improve this manuscript.

## References

[1] A. S. Buch, *A Littlewood–Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78.

[2] Valentin Buciumas and Travis Scrimshaw, *Double Grothendieck polynomials and colored lattice models*, Int. Math. Res. Not. IMRN (2022), no. 10, 7231–7258.

[3] S. Fomin and A. Kirillov, *Yang–Baxter equation, symmetric functions and Grothendieck polynomials*, 1993.

[4] W. Fulton, *Young tableaux. With applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.

[5] Ajeeth Gunna and Paul Zinn-Justin, *Modified honeycombs in K-homology of Grassmannian*, work in progress.

[6] H.Y. Huang, F.Y. Wu, H. Kunz, and D. Kim, *Interacting dimers on the honeycomb lattice: an exact solution of the five-vertex model*, Physica A **228** (1996), 1–32.

[7] A. Knutson and P. Zinn-Justin, *Schubert puzzles and integrability I: invariant trilinear forms*, 2017.

[8] Thomas Lam and Pavlo Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmannians*, Int. Math. Res. Not. IMRN (2007), no. 24, Art. ID rnm125, 48 pages.

[9] Alain Lascoux and Hiroshi Naruse, *Finite sum Cauchy identity for dual Grothendieck polynomials*, Proc. Japan Acad. Ser. A Math. Sci. **90** (2014), no. 7, 87–91.

[10] Alain Lascoux and Marcel-Paul Schützenberger, *Symmetry and flag manifolds*, Invariant Theory (Berlin, Heidelberg) (Francesco Gherardelli, ed.), Springer Berlin Heidelberg, 1983, pp. 118–144.

[11] V. Mangazeev, *On the Yang–Baxter equation for the six-vertex model*, Nuclear Phys. B **882** (2014), 70–96.

[12] Peter J. McNamara, *Factorial Grothendieck polynomials*, Electron. J. Combin. **13** (2006), no. 1, Research Paper 71, 40 pages.

[13] Kohei Motegi and Kazumitsu Sakai, *Vertex models, TASEP and Grothendieck polynomials*, J. Phys. A **46** (2013), article no. 355201 (26 pages).

[14] ———, *K-theoretic boson-fermion correspondence and melting crystals*, J. Phys. A **47** (2014), article no. 445202 (30 pages).

[15] M. Wheeler and P. Zinn-Justin, *Littlewood–Richardson coefficients for Grothendieck polynomials from integrability*, J. Reine Angew. Math. **757** (2019), 159–195.

[16] Damir Yeliussizov, *Duality and deformations of stable Grothendieck polynomials*, J. Algebraic Combin. **45** (2017), no. 1, 295–344.

[17] ———, *Symmetric Grothendieck polynomials, skew Cauchy identities, and dual filtered Young graphs*, J. Comb. Theory, Ser. A **161** (2019), 453–485.

[18] ———, *Enumeration of plane partitions by descents*, J. Combin. Theory Ser. A **178** (2021), article no. 105367 (18 pages).

[19] P. Zinn-Justin, *Six-vertex, loop and tiling models: integrability and combinatorics*, Lambert Academic Publishing, 2009, `http://www.lpthe.jussieu.fr/~pzinn/publi/hdr.pdf`, Habilitation thesis.

[20] ———, *Schur functions and Littlewood–Richardson rule from exactly solvable tiling models*, 2012, `http://www.lpthe.jussieu.fr/~pzinn/semi/berkeley.pdf`, Chern–Simons Research Lectures.

Ajeeth Gunna, School of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia.
*E-mail :* `agunna@student.unimelb.edu.aus`

Paul Zinn-Justin, School of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia.
*E-mail :* `pzinn@unimelb.edu.au`

# PAPER II: CRYSTALS AND INTEGRABLE SYSTEMS FOR EDGE LABELED TABLEAUX

*Séminaire Lotharingien de Combinatoire* **86B** (2022)
Article #61, 12 pp.

*Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)*

# Crystals and integrable systems for edge labeled tableaux

Ajeeth Gunna*[1] and Travis Scrimshaw†[2]

[1]*School of Mathematics and Statistics, University of Melbourne, Parkville, Victoria, Australia*
[2]*OCAMI, Osaka City University, 3–3–138 Sugimoto, Sumiyoshi-ku, Osaka 558–8585, Japan*

**Abstract.** We define an integrable five vertex model whose partition function is the generating function $E^\lambda$ of edge labeled tableau of shape $\lambda$. Using this, we prove a Cauchy-type identity. We give a crystal structure on edge labeled tableau to give a positive Schur expansion of $E^\lambda$.

**Keywords:** edge labeled tableau, integrable system, crystal

## 1 Introduction

Consider the Lie group $G = \mathrm{GL}_n(\mathbb{C})$, and fix a maximal torus $T$ of the diagonal matrices with $B$ being the corresponding Borel of upper triangular matrices. The (complex) Grassmannian $\mathrm{Gr}(k,n)$ is a very important object in algebraic geometry, which can be described as the set of $k$ dimensional hyperplanes in $\mathbb{C}^n$ or as $G/P$, where $P$ is the maximal parabolic of $k \times (n-k)$ block upper triangular matrices. To study its $T$-equivariant cohomology ring $H_T^\bullet(\mathrm{Gr}(k,n))$, the approach of Schubert calculus is to study the Schubert varieties $X_\lambda$, which are Zariski closures of the decomposition of $\mathrm{Gr}(k,n)$ into (left) $B$ orbits. In the nonequivariant case, there is an isomorphism from $H^\bullet(\mathrm{Gr}(k,n))$ to symmetric functions modulo the ideal $(s_\lambda(\mathbf{x}) \mid \lambda \not\subseteq (n-k)^k)$, where $(n-k)^k$ denotes a $k \times (n-k)$ rectangle and $s_\lambda(\mathbf{x})$ is the Schur function and the image of the cohomology class $[X_\lambda]$ for $X_\lambda$. In $H_T^\bullet(\mathrm{Gr}(k,n))$, the factorial Schur function $s_\lambda(\mathbf{x}|\mathbf{a})$ represents $[X_\lambda]$.

The Littlewood–Richardson (LR) coefficients are the structure coefficients for Schur functions $s_\lambda(\mathbf{x})s_\mu(\mathbf{x}) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(\mathbf{x})$ (when $\mathbf{a} = 0$), which have a classical combinatorial description as certain semistandard tableaux of shape $\nu/\lambda$. The problem is more subtle to compute the LR coefficients for the factorial Schurs with a manifestly positive formula. An initial solution given by Molev and Sagan [12], but it is not described in terms of skew tableaux like the usual LR rule. A skew tableau rule was given by Thomas and Yong [17] by introducing edge labeled tableaux of shape $\nu/\lambda$ with certain conditions.

Another natural problem is to compute the dual basis $\widehat{s}_\lambda(\mathbf{x}|\mathbf{a})$ to the factorial Schur functions under the Hall inner product, where $\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x})\rangle = \delta_{\lambda\mu}$. We can motivate this

---

*agunna@student.unimelb.edu.au

†tcscrims@gmail.com. Partially supported by Grant-in-Aid for JSPS Fellows 21F51028.

geometrically with an alternative way to construct the ring of symmetric functions using the homology $\bigoplus_{1 \leq k \leq n} H_{\bullet}(\mathrm{Gr}(k, n))$, where the product corresponds to the induced map from the direct sum of two Grassmannians [9, Sec. 1.1]. In order to get a deformation of symmetric functions from equivariant cohomology desired by Knutson and Lederer [9], we can only use a single circle $\mathbb{S}^1$ action. It was shown in [10, Theorem 8.12] that the Schubert classes correspond to $\widehat{s}_{\lambda}(\mathbf{x}; t)$, which equals $\widehat{s}_{\lambda}(\mathbf{x}|\mathbf{a})$ with $a_i = 0$ for $i \leq 0$ and $a_i = t$ for $i > 0$, by utilizing back stable Schubert calculus. We remark that $\widehat{s}_{\lambda}(\mathbf{x}|\mathbf{a})$ was first studied by Molev [11] and shown to be the generating function of certain tableaux with the weights being rational functions.

Dual bases must satisfy the Cauchy identity [16, Lemma 7.9.2]. Our first main result is that the dual basis $\widehat{s}_{\lambda}$ in finitely many variables, up to a simple overall factor of $\prod_{j=1}^{m} \prod_{k=1}^{m+\lambda_1} (1 + a_k y_j)^{-1}$, is given by the generating function $E^{\lambda}(\mathbf{x}|\mathbf{a})$ of edge labeled tableaux, which we coin the *edge Schur functions*.

**Theorem 1.1.** *Denote* $\mathbf{a}^n := (a_{i-n-1})_{i \in \mathbb{Z}_{>0}}$. *For* $N \geq m + \lambda_1$, *we have*

$$\sum_{\substack{\lambda \\ \ell(\lambda) \leq \min(n,m)}} s_{\lambda}(\mathbf{x}_n | -\mathbf{a}) \prod_{\substack{1 \leq k \leq N \\ 1 \leq j \leq m}} (1 + a_k y_j)^{-1} E^{\lambda}(\mathbf{y}_m | \mathbf{a}^n) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \frac{1}{1 - x_i y_j}. \tag{1.1}$$

Note that the weight is different than in [17] and is reminiscent of the refined weights for refined symmetric Grothendieck polynomials [4] from the K-theory of $\mathrm{Gr}(k, n)$. For our proof, we introduce an integrable lattice model such that the commutation of transfer matrices with the model for factorial Schur functions yields the Cauchy identity in finitely many variables. A consequence of this is $E^{\lambda}(\mathbf{x}|\mathbf{a})$ is a symmetric function.

The next natural question in studying $E^{\lambda}(\mathbf{x}|\mathbf{a})$ is to determine how they expand in terms of Schur functions. We compute this by utilizing our second main result (Theorem 4.1), there exists a $U_q(\mathfrak{sl}_n)$-crystal structure on edge labeled tableaux by breaking the tableau into diagonals (as opposed to rows or columns as detailed in [6, 13]). An immediate consequence is a *positive* Schur expansion of $E^{\lambda/\mu}(\mathbf{x}|\mathbf{a})$ by counting highest weight elements. We also provide an uncrowding algorithm and conclude this is a crystal isomorphism by properties of RSK following [2, 6, 13, 14, 15].

This extended abstract is organized as follows. In Section 2, we give the background on the requisite tableaux and generating functions. In Section 3, we give the lattice model proving Theorem 1.1. In Section 4, we describe the crystal structure on edge labeled tableaux and the corresponding uncrowding algorithm.

After this was submitted, we learned that our lattice model for edge Schur functions (3.1) and the corresponding $R$-matrix (Proposition 3.2) had previously appeared in the preprint of Gorbounov and Korff [5]. We thank the referee for noting this.

**Figure 1:** Crystal $B(\Lambda_1)$ of the natural representation for $U_q(\mathfrak{sl}_n)$.

## 2 Tableaux generating functions

Fix a positive integer $n$, and let $[n] := \{1, \ldots, n\}$. A *partition* $\lambda$ is a weakly decreasing finite sequence of positive integers, and we draw the Young diagram of $\lambda$ using English convention. We identify partitions $\lambda$ with 01-sequences by a 1 for every vertical step and 0 for every horizontal step, read from bottom-left to top-right. Boldface letters will denote an countably infinite sequence of indeterminates unless otherwise stated and a subscript indicates finitely many variables; *e.g.*, $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{x}_n = (x_1, \ldots, x_n, 0, \ldots)$. We make one main exception of $\mathbf{a} := (\ldots, a_{-1}, a_0, a_1, \ldots)$.

Following [18], an *edge labeled tableau* is a filling of the Young diagram by $\mathbb{Z}_{>0}$ and finite subsets of $\mathbb{Z}_{>0}$ on horizontal edges such that rows weakly increase (not including the edges) and vertical edges strictly increase, where for any set $A$ on an edge, an entry above is less than $\min A$ and an entry below is greater than $\max A$. We consider the top of the partition to extend infinitely far to the right, which can also hold edge labels. For skew shapes, we do not allow edge labels only on the *top* row of boxes that have been skewed out. The set of *semistandard tableaux* of (skew) shape $\lambda/\mu$, denoted $\mathrm{SSYT}(\lambda/\mu)$, is the set of edge labeled tableaux of shape $\lambda/\mu$ such that no entry appears on any edge.

The *factorial Schur functions* and *edge Schur functions* are defined as

$$s_{\lambda/\mu}(\mathbf{x}|\mathbf{a}) = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} \prod_{\alpha \in T} (x_\alpha - a_{\alpha+j-i}), \qquad E^{\lambda/\mu}(\mathbf{x}|\mathbf{a}) = \sum_{T \in \mathrm{ELT}(\lambda/\mu)} \prod_{\alpha \in T} x_\alpha \prod_{\ell \in ET} x_\ell a_{j-i},$$

where the product over $\alpha \in T$ (resp. $\ell \in ET$) is all boxes (resp. edge labels in the upper edge of boxes) in $T$, where the box is in the $i$-th row and $j$-th column. When $\mathbf{a} = 0$, then $s_{\lambda/\mu}(\mathbf{x}) = E^{\lambda/\mu}(\mathbf{x}; 0)$ are the usual skew Schur functions.

We recall the crystal structure for $\mathfrak{sl}_n$, the special linear Lie algebra of traceless $n \times n$ matrices (over $\mathbb{C}$). We use the standard identification of partitions such that $\ell(\lambda) \leq n$ with elements in the dominant weight lattice $P^+ = \mathbb{Z}^n_{\geq 0}$ by $\lambda \leftrightarrow \sum_{i=1}^n \lambda_i \epsilon_i$, where $\{\epsilon_i \mid i \in [n]\}$ is the standard basis of $\mathbb{Z}^n$. A *crystal graph* will be an edge-colored by $[n-1]$ colors, weighted, directed graph. A *highest weight crystal* will be a (weakly) connected component of crystal graph for a tensor power of the crystal $B(\Lambda_1)$ given in Figure 1. We define $\mathcal{B} = B(\Lambda_1)^{\otimes k}$ as the crystal graph with vertices $B(\Lambda_1)^k$ and an edge $b_k \otimes \cdots \otimes b_1 \xrightarrow{i} b'_k \otimes \cdots \otimes b'_1$ by the *signature rule*: Replace each $i$ and $i+1$ with $-$ and $+$ respectively, and successively deleting any $(+-)$-pairs (in that order) until obtaining a sequence $- \cdots - + \cdots +$. Let $j_-$ be the index for the rightmost $-$ remaining, and set

$$b'_k \otimes \cdots \otimes b'_{j_-} \otimes \cdots \otimes b'_1 = b_k \otimes \cdots \otimes \boxed{i+1} \otimes \cdots \otimes b_1,$$

where there is no edge if there is no such $-$. Define the *weight* of an element $b \in \mathcal{B}$ as $\prod_{i=1}^{n} x_i^{a_i}$, where $a_i$ is the number of entries equal to $i$. Our tensor product convention follows [3], which is opposite of [7, 8]. See [3] for additional information on crystals.

A *highest weight element* is a source of the crystal graph. For a highest weight crystal, there exists a unique $\lambda \in P^+$ such that $\mathrm{wt}(b) = \mathbf{x}^\lambda$, which is the crystal basis of a quantum group $U_q(\mathfrak{sl}_n)$ module $V(\lambda)$ [8]. Moreover, the *character* of a crystal $\mathcal{B}$ is

$$\operatorname{ch} \mathcal{B} := \sum_{b \in \mathcal{B}} \mathrm{wt}(b).$$

It is a classical fact that $\operatorname{ch} B(\lambda) = s_\lambda(\mathbf{x}_n)$. Hence, we can identify elements of $B(\lambda)$ with $\mathrm{SSYT}(\lambda)$ with max entry $n$ under *admissible reading words* [7, Theorem 7.3.6], where for any fixed box $\mathsf{b}$, we read every box to its northeast after $\mathsf{b}$. We will use a nonstandard reading word by reading along diagonals from bottom-to-top, where along each diagonal we read from bottom-to-top. This gives us an injection $\mathrm{rwd}\colon B(\lambda/\mu) \to B(\Lambda_1)^{\otimes|\lambda/\mu|}$.

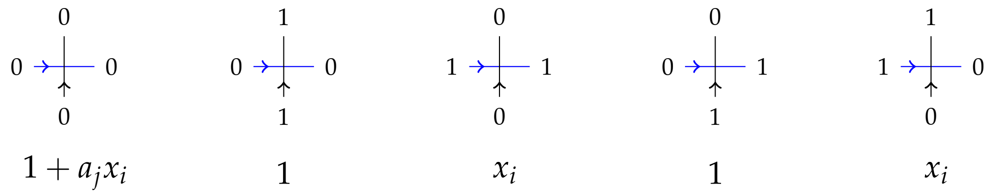
*Example* 2.1. Under the reading word described above, we have

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \longmapsto \boxed{10} \otimes \boxed{7} \otimes \boxed{8} \otimes \boxed{4} \otimes \boxed{9} \otimes \boxed{5} \otimes \boxed{1} \otimes \boxed{6} \otimes \boxed{2} \otimes \boxed{3}\,.$$

## 3 Lattice model

A *state* in a *five vertex model* is a (potentially infinite) square grid with vertices a subset of $\mathbb{Z}^2$ and labels $\{0, 1\}$ on each edge such that around each vertex satisfies one of five possible configurations. We can assign a *Boltzmann weight* to each possible vertex, and the Boltzmann weight of a state is the product of the Boltzmann weights of each vertex. The *partition function* of a vertex model is the sum of the Boltzmann weights of all possible states. This assignment of Boltzmann weights to vertices is called an *L-matrix* (we can consider the Boltzmann weight of the other local configurations to be 0). We can realize $L$-matrix at position $(i, j)$ as a linear map in $\mathrm{End}(H_i \otimes V_j)$, where $H_i \cong \mathbb{C}^2$ is the $i$-th quantum space and $V_j \cong \mathbb{C}^2$ is the $j$-th physical space. For more information, we refer the reader to [1].

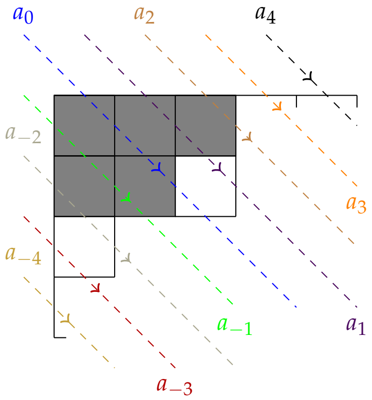
We will now give a lattice model on $[n] \times \mathbb{Z}$ whose partition function is $E^{\lambda/\mu}(\mathbf{x}_n; \mathbf{a})$. The $L$-matrix $L_{ij}$ at position $(i, j)$ is defined as

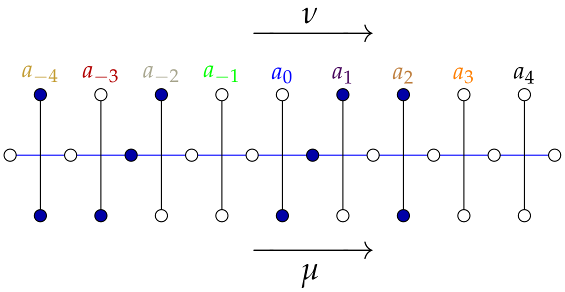


(3.1)

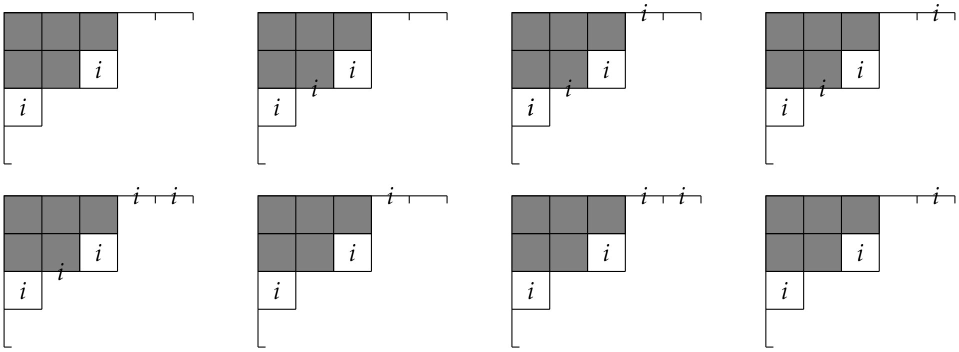
Next, we describe the *boundary conditions*, where the top and bottom edges are the 01-sequence for the partitions $\lambda$ and $\mu$, respectively. To see that the partition function is $E^{\lambda/\mu}(\mathbf{x}_n; \mathbf{a})$, it is sufficient to consider a single row, which is equivalent to restricting to at single letter, say $\mathbf{x}_1$. We identify diagonals of the tableau with the vertical lines in the lattice model. We note that there is a unique state in this model, every vertex outside of the $[-\ell(\lambda), \lambda_1 - 1]$ vertical lines are fixed, and the placement of edge labels correspond to the choice of monomial in $(1 + a_j x_i)$.

*Example* 3.1. We restrict to the finite lattice $[1] \times [-4, 4]$, which means we set $a_i = 0$ for all $i > 4$. For the partitions $\lambda = (3, 3, 1)$ and $\mu = (3, 2)$, the only possible state is





The Boltzmann weight of this state is $(1 + a_{-1}x_i)(1 + a_3x_i)(1 + a_4x_i)x_i^2$. To translate this to edge labeled tableaux, note that we can add an $i$ to the edges along the diagonals with index $-1$, 3 and 4, which are also the upper edges of boxes with those contents.



Note that we have the same partition function with the lattice $[1] \times [-k, 4]$ for any $k \geq 3$.

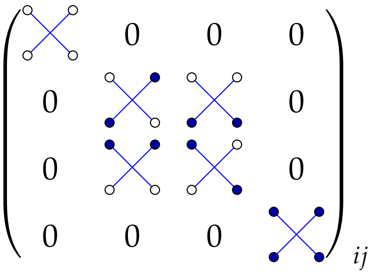
The lattice model perspective also makes it clear how to derive the notion of edge labeled tableau for skew shapes so that we have the branching rule

$$E^{\lambda/\mu}(\mathbf{x}, \mathbf{y}|\mathbf{a}) = \sum_{\mu \subseteq \nu \subseteq \lambda} E^{\lambda/\nu}(\mathbf{y}|\mathbf{a}) E^{\nu/\mu}(\mathbf{x}|\mathbf{a}).$$

This model is *integrable*, which means the following holds.

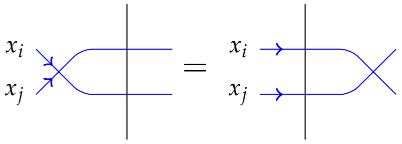
**Proposition 3.2** ([5]). *There exists an R-matrix given by*

$$R_{i,j}(x_i, x_j) =$$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x_i}{x_j} & 0 \\ 0 & 1 & 1 - \frac{x_i}{x_j} & 0 \\ 0 & 0 & 0 & \frac{x_i}{x_j} \end{pmatrix}_{ij} \in \operatorname{End}(H_i \otimes H_j).$$

*satisfying the Yang–Baxter equation: for any fixed boundary, the partition functions are equal:*

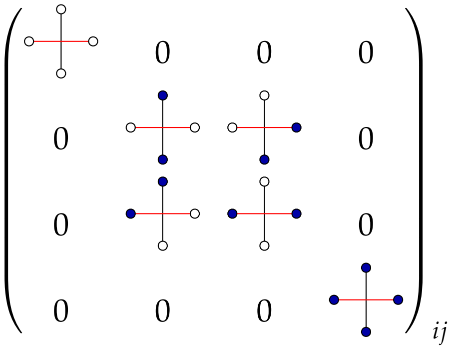


$$\big(R_{ij}(x_i, x_j) L_{i,n}(x_i) L_{j,n}(x_j) = L_{j,n}(x_j) L_{i,n}(x_i) R_{ij}(x_i, x_j)\big).$$

A consequence of Proposition 3.2 is $E^\lambda(\mathbf{x}|\mathbf{a})$ is a symmetric function by repeatedly using the Yang–Baxter equation (this is known as the train argument).

To prove Theorem 1.1, we need a model whose partition function is the factorial Schur function $s_\lambda(\mathbf{x}| - \mathbf{a})$. We will use the model from [19], which is also integrable, with the $l$-matrix

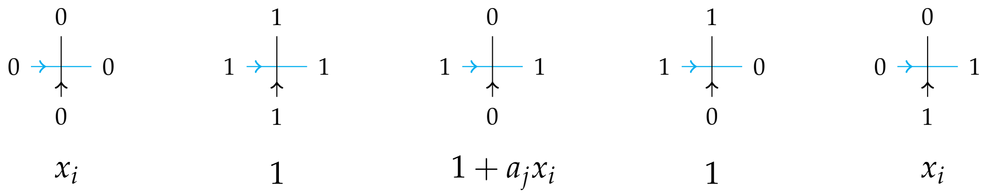
$$l_{i,j} =$$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & x_i + a_j & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \in \operatorname{End}(H_i \otimes V_j).$$

In this model, we use $\mathbb{Z}_{\geq 0} \times [n]$ with the left boundary all being 1.

The goal is to attach the vertex model for $s_\lambda(\mathbf{x}|{-\mathbf{a}})$ with a vertex model for $E^\lambda(\mathbf{x}|\mathbf{a})$, pass them through each other by the train argument, and then be able to easily compute the resulting partition function. However, we cannot use the "natural" model for $E^\lambda(\mathbf{x}|\mathbf{a})$. Instead we use a "dual" model, whose $L^*$-matrix is formed by rotating the vertices 180 degrees and interchanging $0 \leftrightarrow 1$ along the quantum space (horizontal edges):

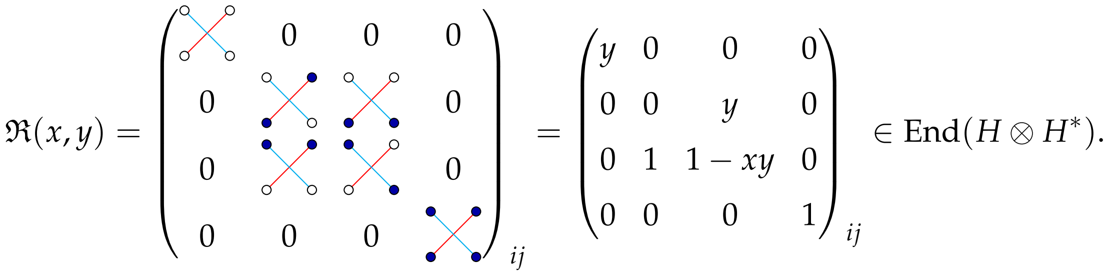


For this model, we note that when we restrict to any sufficiently large $[-N, N] \times [n]$, the left and right boundary conditions are all 1.

We call a single row of a vertex model a *transfer matrix* (with no boundary conditions), and denote the transfer matrix using the $L^*$-matrix (resp. $\ell$-matrix) by $\mathfrak{T}^*$ (resp. $\mathfrak{t}$). Unlike more classical cases of vertex models, our model for $E^{\lambda/\mu}(\mathbf{x}|\mathbf{a})$ depends on the number of $a_i \neq 0$ for $i > 0$. We also will make a technical assumption that $|x_j| < 1$ so that no infinite products can occur. This yields the following key relation.

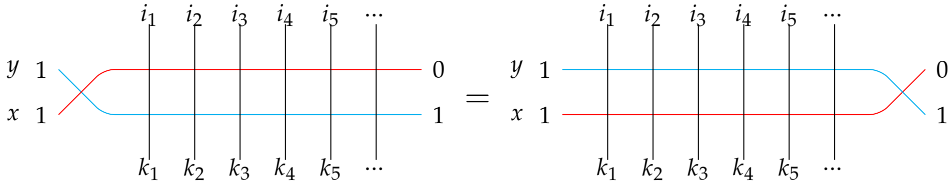
**Proposition 3.3.** *Let $a_i = 0$ for all $i \gg 1$ and $|x| < 1$. Then $\mathfrak{T}^*(y)\mathfrak{t}(x) = (1 - xy)\mathfrak{t}(x)\mathfrak{T}^*(y)$.*

The proof is essentially the existence of a solution of the Yang–Baxter equation with

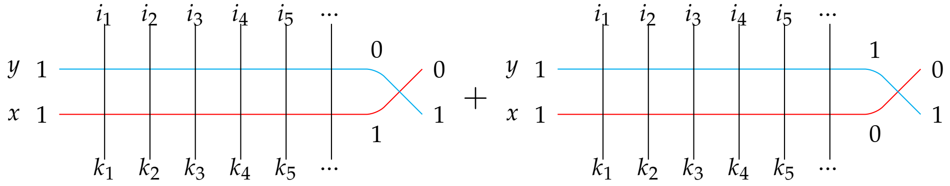


$$\mathfrak{R}(x,y) = \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 1 & 1-xy & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \in \operatorname{End}(H \otimes H^*).$$

Multiplying by the $\mathfrak{R}$-matrix, the train argument yields equal partition functions for

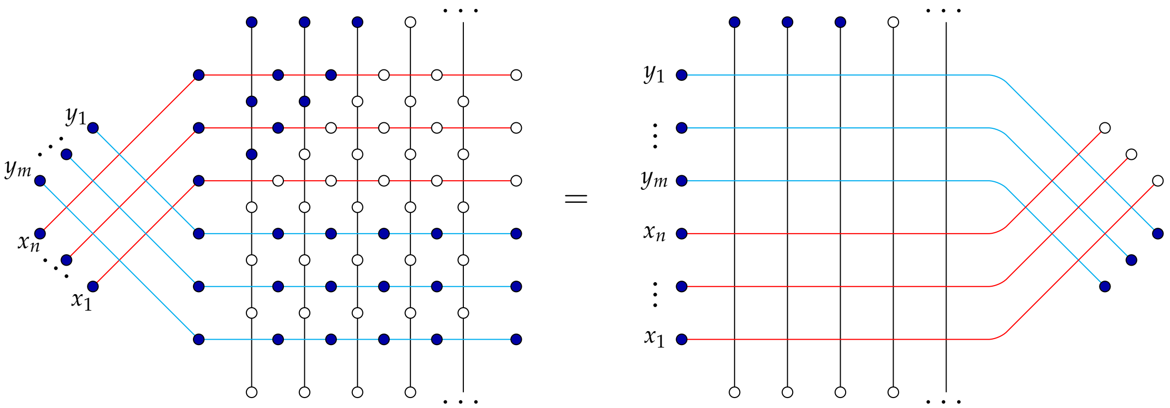


On the left side, as the weight of the $\mathfrak{R}$-matrix is 1, we have $\mathfrak{T}^*(y)\mathfrak{t}(x)$. The right side has potentially two possible configurations:



However, from our assumptions, we note that the Boltzmann weight of for any state of the left configuration must be 0. The claim follows from the weight of the $\mathfrak{R}$-matrix.

To finish the proof of Theorem 1.1 is repeatedly applying Proposition 3.3 and noting there is a unique state on one side that contributes a factor of $\prod_{1\leq k\leq N}\prod_{1\leq j\leq m}(1+a_k y_j)$:



## 4 Crystal structure

We define our crystal structure on edge labeled tableaux by extending the reading for a box b with an entry $b$ and a set $A = \{a_1 < \cdots < a_k\}$ on the edge below b as

$$\underset{A}{\boxed{b}} \iff \boxed{a_k}\otimes\cdots\otimes\boxed{a_1}\otimes\boxed{b} \iff \begin{array}{|c|c|c|c|}\hline b & a_1 & \cdots & a_k \\ \hline\end{array}^{T}.$$

We read the tableau following the reading word rwd using this description for each box.

We define the crystal structure by using the signature rule with this reading word. This generally gives a valid edge labeled tableau with the following exception:

$$\begin{array}{|c|c|}\hline i & \underset{p}{i} \\ \hline\end{array} \xrightarrow{i} \begin{array}{|c|c|}\hline \overset{i}{p} & p \\ \hline\end{array},$$

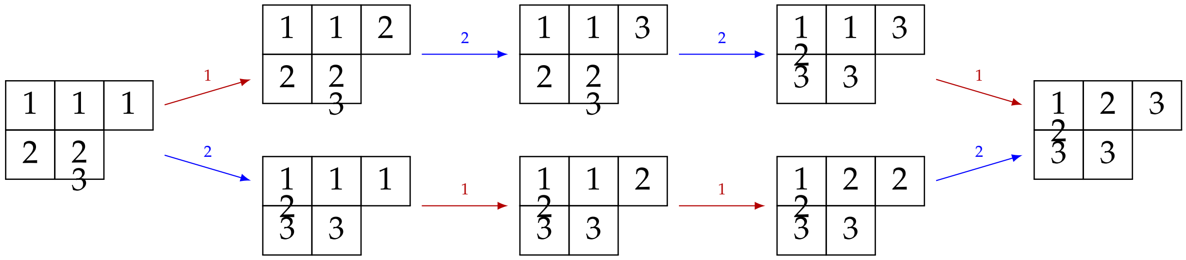
where $p = i+1$. Note that normally we would change the left $i$ to and $i+1$, which would not be an edge-labeled tableau. It is straightforward to see that this is the same operation after taking the reading word, which yields the following.

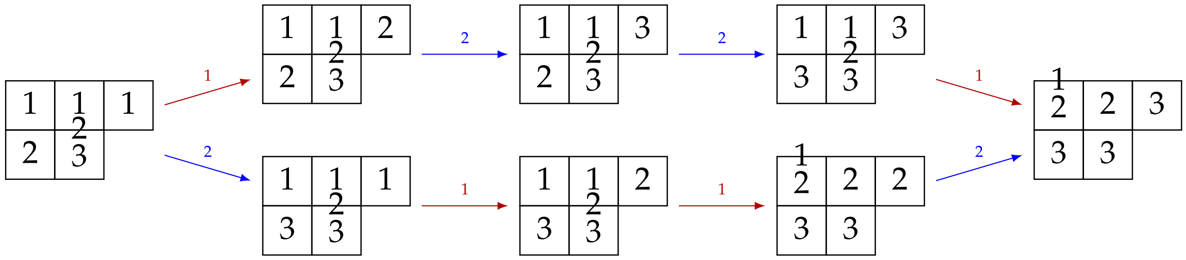
**Theorem 4.1.** *The set of edge labeled tableau of shape $\lambda/\mu$ is a highest weight $U_q(\mathfrak{sl}_n)$-crystal. Moreover, the function $E^{\lambda/\mu}(\mathbf{x}_n|\mathbf{a})$ is Schur positive.*

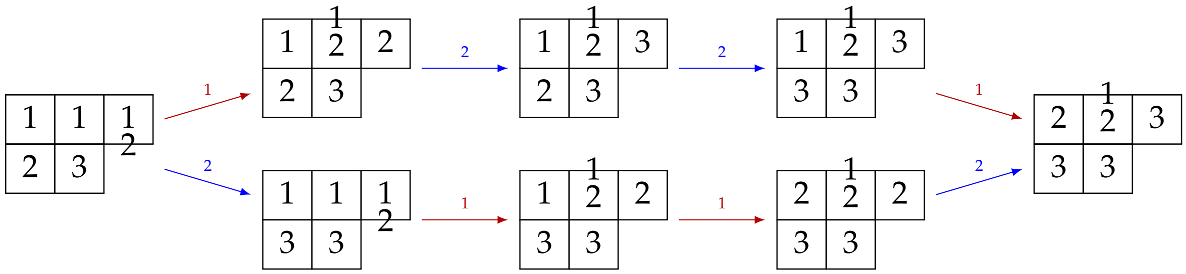
*Example* 4.2. Let $\lambda = (3,2)$. For any $n \geq 3$, we have

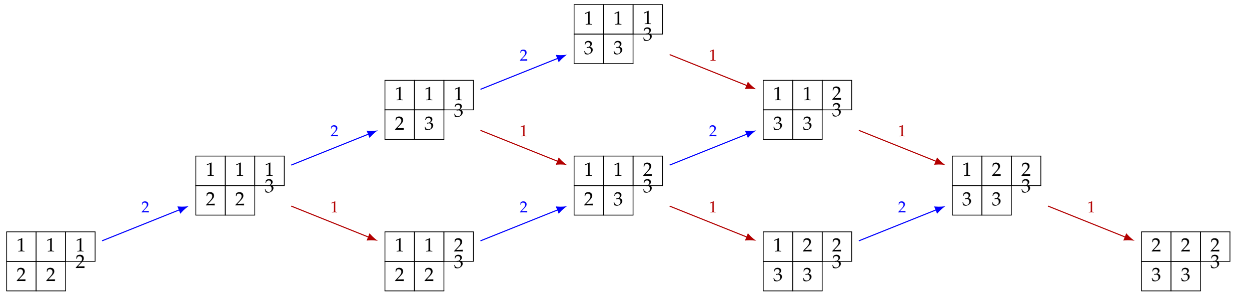
$$E^{32}(\mathbf{x}_n|\mathbf{a}) = s_{32}(\mathbf{x}_n) + (a_{-2}+a_{-1}+a_0+a_1)s_{321}(\mathbf{x}_n) + a_1 s_{33}(\mathbf{x}_n) + \sum_{i>1} a_i s_{42} + HOT.$$

We have the following crystals for the coefficients $a_{-1}$, $a_0$, and $a_1$ for $n = 3$:









Next, we construct an analog of the uncrowding bijection in analogy to [2, 6]. In this case, given our reading word, we will perform the uncrowding along diagonals, which requires a little more care. For simplicity, we index the diagonals so the first diagonal has content $1 - \ell(\lambda)$.

**Definition 4.3** (Uncrowding algorithm)**.** We proceed along diagonals starting from the lower-left box. Start with $(P_0, Q_0) = (\varnothing, \varnothing)$. For the $i$-th diagonal $D_i$, let $P_i = P_{i-1} \xleftarrow{RSK} \operatorname{rwd}(D_i)$ denote the RSK insertion (see, *e.g.*, [16, Ch. 7]). We construct the recording tableau $Q_i$ as the skew shape $\mu^{(i)}/\nu^{(i)}$, where $\mu^{(i)}$ is the shape of $P_i$ and $\nu^{(i)}$ are the boxes of $\lambda$ up to the $i$-th diagonal (counted from the lower-left box) and slid up into a straight shape. The entries of $Q_i$ are those of $Q_{i-1}$ shifted appropriately and then any remaining empty cells are filled with an $i$.

*Example* 4.4. Consider the edge labeled tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & \underset{45}{2} \\ \hline 2 & 2 & 6 \\ \hline 3 & \underset{4}{3} & \multicolumn{1}{c}{} \\ \cline{1-2} \underset{5}{4} & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

Under the uncrowding algorithm, we have

$$(P_0, Q_0) = \emptyset, \emptyset \xleftarrow{RSK} 54 = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}, \begin{array}{|c|}\hline \cdot \\ \hline 1 \\ \hline\end{array} \xleftarrow{RSK} 53 = \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \xleftarrow{RSK} 432$$

$$= \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \xleftarrow{RSK} 21 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \xleftarrow{RSK} 61$$

$$= \begin{array}{|c|c|c|}\hline 1 & 1 & 6 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \multicolumn{1}{c}{} \\ \cline{1-2} \cdot & \cdot & \multicolumn{1}{c}{} \\ \cline{1-2} \cdot & \cdot & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \xleftarrow{RSK} 542 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 3 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & 5 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & \cdot & 6 \\ \hline \cdot & \cdot & \cdot & \multicolumn{1}{c}{} \\ \cline{1-3} \cdot & \cdot & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} \cdot & \cdot & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} = (P_6, Q_6).$$

Now we need to describe the inverse algorithm; in particular, we need to describe which which cells to remove as we will use inverse RSK at each step. We proceed by removing the diagonals in reverse order but starting with the cell at the bottom of the corresponding column. We also remove any cell labeled by $i$ if we are at the $i$-th diagonal from the bottom, the result of which becomes an edge label and can be placed in a unique way such that the result is an edge labeled tableau.

We let $\mathfrak{E}(\lambda/\mu)$ denote the set of recording tableaux obtained by applying the uncrowding algorithm. We do not currently have a characterization of these tableaux other than they will have shape $\lambda/\mu$. We leave this as an open question.

**Theorem 4.5.** *Uncrowding* $\Upsilon\colon \mathrm{ELT}(\lambda) \to \bigsqcup_{\mu \subseteq \lambda} \mathrm{SSYT}(\mu) \times \mathfrak{E}(\lambda/\mu)$ *is a crystal isomorphism, where the crystal operators on the image act only on* $\mathrm{SSYT}(\mu)$.

This gives an alternative proof of Theorem 4.1. It would be good to describe this using a formulation analogous to the alternative descriptions for uncrowding given in [14, 15].

# Acknowledgements

The authors thank J. Lamers and P. Zinn-Justin for many useful conversions. We thank one of the referees for pointing out that our vertex model appeared previously in the preprint of Gorbounov and Korff [5].

# References

[1] R. J. Baxter. *Exactly solved models in statistical mechanics*. Reprint of the 1982 original. London: Academic Press Inc. [Harcourt Brace Jovanovich Publishers], 1989, pp. xii+486.

[2] A. S. Buch. "A Littlewood-Richardson rule for the $K$-theory of Grassmannians". *Acta Math.* **189**.1 (2002), pp. 37–78. DOI.

[3] D. Bump and A. Schilling. *Crystal bases*. Representations and combinatorics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, pp. xii+279. DOI.

[4] M. Chan and N. Pflueger. "Combinatorial relations on skew Schur and skew stable Grothendieck polynomials". *Algebraic Combin.* **4**.1 (2021). DOI.

[5] V. Gorbounov and C. Korff. "Equivariant quantum cohomology and Yang–Baxter algebras". 2014. arXiv:1402.2907.

[6] G. Hawkes and T. Scrimshaw. "Crystal structures for canonical Grothendieck functions". *Algebraic Combin.* **3**.3 (2020), pp. 727–755. DOI.

[7] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*. Vol. 42. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2002, p. 307.

[8] M. Kashiwara. "On crystal bases of the $q$-analogue of universal enveloping algebras". *Duke Math. J.* **63**.2 (1991), pp. 465–516. DOI.

[9] A. Knutson and M. Lederer. "A $K_T$-deformation of the ring of symmetric functions". 2015. arXiv:1503.04070.

[10] T. Lam, S. J. Lee, and M. Shimozono. "Back stable Schubert calculus". *Compos. Math.* **157**.5 (2021), pp. 883–962. DOI.

[11] A. I. Molev. "Comultiplication Rules for the Double Schur Functions and Cauchy Identities". *Electron. J. Comb.* **16**.1 (2009).

[12] A. I. Molev and B. E. Sagan. "A Littlewood–Richardson rule for factorial Schur functions". *Trans. Amer. Math. Soc.* **351**.11 (1999), pp. 4429–4443. DOI.

[13] C. Monical, O. Pechenik, and T. Scrimshaw. "Crystal structures for symmetric Grothendieck polynomials". *Transform. Groups* **26**.3 (2021), pp. 1025–1075. DOI.

[14] J. Morse, J. Pan, W. Poh, and A. Schilling. "A crystal on decreasing factorizations in the 0-Hecke monoid". 2020. arXiv:1911.08732.

[15] J. Pan, J. Pappe, W. Poh, and A. Schilling. "Uncrowding algorithm for hook-valued tableaux". 2020. arXiv:2012.14975.

[16] R. P. Stanley. *Enumerative combinatorics. Vol. 2.* Vol. 62. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge University Press, Cambridge, 1999, pp. xii+581. DOI.

[17] H. Thomas and A. Yong. "A jeu de taquin theory for increasing tableaux, with applications to $K$-theoretic Schubert calculus". *Algebra Number Theory* **3**.2 (2009), pp. 121–148. DOI.

[18] H. Thomas and A. Yong. "Equivariant Schubert calculus and jeu de taquin". *Ann. Inst. Fourier (Grenoble)* **68**.1 (2018), pp. 275–318.

[19] P. Zinn-Justin. "Littlewood–Richardson coefficients and integrable tilings". *Electron. J. Combin.* **16**.1 (2009), Research Paper 12, 33.

# Paper III: Littlewood–Richardson coefficients for spin Hall–Littlewood functions

# LITTLEWOOD–RICHARDSON COEFFICIENTS FOR SPIN HALL–LITTLEWOOD FUNCTIONS

AJEETH GUNNA, MICHAEL WHEELER AND PAUL ZINN-JUSTIN

Abstract. We provide a combinatorial formula for the Littlewood–Richardson (LR) coefficients of spin Hall–Littlewood functions, and factorial versions of them. This is achieved by representing these functions and the LR coefficients as the partition function of a lattice model and applying the underlying Yang-Baxter equation. Our combinatorial expression is in terms of generalised honeycombs; the latter were introduced by Knutson and Tao for ordinary LR coefficients and applied to the computation of Hall polynomials by Zinn–Justin.

## 1. Introduction

1.1. **Spin Hall–Littlewood functions.** In [Bor14], Borodin introduced spin Hall–Littlewood functions ($F_\lambda$) which are symmetric rational functions. They depend on two parameters, $q$ and $s$. At $s = 0$, they reduce to Hall–Littlewood polynomials[Mac98]. Hall–Littlewood polynomials are themselves a generalization of the well-known Schur polynomials. In this paper, we derive a combinatorial formula for the coefficients $C^{\nu}_{\lambda,\mu}$ from the following equation:

$$F_\lambda F_\mu = \sum_\nu C^{\nu}_{\lambda,\mu} F_\nu.$$

Such coefficients are usually referred to as *Littlewood–Richardson coefficients* (LR). These coefficients have various interpretations. For example, the LR coefficients of Schur polynomials appear in the representation theory of $GL(n)$; they also appear in the cohomology of the Grassmannian. Meanwhile, the *Hall polynomials*, which are the LR coefficients of Hall–Littlewood polynomials, count short exact sequences of finite abelian $p$-groups. Therefore, the LR coefficients of $F_\lambda$ are a generalisation of Hall polynomials.

1.2. **Exactly solvable lattice models.** In the past two decades, exactly solvable lattice models have been widely used to study the theory of symmetric functions. In a nutshell, symmetric functions are initially represented as partition functions of a lattice model. By leveraging the underlying Yang–Baxter equation, one can derive many identities, such as symmetry, branching formulas, Cauchy identities, Littlewood identities, etc.

Although the $R$-matrices corresponding to $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$ are sufficient to provide a lattice model formulation for a symmetric polynomial, one cannot compute LR coefficients using the same. In the work by Zinn-Justin [ZJ08], LR coefficients of Schur polynomials were computed using lattice models. The key observation is that we need to consider a lattice model composed of the $R$ matrices of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_3)$ to compute LR coefficients. This approach was later used to compute the LR coefficients of Grothendieck polynomials [WZJ19], Hall–Littlewood polynomials [WZJ16a, ZJ19], and Schubert polynomials [KZJ20], among others.

1.3. **Layout of the paper.** In section 2, we establish our notations and recall the Bosnjak–Mangazeev general solution for the Yang–Baxter equation [BM16]. We define the wave functions of the six-vertex model and inhomogeneous spin Hall–Littlewood functions ($\mathcal{F}_m$) where $m$ is a tuple of positive integers. These inhomogeneous spin Hall–Littlewood functions are symmetric rational functions in two sets of variables, and Spin Hall–Littlewood functions and factorial Hall–Littlewood polynomials are obtained as its specializations.

In section 3, we describe our three theorems. Our theorem computes a puzzle formula for the LR coefficients of the wave functions of the six-vertex model, spin Hall–Littlewood functions, and Hall–Littlewood polynomials.

In section 4, we introduce our hexagonal lattice made up of the $R$-matrices of $\mathcal{U}_q(\mathfrak{sl}_3)$. We demonstrate that the partition function of this lattice equals the product of two spin Hall–Littlewood functions. Subsequently, the partition function of the same lattice provides the desired combinatorial objects multiplied by a spin Hall–Littlewood function. Our technique in this paper appears to be new and is reminiscent of the standard arguments used to prove Cauchy identities using the Yang–Baxter equation. In the next section, section 5, we consider the most general hexagonal lattice model to derive the following equation:

$$\begin{aligned}&\mathcal{F}_m(x_1,\dots,x_n;y_1,\dots,y_P;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P})\mathcal{F}_l(x_1,\dots,x_n;z_0,z_1,\dots,z_N;q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N})\\ &\qquad = \sum_k \mathcal{C}^k_{l,m}(y_1,\dots,y_P;z_0,z_1,\dots,z_N;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P};q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N})\\ &\qquad\qquad \mathcal{F}_k(x_1,\dots,x_n;y_1,\dots,y_n;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P})\end{aligned}\tag{1.1}$$

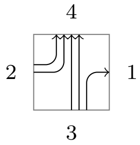
Finally, in section 6, we make certain substitutions in the above equation to obtain our main theorems.

## 2. Preliminaries

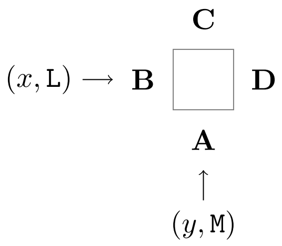
2.1. **Lattice model.** We consider a lattice model made up of horizontal lines, which are oriented from left to right, and vertical lines, oriented from bottom to top, with their intersections forming vertices. We attach variables to both vertical and horizontal lines.

We decorate the edges with labels, typically a non-negative integer or a tuple of non-negative integers, depending on the context. We refer to the labelling of the edges of all vertices in a lattice as a configuration. We attach a weight to each vertex depending on its respective row and column. These weights are non-zero only when the conservation is satisfied i.e., the sum of the labels entering from the bottom and left is equal to the sum of the labels on the top and right. A configuration is only valid when the conservation is satisfied at each vertex. For every such configuration, we get its weight by multiplying the weight of each of the vertices. A partition function of a lattice model with a fixed boundary is defined as a summation of the weights of all possible configurations.

Graphically, we draw a vertex as a tile, with particles entering from the bottom and the left, and exiting through the top and right. Below is an example of a vertex with labelled edges.



2.2. **Yang–Baxter equation.** We recall the Bosnjak–Mangazeev solution of the Yang–Baxter equation. Here, $m$ denotes the rank of the quantized affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}_{m+1})$, and $\mathtt{L},\mathtt{M},\mathtt{N}$ be positive numbers that denote the spin of a line, and $x, y, z$ be the variables attached to lines. For an $m$-tuple of non-negative integers $\mathbf{A}$, we denote $|\mathbf{A}|$ as the sum of all the coordinates of $\mathbf{A}$. For a vertex of the type:



$$= W_{\mathtt{L},\mathtt{M}}\left(\frac{x}{y}; q; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\right)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are $m$-tuples of non-negative integers are said to satisfy the Yang–Baxter equation when

$$\begin{aligned}\sum_{\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3} & W_{\mathtt{L},\mathtt{M}}\left(\frac{x}{y}; q; \mathbf{A}_2, \mathbf{A}_1, \mathbf{C}_2, \mathbf{C}_1\right) W_{\mathtt{L},\mathtt{N}}\left(\frac{x}{z}; q; \mathbf{A}_3, \mathbf{C}_1, \mathbf{C}_3, \mathbf{B}_1\right) W_{\mathtt{M},\mathtt{N}}\left(\frac{y}{z}; q; \mathbf{C}_3, \mathbf{C}_2, \mathbf{B}_3, \mathbf{B}_2\right) \\ = \sum_{\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3} & W_{\mathtt{M},\mathtt{N}}\left(\frac{y}{z}; q; \mathbf{A}_3, \mathbf{A}_2, \mathbf{C}_3, \mathbf{C}_2\right) W_{\mathtt{L},\mathtt{N}}\left(\frac{x}{z}; q; \mathbf{C}_3, \mathbf{A}_1, \mathbf{B}_3, \mathbf{C}_1\right) W_{\mathtt{L},\mathtt{M}}\left(\frac{x}{y}; q; \mathbf{C}_2, \mathbf{C}_1, \mathbf{B}_2, \mathbf{B}_1\right).\end{aligned} \tag{2.1}$$

where $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ are fixed $m$-tuple of non-negative integers, and the summation on both sides of the equation is over triples of $m$-tuples $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ of non-negative integers. For vertices where $|\mathbf{A}|, |\mathbf{C}| > \mathtt{M}$ or $|\mathbf{B}|, |\mathbf{D}| > \mathtt{L}$, the weights are identically zero.

**Theorem 2.1** ( [BM16]). *For any two tuples $\lambda$, $\mu$ such that $\lambda_i \leq \mu_i$ for all $1 \leq i \leq m$ , we define the functions*

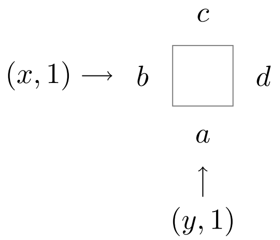
$$\Phi(\lambda, \mu; x, y) := \frac{(x; q)_{|\lambda|}(y/x; q)_{|\mu-\lambda|}}{(y; q)_{|\mu|}} (y/x)^{|\lambda|} \left(q^{\sum\limits_{i<j}(\mu_i - \lambda_i)\lambda_j}\right) \prod_{i=1}^{n} \binom{\mu_i}{\lambda_i}_q .$$

*Then the weights defined as*

$$\begin{aligned}W_{\mathtt{L},\mathtt{M}}(x; q; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) = \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} x^{|\mathbf{D}-\mathbf{B}|} q^{|\mathbf{A}\mathtt{L}-\mathbf{D}\mathtt{M}|} \\ \times \sum_{\mathbf{P}} \Phi(\mathbf{C}-\mathbf{P}, \mathbf{C}+\mathbf{D}-\mathbf{P}; q^{\mathtt{L}-\mathtt{M}}x, q^{-\mathtt{M}}x) \Phi(\mathbf{P}, \mathbf{B}; q^{-\mathtt{L}}/x, q^{-\mathtt{L}})\end{aligned} \tag{2.2}$$

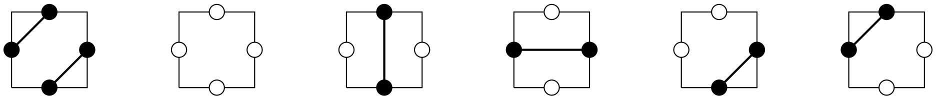
*where the summation is over $m$-tuples $\mathbf{P} = (\mathbf{P}_1, \ldots, \mathbf{P}_m)$ such that $0 \leq \mathbf{P}_i \leq \min(\mathbf{B}_i, \mathbf{C}_i)$, satisfy the Yang–Baxter equation.*

2.3. **Six vertex model and its wave functions.** In this subsection, we consider weights of the following type:



$$= W_{1,1}\left(\frac{x}{y}; q; a, b, c, d\right)$$

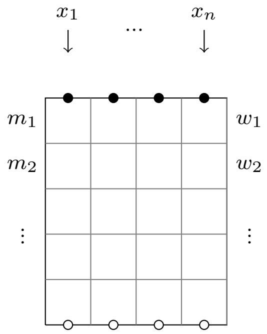
where each of $a, b, c, d$ is either 1 or 0. Graphically, we use (●) to indicate that the corresponding edge label 1 and a (○) when the edge label is zero. Then there are six vertices where the conservation is satisfied:



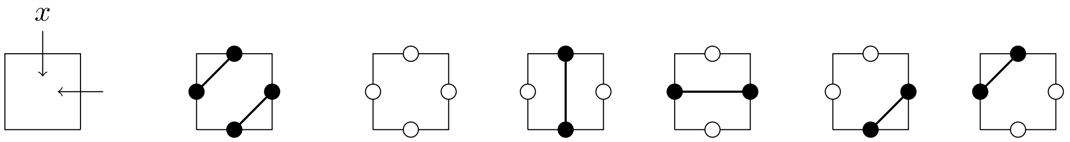
(2.3)

**Definition 2.2.** For two binary strings $m$ and $w$ where $m$ has $n$ more particles than $w$, we define the rational function $\mathrm{H}_{m/w}(x_1, \ldots, x_n)$ as partition function of

$$\mathrm{H}_{m/w}(x_1, \ldots, x_n) := \tag{2.4}$$



with the following weights:



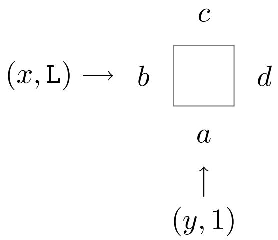
| | | | | | | |
|---|---|---|---|---|---|---|
| | $1$ | $1$ | $\frac{1-x}{1-qx}$ | $\frac{q\,(1-x)}{1-qx}$ | $\frac{(1-q)x}{1-qx}$ | $\frac{(1-q)}{1-qx}$ |

(2.5)

When $w$ is entirely holes, we denote it by $\mathrm{H}_m$. As a consequence of the Yang–Baxter equation, these rational functions are invariant under permutations of the variables. These functions have been extensively studied in the literature.

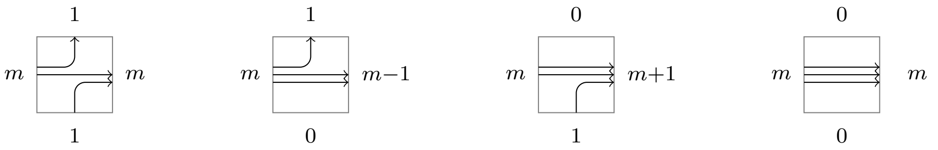
**Remark 2.3.** At $q = 0$, we recover symmetric Grothendieck polynomials. They are symmetric polynomials arising from the $K$-theory of Grassmanians. In the recent literature, many papers have explored the application of lattice models to the study of these polynomials. In [WZJ19], Wheeler and Zinn-Justin derived puzzles for the structure constants of these polynomials using solvable lattice models.

2.4. **Higher spin vertex model.** In this subsection, we define functions by considering weights of the following type:



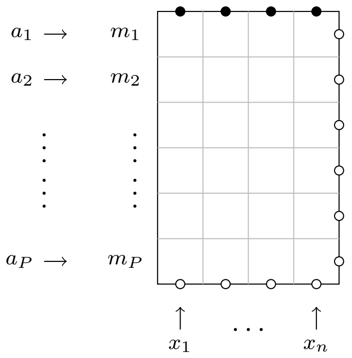
$$= W_{\mathtt{L},1}\left(\frac{x}{y}; q; a, b, c, d\right)$$

where each of $a, c$ is either 1 or 0 and $b, d$ are positive integers bounded by `L`. Then there are four types of vertices where the conservation is satisfied:



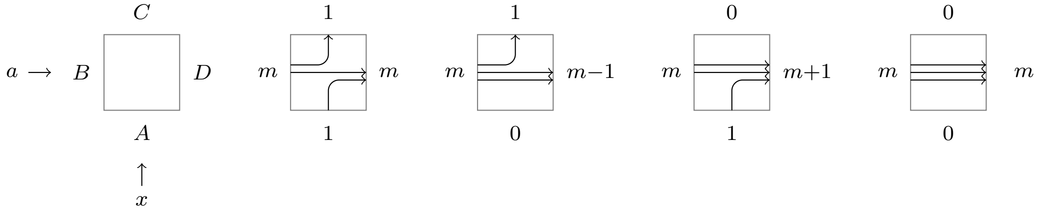
(2.6)

**Definition 2.4.** Fix positive integers $P$ and $n$. Let $m = (m_1, \ldots, m_P)$ be a $P$-tuple of non-negative integers such that $\sum_{i=1}^{P} m_i = n$. We define $\mathcal{F}_m(x_1, \ldots x_n; a_1, \ldots, a_P; s)$ as the partition function of



(2.7)

with weights:



(2.8)

$$\frac{1 - q^m x a}{x - s} \qquad \frac{1 - q^m}{x - s} \qquad \frac{(1 - q^m a s)x}{x - s} \qquad \frac{x - q^m s}{x - s}$$

These functions are symmetric in $x$ variables. Similar to many families of symmetric functions they satisfy many interesting identities like Cauchy identity etc. In this paper, we focus on the structure constants for certain specialisations of definition 2.4.

**Definition 2.5.** We define the spin Hall–Littlewood functions by setting $a_i = t$, a formal parameter, in the $\mathcal{F}$:

$$\mathrm{F}_l(x_1, \ldots, x_n; t, s) = \mathcal{F}_l(x_1, \ldots, x_n; t, \ldots, t; s). \tag{2.9}$$

This definition of spin Hall–Littlewood function is not as in the literature [Bor14]. However, we can recover it by simply setting $t = s$. By setting $t = s = 0$, we recover the lattice model for Hall–Littlewood polynomials introduced in [WZJ16b], with inverted spectral variables.

**Remark 2.6.** The spin Hall–Littlewood functions considered in this paper reduce to weak dual Grothendieck polynomials ($j_\lambda$ in the notation of [GZJ20]) upon setting $s = q = 0$. We leave the details to the interested reader.

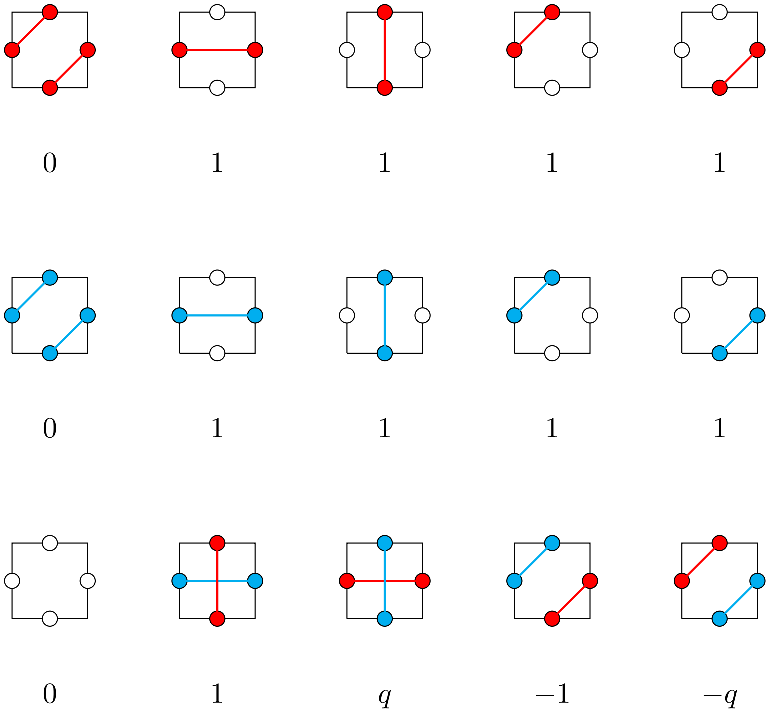
**Definition 2.7.** We define the factorial Hall–Littlewood polynomials by setting $s = 0$ in $\mathcal{F}$:

$$\mathrm{P}_l(x_1, \ldots, x_n; a_1, \ldots, a_P) = \lim_{s \to 0} \mathcal{F}_l(x_1, \ldots, x_n; a_1, \ldots, a_P; s). \tag{2.10}$$

In this paper, we derive combinatorial formulae in terms of *puzzles* for the structure constants for the spin Hall–Littlewood functions and factorial Hall–Littlewood polynomials.

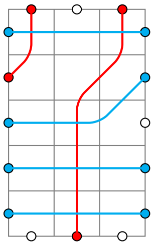
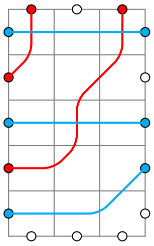
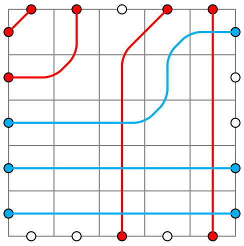
## 3. Main Theorems

3.1. **Puzzles for the wave functions of the six vertex model.** Consider the following tiles. Every tile has a weight associated with it.



(3.1)

In this paper, a *puzzle* is a tiling of a rectangular region with the above tiles while ensuring the continuity of red and blue lines. In these puzzles, red particles enter from the left and bottom, while blue particles enter only from the left. The red particles exit through the upper boundary and the blue particles exit through the right. Let $P$ be the breadth of the puzzle and $n$ be the number of red particles entering from the left. In other words, the left boundary of a puzzle is a binary string of red and blue particles, and the right boundary is a binary string of blue particles and holes. We give three examples to illustrate puzzles.

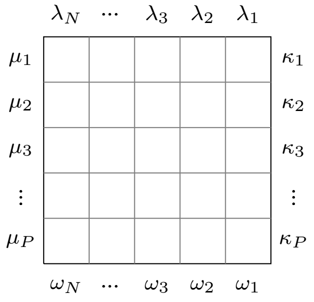




**Theorem 3.1.** *For positive integers $P$, $N$ and $n$, consider binary strings $m, k \in \{\circ, \bullet\}^P$ and $w, l \in \{\circ, \bullet\}^N$ where $l$ has exactly $n$ more particles than $w$. Then we have:*

$$\mathrm{H}_m(x_1, \ldots, x_n)\mathrm{H}_{l/w}(x_1, \ldots, x_n) = \sum_k C^{k,w}_{l,m} \mathrm{H}_k(x_1, \ldots, x_n) \tag{3.2}$$

*where*

$$C^{k,w}_{l,m} =$$



(3.3)

$$\lambda_i = \begin{cases} {\color{red}\bullet} & \textit{if } l_i = \bullet \\ \circ & \textit{if } l_i = \circ \end{cases} \qquad \omega_i = \begin{cases} {\color{red}\bullet} & \textit{if } w_i = \bullet \\ \circ & \textit{if } w_i = \circ \end{cases}$$

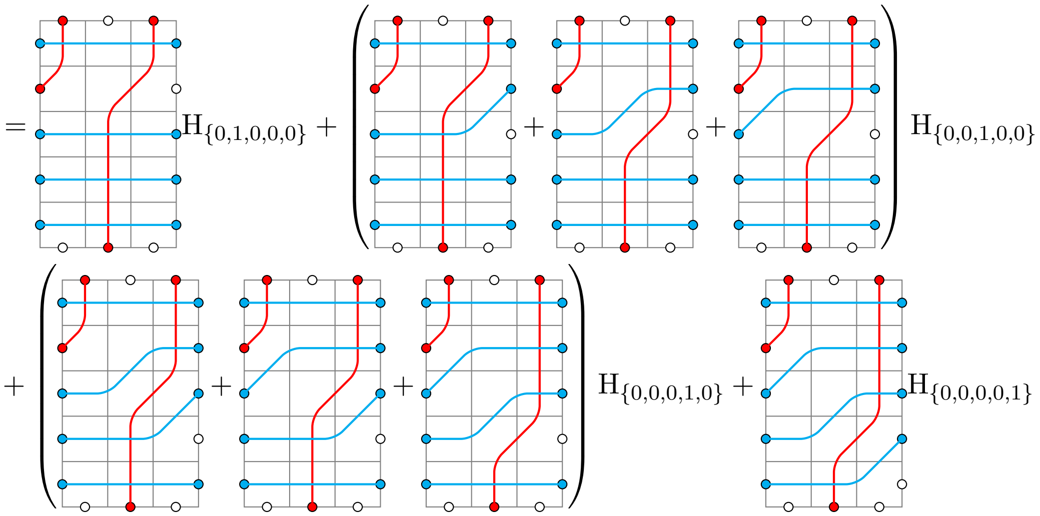
$$\mu_i = \begin{cases} {\color{red}\bullet} & \textit{if } m_i = \bullet \\ {\color{cyan}\bullet} & \textit{if } m_i = \circ \end{cases} \qquad \kappa_i = \begin{cases} \circ & \textit{if } k_i = \bullet \\ {\color{cyan}\bullet} & \textit{if } k_i = \circ \end{cases}$$

*Furthermore,* $C^{k,w}_{l,m}$ *are polynomials in* $q$ *with positive coefficients, up to an overall sign.*

Below, we provide two examples to illustrate the theorem mentioned above.

**Example 3.2.** For $P = 5$, $N = 3$ and $n = 1$. The product $\mathrm{H}_{\{0,1,0,0,0\}}\mathrm{H}_{\{1,0,1\}/\{0,1,0\}}$ has the following expansion:

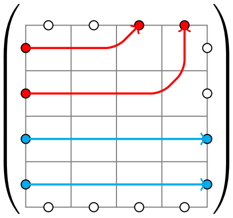
$$\mathrm{H}_{\{0,1,0,0,0\}}\mathrm{H}_{\{1,0,1\}/\{0,1,0\}} = \mathrm{H}_{\{0,1,0,0,0\}} - (1 + q + q)\mathrm{H}_{\{0,0,1,0,0\}} + (q + q^2 + q)\mathrm{H}_{\{0,0,0,1,0\}} - q^2\mathrm{H}_{\{0,0,0,0,1\}}$$



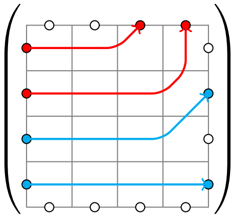
**Example 3.3.** For $P, N = 4$ and $n = 2$, we compute

$$\mathrm{H}_{\{1,1,0,0\}}\mathrm{H}_{\{1,1,0,0\}} = \mathrm{H}_{\{1,1,0,0\}} - q\mathrm{H}_{\{1,0,1,0\}} - q^2\mathrm{H}_{\{0,1,0,1\}} + q^3\mathrm{H}_{\{0,0,1,1\}}$$

$=$

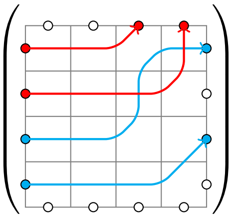


$\mathrm{H}_{\{0,0,1,1\}} +$

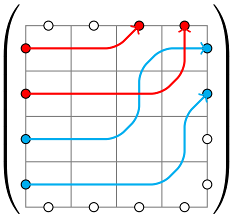


$\mathrm{H}_{\{0,1,0,1\}}$

$+$

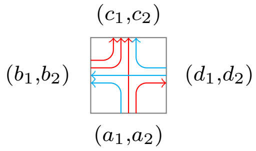


$\mathrm{H}_{\{1,0,1,0\}} +$



$\mathrm{H}_{\{1,1,0,0\}}$

3.2. **Puzzles for spin Hall–Littlewood functions and factorial Hall–Littlewood polynomials.** To give a formula for the structure constants for spin Hall–Littlewood functions, we need to consider a slightly modified version of the puzzles presented in the previous subsection. Let us begin with the description of the tiles necessary to construct these puzzles. Every edge of a tile is decorated with an ordered pair of non-negative integers.

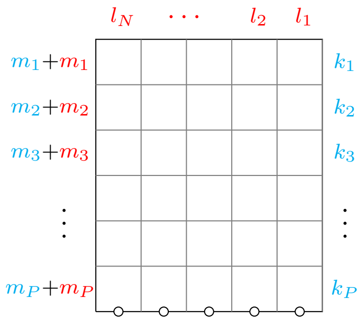


(3.4)

We take the weight of such tiles to be identically zero whenever either $a_1 + b_1 \neq c_1 + d_1$ or $a_2 + d_2 \neq b_2 + c_2$. In other words, the first particles (red particles) enter from the left and bottom, exiting through the top and the right edge, while the second particles (blue particles) enter from the bottom and the right, exiting through the left and the top edge.

Analogous to the previous subsection, a puzzle is a tiling of a rectangular region with the above-mentioned tiles. For a positive integer $k$, we write $k$ and $k$ to denote $(k, 0)$ and $(0, k)$.

For three compositions $l = (l_1, \ldots, l_N)$, $m = (m_1, \ldots, m_N)$, and $k = (k_1, \ldots, k_N)$ such that $\sum_i l_i = \sum_i m_i = \sum_i k_i = n$, we consider puzzles of the following type:



(3.5)

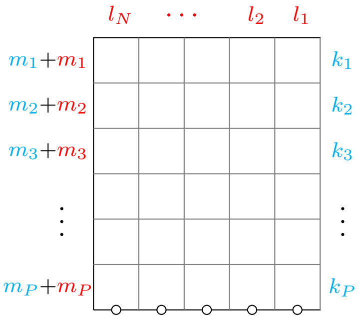
Graphically, we can interpret the boundary conditions of the puzzle as follows: the blue particles enter only from the right and exit through the left boundary while moving upwards and to the left, while the red particles enter from the left and exit through the top while moving upwards and to the right. Before we state our theorem, let us pause to give an example.

**Theorem 3.4.** *We have the following product rule for spin Hall–Littlewood functions:*

$$\mathrm{F}_m(x_1,\dots,x_n;t,s)\mathrm{F}_l(x_1,\dots,x_n;t,s) = \left(\prod_{i=1}^{N}\frac{(q;q)_{l_i}}{(ts;q)_{l_i}}\right)\sum_k \mathcal{C}^k_{l,m}(t,s)\mathrm{F}_k(x_1,\dots,x_n;t,s) \tag{3.6}$$

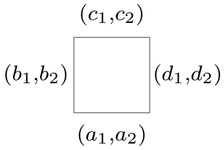
*where*

$$\mathcal{C}^k_{l,m}(t,s) = \tag{3.7}$$



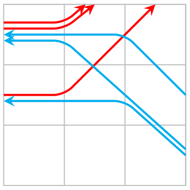
*with weights:*

$$\mathcal{W}_{\mathtt{L},\mathtt{M}}$$

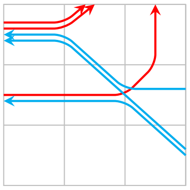


$$= \mathbf{1}_{a_1+b_1=c_1+d_1}\mathbf{1}_{a_2+d_2=b_2+c_2}q^{d_1+b_2-d_2-b_1}(-t)^{d_2-d_1}(-t)^{-c_2}$$
$$\frac{(q^{-1};q^{-1})_{b_2-b_1}}{(q^{-1};q^{-1})_{d_2-d_1}}\sum_{\substack{0\leq p_1\leq\min(b_1,c_1)\\ 0\leq p_2\leq c_2}} q^{d_1(c_2-p_2)}q^{p_1+p_2+p_2(b_1-p_1)}\frac{\prod_{i=1}^{p_1+p_2}(1-q^{-1}tsq^{i-1})}{\prod_{i=1}^{p_1+p_2+b_2-b_1}(ts-q^{1-i})}$$
$$\Bigg(\mathbf{1}_{c_1+c_2-p_1-p_2\leq d_2-d_1}\prod_{i=1}^{c_1+c_2-p_1-p_2}(1-q^i)\prod_{i=1}^{d_2-d_1-c_1-c_2+p_1+p_2}(1-qq^{1-i})$$
$$+\mathbf{1}_{c_1+c_2-p_1-p_2>d_2-d_1}\prod_{i=c_1+c_2-p_1-p_2-d_2+d_1+1}^{c_1+c_2-p_1-p_2}(1-q^i)\Bigg)$$
$$\binom{c_1+d_1-p_1}{c_1-p_1}_q\left(\frac{\prod_{i=1}^{c_2-p_2}(ts-q^{i-d_2})}{(q;q)_{c_2-p_2}}\right)\binom{b_1}{p_1}_q\left(\frac{\prod_{i=1}^{p_2}(ts-q^{i-b_2-p_2})}{(q;q)_{p_2}}\right) \tag{3.8}$$

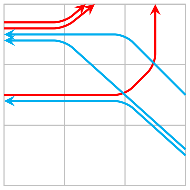
**Example 3.5.** For $l=(1,2,0)$, $m=(2,1,0)$ and $k=(0,1,2)$, we have three puzzles with non-trivial weight. We list the puzzles with their weights given below.



$$\frac{(q+1)t\left(1-q^2st\right)}{1-st}$$



$$\frac{t(1-q^2st)}{1-st}$$



$$\frac{(1+q)t(1-qst)^2}{(1-st)^2}$$

**Remark 3.6.** It is worth noticing that even though spin Hall–Littlewood functions are rational functions, the expansion of their product as a linear combination of them is finite.

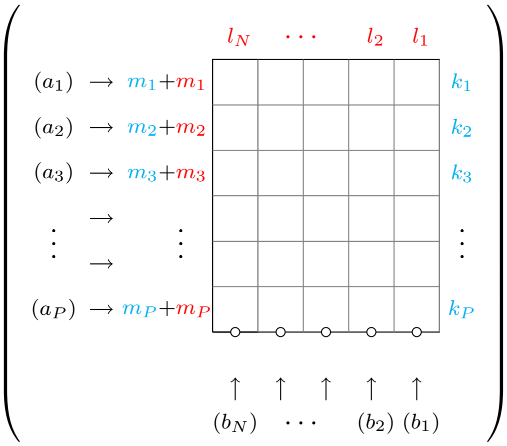
**Theorem 3.7.** *We have the following product rule for factorial Hall–Littlewood polynomials:*

$$\mathrm{P}_m(x_1,\dots,x_n;a_1,\dots,a_P)\mathrm{P}_l(x_1,\dots,x_n;b_1,\dots,b_N)$$

$$= \left(\prod_{i=1}^{N}(q;q)_{l_i}\right)\sum_k \mathcal{C}^k_{l,m}(a_1,\dots,a_P;b_1,\dots,b_N)\mathrm{P}_k(x_1,\dots,x_n;a_1,\dots,a_n) \quad (3.9)$$

*where*

$$\mathcal{C}^k_{l,m}(a_1,\dots,a_P;b_1,\dots,b_N) = \quad (3.10)$$



*with weights:*

$$\mathcal{W}_{\mathtt{L},\mathtt{M}}\left(\begin{array}{c} (c_1,c_2) \\ a \leftarrow (b_1,b_2)\ \boxed{\phantom{x}}\ (d_1,d_2) \\ (a_1,a_2) \\ \uparrow \\ b \end{array}\right) = \mathbf{1}_{a_1+b_1=c_1+d_1}\mathbf{1}_{a_2+d_2=b_2+c_2}\, q^{d_1+b_2-d_2-b_1}(-1)^{d_2-d_1}(-a)^{-c_2}$$

$$\frac{(q^{-1};q^{-1})_{b_2-b_1}}{(q^{-1};q^{-1})_{d_2-d_1}} \sum_{\substack{0\le p_1\le \min(b_1,c_1) \\ 0\le p_2\le c_2}} q^{d_1(c_2-p_2)} q^{p_1+p_2+p_2(b_1-p_1)} \frac{1}{\prod_{i=1}^{p_1+p_2+b_2-b_1}(-q^{1-i})}$$

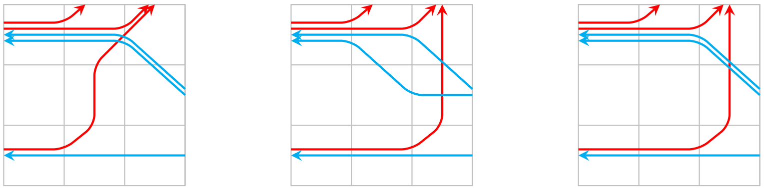
$$\Bigg(\mathbf{1}_{c_1+c_2-p_1-p_2\le d_2-d_1} \prod_{i=1}^{c_1+c_2-p_1-p_2} (a-bq^i) \prod_{i=1}^{d_2-d_1-c_1-c_2+p_1+p_2} (a-bqq^{1-i})$$

$$+\mathbf{1}_{c_1+c_2-p_1-p_2> d_2-d_1} \prod_{i=c_1+c_2-p_1-p_2-d_2+d_1+1}^{c_1+c_2-p_1-p_2} (a-bq^i)\Bigg)$$

$$\binom{c_1+d_1-p_1}{c_1-p_1}_q \left(\frac{\prod_{i=1}^{c_2-p_2}(-q^{i-d_2})}{(q;q)_{c_2-p_2}}\right)\binom{b_1}{p_1}_q\left(\frac{\prod_{i=1}^{p_2}(-q^{i-b_2-p_2})}{(q;q)_{p_2}}\right) \tag{3.11}$$

Below is an example of puzzles that compute the structure constants for factorial Hall–Littlewood polynomials:

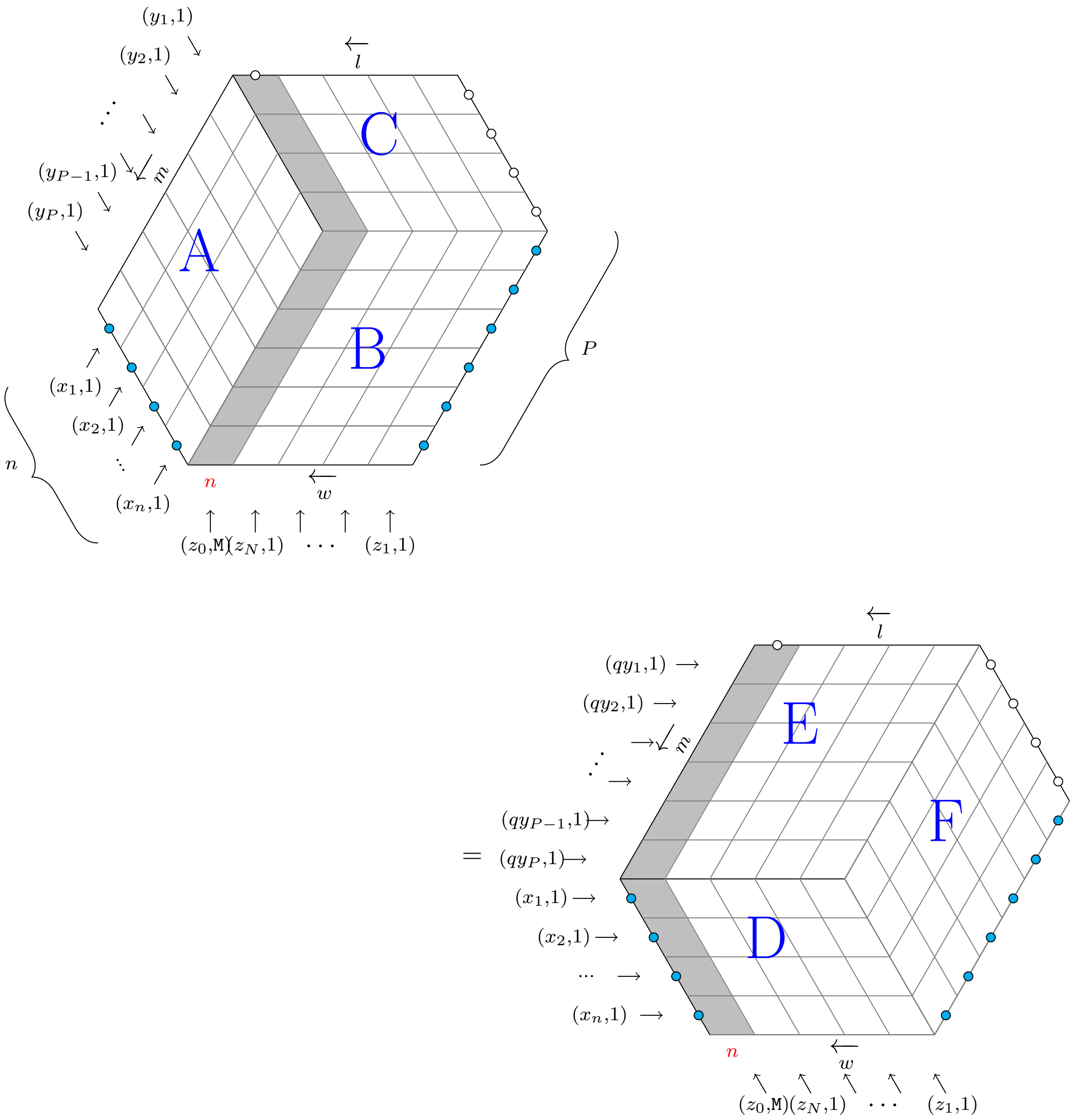
**Example 3.8.** For $l=(2,1,0)$, $m=(2,0,1)$ and $k=(0,2,1)$, we have three puzzles with non-trivial weight. We list the puzzles with their weights given below.



$a_3(a_3-b_1)$ $(1+q)a_3(a_2-qb_1)$ $(1+q)a_1a_3$

## 4. Proof

To prove 3.1, we study a lattice model built from the weights of $\mathcal{U}_q(\widehat{\mathfrak{sl}_3})$. We use two types of vertices $W_{\mathtt{L},\mathtt{M}}(x;q;\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ where $\mathtt{L}=\mathtt{M}=1$; and $\mathtt{L}=1$ with generic $\mathtt{M}$. When $\mathtt{L}=1$ and $\mathtt{M}$ is generic, we fill the vertex with gray. We begin by considering the Yang–Baxter equation illustrated below. We prove our main theorem by computing the partition functions of both sides of (4.1).

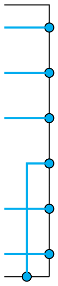


(4.1)

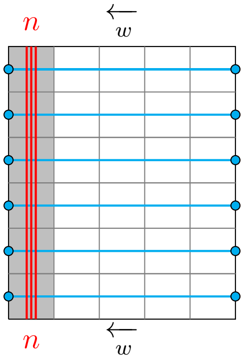
Before we proceed further, we establish/recall the necessary notation and conventions. We use a red particle (●) to denote $(1,0)$, a blue particle (●) for $(0,1)$, and a white particle (also referred as a hole) (○) for $(0,0)$, and $n$ to denote $(n,0)$. We denote the partition function of a region by $\mathcal{Z}$, using the name of the region as a subscript.

We select $m$ to be a $P$-tuple made up of blue particles and holes, where the number of blue particles equals $P-n$. As a result, only blue particles enter A. At the bottom boundary of B, we have $(n,0)$ at the leftmost edge, which can be thought of as a reservoir of red particles. Moreover, $w$ and $l$ are binary strings of red particles and holes where $l$ has exactly $n$ more red particles. We begin by computing the partition functions for each region.

4.1. **Region** $B$**.** We begin by analyzing the consequences of the boundary of A. Observe that only blue particles enter A. These particles can enter B or C.



If they enter C, they cannot exit because the top and right boundaries do not contain blue particles. Let us recall that we have $P$ number of blue particles that enter A, which is equal to the number of rows of B. Hence, the left boundary of B is fixed as shown in the picture above. All these particles entering B have to exit through the right boundary, which freezes the entire region. To see this, observe that a blue particle has to exit from the last row. Since no blue particles are entering from below, the blue particle entering from the left must exit from the last row. Additionally, the red particles entering from below cannot turn right, since the horizontal edges in B can contain almost one particle. By applying the same argument to each row, we can conclude that the whole B is frozen, as shown in the picture below.

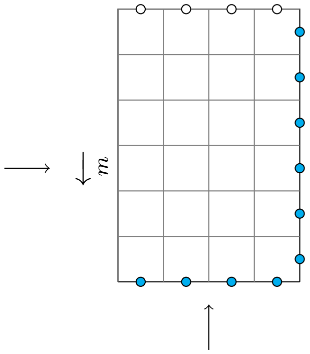


We have:

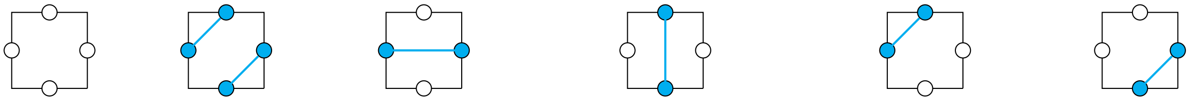
$$\mathcal{Z}_{\mathrm{B}} = \prod_{i=1}^{P} \frac{1-q^{-1}z_0y_i^{-1}}{1-q^{\mathtt{M}-1}z_0y_i^{-1}} \prod_{i=1}^{P}\prod_{j=1}^{N} \frac{1-q^{-1}z_jy_i^{-1}}{1-z_jy_i^{-1}} \tag{4.2}$$

4.2. **Region** *A***.** From the discussion above, it is clear that four boundaries of A are fixed. Blue particles enter from the bottom boundary which combines with the particles from the left and they exit through the right boundary.

$$\mathcal{Z}_{\mathrm{A}} = \tag{4.3}$$



Then the weight of A is the partition function of the above lattice with the tiles given below:



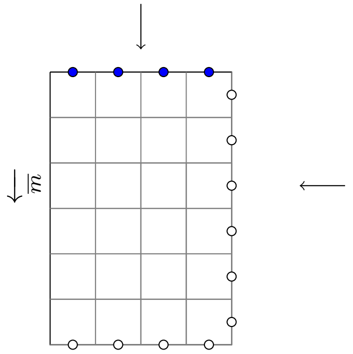
$$1 \qquad 1 \qquad \frac{1-xy^{-1}q^{-1}}{1-xy^{-1}} \qquad \frac{q\left(1-xy^{-1}q^{-1}\right)}{1-xy^{-1}} \qquad \frac{q^{-1}(1-q)xy^{-1}}{1-xy^{-1}} \qquad \frac{(1-q)}{1-xy^{-1}} \tag{4.4}$$

We claim that

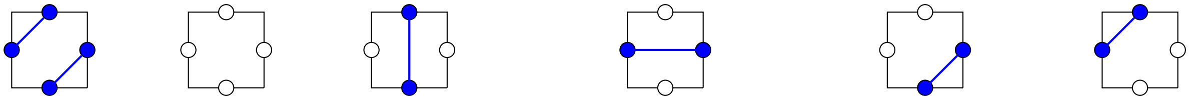
$$\mathcal{Z}_{\mathrm{A}} = \mathrm{H}_{\overline{m}}(x_1, \dots, x_n; y_1, \dots, y_P) \tag{4.5}$$

where $\overline{m}$ is obtained by complementing the edge labels i.e., by replacing every blue particle with a white particle and vice-versa. To prove this, we redraw the vertices and the lattice concerning A by complementing the particles. We then have:

$$\mathcal{Z}_{\mathrm{A}} = \tag{4.6}$$



with the following weights:

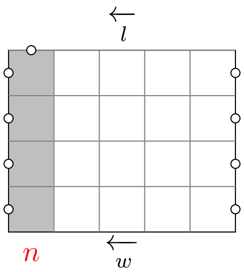


$$1 \qquad 1 \qquad \frac{1-xy^{-1}q^{-1}}{1-xy^{-1}} \qquad \frac{q\left(1-xy^{-1}q^{-1}\right)}{1-xy^{-1}} \qquad \frac{q^{-1}(1-q)xy^{-1}}{1-xy^{-1}} \qquad \frac{(1-q)}{1-xy^{-1}} \tag{4.7}$$

From definition 2.2, we conclude that

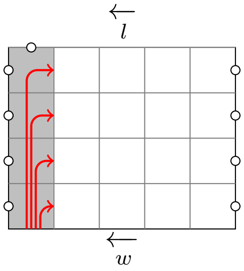
$$\mathcal{Z}_{\mathrm{A}} = \mathrm{H}_{\overline{m}}(x_1, \dots, x_n; y_1, \dots, y_P). \tag{4.8}$$

4.3. **Region *C*.** We observe that the left and bottom boundaries of C are fixed. As B is completely frozen, the bottom boundary of C remains the same as that of B. As no particles from A enter C, the left boundary is fixed to all holes.



(4.9)

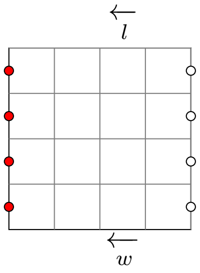
Observe that $n$ red particles enter from the bottom of the first column and none exit through the top edge of it. Given that there are $n$ rows, a particle has to turn right in every row, as shown in the picture below.



(4.10)

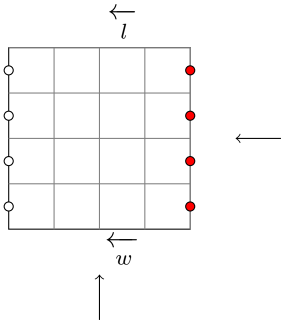
With the fixed weight of the first column, we get that:

$$\mathcal{Z}_{\mathrm{C}} = \frac{(q;q)_n}{\prod_{i=1}^{n}(1-q^{\mathtt{M}}z_0x_i^{-1})} \times \qquad (4.11)$$

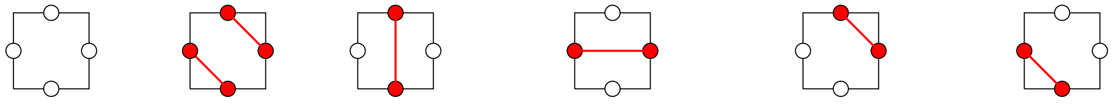


We denote the partition function of the remaining lattice of C as $\widetilde{\mathrm{H}}_{l/w}(x_1,\ldots,x_n;z_1,\ldots,z_N)$. To compare H and $\widetilde{\mathrm{H}}$, it is convenient to give an alternate presentation for $\widetilde{\mathrm{H}}$ by complimenting the particles on horizontal edges.

$$\widetilde{\mathrm{H}}_{l/w}(x_1,\ldots,x_n;z_1,\ldots,z_N) = \qquad (4.12)$$



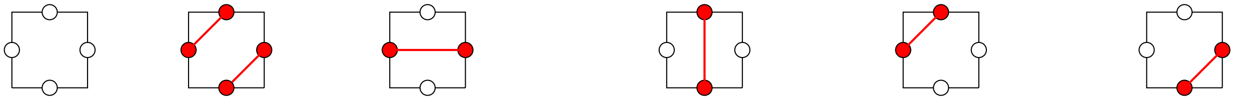
with the following vertices:



(4.13)

| | | | | | |
|---|---|---|---|---|---|
| $1$ | $q$ | $\dfrac{1-qzx^{-1}}{1-zx^{-1}}$ | $\dfrac{1-qzx^{-1}}{1-zx^{-1}}$ | $\dfrac{(1-q)zx^{-1}}{1-zx^{-1}}$ | $\dfrac{1-q}{1-zx^{-1}}$ |

We now explore the relationship between H and $\widetilde{\mathrm{H}}$. To the weights above, multiply by $q^{-1}$ when the left edge is occupied with a red particle, and apply a negative sign to the last two vertices. And then rotate them by 90° anti-clockwise. For convenience, let us reproduce the vertices after applying these operations.



$$1 \qquad 1 \qquad \frac{q(1-q^{-1}xz^{-1})}{1-xz^{-1}} \qquad \frac{(1-q^{-1}xz^{-1})}{1-xz^{-1}} \qquad \frac{(1-q)}{1-xz^{-1}} \qquad \frac{(1-q)xz^{-1}q^{-1}}{1-xz^{-1}} \tag{4.14}$$

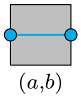
These weights are identical to the weights used in evaluating $\mathcal{Z}_{\mathrm{A}}$ eq. (4.7). We then deduce that:

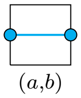
$$(-1)^n q^{-\sum\limits_{i=1}^{N}(i-1)(l_i-w_i)} \widetilde{\mathrm{H}}_{l/w}(x_1,\dots,x_n;z_1,\dots,z_N) = \mathrm{H}_{l/w}(x_1,\dots,x_n;z_1,\dots,z_N). \tag{4.15}$$

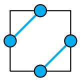
Finally, we get:

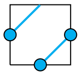
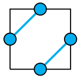
$$\mathcal{Z}_{\mathrm{C}} = \frac{(-1)^n q^{\sum\limits_{i=1}^{N}(i-1)(l_i-w_i)}(q;q)_n}{\prod_{i=1}^{n}(1-q^{\mathtt{M}}z_0x_i^{-1})} \mathrm{H}_{l/w}(x_1,\dots,x_n;z_1,\dots,z_N) \tag{4.16}$$

4.4. **Region** $D$**.** We have $n$ blue particles entering from the left boundary of D. As these particles can either enter F and E, we have many non-trivial configurations. However, we make certain assumptions that restrict the particles from entering E. Consequently, F is frozen with a trivial weight.

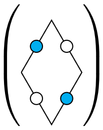
Before proceeding, we need to appropriately normalise the weights within E. Observe that, as we are interested in the structure constants, we need to divide the right side of (4.1) with $\mathcal{Z}_{\mathrm{B}}$. This is equivalent to saying that the weights of the Region E are normalized such that the weight of 

and 

(weights of these vertices are independent of the bottom and the top label) is 1. With this normalization, the weight of  is $\dfrac{1-z_jy_i^{-1}}{1-q^{-1}z_jy_i}$. Suppose that, for a large enough $P$, there exists a positive integer $t$ such that $m_i =$ ● for all $i \geq t$.

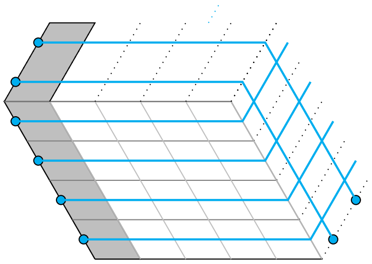
Suppose a blue particle from D enters E in column $j$ from the left. This action causes the blue particles entering from the left of E to turn upwards. We count the number of  vertices that appear. Let us consider the worst-case scenario where no such vertices occur. In the last row of E, the particle entering from the left turns upwards in a column to the left of the $j$-th column. Similarly, in the $(P-1)^{th}$ row, the particle from the left must turn upwards in a column preceding the column in which the blue particle, from $P^{th}$ row, entered. Since there are only a finite number of columns, the particle entering from the $(P-N)^{th}$ row has to turn upwards in the first column. This implies that there are at least $(P-N-t)$  vertices.

Assume that $\left|\dfrac{1-z_jy_i^{-1}}{1-q^{-1}z_jy_i}\right| < 1$. As we take $P \to \infty$, the assumption implies that the overall weight of such a configuration is 0.

Next, we need to determine which configurations survive as we let $P \to \infty$. Based on the above analysis, all blue particles from D must enter F. Now, let us examine the right boundary of E. Suppose there is a hole (white particle) on the right boundary in the $j^{th}$ row, and there are $k$ blue particles entering F from rows 1 to $j$. This white particle can only exit through the top boundary of F. This means there must be at least $k$ number of  vertices.

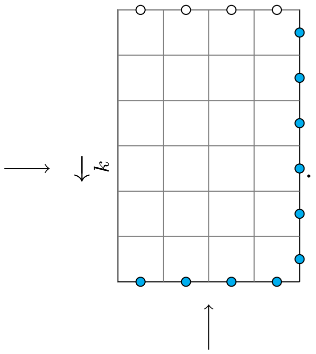
Similar to the previous analysis, we assume that $\left|\frac{1-xy^{-1}q^{-1}}{1-xy^{-1}}\right| < 1$. This implies that as $P \to \infty$, there can only be a finite number of particles entering F through the rows preceding the row with a white particle on the right boundary.

Therefore, in the limit $P \to \infty$, the overall weight of a configuration is non-zero only when there exists a positive integer $t$ such that the right boundary of E in all the rows below row $t$ is a particle, as shown in the picture below.



4.5. **Region $F$.** As all the blue particles from D enters F, this freezes the bottom boundary (or south-west boundary of F in eq. (4.1)). This forces the right boundary to be a binary string of blue particles and holes with exactly $n$ holes. For a fixed $k$, we have:

$$\mathcal{Z}_{\mathrm{F}} =$$



. (4.17)

Using similar reasoning as in (4.5), we conclude that:

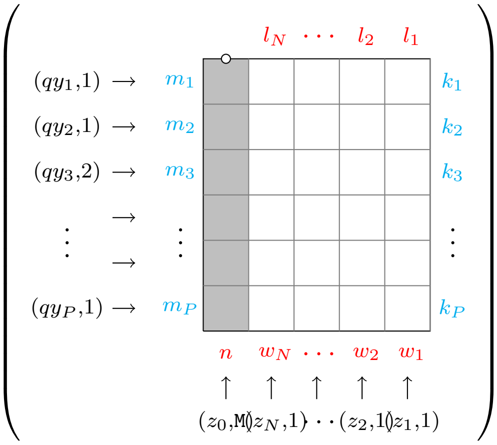
$$\mathcal{Z}_{\mathrm{F}} = \mathrm{H}_{\overline{k}}(x_1,\dots,x_n;y_1,\dots,y_P). \tag{4.18}$$

4.6. **Final equation.** Let $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{y} = (y_1,\dots,y_P)$ and $\mathbf{z} = (z_1,\dots,z_N)$. Putting all together, the original equation eq. (4.1) gives us the following:

$$\left(\frac{(q;q)_n}{\prod_{i=1}^n(1-z_0x_i^{-1})}\right)\left((-1)^n q^{\sum\limits_{i=1}^{N}(i-1)(l_i-w_i)}\right)\mathrm{H}_{l/w}(\mathbf{x};\mathbf{z})\mathrm{H}_{\overline{m}}(\mathbf{x};\mathbf{y}) = \sum_k \mathcal{Z}^{k,w}_{l,m}(\mathbf{y};\mathbf{z})\mathrm{H}_{\overline{k}}(\mathbf{x};\mathbf{y}) \tag{4.19}$$

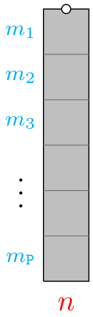
where

$$\mathcal{Z}^{k,w}_{l,m}(\mathbf{y};\mathbf{z}) = \left(\prod_{i=1}^{P} \frac{1-q^{\mathtt{M}-1}z_0 y_i^{-1}}{1-q^{-1}z_0 y_i^{-1}}\right)\left(\prod_{i=1}^{P}\prod_{j=1}^{N} \frac{1-z_j y_i^{-1}}{1-q^{-1}z_j y_i}\right)$$

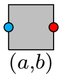


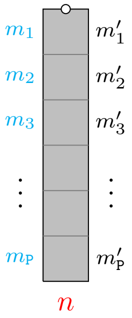
(4.20)

4.7. **Main theorem.** Our theorem 3.1 is obtained by specializing (4.19). We set $y_i = q^{-1}$, $z_i = q^{-1}$ for all $i \geq 1$, and $z_0 = 0$. Let us first analyze the consequences of these specializations on the first column of the lattice component of $\mathcal{Z}^{k,w}_{l,m}$.



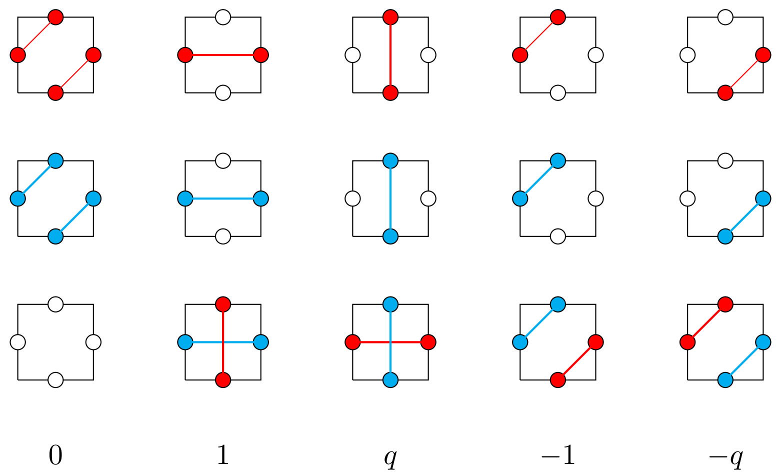
(4.21)

Firstly, recall that $m$ has $n$ holes and $P-n$ blue particles, and we have $n$ red particles entering from the bottom and none exiting from the top. This implies that every right edge has to contain a red or a blue particle. Observe that the weight of the vertex , which is equal to $\dfrac{z_0(1-q^a)q^b}{z_0 q^{\mathtt{M}} - yq}$, vanishes when $z_0 = 0$. This prevents the blue particles from moving upwards. Therefore, we have:



$$= (q;q)_n \qquad (4.22)$$

where $m'_k = \begin{cases} 🔵 & \text{if } m_k = 🔵 \\ 🔴 & \text{if } m_k = ○ \end{cases}$. The tiles of the lattice corresponding to the puzzles and their weights, after the specializations, are given below:



(4.23)

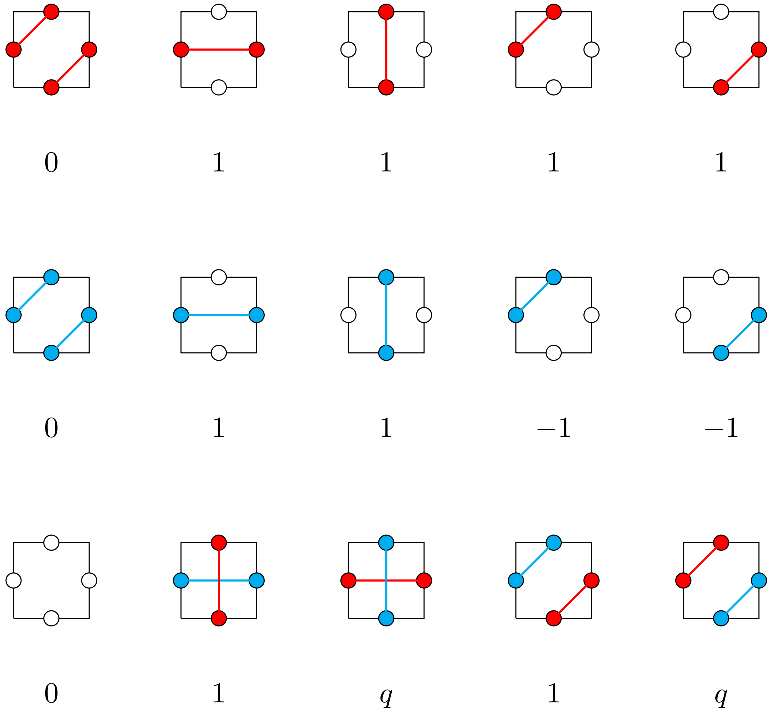
It remains to show that the factor $\left((-1)^n q^{-\sum_{i>0}(i-1)(l_i-w_i)}\right)$ can be incorporated into the puzzle weights. Observe that when a red particle exits through the $i^{th}$ column from the right, it implies that there are $N-i$ tiles with that red particle on the right edge. We can conclude that:

$$-\left(\sum_{i>0}(i-1)l_i\right) = \#\,\square\!{\color{red}\bullet} - n(N) + n.$$

Using the observation above, we can incorporate the factor into the puzzle weights with the following operations.

(i) If the right edge contains a red particle, multiply the weight by $q$,
(ii) If the left edge does not contain a blue particle, multiply the weight by $q^{-1}$ which gives an overall factor of $q^{-n(N)}$, and
(iii) If the top edge contains a red particle, multiply the weight by $-q$; if the bottom edge contains a red particle, divide the weight by $-q$ which gives an overall factor of $(-q)^n$.

We list the tiles with their weights after performing the above-mentioned operations:

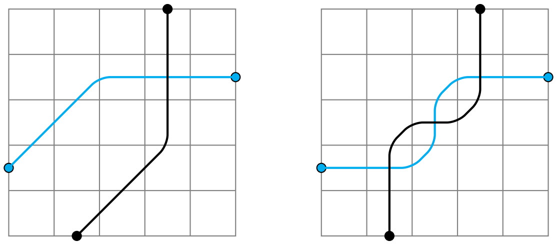


(4.24)

Finally, to get the tiles eq. (3.1) multiply each weight with $(-1)$ whenever there is a blue particle on the top edge, and also whenever there is a blue particle on the bottom edge. As $n$ blue particles enter from the right and exit through the left, this operation does not alter the partition function of a puzzle.

4.8. **Positivity.** To see positivity, we work with weights (4.23). We argue that for a fixed boundary, all puzzles have the same overall sign. This follows as a consequence of the weights of the tiles with identical labels on all edges being 0 (i.e., tiles in the first column (4.23)). We make the following observation.

For convenience, we draw holes as black particles. We refer to the tiles in the first column of eq. (4.23) as **a** vertices, those in the second and third columns as **b** vertices, and those in the fourth and fifth column as **c** vertices. Pick a particle entering from the left and a particle entering from the bottom. These two particles exit from specific positions. Then in all the configurations, the number of **b** type vertices involving the strings corresponding to these two particles is always either odd or even, as shown in the picture below. As the weights of the **a** vertices are 0, this implies that for a fixed boundary all the configurations have either an odd number or an even number of **c** tiles.



## 5. Puzzles for spin Hall-Littlewood functions

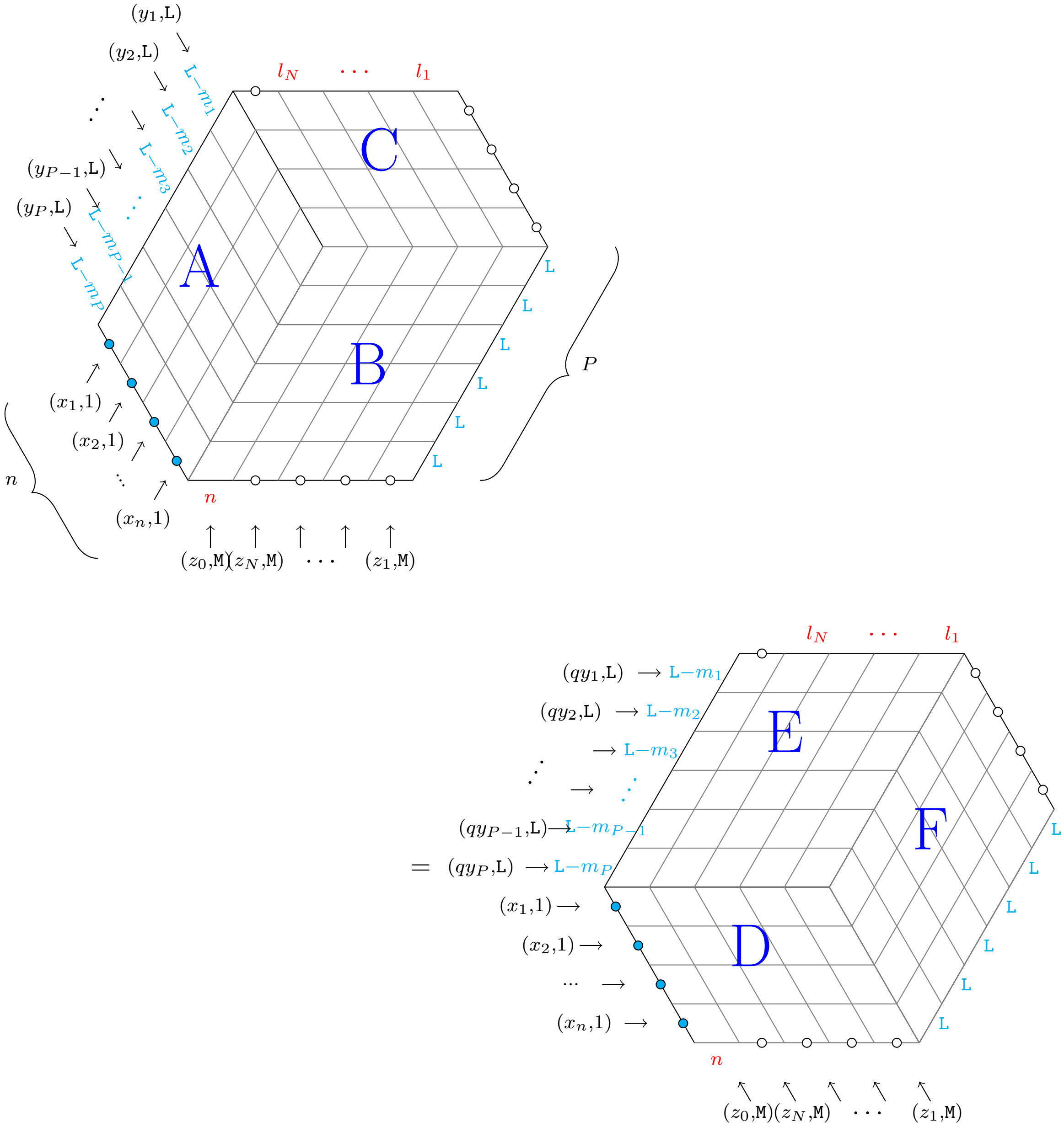
We follow the same argument in deriving the puzzles for the spin Hall–Littlewood functions as we did in the previous section. We shall only focus on the modifications and keep our explanations to a minimum to avoid repeating ourselves.

### 5.1. Notation.

Recall that positive integers $\mathtt{L}, \mathtt{M}$ are associated with the spin of the lines attached to the variables $y, z$. We use a white bullet to denote $(0,0)$, for a positive integer $k$, we write $k$ to denote $(k,0)$, and $k$ to denote $(0,k)$. For this section, we depict all vertices without any shading. There are three types of vertices that we use to construct the lattice model based on the spin. It is evident from the graphical notation to see the spin of the lines.

### 5.2. Lattice model.

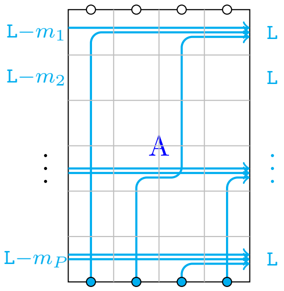
We consider the following higher spin $\mathcal{U}_q(\widehat{\mathfrak{sl}}_3)$ lattice model:



(5.1)

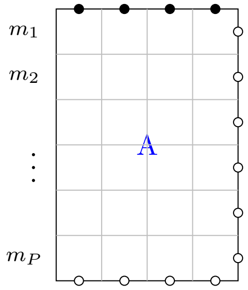
where $\sum_{i=1}^{P} m_i = \sum_{i=1}^{N} l_i = n$.

5.3. **Region** $A$**.** Only blue particles enter A through the left and bottom boundaries. Given the boundary of C, they all have to enter B. In total, there are $\mathtt{L}P$ blue particles entering the A. Recall that the horizontal edges in A and B can carry almost $\mathtt{L}$ particles. This implies that $\mathtt{L}$ number of particles exit from A through every row. A typical configuration in A is given below.

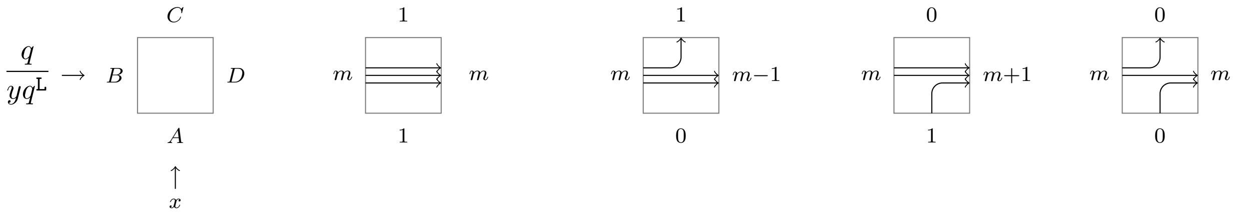


By complementing the particles on the vertical edges with $\mathtt{L}$, and swapping 1 to 0 and vice versa on the horizontal edges, we get the following:

$$\mathcal{Z}_{\mathrm{A}}(x_1,\ldots,x_n;y_1,\ldots,y_P;q^{-\mathtt{L}}) = \tag{5.2}$$



with weights:

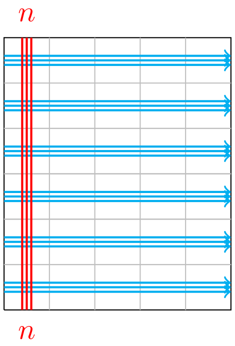


$$\frac{1-q^m xy}{(-yq^{\mathtt{L}})(x-y^{-1}q^{-\mathtt{L}})} \qquad \frac{1-q^m}{(-yq^{\mathtt{L}})(x-y^{-1}q^{-\mathtt{L}})} \qquad \frac{(1-q^{m-\mathtt{L}})x}{x-y^{-1}q^{-\mathtt{L}}} \qquad \frac{x-q^m y^{-1}q^{-\mathtt{L}}}{x-y^{-1}q^{-\mathtt{L}}} \tag{5.3}$$

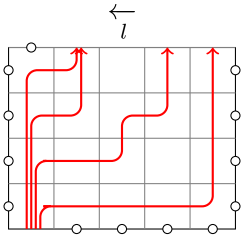
Suppose that the spin of the lines attached to $y$ variables are all different. Then the function $\mathcal{Z}_{\mathrm{A}}$ is dependent $x$ variables, $y$ variables and also in $q^{-\mathtt{L}_i}$. This makes it possible to take different limits.

$$\prod_{i=1}^{P}(-y_i q^{\mathtt{L}_i})^{\sum_{j=i}^{P} m_j}\mathcal{Z}_{\mathrm{A}} = \mathcal{F}_m(x_1,\ldots,x_n;y_1,\ldots,y_P;q^{-\mathtt{L}_1},\ldots,,q^{-\mathtt{L}_P}) \tag{5.4}$$

5.4. **Region** $B$**.** From the discussion concerning A, both left and right boundaries are fixed and are identical. We show that the entire region is frozen. To see this, we begin with the last row. In the last row, we have $\mathtt{L}$ blue particles entering from the left and the same exiting through the right. As no other blue particles are entering from the bottom, we get that those blue particles that entered are the ones that exit. As each of the vertical edges can carry almost $\mathtt{L}$ particles, we get that the entire last row is frozen. Using the same reasoning from every row, we conclude that the entire region is frozen. The weights of the vertex where the vertical labels are both $(0,\mathtt{L})$, then the weight is independent of the top and bottom labels. We chose the normalisation so that the weight of B is trivial and equal to 1.

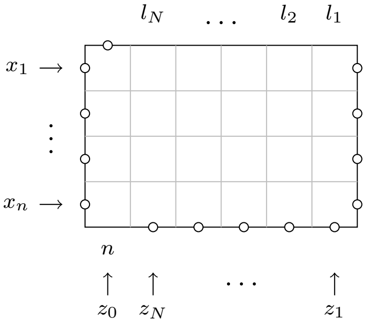


5.5. **Region** $C$**.** As B is frozen, the bottom boundary of C is the same as the bottom boundary of B. Recall that particles from A do not enter C, this fixes the labels on the vertical edges of the left boundary of C to be $(0, 0)$. This implies that in C we only have red particles. Therefore, $n$ red particles enter C from the bottom boundary of the first column. An example of a typical configuration is given below.

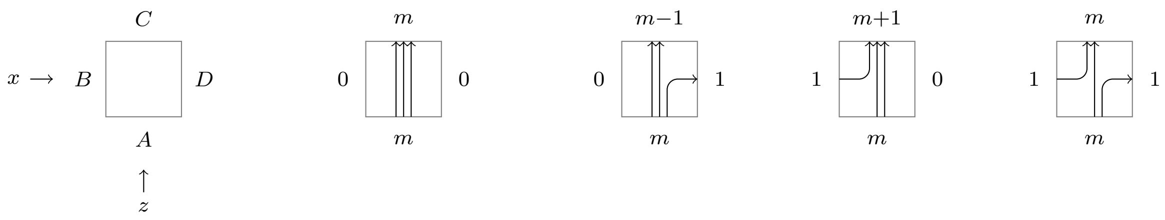


Then we can compute $\mathcal{Z}_{\mathrm{C}}$ as partition function:

$$\mathcal{Z}_{\mathrm{C}}(x_1, \dots, x_n; z_0, \dots, z_N; q^{-\mathtt{M}}) = \qquad (5.5)$$



with weights:



$$\frac{(-zq^{\mathtt{M}})(1-q^{m}xz^{-1}q^{-\mathtt{M}})}{x-z} \qquad \frac{(1-q^{m})x}{x-z} \qquad \frac{(-zq^{\mathtt{M}})(1-q^{m-\mathtt{M}})}{x-z} \qquad \frac{x-zq^{m}}{x-z} \tag{5.6}$$

By choosing the spin of the lines attached to $z$ variables to be all different, we define the following function:

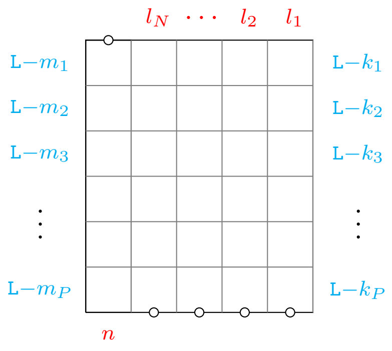
$$\prod_{i=1}^{N}(-z_i^{-1}q^{-\mathtt{M}_i})^{\sum_{j=i}^{N} l_j}\mathcal{Z}_{\mathrm{C}} = \mathcal{G}_l(x_1,\ldots,x_n;z_0,z_1,\ldots,z_N;q^{-\mathtt{M}_1},\ldots,q^{-\mathtt{M}_N}) \tag{5.7}$$

5.6. **Region** $D$**.** We assume that $\left|\frac{1-q^{\mathtt{L}}yz}{1-yz}\cdot\frac{(1-xy)}{1-xyq^{\mathtt{L}}}\right| < 1$. With these assumptions and applying the same argument as in section 4.4, we conclude that D is entirely frozen, where all the blue particles travel across into F, with a trivial weight.

5.7. **Region** $F$**.** Let us begin by analysing the boundaries of F. As a consequence of D being frozen, the bottom boundary of F is fixed i.e., a blue particle enters from the bottom boundary in every column. Observe that $\mathtt{L}$ particles exit from every row through the right boundaries. This implies that the labels of the edges of the left boundary are of the form $(0, \mathtt{L}-k_i)$ where $\sum_i k_i = n$. Observe that these boundaries as precisely the same as the boundaries of A. As the weight used in A and F are same, we conclude that:

$$\prod_{i=1}^{P}(-y_iq^{\mathtt{L}_i})^{\sum_{j=i}^{P} k_j}\mathcal{Z}_{\mathrm{F}} = \mathcal{F}_k(x_1,\ldots,x_n;y_1,\ldots,y_P;q^{-\mathtt{L}_1},\ldots,,q^{-\mathtt{L}_P}) \tag{5.8}$$

5.8. **Region** $E$**.** Finally, we are left with E. All boundaries are fixed except for the right boundary. However, this boundary is the same as the left boundary of F. For an arbitrary boundary $k$, E is of the following:

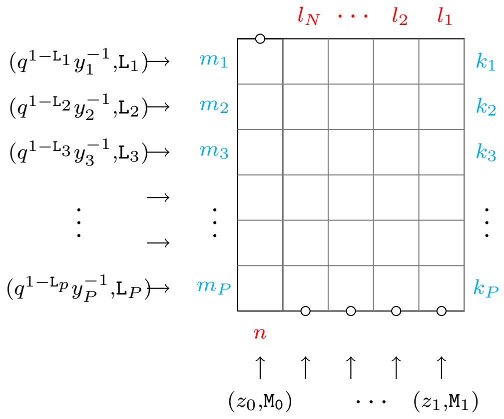


(5.9)

where $\sum_i k_i = n$. As usual, it is convenient to redraw the lattice by complementing particles on both horizontal and vertical edges with their corresponding spins. Finally, we get that:

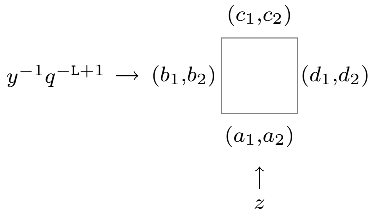
$$\mathcal{Z}_{\mathrm{E}}(y_1,\dots,y_P;z_0,z_1,\dots,z_N;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P};q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N})$$

$$=\sum_k \left( \vphantom{\sum} \right. \tag{5.10}$$



with the following weights:

$$W_{\mathtt{L},\mathtt{M}}\left( \vphantom{\sum} \right.$$



$$= \mathbf{1}_{a_1+b_1=c_1+d_1}\mathbf{1}_{a_2+d_2=b_2+c_2}q^{d_1+b_2-d_2-b_1}z^{d_2-d_1}q^{\mathtt{L}c_2}q^{-\mathtt{M}(d_1-d_2)}$$

$$\frac{(q^{-1};q^{-1})_{b_2-b_1}}{(q^{-1};q^{-1})_{d_2-d_1}}\sum_{\substack{0\le p_1\le\min(b_1,c_1)\\ \\ 0\le p_2\le c_2}} q^{d_1(c_2-p_2)}q^{p_1+p_2+p_2(b_1-p_1)}\frac{\prod_{i=1}^{p_1+p_2}(1-q^{-1}zyq^{i-1})}{\prod_{i=1}^{p_1+p_2+b_2-b_1}(yz-q^{1-i})}$$

$$\left(\mathbf{1}_{c_1+c_2-p_1-p_2\le d_2-d_1}\prod_{i=1}^{c_1+c_2-p_1-p_2}(y-q^{-\mathtt{M}+1}z^{-1}q^{i-1})\prod_{i=1}^{d_2-d_1-c_1-c_2+p_1+p_2}(y-q^{-\mathtt{M}+1}z^{-1}q^{1-i})\right.$$

$$\left.+\mathbf{1}_{c_1+c_2-p_1-p_2>d_2-d_1}\prod_{i=c_1+c_2-p_1-p_2-d_2+d_1+1}^{c_1+c_2-p_1-p_2}(y-q^{-\mathtt{M}+1}z^{-1}q^{i-1})\right)$$

$$\binom{c_1+d_1-p_1}{c_1-p_1}_q\left(\frac{\prod_{i=1}^{c_2-p_2}(q^{-\mathtt{L}}-q^{i-d_2})}{(q;q)_{c_2-p_2}}\right)\binom{b_1}{p_1}_q\left(\frac{\prod_{i=1}^{p_2}(q^{-\mathtt{L}}-q^{i-b_2-p_2})}{(q;q)_{p_2}}\right) \tag{5.11}$$

5.9. **Final equation.** The Yang–Baxter equation eq. (5.1), gives the following relation among the partition functions of various regions:

$$\mathcal{Z}_{\mathrm{A}}\frac{\mathcal{Z}_{\mathrm{C}}}{\mathcal{Z}_{\mathrm{D}}}=\frac{\mathcal{Z}_{\mathrm{E}}}{\mathcal{Z}_{\mathrm{B}}}\mathcal{Z}_{\mathrm{F}} \tag{5.12}$$

By plugging in the actual functions in the above relation, we get the following equation which governs the product of two functions:

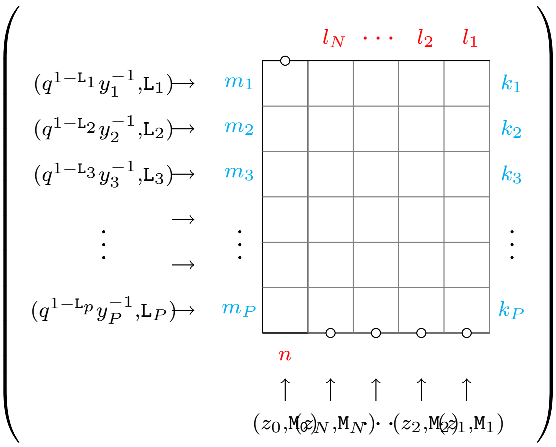
$$\mathcal{F}_m(x_1,\dots,x_n;y_1,\dots,y_P;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P})\mathcal{G}_l(x_1,\dots,x_n;z_0,z_1,\dots,z_N;q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N})$$

$$=\sum_k\prod_{i=1}^{P}(-y_iq^{\mathtt{L}_i})^{\sum_{j=i}^{P}m_i}\prod_{i=1}^{P}(-y_iq^{\mathtt{L}_i})^{-\sum_{j=i}^{P}k_i}\prod_{i=1}^{N}(-z_i^{-1}q^{-\mathtt{M}_i})^{\sum_{j=i}^{N}l_j}$$

$$\mathcal{C}_{l,m}^{k}(y_1,\dots,y_P;z_0,z_1,\dots,z_N;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P};q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N})\mathcal{F}_k(x_1,\dots,x_n;y_1,\dots,y_n;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P}) \tag{5.13}$$

where

$$\mathcal{C}_{l,m}^{k}(y_1,\dots,y_P;z_0,z_1,\dots,z_N;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P};q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_P})=$$



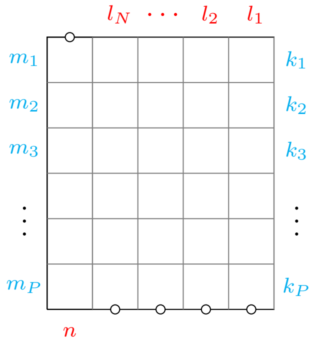
(5.14)

with weights eq. (5.11).

5.10. **Conjugation.** We show that the factor

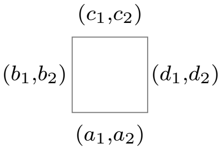
$$\prod_{i=1}^{N}(-z_i^{-1}q^{-\mathtt{M}_i})^{\sum_{j=i}^{N}l_j}\prod_{i=1}^{P}(-y_iq^{\mathtt{L}_i})^{\sum_{j=i}^{P}m_j-k_j}. \tag{5.15}$$

in eq. (5.13) can be incorporated into the weights eq. (5.11). We show that the partition function of the lattice



(5.16)

with weight of a vertex in $i^{th}$ row and $j^{th}$ column:



$$= (-z_j^{-1}q^{-\mathtt{M}_j})^{d_2-d_1}(-y_i^{-1}q^{-\mathtt{L}_i})^{c_2} \tag{5.17}$$

is equal to $\prod_{i=1}^{P}(-y_iq^{-\mathtt{L}_i})^{\sum_{j=i}^{P}k_i-m_i}\prod_{i=1}^{N}(-z_iq^{-\mathtt{M}_i})^{\sum_{j=i}^{N}l_j}$.

Consider a column in the above lattice. Then the summation of $(d_2-d_1)$ of all the vertices in that column is the difference between the number of blue particles and the of red particles on the right boundary of that column. In the first column, we have precisely $n$ red and $n$ blue particles, and hence we do not have any contribution of $(-z_0^{-1})q^{-\mathtt{M}_0}$. Similarly, let us look at the right boundary of the $i^{th}$ column from the right whose top label is $l_i$. On the right boundary of this column, we have $n$ blue particles and $n-\sum_{i=1}^{i-1}l_i=\sum_{j=i}^{N}l_i$ red particles.

On the other hand, let us look at the overall factor of $i^{th}$ row from the bottom. Recall that the left and the right boundary are $m_i$ and $k_i$. The bottom $i$ rows can be interpreted as $\sum_{j=i}^{P}k_i$ blue particles enter from the right while $\sum_{j=i}^{P}m_i$ particles exit through the left. Then the number of particles on the top edge is precisely $\sum_{j=i}^{P}k_i-m_i$. Therefore by recording the blue particles on the top edge of $i^{th}$ row from the bottom is $(-y_iq^{-\mathtt{L}_i})$ is $\prod_{i=1}^{P}(-y_iq^{-\mathtt{L}_i})^{\sum_{j=i}^{P}k_i-m_i}$.

After this conjugation, we can rewrite eq. (5.13) as follows:

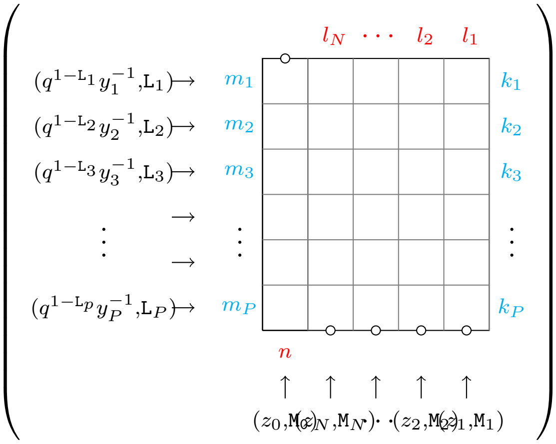
$$\mathcal{F}_m(x_1,\dots,x_n;y_1,\dots,y_P;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P})\mathcal{G}_l(x_1,\dots,x_n;z_0,z_1,\dots,z_N;q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N})$$

$$=\sum_k \mathcal{C}_{l,m}^k(y_1,\dots,y_P;z_0,z_1,\dots,z_N;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P};q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N})$$

$$\mathcal{F}_k(x_1,\dots,x_n;y_1,\dots,y_n;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P}) \tag{5.18}$$

where

$$\mathcal{C}_{l,m}^k(y_1,\dots,y_P;z_0,z_1,\dots,z_N;q^{-\mathtt{L}_1},\dots,q^{-\mathtt{L}_P};q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_P})=$$



(5.19)

with weights:

$$\mathcal{W}_{\mathtt{L},\mathtt{M}}\left(\begin{array}{ccc} & (c_1,c_2) & \\ y^{-1}q^{-\mathtt{L}+1} \to (b_1,b_2) & \boxed{\phantom{XX}} & (d_1,d_2) \\ & (a_1,a_2) & \\ & \uparrow & \\ & z & \end{array}\right) = \mathbf{1}_{a_1+b_1=c_1+d_1}\mathbf{1}_{a_2+d_2=b_2+c_2}\, q^{d_1+b_2-d_2-b_1}(-1)^{d_2-d_1}(-y)^{-c_2}$$

$$\frac{(q^{-1};q^{-1})_{b_2-b_1}}{(q^{-1};q^{-1})_{d_2-d_1}} \sum_{\substack{0\le p_1\le \min(b_1,c_1)\\ 0\le p_2\le c_2}} q^{d_1(c_2-p_2)} q^{p_1+p_2+p_2(b_1-p_1)} \frac{\prod_{i=1}^{p_1+p_2}(1-q^{-1}zyq^{i-1})}{\prod_{i=1}^{p_1+p_2+b_2-b_1}(yz-q^{1-i})}$$

$$\Bigg(\mathbf{1}_{c_1+c_2-p_1-p_2\le d_2-d_1} \prod_{i=1}^{c_1+c_2-p_1-p_2} \left(y-q^{-\mathtt{M}+1}z^{-1}q^{i-1}\right) \prod_{i=1}^{d_2-d_1-c_1-c_2+p_1+p_2} \left(y-q^{-\mathtt{M}+1}z^{-1}q^{1-i}\right)$$

$$+\mathbf{1}_{c_1+c_2-p_1-p_2>d_2-d_1} \prod_{i=c_1+c_2-p_1-p_2-d_2+d_1+1}^{c_1+c_2-p_1-p_2} \left(y-q^{-\mathtt{M}+1}z^{-1}q^{i-1}\right)\Bigg)$$

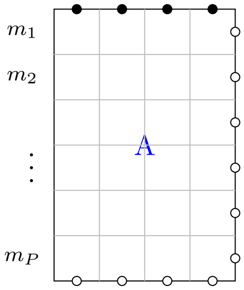
$$\binom{c_1+d_1-p_1}{c_1-p_1}_q \left(\frac{\prod_{i=1}^{c_2-p_2}(q^{-\mathtt{L}}-q^{i-d_2})}{(q;q)_{c_2-p_2}}\right) \binom{b_1}{p_1}_q \left(\frac{\prod_{i=1}^{p_2}(q^{-\mathtt{L}}-q^{i-b_2-p_2})}{(q;q)_{p_2}}\right) \tag{5.20}$$

## 6. Limits

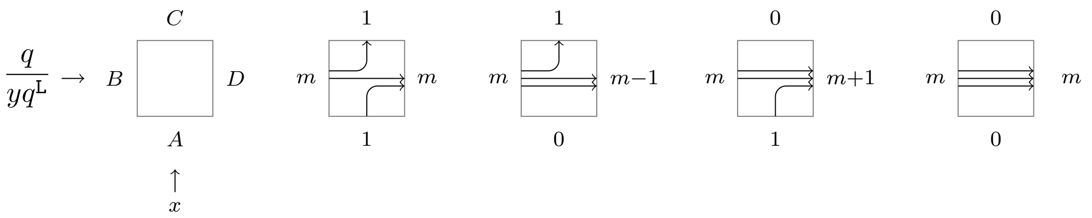
In this section, we take certain limits of eq. (5.18) to obtain a product formula for the functions of interest. We begin by deriving a relation between $\mathcal{F}$ and $\mathcal{G}$.

6.1. **Limits corresponding to $\mathcal{F}$.** For convenience, let us recall the definition of $\mathcal{F}$ from the previous section. Recall that:

$$\mathcal{F}_m(x_1,\ldots,x_n;y_1,\ldots,y_P;q^{-\mathtt{L}_1},\ldots,q^{-\mathtt{L}_P}) = \tag{6.1}$$

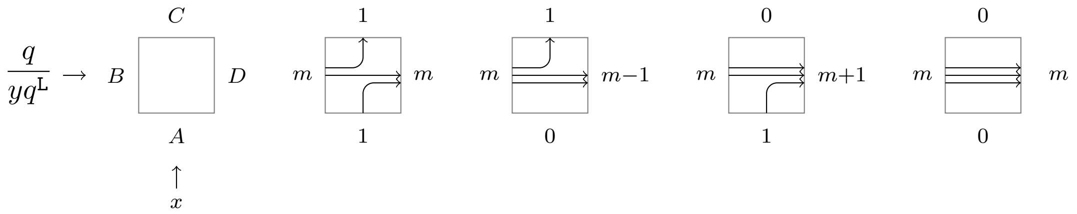


with weights:



$$\frac{1-q^m xy}{(x-y^{-1}q^{-\mathtt{L}})} \qquad \frac{1-q^m}{(x-y^{-1}q^{-\mathtt{L}})} \qquad \frac{(1-q^{m-\mathtt{L}})x}{x-y^{-1}q^{-\mathtt{L}}} \qquad \frac{x-q^m y^{-1}q^{-\mathtt{L}}}{x-y^{-1}q^{-\mathtt{L}}} \tag{6.2}$$

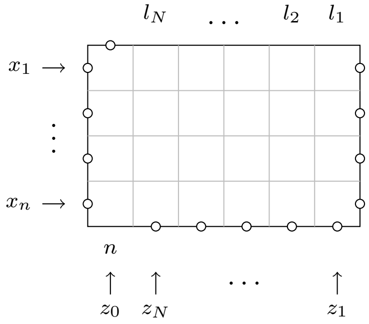
To obtain the functions definition 2.4, we need to take the following limits: $q^{-\mathtt{L}_i} \mapsto sa_i$ and $y_i \mapsto a_i$. Let us reproduce the weights:



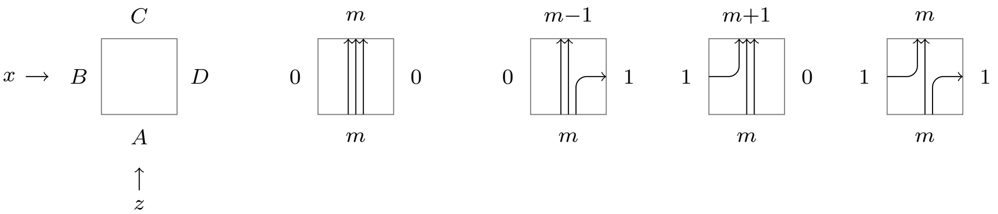
$$\frac{1-q^m xa}{x-s} \qquad \frac{1-q^m}{x-s} \qquad \frac{(1-q^m as)x}{x-s} \qquad \frac{x-q^m s}{x-s} \tag{6.3}$$

6.2. **Limits corresponding to $\mathcal{G}$.** As a first step, we take $z_0 = 0$ in the definition of $\mathcal{G}$.

$$\mathcal{G}_l(x_1,\dots,x_n;z_0,\dots,z_N;q^{-\mathtt{M}_0},\dots,q^{-\mathtt{M}_N}) = \tag{6.4}$$



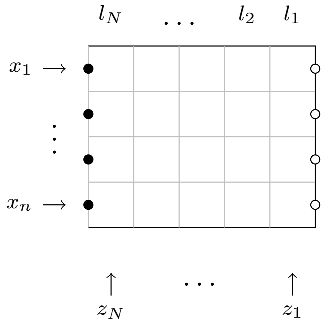
with weights:



$$\frac{(1-q^m xz^{-1}q^{-\mathtt{M}})}{x-z} \qquad \frac{(1-q^m)x}{x-z} \qquad \frac{(1-q^{m-\mathtt{M}})}{x-z} \qquad \frac{x-zq^m}{x-z} \tag{6.5}$$

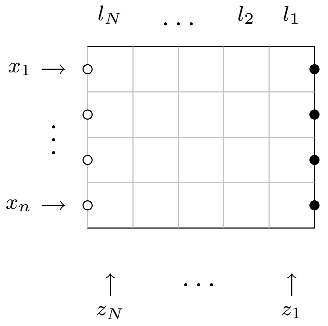
Observe that $n$ particles enter through the first column and none exit through it. As there are $n$ rows, this enforces that a particle turns right in every row of the first column. Therefore, as we set $z_0 = 0$, the overall weight of the first column is $(q;q)_n$.

$$\mathcal{G}_l(x_1,\dots,x_n;0,z_1,\dots,z_N;q^{-\mathtt{M}_0},\dots,q^{-\mathtt{M}_N}) = (q;q)_n \quad \times \tag{6.6}$$

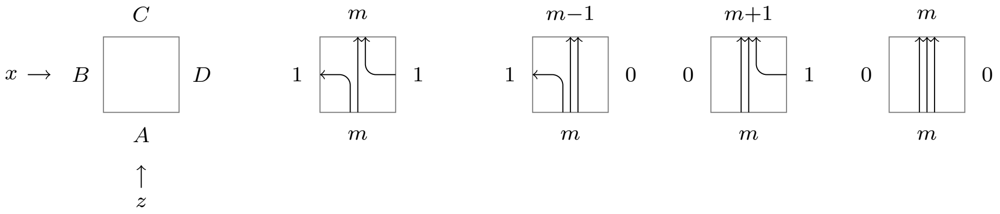


Let us redraw the above lattice by complementing the particles on the horizontal edges. Then

$$\mathcal{G}_l(x_1,\dots,x_n;0,z_1,\dots,z_N;q^{-\mathtt{M}_0},q^{-\mathtt{M}_1},\dots,q^{-\mathtt{M}_N}) = (q;q)_n \quad \times \tag{6.7}$$

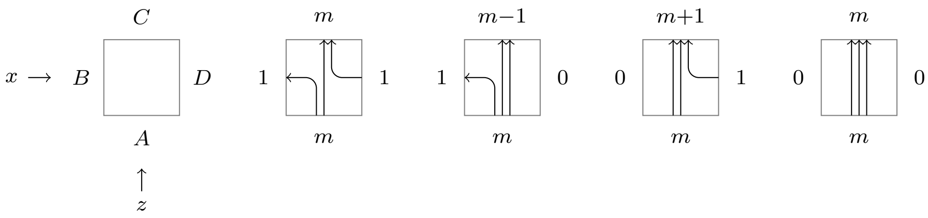


with weights:



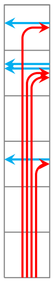
$$\frac{(1-q^m x z^{-1} q^{-\mathtt{M}})}{x-z} \qquad \frac{(1-q^m)x}{x-z} \qquad \frac{(1-q^{m-\mathtt{M}})}{x-z} \qquad \frac{x-zq^m}{x-z} \tag{6.8}$$

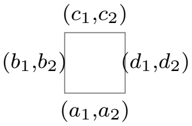
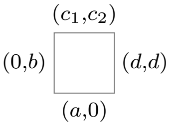
Similar to the earlier case, we take the following limits: $q^{-\mathtt{M}_i} = b_i s$ and $z_i = s$ which we denote as $\mathcal{G}(x_1,\dots,x_n;a_1,\dots,a_P;s)$. Let us reproduce the weights:



$$\frac{(1-q^m xb)}{x-s} \qquad \frac{(1-q^m)x}{x-s} \qquad \frac{(1-q^m bs)}{x-s} \qquad \frac{x-sq^m}{x-s} \tag{6.9}$$

6.3. **$z_0 = 0$ limit for the puzzles.** We argue that setting $z_0 = 0$ in eq. (5.19) results in the first column freezing with weight $(q;q)_n$. To show this, we explicitly compute the weights for the type of vertices in the initial column of the puzzles. We prove that when a blue particle entering from the right turns upwards, the weight of such a vertex has the factor $z_0^{\alpha}$ for some $\alpha > 0$. Subsequently, by setting $z_0 = 0$, the weight of such vertices vanishes. Therefore, the first column freezes, with all blue particles traversing across, see the picture below.



Consider the general vertex . For a vertex appearing in the first column, we have $b_1 = 0$. Using eq. (5.9), we conclude that $d_1 = d_2$ for all the vertices in the first column. For the vertex at the bottom of the first column, we have $a_2 = 0$. We show that the weight of the bottom vertex vanishes as $z_0 = 0$ unless $c_2 = 0$. Using this fact, we can assume that $a_2 = 0$ for every other vertex in the first column. In conclusion, we need to show that the weight of the  vanishes as $z_0 = 0$ unless $c_2 = 0$. To further simplify the computations, we can take the limit $q^{-\mathtt{M}_0} \mapsto 0$. This does not affect any other regions: B and D are normalised so that the weights of the vertices in the first column are 1 and C remains unaffected which can be seen from the discussion in section 6.2. Let us recall the weights in their general form eq. (5.19):

$$\mathcal{W}_{\mathtt{L},\mathtt{M}}\left(\begin{array}{cccc} & & (c_1,c_2) & \\ y^{-1}q^{-\mathtt{L}+1} \to & (b_1,b_2) & \boxed{\phantom{xx}} & (d_1,d_2) \\ & & (a_1,a_2) & \\ & & \uparrow & \\ & & z & \end{array}\right) = \mathbf{1}_{a_1+b_1=c_1+d_1}\mathbf{1}_{a_2+d_2=b_2+c_2}\, q^{d_1+b_2-d_2-b_1}(-1)^{d_2-d_1}(-y)^{-c_2}$$

$$\frac{(q^{-1};q^{-1})_{b_2-b_1}}{(q^{-1};q^{-1})_{d_2-d_1}} \sum_{\substack{0\le p_1\le \min(b_1,c_1)\\ 0\le p_2\le c_2}} q^{d_1(c_2-p_2)} q^{p_1+p_2+p_2(b_1-p_1)} \frac{\prod_{i=1}^{p_1+p_2}(1-q^{-1}zyq^{i-1})}{\prod_{i=1}^{p_1+p_2+b_2-b_1}(yz-q^{1-i})}$$

$$\Bigg(\mathbf{1}_{c_1+c_2-p_1-p_2\le d_2-d_1} \prod_{i=1}^{c_1+c_2-p_1-p_2} \left(y-q^{-\mathtt{M}+1}z^{-1}q^{i-1}\right) \prod_{i=1}^{d_2-d_1-c_1-c_2+p_1+p_2} \left(y-q^{-\mathtt{M}+1}z^{-1}q^{1-i}\right)$$

$$+\mathbf{1}_{c_1+c_2-p_1-p_2>d_2-d_1} \prod_{i=c_1+c_2-p_1-p_2-d_2+d_1+1}^{c_1+c_2-p_1-p_2} \left(y-q^{-\mathtt{M}+1}z^{-1}q^{i-1}\right)\Bigg)$$

$$\binom{c_1+d_1-p_1}{c_1-p_1}_q \left(\frac{\prod_{i=1}^{c_2-p_2}(q^{-\mathtt{L}}-q^{i-d_2})}{(q;q)_{c_2-p_2}}\right) \binom{b_1}{p_1}_q \left(\frac{\prod_{i=1}^{p_2}(q^{-\mathtt{L}}-q^{i-b_2-p_2})}{(q;q)_{p_2}}\right) \tag{6.10}$$

To the weights above, we take $q^{-\mathtt{M}} \mapsto 0$:

$$\mathcal{W}_{\mathtt{L},\mathtt{M}}\left(\begin{array}{cccc} & & (c_1,c_2) & \\ y^{-1}q^{-\mathtt{L}+1} \to & (0,b) & \boxed{\phantom{xx}} & (d,d) \\ & & (a,0) & \\ & & \uparrow & \\ & & z & \end{array}\right) = \mathbf{1}_{a=c_1+d}\mathbf{1}_{d=b+c_2}\, q^{b}(-y)^{-c_2}\frac{(q^{-1};q^{-1})_b}{(q;q)_{c_2}}\binom{c_1+d}{c_1}_q \prod_{i=1}^{c_2}(q^{-\mathtt{L}}-q^{i-d})$$

$$q^{\frac{2c_2-d+d^2}{2}}(-1)^{d+c_2}\sum_{p_2=0}^{c_2}(-1)^{-p_2}q^{\frac{(c_2-p_2)(c_2-p_2-1)}{2}}\frac{\prod_{i=1}^{p_2}(1-q^{-1}zyq^{i-1})}{\prod_{i=1}^{p_2+d-c_2}(1-yzq^{i-1})}\left(\frac{(q;q)_{c_2}}{(q;q)_{c_2-p_2}(q;q)_{p_2}}\right) \tag{6.11}$$

By applying the binomial theorem, we get:

$$= \mathbf{1}_{a=c_1+d}\mathbf{1}_{d=b+c_2} q^b (-y)^{-c_2} \frac{(q^{-1};q^{-1})_b}{(q;q)_{c_2}} \binom{c_1+d}{c_1}_q \prod_{i=1}^{c_2} (q^{-\mathtt{L}} - q^{i-d})$$

$$q^{\frac{2c_2-d+d^2}{2}} (-1)^{d+c_2} \frac{\left(\prod_{i=1}^{c2} q^{i-2} yz\right)}{(yz;q)_d} \frac{(q;q)_d}{(q;q)_{d-c_2}} \tag{6.12}$$

$$= \mathbf{1}_{a=c_1+d}\mathbf{1}_{d=b+c_2} \frac{(q^{-1};q^{-1})_b}{(q;q)_{c_2}} \binom{c_1+d}{c_1}_q \prod_{i=1}^{c_2} (q^{-\mathtt{L}} - q^{i-d})$$

$$q^{\frac{2c_2-d+d^2}{2}+b} (-1)^d z^{c_2} \frac{\left(\prod_{i=1}^{c2} q^{i-2}\right)}{(yz;q)_d} \frac{(q;q)_d}{(q;q)_{d-c_2}} \tag{6.13}$$

Therefore, the weight of the vertices in the first column vanishes as $z_0 = 0$ unless $c_2 = 0$. When $c_2 = 0$, we get that $b = d$ and $c_1 = a - d$ and the weights reduce to:

$$= \mathbf{1}_{a_1=c_1+d}\mathbf{1}_{d=b} \frac{(q;q)_a}{(q;q)_{a-b}} \frac{1}{(yz;q)_b} \tag{6.14}$$

As we set $z_0$ the weight, by telescoping, of the first column is $(q;q)_n$.

6.4. **Relations between $\mathcal{F}$ and $\mathcal{G}$.** In order to get a product rule from eq. (5.18), we need find a relation between $\mathcal{F}(x_1,\dots,x_n;a_1,\dots,a_P;s)$ and $\mathcal{G}(x_1,\dots,x_n;a_1,\dots,a_P;s)$. A relation for general secondary variables is not possible. However, in some special cases, they are equal up to a factor.

6.4.1. *Limit to the spin Hall–Littlewood functions.* After setting $a_i = b_i = t$ in the weights eq. (6.9) and eq. (6.3), we get that

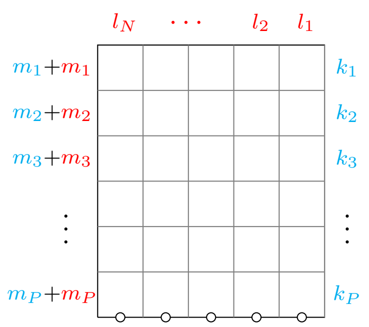
$$\mathrm{F}_m(x_1,\dots,x_n;t,s) = \mathcal{F}_m(x_1,\dots,x_n;t,\dots,t;s) = \prod_{i=1}^{N} \frac{(q;q)_{l_i}}{(ts;q)_{l_i}} \mathcal{G}(x_1,\dots,x_n;t,\dots,t,s) \tag{6.15}$$

We can then perform the same limits in (5.18) to get the product rule for spin Hall–Littlewood functions:

$$\mathrm{F}_m(x_1,\dots,x_n;t,s)\mathrm{F}_l(x_1,\dots,x_n;t,s) = \left(\prod_{i=1}^{N} \frac{(q;q)_{l_i}}{(ts;q)_{l_i}}\right) \sum_k \mathcal{C}^k_{l,m}(t,s)\mathrm{F}_k(x_1,\dots,x_n;t,s) \tag{6.16}$$

where

$$\mathcal{C}^k_{l,m}(t,s) = \tag{6.17}$$



with weights:

$$\mathcal{W}_{\mathtt{L},\mathtt{M}}\left(\begin{array}{ccc} & (c_1,c_2) & \\ (b_1,b_2) & \square & (d_1,d_2) \\ & (a_1,a_2) & \end{array}\right) = \mathbf{1}_{a_1+b_1=c_1+d_1}\mathbf{1}_{a_2+d_2=b_2+c_2}\, q^{d_1+b_2-d_2-b_1}(-t)^{d_2-d_1}(-t)^{-c_2}$$

$$\frac{(q^{-1},q^{-1})_{b_2-b_1}}{(q^{-1},q^{-1})_{d_2-d_1}} \sum_{\substack{0\le p_1\le \min(b_1,c_1)\\ 0\le p_2\le c_2}} q^{d_1(c_2-p_2)}q^{p_1+p_2+p_2(b_1-p_1)}\frac{\prod_{i=1}^{p_1+p_2}(1-q^{-1}tsq^{i-1})}{\prod_{i=1}^{p_1+p_2+b_2-b_1}(ts-q^{1-i})}$$

$$\Bigg(\mathbf{1}_{c_1+c_2-p_1-p_2\le d_2-d_1}\prod_{i=1}^{c_1+c_2-p_1-p_2}\left(1-q^i\right)\prod_{i=1}^{d_2-d_1-c_1-c_2+p_1+p_2}\left(1-qq^{1-i}\right)$$

$$+\mathbf{1}_{c_1+c_2-p_1-p_2>d_2-d_1}\prod_{i=c_1+c_2-p_1-p_2-d_2+d_1+1}^{c_1+c_2-p_1-p_2}\left(1-q^i\right)\Bigg)$$

$$\binom{c_1+d_1-p_1}{c_1-p_1}_q\left(\frac{\prod_{i=1}^{c_2-p_2}(ts-q^{i-d_2})}{(q;q)_{c2-p2}}\right)\binom{b_1}{p_1}_q\left(\frac{\prod_{i=1}^{p_2}(ts-q^{i-b_2-p_2})}{(q;q)_{p_2}}\right) \tag{6.18}$$

6.4.2. *Limit to the factorial Hall–Littlewood polynomials.* After setting $s=0$ in the weights eq. (6.9) and eq. (6.3), we get the following relation:

$$\mathrm{P}_m(x_1,\dots,x_n;a_1,\dots,a_P)=\lim_{s\to 0}\mathcal{F}_m(x_1,\dots,x_n;a_1,\dots,a_P;s)=\prod_{i=1}^{N}(q;q)_{l_i}\lim_{s\to 0}\mathcal{G}(x_1,\dots,x_n;a_1,\dots,a_P;s) \tag{6.19}$$

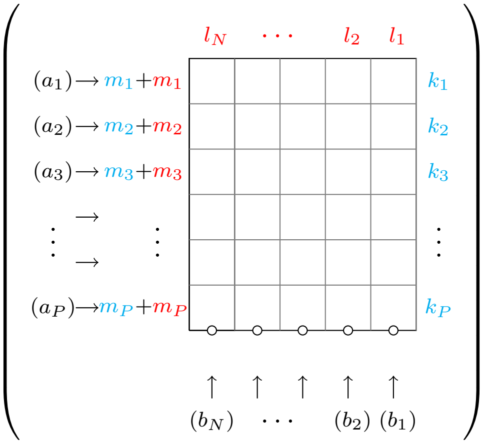
As earlier, by performing the necessary limits, we get the following product rule for factorial Hall–Littlewood polynomials;

$$\mathrm{P}_m(x_1,\dots,x_n;a_1,\dots,a_P)\mathrm{P}_l(x_1,\dots,x_n;b_1,\dots,b_N)$$

$$=\left(\prod_{i=1}^{N}(q;q)_{l_i}\right)\sum_k \mathcal{C}^k_{l,m}(a_1,\dots,a_P;b_1,\dots,b_N)\mathrm{P}_k(x_1,\dots,x_n;a_1,\dots,a_n) \tag{6.20}$$

where

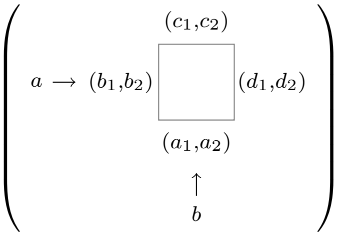
$$\mathcal{C}^{k}_{l,m}(a_1,\dots,a_P;b_1,\dots,b_N) =$$



(6.21)

with weights:

$\mathcal{W}_{\mathtt{L},\mathtt{M}}$



$$= \mathbf{1}_{a_1+b_1=c_1+d_1}\mathbf{1}_{a_2+d_2=b_2+c_2}\, q^{d_1+b_2-d_2-b_1}(-1)^{d_2-d_1}(-a)^{-c_2}$$

$$\frac{(q^{-1},q^{-1})_{b_2-b_1}}{(q^{-1},q^{-1})_{d_2-d_1}} \sum_{\substack{0\le p_1\le \min(b_1,c_1)\\ 0\le p_2\le c_2}} q^{d_1(c_2-p_2)} q^{p_1+p_2+p_2(b_1-p_1)} \frac{1}{\prod_{i=1}^{p_1+p_2+b_2-b_1}(-q^{1-i})}$$

$$\left(\mathbf{1}_{c_1+c_2-p_1-p_2\le d_2-d_1} \prod_{i=1}^{c_1+c_2-p_1-p_2} (a-bq^i) \prod_{i=1}^{d_2-d_1-c_1-c_2+p_1+p_2} (a-bqq^{1-i})\right.$$

$$\left. +\mathbf{1}_{c_1+c_2-p_1-p_2> d_2-d_1} \prod_{i=c_1+c_2-p_1-p_2-d_2+d_1+1}^{c_1+c_2-p_1-p_2} (a-bq^i)\right)$$

$$\binom{c_1+d_1-p_1}{c_1-p_1}_q \left(\frac{\prod_{i=1}^{c_2-p_2}(-q^{i-d_2})}{(q;q)_{c2-p2}}\right) \binom{b_1}{p_1}_q \left(\frac{\prod_{i=1}^{p_2}(-q^{i-b_2-p_2})}{(q;q)_{p_2}}\right) \tag{6.22}$$

## References

[BM16] Gary Bosnjak and Vladimir V Mangazeev. Construction of R-matrices for symmetric tensor representations related to $\mathcal{U}_q(\widehat{\mathfrak{sl}_n})$. *Journal of Physics A: Mathematical and Theoretical*, 49(49):495204, 2016.

[Bor14] Alexei Borodin. On a family of symmetric rational functions. *Advances in Mathematics*, 306, 10 2014.

[GZJ20] Ajeeth Gunna and Paul Zinn-Justin. Vertex models for canonical Grothendieck polynomials and their duals. *Algebraic Combinatorics*, 2020.

[KZJ20] Allen Knutson and Paul Zinn-Justin. Schubert puzzles and integrability I: invariant trilinear forms, 2020.

[Mac98] I.G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford classic texts in the physical sciences. Clarendon Press, 1998.

[WZJ16a] Michael Wheeler and Paul Zinn-Justin. Hall polynomials, inverse Kostka polynomials and puzzles. *J. Comb. Theory, Ser. A*, 159:107–163, 2016.

[WZJ16b] Michael Wheeler and Paul Zinn-Justin. Refined Cauchy/Littlewood identities and six-vertex model partition functions: III. Deformed bosons. *Advances in Mathematics*, 299:543–600, 2016.

[WZJ19] Michael Wheeler and Paul Zinn-Justin. Littlewood-Richardson coefficients for Grothendieck polynomials from integrability. *J. Reine Angew. Math.*, 757:159–195, 2019.

[ZJ08] Paul Zinn-Justin. Littlewood-Richardson coefficients and integrable tilings. *Electron. J. Comb.*, 16, 2008.

[ZJ19] Paul Zinn-Justin. Honeycombs for Hall polynomials. *Electron. J. Comb.*, 27:2, 2019.

Ajeeth Gunna, Michael Wheeler, Paul Zinn-Justin, School of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia.

*Email address*: `agunna@student.unimelb.edu.au,wheelerm@unimelb.edu.au,pzinn@unimelb.edu.au`

# Paper IV: Shuffle algebras, Lattice paths and Macdonald functions

# SHUFFLE ALGEBRAS, LATTICE PATHS AND MACDONALD FUNCTIONS

ALEXANDR GARBALI AND AJEETH GUNNA

Abstract. We consider partition functions on the $N \times N$ square lattice with the local Boltzmann weights given by the $R$-matrix of the $U_t(\widehat{sl}(n+1|m))$ quantum algebra. We identify boundary states such that the square lattice can be viewed on a conic surface. The partition function $Z_N$ on this lattice computes the weighted sum over all possible closed coloured lattice paths with $n+m$ different colours: $n$ "bosonic" colours and $m$ "fermionic" colours. Each bosonic (fermionic) path of colour $i$ contributes a factor of $z_i$ ($w_i$) to the weight of the configuration. We show the following:

i) $Z_N$ is a symmetric function in the spectral parameters $x_1 \dots x_N$ and generates basis elements of the commutative trigonometric Feigin–Odesskii shuffle algebra. The generating function of $Z_N$ admits a shuffle-exponential formula analogous to the Macdonald Cauchy kernel.

ii) $Z_N$ is a symmetric function in two alphabets $(z_1 \dots z_n)$ and $(w_1 \dots w_m)$. When $x_1 \dots x_N$ are set to be equal to the box content of a skew Young diagram $\mu/\nu$ with $N$ boxes the partition function $Z_N$ reproduces the skew Macdonald function $P_{\mu/\nu}[w-z]$.

## 1. Introduction

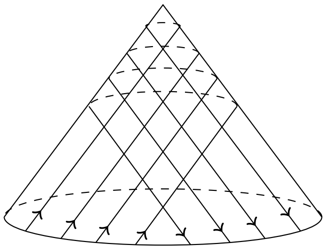
Lattice partition functions with Yang–Baxter (YB) integrable Boltzmann weights have been widely used in a variety of problems in integrability. These include diagonalization and correlation functions in XXZ type Hamiltonians [3, 28, 41, 39, 29, 42, 26, 31], applications to symmetric functions [12, 22, 43, 25, 8, 11, 35, 23, 10, 9], integrable probability [13, 6, 30, 7, 1] and other areas of mathematical physics. More recently it was shown that the Feigin–Odesskii shuffle algebra $\mathcal{A}^{\circ}$ can be realized using lattice partition functions associated to the vertex model of $U_t(\widehat{sl}_n)$ [38, 24]. In the present work we find further connections between this shuffle algebra and lattice partition functions.

Following [24] we consider domain-wall boundary partition functions of vertex models with a special relation between the spectral parameters. These partition functions are symmetric rational functions in the spectral parameters which satisfy the wheel conditions of the shuffle algebra $\mathcal{A}^{\circ}$, an algebra of symmetric functions with a special product called the shuffle product. Therefore the domain-wall partition functions represent certain elements of $\mathcal{A}^{\circ}$. We use them as building blocks to design lattice partition functions which realize shuffle products of elements of $\mathcal{A}^{\circ}$. As an application of this result we compute vertex model partition functions on the cone with the Boltzmann weights determined by the $R$-matrix of the $U_t(\widehat{sl}(n+1|m))$ algebra. The generating function of these partition functions is given by the mixed Cauchy kernel [15], an object similar to the Cauchy kernel in the Macdonald theory. We then specialize the spectral parameters to particular values and obtain a lattice path formula for the skew Macdonald functions.

In our construction the appearance of the Macdonald symmetric functions as lattice partition functions is quite different in spirit from the existing lattice constructions of symmetric functions which we mentioned above. It is analogous to the computation of the off-shell Bethe vectors (the mixed Cauchy kernel) in integrable models by means of the algebraic Bethe Ansatz and computation of the eigenvectors (Macdonald functions) by demanding that the spectral parameters satisfy Bethe equations. For example, in [16] the mixed Cauchy kernel was used as the off-shell Bethe vector in the diagonalization problem of the integrable model associated to the quantum toroidal $gl_1$ algebra.

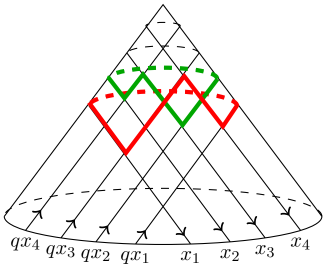
### 1.1. Conic partition function.

Consider a cone pointing upwards and the square lattice drawn on it. More specifically, we draw $N$ directed parallel lines which wrap around the tip of the cone and then self-intersect:



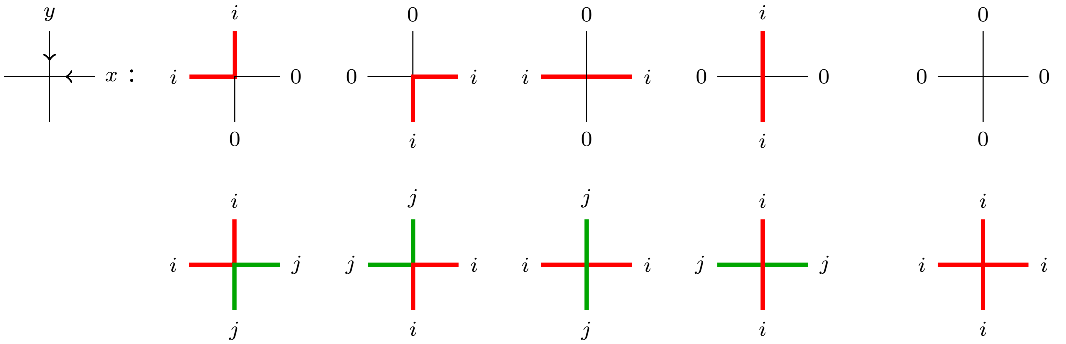
(1.1)

On this square lattice, we draw closed coloured paths where the colours are labelled by $1 \dots n+m$ and the label 0 is associated to the absence of a path. A path is drawn as follows: pick a colour $i$ and an intersection point of any two lines, and start drawing a path of colour $i$ from this intersection point in any of the two directions indicated by the arrows. By repeating the same process, we draw a continuous path. For such a path to be closed, it has to end where it started and for that it has to wrap around the cone. After a closed path is drawn one can continue drawing other coloured paths, ensuring that no two paths share the same edge, although they can occupy the same vertex.



(1.2)

Such configurations $C$ carry a weight $W_C$ which is computed by multiplying the values of all local Boltzmann weights[1]:



$$\frac{1-t}{1-tx/y} \qquad \frac{(1-t)x/y}{1-tx/y} \qquad \frac{t(1-x/y)}{1-tx/y} \qquad \frac{1-x/y}{1-tx/y} \qquad \begin{cases} 1 & i \leq n \\ \dfrac{x/y-t}{1-tx/y} & i > n \end{cases} \tag{1.3}$$

where red "$i$" and green "$j$" can be replaced by any pair of colours such that the condition $i < j$ is preserved. In our conventions "0" is considered as the greatest colour. The vertices in the second row and last column are interpreted as two paths of colour $i \in \{0 \dots n+m\}$ touching each other but not intersecting. The weight of this vertex depends on the colour label $i$ and for this reason we distinguish "bosonic" ($i \leq n$) and "fermionic" paths ($i > n$). The collection of all global configurations on the $N$ by $N$ lattice (1.1) with such bosonic and fermionic paths is denoted by $\Omega_N^{(n,m)}$. In addition to the local

---

[1] These Boltzmann weights correspond to the super-symmetric $R$-matrix of the algebra $U_t(\widehat{sl}(n+1|m))$ which was computed in [4]. Our convention for the weights correspond to [43, 1].

weights (1.3) each global path configuration is multiplied by a factor which accounts for the loop content of the configuration. The partition function is defined by:

$$Z_N = \sum_{C \in \Omega_N^{(n,m)}} z_0^{N-\Lambda(C)} \prod_{i=1}^{n} z_i^{\Lambda_i(C)} \prod_{i=1}^{m} (-w_i)^{\Lambda_{n+i}(C)} \times W_C \tag{1.4}$$

where $\Lambda(C)$ denotes the total number of loops in $C$ and $\Lambda_j(C)$ denotes the total number of loops of colour $j$ in $C$. Therefore the new variables $z_i$ count bosonic loops of colours $i$ and the variables $w_i$ count fermionic loops of colours $n+i$. The factor $z_0$ can be viewed as counting the "empty" loops cycling around the cone. The loop content can be conveniently represented by two non-negative integer compositions $\kappa = (\kappa_1 \dots \kappa_n)$ and $\lambda = (\lambda_1 \dots \lambda_m)$ in which $\kappa_i$ counts the number of loops of colour $i = 1 \dots n$ and $\lambda_i$ counts the number of loops of colour $i = n+1 \dots n+m$. Therefore we can write:

$$Z_N = \sum_{\kappa,\lambda:\ |\kappa|+|\lambda| \leq N} z_0^{N-|\kappa|-|\lambda|} z^{\kappa} w^{\lambda} Z_{N,\kappa,\lambda} \tag{1.5}$$

where $z^{\kappa} = z_1^{\kappa_1} \cdots z_n^{\kappa_n}$, $|\kappa|$ denotes the total number of loops in $\kappa$ and similarly for $\lambda$. Therefore $Z_{N,\kappa,\lambda}$ is the conic partition function with the loop content given by $\kappa$ and $\lambda$. Let us consider an example. Set $N = 4, n = 1, m = 1$ and suppose "red colour = 1" and "green colour = 2", then (1.2) represents a valid configuration and we can compute its contribution to the partition function $Z_4$:

$$z_0^2 z_1 w_1 \times \frac{\left(1-\frac{1}{q}\right)(1-t)^6 t^2 \left(1-\frac{x_2}{qx_3}\right)\left(1-\frac{x_1}{qx_4}\right)\left(1-\frac{x_4}{qx_2}\right)}{q^4 \left(1-\frac{tx_3}{qx_2}\right)\left(1-\frac{tx_4}{qx_1}\right)\left(1-\frac{tx_4}{qx_2}\right)\left(1-\frac{tx_4}{qx_3}\right) \prod_{i\in 3,4} \prod_{j \in 1,2,3} \left(1-\frac{tx_j}{qx_i}\right)}$$

We find that the partition function $Z_N$ can be computed for an arbitrary choice of $n, m$ in terms of the *shuffle product* $*$. For two symmetric functions $F(x_1 \dots x_k)$ and $G(x_1 \dots x_l)$ we have:

$$F(x_1 \dots x_k) * G(x_1 \dots x_l) = \sum_{\substack{S \subseteq [1 \dots k+l] \\ |S|=k}} F(x_S) G(x_{S^c}) \prod_{\substack{i \in S \\ j \in S^c}} \zeta\left(\frac{x_i}{x_j}\right) \tag{1.6}$$

where $S^c$ denotes the complement of the subset $S$ and $|S| = k$ means that the sum runs over subsets of length $k$; the function $\zeta$ is given by:

$$\zeta(x) := \frac{(1-qx)(1-t^{-1}x)}{(1-x)(1-qt^{-1}x)} \tag{1.7}$$

We define the *shuffle-exponential*: $\exp_*(A) := 1 + A + \frac{1}{2!} A * A + \frac{1}{3!} A * A * A + \cdots$.

**Theorem 1.1.** *The generating function:*

$$Z(v) = \sum_{N=0}^{\infty} v^N Z_N \tag{1.8}$$

*is given by:*

$$Z(v) = \exp_*\left(\sum_{k>0} \frac{v^k}{k} \left(\sum_{i=1}^{m} w_i^k - \sum_{i=1}^{n} z_i^k - \frac{q^k - t^k}{1-t^k} z_0^k\right) L_k\right) \tag{1.9}$$

*where $L_k = L_k(x_1 \dots x_k)$ is a two-colour (one bosonic and one fermionic) conic partition function which can be defined by a formula similar to* (1.4) *but with a different loop counting weight:*

$$L_k := \sum_{C \in \Omega_k^{(1,1)}:\ \Lambda(C)=k} (-1)^{\Lambda_2(C)} \Lambda_2(C) \times W_C \tag{1.10}$$

This result implies that $Z_N$ is symmetric in $(z_1 \dots z_n)$ and separately in $(w_1 \dots w_m)$ since the dependence on $z$'s and $w$'s in the exponent in (1.9) is given by the power sum symmetric functions $p_k(z)$ and $p_k(w)$. The dependence of $Z_N$ on the spectral parameters $(x_1 \dots x_N)$ enters through the shuffle product of the functions $L_k(x_1 \dots x_k)$ in the view of the definition of $\exp_*$. Thus (1.9) allows us to express the partition function $Z_N$ (which is associated to the vertex model of $U_t(\widehat{sl}(n+1|m))$ in terms of the partition functions $L_k$ (which is associated to the vertex model of $U_t(\widehat{sl}(1+1|1))$. The function $L_k$ in turn can be decomposed further:

$$L_k = \sum_{j=0}^{k} (-1)^j j Z^b_{k-j} * Z^f_j \tag{1.11}$$

where $Z^b_k$ and $Z^f_k$ are the six vertex bosonic and fermionic domain-wall partition functions. The functions $L_k$ admit an explicit symmetrization formula:

$$L_k = \frac{-(1-q)^k(1-t)^k}{(1-q^k)(q-t)^k} \mathrm{Sym} \left( \frac{\sum_{j=0}^{k-1} (q/t)^j x_{j+1}/x_1}{\prod_{j=1}^{k-1} (1-(q/t)x_{j+1}/x_j)} \prod_{1 \le i < j \le k} \zeta(x_i/x_j) \right) \tag{1.12}$$

where Sym is defined in (3.5). This symmetrization formula appeared in [36] as an explicit expression for a family of elements of $\mathcal{A}^\circ$ which are in some sense analogous to the power sums symmetric functions. Because of this connection $Z(v)$ in (1.9) can be viewed as a generalized mixed Cauchy kernel (see [15]).

1.2. **Skew Macdonald functions.** The algebra $\mathcal{A}$ is well studied and has several representations [14, 40, 19, 36]. Consider a skew Young diagram of $\mu/\nu$ (in the English convention) with $k$ boxes $\square \in \mu/\nu$ which are labelled by $1 \dots k$ in the reading order. We identify $\square = (a, b)$ where $a$ is the column index and $b$ is the row index of the box $\square \in \mu/\nu$. Let $\chi_\square = q^{a-1}t^{1-b}$ be the content of the box $\square \in \mu/\nu$ and denote by $\chi_1 \dots \chi_k$ the contents of all $k$ boxes. The algebra $\mathcal{A}$ has a matrix representation $f \mapsto M$ in which an element $f(x_1 \dots x_k)$ is mapped to a matrix $M$ whose non-zero matrix elements $M_{\mu,\nu}$ are those for which $\mu/\nu$ represents a skew Young diagram with $k$ boxes and:

$$M_{\mu,\nu} = f(\chi_1 \dots \chi_k)$$

This representation of the shuffle algebra is intimately related to the theory of Macdonald functions $P_\lambda$. Using this representation we obtain the following connection between $Z_N$ and Macdonald functions.

**Theorem 1.2.** *Let $\mu/\nu$ be a skew Young diagram with $N$ boxes and $Z_N(x_1 \dots x_N; z, w)$ be the partition function defined by* (1.4) *then:*

$$Z_N(\chi_1 \dots \chi_N; z, w) = a^{-1}_{\mu,\nu} P_{\mu/\nu} \left[ w - z - \frac{q-t}{1-t} z_0 \right] \tag{1.13}$$

*where we used the plethystic notation* (6.10) *and $a_{\mu,\nu}$ is a combinatorial coefficient given in* (6.13). *When $z_i = 0$ the above formula implies:*

$$P_{\mu/\nu}(w_1 \dots w_m) = a_{\mu,\nu} Z_N(\chi_1 \dots \chi_N; 0, w) \tag{1.14}$$

This theorem connects the lattice partition function on the cone $Z_N$ and the skew Macdonald functions. We may ask for a conic partition function with a fixed colour content of the paths. This amounts to asking for the coefficient of a specific monomial in $z$'s and $w$'s in (1.4). Set $z_i = 0$ and for a non-negative integer composition $\lambda = (\lambda_1 \dots \lambda_m)$ denote $Z_\lambda := Z_{N,\varnothing,\lambda}$. We obtain a lattice partition function representation of the monomial expansion of the skew Macdonald functions:

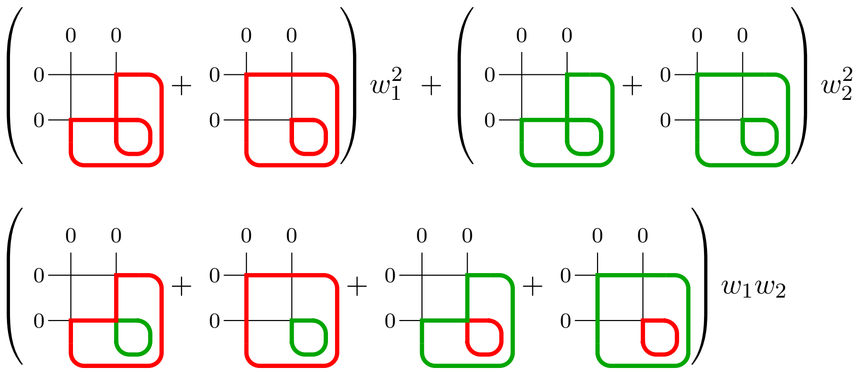
$$P_{\mu/\nu}(w_1 \dots w_m) = a_{\mu,\nu} \sum_{\lambda:\ |\lambda|=N} w^\lambda Z_\lambda(\chi_1 \dots \chi_N) \tag{1.15}$$

Let us consider an example (see Section 6.3). We compute $P_{(2,1)/(1)}(w_1, w_2)$ with the above formula:

$$a^{-1}_{(2,1),(1)}P_{(2,1)/(1)}(w_1, w_2) = Z_{(2,0)}(q, t^{-1})w_1^2 + Z_{(0,2)}(q, t^{-1})w_2^2 + Z_{(1,1)}(q, t^{-1})w_1w_2$$

Each of the monomial coefficients corresponds to specific loop configurations. We represent them using planar diagrams which are equivalent to the diagrams on the cone[2]:

$$a^{-1}_{(2,1),(1)}P_{(2,1)/(1)}(w_1, w_2) =$$



Using the Boltzmann weights (1.3), with $n = 0$ and the appropriate choice of the spectral parameters, and the coefficient $a_{(2,1),(1)}$, given in (6.16), we recover the skew Macdonald function $P_{(2,1)/(1)}(w_1, w_2)$:

$$P_{(2,1)/(1)}(w_1, w_2) = w_1^2 + \frac{(1-t)(2+q+t+2qt)}{1-qt^2}w_1w_2 + w_2^2$$

1.3. **Overview of the paper.** In Section 2 we give the background on the theory of symmetric functions. In Section 3 we review some aspects of the trigonometric Feigin–Odesskii shuffle algebra. In Section 4 we compute the conic partition for the six vertex model which is based on the $R$-matrix of $U_t(\widehat{sl}(2))$. In Section 5 we extend the results of Section 4 to the case of $U_t(\widehat{sl}(n+1|m))$. In Section 6 we apply our results to the problem of computing the skew Macdonald functions.

## 2. Background on symmetric functions

In this section we recall some basic facts from the theory of symmetric functions which will be required in the rest of the paper. All of the background material in this section can be found in [32].

2.1. **Partitions.** Let $\lambda = (\lambda_1, \lambda_2, \dots)$, s.t. $\lambda_i \geq \lambda_{i+1}$ for all $i$, be an integer partition of $N$ and write $\lambda \vdash N$. The length of $\lambda$ is equal to the number of non-zero parts of $\lambda$ and is denoted by $\ell(\lambda)$. The sum of all parts $\lambda_i$ of $\lambda \vdash N$ is denoted by $|\lambda|$ and is equal to $N$. The multiplicity vector $m(\lambda) = (m_1(\lambda), m_2(\lambda)\dots)$ is composed of integers $m_k(\lambda)$ which count how many parts of $\lambda$ are equal to $k$. In the context of integer sequences the notation $k^l$ means the sequence $k \dots k$ which has $l$ repeats of $k$. We can write $\lambda = (\lambda_1^{m_{\lambda_1}} \dots 2^{m_2}, 1^{m_1})$, where $m_k = m_k(\lambda)$. Integer partitions can be partially ordered using the dominance ordering: $\lambda \geq \mu$ when $\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$ for all $k > 0$.

A partition $\lambda$ is identified with the Young diagram where rows of boxes are placed horizontally and are non-increasing from top to bottom. By $\lambda'$ we denote the dual partition to the partition $\lambda$ which corresponds to the Young diagram which has rows and columns exchanged compared to $\lambda$. A box of a Young diagram is denoted by $\square$ and is identified with its coordinate $\square = (i, j)$, where the row index $i$ increases downwards and column index $j$ increases rightwards. The arm and leg functions $a_\lambda(\square)$, $l_\lambda(\square)$ are defined by

$$a_\lambda(\square) = \lambda_i - j, \quad l_\lambda(\square) = \lambda'_j - i$$

The summations or products over $\square \in \lambda$ mean that $\square$ runs over all the boxes in the Young diagram of the partition $\lambda$. For two partitions $\lambda$ and $\mu$ such the $\lambda_i \geq \mu_i$, for all $i$, we define the skew partition $\lambda/\mu$ and the corresponding Young diagram.

---

[2] If we rotate the planar pictures by 135 degrees counterclockwise we can place them on the lattice drawn on the cone (1.1).

2.2. **Basic symmetric functions.** Let $q, t$ be two formal variables and consider the ring $\Lambda$ of functions which are symmetric in the alphabet $(x) = (x_1 \dots x_k)$, for a fixed $k \geq 0$, with coefficients in $\mathbb{F} := \mathbb{Q}(q,t)$. A basic family of symmetric functions in this ring is the set of monomial symmetric functions $m_\lambda(x)$ which are labelled by integer partitions $\lambda$:

$$m_\lambda = \sum_\alpha x^\alpha \tag{2.1}$$

where the sum runs over all distinct permutations $\alpha$ of $\lambda$, $x^\alpha := \prod_{i=1}^{\ell(\lambda)} x_i^{\alpha_i}$. As in (2.1) we will often drop the explicit dependence on the alphabet. Another important family is the power sums symmetric functions:

$$p_r = \sum_{i=1}^{k} x_i^r, \qquad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}} \tag{2.2}$$

The two families $m_\lambda$ and $p_\lambda$ form bases in the ring $\Lambda$. Using the power sums we can generate new families of symmetric functions via exponential generating functions of the form:

$$\exp\left(\sum_{r>0} c_r v^r p_r\right)$$

for some choices of coefficients $c_r$. For our purposes we will need to consider three such families of functions $e_j$, $g_j$ and $g_j^*$, with $j = 0, 1, 2 \dots$:

$$\sum_{j=0}^{\infty} v^j e_j = \exp\left(\sum_{r>0} \frac{(-1)^{r+1}}{r} v^r p_r\right) \tag{2.3}$$

$$\sum_{j=0}^{\infty} v^j g_j = \exp\left(\sum_{r>0} \frac{1}{r}\frac{1-t^r}{1-q^r} v^r p_r\right) \tag{2.4}$$

$$\sum_{j=0}^{\infty} v^j g_j^* = \exp\left(-\sum_{r>0} \frac{1}{r}\frac{1-t^r}{1-q^r} v^r p_r\right) \tag{2.5}$$

These formulas define the elementary symmetric functions $e_j$ and two symmetric functions $g_j$ and $g_j^*$ (see [32, Ch.VI, §2]). By expanding the exponentials we can write explicit expansions of $e_j, g_j$ and $g_j^*$ in the power sums basis:

$$e_j = \sum_{\lambda \vdash j} \frac{(-1)^{j+\ell(\lambda)}}{\lambda!} \prod_{r \in \lambda} \frac{1}{r} \cdot p_\lambda, \qquad g_j = \sum_{\lambda \vdash j} \frac{1}{\lambda!} \prod_{r\in\lambda} \frac{1}{r}\frac{1-t^r}{1-q^r} \cdot p_\lambda, \qquad g_j^* = \sum_{\lambda \vdash j} \frac{(-1)^{\ell(\lambda)}}{\lambda!} \prod_{r\in\lambda} \frac{1}{r}\frac{1-t^r}{1-q^r} \cdot p_\lambda$$

where $\lambda! := m_1(\lambda)! m_2(\lambda)! \cdots$ and the products over $r \in \lambda$ run over the parts of $\lambda$. The symmetric functions $e_j, g_j$ and $g_j^*$ can be used to write new bases:

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_{\ell(\lambda)}}, \qquad g_\lambda = g_{\lambda_1} \cdots g_{\lambda_{\ell(\lambda)}}, \qquad g_\lambda^* = g_{\lambda_1}^* \cdots g_{\lambda_{\ell(\lambda)}}^* \tag{2.6}$$

2.3. **Macdonald functions.** Define the scalar product using the power sums basis:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \lambda! \prod_{r \in \lambda} r \frac{1-q^r}{1-t^r} \cdot \delta_{\lambda,\mu} \tag{2.7}$$

Recall [32, Ch.VI, §4] that the Macdonald functions $P_\lambda$ are the unique symmetric functions in $\Lambda$ that satisfy the following conditions:

$$P_\lambda(x;q,t) = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu, \qquad \langle P_\mu, P_\nu \rangle = 0, \qquad \text{for } \mu \neq \nu$$

where $c_{\lambda,\mu} \in \mathbb{F}$ are some coefficients and $\mu < \lambda$ in the sense of the dominance partial ordering. The Macdonald functions are self-dual w.r.t. the scalar product (2.7) up to a constant:

$$\langle P_\mu, Q_\lambda \rangle = \delta_{\mu,\lambda}, \qquad Q_\lambda := b_\lambda P_\lambda, \tag{2.8}$$

where the coefficient $b_\lambda \in \mathbb{F}$ is defined in terms two other coefficients $c_\lambda$ and $c'_\lambda$:

$$b_\lambda := \frac{c_\lambda}{c'_\lambda}, \qquad c_\lambda := \prod_{\Box\in\lambda}(1 - q^{a_\lambda(\Box)} t^{l_\lambda(\Box)+1}), \qquad c'_\lambda := \prod_{\Box\in\lambda}(1 - q^{a_\lambda(\Box)+1} t^{l_\lambda(\Box)}) \tag{2.9}$$

Consider $P_\lambda$ with the Young diagram of $\lambda$ given by a single column, in this case it is known that:

$$P_{(1^k)} = e_k \tag{2.10}$$

We will require the following *Pieri formula* [32, Ch.VI, §6] for Macdonald functions:

$$e_j P_\mu = \sum_\lambda \psi'_{\lambda/\mu} P_\lambda \tag{2.11}$$

where the summation runs over $\lambda$ such that the skew partitions $\lambda/\mu$ are all vertical strips with $j$ boxes (i.e. $\lambda/\mu$ does not contain more than one box in a single row). The expansion coefficients $\psi'_{\lambda/\mu}$ are defined by:

$$\psi'_{\lambda/\mu} := \prod_{i,j} \frac{(1 - q^{\mu_i-\mu_j} t^{j-i-1})(1 - q^{\lambda_i-\lambda_j} t^{j-i+1})}{(1 - q^{\mu_i-\mu_j} t^{j-i})(1 - q^{\lambda_i-\lambda_j} t^{j-i})} \tag{2.12}$$

where the product is taken over all $i, j$ such that $i < j$ and $\lambda_i = \mu_i$, $\lambda_j = \mu_j + 1$.

Next we recall [32, Ch.VI, §7] the Littlewood–Richardson coefficients $f^\mu_{\lambda,\nu} = f^\mu_{\lambda,\nu}(q,t)$ and the skew Macdonald functions $P_{\mu/\nu}$. The Littlewood–Richardson coefficients are the expansion coefficients of $P_\lambda P_\nu$ in the Macdonald basis:

$$P_\lambda P_\nu = \sum_\mu f^\mu_{\lambda,\nu} P_\mu \tag{2.13}$$

With these coefficients we can define the skew Macdonald functions which are symmetric functions labelled by skew partitions:

$$P_{\mu/\nu} = \sum_\lambda \frac{b_\lambda b_\nu}{b_\mu} f^\mu_{\lambda,\nu} P_\lambda \tag{2.14}$$

2.4. **Cauchy kernel.** Consider two alphabets $(x) = (x_1 \dots x_k)$ and $(y) = (y_1 \dots y_l)$ and define the Macdonald *Cauchy kernel* [32, Ch.VI, §2]:

$$\Pi(x,y) := \exp\left(\sum_{r>0} \frac{1}{r}\frac{1-t^r}{1-q^r} p_r(x) p_r(y)\right) \tag{2.15}$$

For any pair of bases $u_\lambda$ and $u^*_\lambda$ which are dual w.r.t. the scalar product (2.7), i.e. $\langle u_\lambda, u^*_\mu \rangle_{q,t} = \delta_{\lambda,\mu}$, we have:

$$\Pi(x,y) = \sum_\lambda u_\lambda(x) u^*_\lambda(y) \tag{2.16}$$

An important role in our paper is played by two pairs of dual bases. One pair is given by $m_\lambda$ and $g_\lambda$ and the other one by the Macdonald functions $P_\lambda$ and $Q_\lambda$:

$$\Pi(x,y) = \sum_\lambda m_\lambda(x) g_\lambda(y) = \sum_\lambda P_\lambda(x) Q_\lambda(y) \tag{2.17}$$

## 3. The shuffle algebra

In this section, we introduce the trigonometric Feigin–Odesskii shuffle algebra [40, 19, 14, 36] which we denote by $\mathcal{A}$. This is an algebra of symmetric rational functions with a multiplication which is non-commutative in general. The algebra $\mathcal{A}$ contains a commutative subalgebra [14], which we denote by $\mathcal{A}^\circ$. We will focus on this commutative subalgebra. We present several examples of families of elements of $\mathcal{A}^\circ$ which were considered in the literature in various contexts. Among these examples we have the elements (completely factorized products) which were used in [14] for computations related to the commutative algebra $\mathcal{A}^\circ$ and the elements (Izergin-type determinants) which are related to domain-wall partition functions as discussed in Sections 4 and 5. In this section we recall a particular representation of the algebra $\mathcal{A}$ [40, 19, 14] which gives rise to an isomorphism between $\mathcal{A}^\circ$ and the ring of symmetric functions $\Lambda$. This isomorphism is a key tool which will help us to relate the conic partition function $Z_N$ (1.4) with the skew Macdonald functions in Section 6.

In this section it will be convenient to use three parameters which are related to $q$ and $t$:

$$q_1 = q, \quad q_2 = t^{-1}, \quad q_3 = tq^{-1} \tag{3.1}$$

### 3.1. Definition of the shuffle algebra $\mathcal{A}$.

The shuffle algebra $\mathcal{A}$ is a vector space whose elements are symmetric rational functions. Their properties are determined by the function:

$$\zeta(x) := \frac{(1-qx)(1-t^{-1}x)}{(1-x)(1-qt^{-1}x)} \tag{3.2}$$

**Definition 3.1.** *Consider the vector space of symmetric rational functions:*

$$\mathcal{V} = \bigoplus_{k\geq 0} \mathbb{F}(x_1 \dots x_k)^{\mathcal{S}_k} \tag{3.3}$$

*Endow $\mathcal{V}$ with an algebra structure given by the shuffle product $*$. For $F(x_1 \dots x_k) \in \mathcal{V}$ and $G(x_1 \dots x_l) \in \mathcal{V}$ we have:*

$$F(x_1 \dots x_k) * G(x_1 \dots x_l) = \frac{1}{k!l!} \mathrm{Sym}\ F(x_1 \dots x_k) G(x_{k+1} \dots x_{k+l}) \prod_{\substack{i \in 1,\dots k \\ j \in k+1,\dots k+l}} \zeta\left(\frac{x_i}{x_j}\right) \tag{3.4}$$

*where:*

$$\mathrm{Sym}\ P(x_1 \dots x_k) = \sum_{\sigma \in \mathcal{S}_k} P(x_{\sigma(1)} \dots x_{\sigma(k)}) \tag{3.5}$$

*The shuffle algebra $\mathcal{A} \subset \mathcal{V}$ is defined as the set of rational functions of the form:*

$$F(x_1 \dots x_k) = \frac{f(x_1 \dots x_k)}{\prod_{1 \leq i \neq j \leq k} (x_i - qt^{-1} x_j)}, \qquad f(x_1 \dots x_k) \in \mathbb{F}[x_1^{\pm 1} \dots x_k^{\pm 1}]^{\mathcal{S}_k} \tag{3.6}$$

*where $f(x_1 \dots x_k)$ satisfies the wheel conditions:*

$$f(x_1 \dots x_k) = 0 \quad \textit{if} \quad (x_i, x_j, x_k) = (x, qt^{-1}x, t^{-1}x) \quad \textit{or} \quad (x_i, x_j, x_k) = (x, qt^{-1}x, qx) \tag{3.7}$$

The shuffle product (3.4) is such that the product $F * G$ satisfies the wheel conditions if $F$ and $G$ do. Note that (3.4) can also be written using a sum over subsets:

$$F(x_1 \dots x_k) * G(x_1 \dots x_l) = \sum_{\substack{S \subseteq [1 \dots k+l] \\ |S| = k}} F(x_S) G(x_{S^c}) \prod_{\substack{i \in S \\ j \in S^c}} \zeta\left(\frac{x_i}{x_j}\right) \tag{3.8}$$

where the condition $|S| = k$ fixes the size of the subsets, $S^c \subseteq [1 \dots k+l]$ refers to the subset complement to $S$ and its size must be equal to $l$. The algebra $\mathcal{A}$ is graded by the number of arguments $\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}_k$,

$F(x_1 \dots x_k) \in \mathcal{A}_k$. Consider a subalgebra $\mathcal{A}^\circ \subset \mathcal{A}$ of the elements $F \in \mathcal{A}_k$ for which the two limits:

$$\lim_{\xi \to 0} F(\xi x_1 \dots \xi x_r, x_{r+1} \dots x_k) \tag{3.9}$$

$$\lim_{\xi \to \infty} F(\xi x_1 \dots \xi x_r, x_{r+1} \dots x_k) \tag{3.10}$$

exist and coincide for all fixed $r = 1 \dots k$. This subalgebra splits into components of fixed degree in the same way as $\mathcal{A}$: $\mathcal{A}^\circ = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathcal{A}^\circ_k$. We have the following Proposition due to [14].

**Proposition 3.2.** *The algebra $(\mathcal{A}^\circ, *)$ is commutative and the dimension of the graded subspace $\mathcal{A}^\circ_k$ is equal to $p(k)$, the number of partitions of $k$.*

3.2. **Basic elements of $\mathcal{A}^\circ$.** Let us define the elements $S_k$ of $\mathcal{A}^\circ$ which play the same role as the power sums in the ring of symmetric functions. These functions were introduced and studied in [36].

**Definition 3.3.** *Define $S_k = S_k(x_1 \dots x_k) \in \mathcal{A}^\circ_k$:*

$$S_k := \frac{(1-q)^k(1-t^{-1})^k}{(t-q)^k(1-t^{-k})} \mathrm{Sym} \left( \frac{\sum_{j=0}^{k-1} (q/t)^j x_{j+1}/x_1}{\prod_{j=1}^{k-1} (1-(q/t) x_{j+1}/x_j)} \prod_{1 \leq i < j \leq k} \zeta(x_i/x_j) \right), \qquad k = 0, 1, 2, \dots \tag{3.11}$$

**Definition 3.4.** *Define $E_k(q_a) = E_k(q_a; x_1 \dots x_k) \in \mathcal{A}^\circ_k$:*

$$E_k(q_a) := \prod_{1 \leq i < j \leq k} \frac{(x_i - q_a x_j)(x_i - q_a^{-1} x_j)}{(x_i - q t^{-1} x_j)(x_i - t q^{-1} x_j)}, \qquad a = 1, 2, 3 \tag{3.12}$$

We note that $E_k(t/q) = 1$ but as an element in $\mathcal{A}^\circ_k$ it has to be viewed as a function of $k$ arguments $x_1 \dots x_k$. The factorized elements $E_k(q_a)$ and their elliptic generalizations were proposed in [18]. Since $E_k(q_a)$ are completely factorized they are very useful for computational purposes (see [14]). In particular, it is easy to check that $E_k(q_a) \in \mathcal{A}_k$. Such factorized elements also play important roles in other commutative shuffle algebras [20, 17].

**Definition 3.5.** *Let $(a, b, c)$ be a permutation of $(1, 2, 3)$, define $H_k(q_a) = H_k(q_a; x_1 \dots x_k) \in \mathcal{A}^\circ_k$:*

$$H_k(q_a) := (q_a q t^{-1})^{k(k-1)/2} \frac{\prod_{1 \leq i,j \leq k} (x_i - q_b x_j)(x_j - q_c x_i)}{\prod_{1 \leq i \neq j \leq k} (x_i - x_j)(x_i - q t^{-1} x_j)} \det_{1 \leq i,j \leq k} \frac{1}{(x_i - q_b x_j)(x_j - q_c x_i)} \tag{3.13}$$

The elements $H_k(q_a)$ where discussed in [24] in connection with coloured lattice models. From Definition 3.5 one can see that these element are members of $\mathcal{A}_k$ by expanding the determinant and applying the wheel conditions (3.7). Alternatively one can express $H_k(q_a)$ in terms of $E_k$ as in (3.18) below. All families of functions listed in Definitions 3.3-3.5 are members of $\mathcal{A}^\circ$ and thus they mutually commute with respect to the shuffle product. Therefore they can be shuffle multiplied and ordered and thus give bases in $\mathcal{A}^\circ$. For $\lambda \vdash n$, $a \in \{1, 2, 3\}$, we have:

$$v_\lambda = v_{\lambda_1} * v_{\lambda_2} * \dots * v_{\lambda_{\ell(\lambda)}}, \qquad v = S, E(q_a), H(q_a)$$

In order to exponentially generate elements of $\mathcal{A}^\circ$ we introduce the shuffle version of the exponential function:

$$\exp_*(A) := 1 + A + \frac{1}{2!} A * A + \frac{1}{3!} A * A * A + \dots \tag{3.14}$$

Define the generating functions:

$$E(v; q_a) := \sum_{k=0}^{\infty} v^k E_k(q_a), \qquad H(v; q_a) := \sum_{k=0}^{\infty} v^k H_k(q_a) \tag{3.15}$$

**Lemma 3.6.** *Let $a \in \{1,2,3\}$. The generating functions $E_k(q_a)$ and $H_k(q_a)$ are equal to shuffle-exponentials:*

$$E(v;q_a) = \exp_* \left( \sum_{r>0} \frac{(-1)^{r+1}}{r} \frac{1-q_a^r}{1-q^r} \frac{(t-q)^r}{(1-q_a)^r} v^r S_r \right) \tag{3.16}$$

$$H(v;q_a) = \exp_* \left( \sum_{r>0} \frac{1}{r} \frac{1-q_a^r}{1-q^r} \frac{(t-q)^r}{(1-q_a)^r} v^r S_r \right) \tag{3.17}$$

*Proof.* The generating function $E(v;p)$ was computed and written in the exponential form (3.16) in [37]. The generating function $H(v;q_a)$ follows from the quadratic identity:

$$H_k(q_a) = \sum_{r=0}^{k} q_c^{k-r} \left( \frac{1-q_b}{1-q_b q_c} \right)^{k-r} \left( \frac{1-q_c}{1-q_b q_c} \right)^{r} E_{k-r}(q_b) * E_r(q_c) \tag{3.18}$$

where $(a,b,c)$ is a permutation of $(1,2,3)$. We prove (3.18) in Appendix A. By summing (3.18) over $k$ with $v^k$ we obtain a product of two shuffle-exponentials on the right hand side of the resulting equation. These exponentials combine and produce the r.h.s. of (3.17). $\square$

3.3. **Evaluation representation of the shuffle algebra $\mathcal{A}$.** Recall that $\square = (a,b)$ denotes a box in the Young diagram of $\lambda \vdash k$ located in the $a$-th column and $b$-th row. Define $\chi_\square$ to be the content of the box $\square$:

$$\chi_\square = q^{a-1} t^{1-b}, \qquad \square \in \lambda \tag{3.19}$$

Consider an example of $\lambda = (532)$ and in each box of the Young diagram of $\lambda$ write its content:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & q & q^2 & q^3 & q^4 \\ \hline t^{-1} & qt^{-1} & q^2t^{-1} \\ \cline{1-3} t^{-2} & qt^{-2} \\ \cline{1-2} \end{array} \tag{3.20}$$

**Definition 3.7.** *Let $\lambda \vdash k$ be a partition, we say that $\square \in \lambda$ is the $i$-th box of $\lambda$ if $\square$ is located on the $i$-th position in the reading order*[3]*. For a function $f(x_1 \dots x_k)$ we define evaluations $\mathrm{ev}_\lambda$:*

$$\mathrm{ev}_\lambda : \quad f(\dots x_i \dots) \mapsto f(\dots \chi_\square \dots) \tag{3.21}$$

*which means that each $x_i$ is replaced with the content of the $i$-th box of $\lambda$.*

For example, if we need to compute $\mathrm{ev}_\lambda(f(\dots x_i \dots))$ with $\lambda = (532) \vdash 10$ we first assign $x$'s to the boxes as follows:

$$\begin{array}{|c|c|c|c|c|} \hline x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_6 & x_7 & x_8 \\ \cline{1-3} x_9 & x_{10} \\ \cline{1-2} \end{array} \tag{3.22}$$

and then substitute for $x$'s the values of the contents of their boxes. We would like to apply $\mathrm{ev}_\lambda$ to symmetric functions in $\mathcal{A}$, however, due to the poles at $x_i = qt^{-1}x_j$ in (3.6) we need to take extra care. This can be realized using an intermediate step:

$$\mathrm{ev}_\lambda(f(x_1 \dots x_k)) = \mathrm{ev}^y \left( \mathrm{ev}_\lambda^x (f(x_1 \dots x_k)) \right), \qquad \lambda \vdash k \tag{3.23}$$

---

[3] Since we will be working with symmetric functions $f(x_1 \dots x_k)$ the order in which we associate $x_i$ with a particular $\square$ in $\lambda$ does not matter but it is convenient to fix it.

where

$$\mathrm{ev}^x_\lambda(f(x_1\dots x_k)) = f(y_1, qy_1\dots q^{\lambda_1-1}y_1, y_2, qy_2\dots q^{\lambda_2-1}y_2\dots y_{\ell(\lambda)}, qy_{\ell(\lambda)}\dots q^{\lambda_{\ell(\lambda)}-1}y_{\ell(\lambda)}) \tag{3.24}$$

$$\mathrm{ev}^y(g(y_1\dots y_j)) = g(y, t^{-1}y\dots t^{j-1}y) \tag{3.25}$$

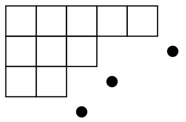
Note that for $F \in \mathcal{A}_k$, $\mathrm{ev}^x_\lambda(F(x_1\dots x_k))$ will produce a function of $y_1\dots y_{\ell(\lambda)}$ which has no poles at $y_i = t^{-1}y_j$ due to the wheel conditions (3.7). The evaluation maps (3.21), (3.24) and (3.25) can be extended to the skew diagrams $\lambda/\mu$, s.t. $|\lambda|-|\mu| = k$:

$$\mathrm{ev}_{\lambda/\mu}: \quad f(\dots x_i\dots) \mapsto f(\dots \chi_\Box \dots) \tag{3.26}$$

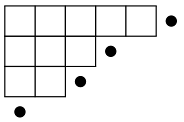
and in the skew analogue of (3.22) $x$'s are distributed in the reading order similar to (3.22).

Let $\mathrm{o}(\lambda)$ and $\mathrm{i}(\lambda)$ be the sets of coordinates of boxes corresponding to outer and inner corners of a Young diagram of $\lambda$ respectively. In order to explain this more precisely we consider an example with $\lambda = (532)$. The locations of the $\mathrm{o}(\lambda)$ boxes and $\mathrm{i}(\lambda)$ boxes are indicated on the left and on the right pictures respectively:

$\mathrm{o}(\lambda)$ :



$\mathrm{i}(\lambda)$ :



In this example we have $\mathrm{o}(\lambda) = \{(2,6),(3,4),(4,3)\}$ and $\mathrm{i}(\lambda) = \{(1,6),(2,4),(3,3),(4,1)\}$.

**Definition 3.8.** *Define a combinatorial factor* $d_{\lambda/\mu}$*:*

$$d_{\lambda/\mu} := \left(\frac{(1-q)(1-t^{-1})}{1-qt^{-1}}\right)^{|\lambda/\mu|} \prod_{\Box\in\lambda/\mu} \frac{\prod_{\Box'\in\mathrm{o}(\lambda)}(1-\chi_{\Box'}/\chi_\Box)}{\prod_{\Box'\in\mathrm{i}(\lambda)}(1-\chi_{\Box'}/\chi_\Box)}$$

Let us turn to the *evaluation representation* [19, 40, 37] of the shuffle algebra $\mathcal{A}$ on a graded vector space $\mathcal{F}$ whose basis vectors (assumed to be orthonormal) are labelled by integer partitions and the grading is determined by the weight of partitions.

**Proposition 3.9.** *To* $F(x_1\dots x_k) \in \mathcal{A}_k$ *we associate an infinite dimensional matrix whose rows and columns are labelled by partitions* $\lambda$ *and* $\mu$*:*

$$\langle\lambda|\, F(x_1\dots x_k)\, |\mu\rangle = \delta_{|\lambda/\mu|=k}\, d_{\lambda/\mu}\mathrm{ev}_{\lambda/\mu}(F(x_1\dots x_k)) \tag{3.27}$$

*We set the right hand side of* (3.27) *to zero if* $\lambda/\mu$ *is not a skew partition. The map from* $\mathcal{A}_k$ *to such matrices defines a representation of* $\mathcal{A}$.

Let us sketch the proof of this statement. We need to show that:

$$\langle\lambda|\, F(x_1\dots x_k) * G(x_1\dots x_l)\, |\mu\rangle = \sum_\nu \langle\lambda|\, F(x_1\dots x_k)\, |\nu\rangle\, \langle\nu|\, G(x_1\dots x_l)\, |\mu\rangle \tag{3.28}$$

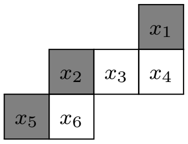
Due to (3.27) the summation on the r.h.s. of (3.28) contains non-zero terms only for such partitions $\nu$ that both $\lambda/\nu$ and $\nu/\mu$ correspond to skew Young diagrams with $k$ and $l$ boxes respectively. In order to see that the same summation occurs on the l.h.s. write the shuffle product in the form (3.8) and consider the map $\mathrm{ev}_{\lambda/\mu}$ applied to each summand:

$$\mathrm{ev}_{\lambda/\mu}\left(F(x_1\dots,x_k) * G(x_1\dots,x_l)\right) = \sum_{\substack{S\subseteq[1\dots k+l]\\|S|=k}} \mathrm{ev}_{\lambda/\mu}\left(F(x_S)G(x_{S^c})\prod_{\substack{i\in S\\j\in S^c}}\zeta\left(\frac{x_i}{x_j}\right)\right) \tag{3.29}$$

Fix an $S \in [1 \dots k+l]$ and compute the corresponding term on the r.h.s. of (3.29). The evaluation of $\zeta$ factors reads:

$$\mathrm{ev}_{\lambda/\mu}\zeta\left(\frac{x_i}{x_j}\right) = \zeta\left(\frac{\chi_{\square_i}}{\chi_{\square_j}}\right) = \zeta\left(q^{a_i-a_j}t^{b_j-b_i}\right) = 0 \quad \text{if} \begin{cases} a_j - a_i = 1 & b_i = b_j \\ b_j - b_i = 1 & a_i = a_j \end{cases}, \qquad i \in S,\ j \in S^c$$

where $\square_i = (a_i, b_i)$ and $\square_j = (a_j, b_j)$ are the $i$-th and $j$-th boxes of $\lambda/\mu$ respectively. This means that if the boxes $\square_i$ and $\square_j$ are located in the same row ($b_i = b_j$) and are bordering each other then $\square_i$ must be on the right to $\square_j$ for the $\zeta$ factor not to vanish. It also means that if the boxes $\square_i$ and $\square_j$ are located in the same column ($a_i = a_j$) and are bordering each other then $\square_i$ must be on the bottom to $\square_j$ for the $\zeta$ factor not to vanish. Together these conditions mean that, if the $x$'s are distributed over the boxes of $\lambda/\mu$ in the reading order, the non-zero terms of $\mathrm{ev}_{\lambda/\mu}(F * G)$ are such that the sets $S$ and $S^c$ split the boxes of $\lambda/\mu$ into two regions whose boundary defines another partition $\nu$. For $\lambda = (442)$ and $\mu = (31)$, $k = 3$ and $l = 3$ an example of a term, on the r.h.s. of (3.29), which gives a non-zero contribution under $\mathrm{ev}_{\lambda/\mu}$ is:



$$S = \{3,4,6\}, \quad S^c = \{1,2,5\} \tag{3.30}$$

where in the diagram we shaded those $x$'s which correspond to the set $S^c$. The partition $\nu$ corresponding to this term is $\nu = (421)$. Continuing with (3.29) in this particular case we have:

$$\mathrm{ev}_{(442)/(31)}\left(F(x_1,x_2,x_3) * G(x_1,x_2,x_3)\right) = \cdots + \mathrm{ev}_{(442)/(31)}\left(F(x_S)G(x_{S^c}) \prod_{\substack{i\in S\\ j\in S^c}} \zeta\left(\frac{x_i}{x_j}\right)\right) + \cdots$$

$$= \cdots + \mathrm{ev}_{(442)/(421)}\left(F(x_1,x_2,x_3)\right)\mathrm{ev}_{(421)/(31)}\left(G(x_1,x_2,x_3)\right)\mathrm{ev}_{(442)/(31)}\left(\prod_{\substack{i\in S\\ j\in S^c}} \zeta\left(\frac{x_i}{x_j}\right)\right) + \cdots$$

In order to verify (3.28) it remains to check that:

$$\mathrm{ev}_{\lambda/\mu}\left(\prod_{\substack{i\in S\\ j\in S^c}} \zeta\left(\frac{x_i}{x_j}\right)\right) = \frac{d_{\lambda/\nu}d_{\nu/\mu}}{d_{\lambda/\mu}}$$

3.4. **Isomorphism of $\mathcal{A}^\circ$ and $\Lambda$.** Let us now restrict our attention to $\mathcal{A}^\circ$ and derive a graded algebra isomorphism between $\mathcal{A}^\circ$ and $\Lambda$ [19]. The representation (3.27) allows us to construct the vector space $\mathcal{F}$ as a module starting from the *vacuum vector* $|\varnothing\rangle$. By (3.27) we have:

$$G(x_1 \dots x_k)\,|\varnothing\rangle = \sum_{\lambda \vdash k} d_\lambda \mathrm{ev}_\lambda(G(x_1 \dots x_k))\,|\lambda\rangle, \qquad G \in \mathcal{A}^\circ_k \tag{3.31}$$

By choosing a basis in $\mathcal{A}^\circ_k$ we can write $p(k)$ such equations whose left hand sides would be linearly independent from each other. Thus we can solve this system for $|\lambda\rangle$ in terms of the elements of $\mathcal{A}^\circ_k$. In other words there exists an element $F_\lambda \in \mathcal{A}^\circ_k$ such that:

$$F_\lambda(x_1,\dots,x_k)\,|\varnothing\rangle = |\lambda\rangle, \qquad \lambda \vdash k \tag{3.32}$$

Applying the dual vector $\langle\mu|$ to this equation and using (3.27) gives us:

$$\mathrm{ev}_\mu(F_\lambda(x_1,\dots,x_k)) = \delta_{\lambda,\mu}\frac{1}{d_\lambda} \tag{3.33}$$

In particular, these equations can be used to compute $F_\lambda$. From (3.31) and (3.32) it follows that for any element of $\mathcal{A}^\circ_k$ we have the expansion in the basis of $F_\lambda$:

$$G(x_1 \dots x_k) = \sum_{\lambda \vdash k} d_\lambda \mathrm{ev}_\lambda (G(x_1 \dots x_k)) F_\lambda, \qquad G \in \mathcal{A}^\circ_k \tag{3.34}$$

Recall the coefficients $\psi'_{\lambda/\mu}$ and $c_\lambda$ from (2.12) and (2.9) respectively and define:

$$n(\lambda) := \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i \tag{3.35}$$

We have the following two Lemmas [19].

**Lemma 3.10.** *The shuffle product $E_k(q) * F_\mu$ expands in the basis of $F_\lambda$ as follows:*

$$E_k(q) * F_\mu = \sum_\lambda \phi_{\lambda/\mu} F_\lambda, \tag{3.36}$$

*where the summation runs over $\lambda$ such that the skew partitions $\lambda/\mu$ are all vertical strips with $k$ boxes and*

$$\phi_{\lambda/\mu} := (1-t)^k q^{n(\lambda')-n(\mu')} \frac{c_\mu}{c_\lambda} \psi'_{\lambda/\mu} \tag{3.37}$$

Let us sketch the proof of this statement. Due to (3.32) showing (3.36) is equivalent to showing:

$$E_k(q) \left|\mu\right\rangle = \sum_\lambda \phi_{\lambda/\mu} \left|\lambda\right\rangle \tag{3.38}$$

By (3.27) we have:

$$\phi_{\lambda/\mu} = d_{\lambda/\mu} \mathrm{ev}_{\lambda/\mu} \left(E_k(q)\right) \tag{3.39}$$

Recall that ev is computed in two steps (3.24) and (3.25). Compute $\mathrm{ev}^x_{\lambda/\mu}$:

$$\mathrm{ev}^x_{\lambda/\mu} \left( \prod_{1 \le i < j \le k} \frac{(x_i - qx_j)(x_i - q^{-1}x_j)}{(x_i - qt^{-1}x_j)(x_i - tq^{-1}x_j)} \right) = 0 \quad \text{if } \exists\, \Box, \Box' \in \lambda/\mu, \text{ s.t.: } \chi_\Box / \chi_{\Box'} = q$$

This implies that the skew partition $\lambda/\mu$ cannot contain more than one box in a single row and thus is a vertical strip. This determines that the summation set over $\lambda$ in (3.38) and (3.36) must be given by the set of all vertical strips with $k$ boxes. Computing $\mathrm{ev}_{\lambda/\mu}$ of $E_k(q)$ gives:

$$\mathrm{ev}_{\lambda/\mu} \left(E_k(q)\right) = t^{-k(k-1)/2} \frac{(1-qt^{-1})^k}{(1-q)^k} \prod_{\Box, \Box' \in \lambda/\mu} \frac{1 - q\chi_{\Box'}/\chi_\Box}{1 - qt^{-1}\chi_{\Box'}/\chi_\Box} \tag{3.40}$$

In order to complete the proof one needs to multiply the expression on the r.h.s. in (3.40) by $d_{\lambda/\mu}$ and compare it with the definition of $\phi$ from (3.37).

**Lemma 3.11.** *We have an isomorphism $\iota$ of algebras $\mathcal{A}^\circ$ and $\Lambda$ given by matching $F_\lambda$ with Macdonald functions $P_\lambda$:*

$$\iota: \quad F_\lambda \mapsto \frac{c_\lambda}{q^{n(\lambda')}(1-t)^{|\lambda|}} P_\lambda \tag{3.41}$$

*Proof.* We consider (3.41) to be a linear map and show that it is an isomorphism. A particular case of the map (3.41) is when $\lambda = (1^k)$. In this case the Macdonald function coincides with the elementary symmetric function $P_{(k)} = e_k$. Consider (3.36) with $\mu = \varnothing$, in this case the r.h.s. of (3.36) contains a single term with $\lambda = (1^k)$. By (3.32) we have $F_\varnothing = 1$ and computing $\phi_{(1)^k/\varnothing}$ gives:

$$F_{(1^k)} = \prod_{i=1}^{k} \frac{1-t^i}{1-t} E_k(q) \tag{3.42}$$

Therefore a special case of the map (3.41) is:

$$\iota : \quad E_k(q) \mapsto e_k \tag{3.43}$$

where the factors depending on $t$ in (3.42) canceled with the factors from (3.41). Next we compute:

$$\iota\left(E_k(q) * F_\mu\right) = \sum_\lambda \phi_{\lambda/\mu} \iota\left(F_\lambda\right) = \frac{c_\mu}{q^{n(\mu')}(1-t)^{|\mu|}} \sum_\lambda \psi_{\lambda'/\mu'} P_\lambda = \frac{c_\mu}{q^{n(\mu')}(1-t)^{|\mu|}} e_k P_\mu = \iota\left(E_k(q)\right) \iota\left(F_\mu\right)$$

where we used (3.36), (3.37) and (3.41) in the third equality we used the Pieri formula (2.11). Consider an expansion of $E_\lambda(q)$ in the basis $F_\mu$ and let $C_{\lambda,\mu}$ denote the expansion coefficients, then compute using linearity of $\iota$ and the above equation:

$$\begin{aligned}\iota\left(E_k(q) * E_\lambda(q)\right) = \sum_\mu C_{\lambda,\mu}\, \iota\left(E_k(q) * F_\mu\right) &= \sum_\mu C_{\lambda,\mu}\, \iota\left(E_k(q)\right) * \iota\left(F_\mu\right) \\ &= \iota\left(E_k(q)\right) * \iota\left(\sum_\mu C_{\lambda,\mu} F_\mu\right) = \iota\left(E_k(q)\right) \iota\left(E_\lambda(q)\right)\end{aligned}$$

From this we have $\iota(E_\lambda(q)) \mapsto e_\lambda$ and by dimensionality argument we have an isomorphism. □

**Corollary 3.12.** *For the functions* $S_k$, $E_k(t^{-1}), H_k(t^{-1})$ *we have:*

$$\iota : \quad S_k \mapsto \frac{(1-q)^k}{(t-q)^k} p_k \tag{3.44}$$

$$\iota : \quad E_k(t^{-1}) \mapsto \frac{(1-q)^k}{(1-t^{-1})^k} g_k \tag{3.45}$$

$$\iota : \quad H_k(t^{-1}) \mapsto \frac{(1-q)^k}{(1-t^{-1})^k} g_k^* \tag{3.46}$$

*Proof.* The first map (3.44) follows from (3.43) and matching the generating function of $E_k(q)$ and the generating function (2.3) of the elementary symmetric functions $e_k$:

$$\iota\left(E(v;q)\right) = \exp\left(\sum_{r>0} \frac{(-1)^{r+1}}{r} v^r p_r\right)$$

Due to (3.16) and the fact that $\iota$ is an isomorphism it implies (3.44). The two other maps (3.45) and (3.46) can be verified by applying $\iota^{-1}$ to the generating functions of $E_k(t^{-1})$ and $H_k(t^{-1})$ and then comparing the result with the generating functions (2.4) and (2.5) of $g_k$ and $g_k^*$. □

In Lemma 3.11 and Corollary 3.12 we omitted the dependence of symmetric functions of $\Lambda$ on the alphabet. This will be important in a later section. Consider a pair $F \in \mathcal{A}^\circ$ and $f \in \Lambda$ such that $\iota(F) = f$ and let $\Lambda$ be the ring of symmetric function in the alphabet $(z) = (z_1 \dots z_l)$, then we will write:

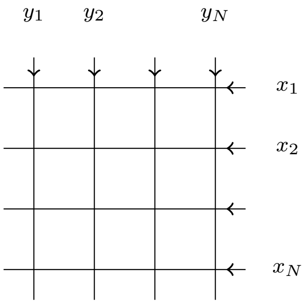
$$\iota_z(F) = f(z_1 \dots z_l)$$

## 4. Six vertex model and the shuffle algebra $\mathcal{A}^\circ$

In this section we explain one of the main results of the paper using the example of the six vertex model. We start with the Boltzmann weights of the fundamental $R$-matrix of $U_t(\widehat{sl}(2))$ and consider the associated square lattice partition functions. The configurations of these partition functions involve lattice paths of a single colour, labelled "1". The conic partition function, which is discussed in the introduction, can be expressed as a sum of planar partition functions with identified boundary conditions. In the algebraic language the conic partition function equals to the trace of a product of $R$-matrices. We use the notion of the *F-basis* to rewrite this trace as a symmetrization, w.r.t. the spectral parameters $x_1 \dots x_N$, of a fixed partition function with ordered labels on the boundaries. This produces a shuffle product formula for the conic partition function and leads us to a proof of Theorem 1.1 in the special case of $m = 0$ and $n = 1$.

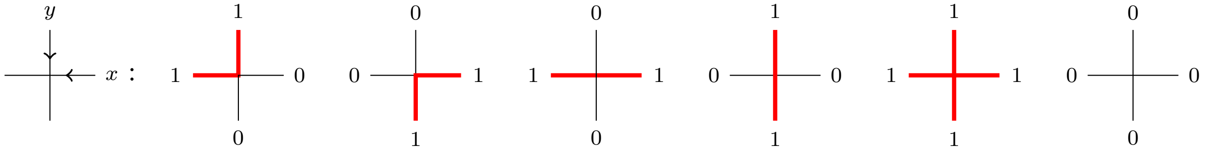
### 4.1. The six vertex model.

We consider a grid made up of a finite number of horizontal lines oriented from right to left and the same number of vertical lines oriented from top to bottom. We attach *spectral* the parameter $x_i(y_i)$ to each $i^{th}$ horizontal (vertical) line counting from the top (left):



(4.1)

We are interested in the special case $y_i = qx_i$, however, the more general case which involves the $y$ parameters will be useful for computational purposes. We refer to an intersection of a horizontal and vertical line as a vertex. In this section every edge of a vertex can be labelled either 0 or 1 and to the edges carrying the label 1 we will associate a red path. Specifying the boundary conditions in (4.1) means assigning labels to the $4N$ external edges. To every vertex, depending on the local configuration, we attach a Boltzmann weight:

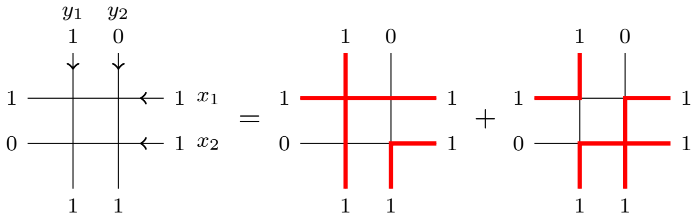


$$\frac{1-t}{1-tx/y} \qquad \frac{(1-t)x/y}{1-tx/y} \qquad \frac{t(1-x/y)}{1-tx/y} \qquad \frac{1-x/y}{1-tx/y} \qquad 1 \qquad 1$$

(4.2)

and the Boltzmann weights of all other local path configurations being zero. The leftmost vertex in (4.2) with oriented lines and unspecified edge labels represents the collection of all vertices with their Boltzmann weights and will be referred to as the graphical representation of the $\check{R}$-matrix of the six vertex model. When the values of the external edges of the graphical $\check{R}$-matrix are specified we will identify them with the corresponding Boltzmann weights in (4.2). We can join edges of several graphical $\check{R}$-matrices together in which case we will assume that the values at the joined edges are summed over.

Consider an example of (4.1) with $N = 2$ and a choice of boundary conditions:



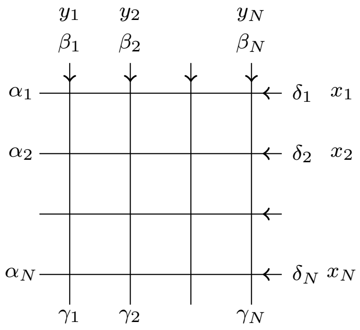
(4.3)

$$= \frac{t\,(1-x_1/y_2)\,(1-x_2/y_1)\,(1-t)x_2/y_2}{(1-tx_1/y_2)\,(1-tx_2/y_1)\,(1-tx_2/y_2)} + \frac{(1-t)^3 x_1/y_2\, x_2/y_1}{(1-tx_1/y_1)\,(1-tx_1/y_2)\,(1-tx_2/y_1)}$$

In this case, if we sum over all possible labels of the internal edges on the l.h.s. of (4.3) we will find two configurations with non-zero Boltzmann weights which we computed using (4.2) in the second line.

**Definition 4.1.** *Let $\alpha, \beta, \gamma, \delta \in \{0,1\}^N$ be collections of 0's and 1's. The six vertex partition function $Z^{\beta,\delta}_{\alpha,\gamma}(x;y) := Z^{\beta,\delta}_{\alpha,\gamma}(x_1 \dots x_N; y_1 \dots y_N)$ is defined as the rational function in the spectral parameters equal to the weighted sum over all possible six vertex configurations with the boundary conditions as specified below:*

$$Z^{\beta,\delta}_{\alpha,\gamma}(x;y) = \tag{4.4}$$



Let us turn to the algebraic picture. We define the six vertex $\check{R}$-matrix:

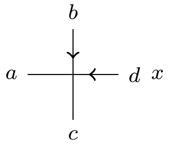
$$\check{R}(x/y) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(1-t)x/y}{1-tx/y} & \frac{1-x/y}{1-tx/y} & 0 \\ 0 & \frac{t(1-x/y)}{1-tx/y} & \frac{1-t}{1-tx/y} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

This matrix acts in $V_y \otimes V_x$ with $V_y, V_x \simeq \mathbb{C}^2$. Let $P$ be the permutation matrix acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$ by swapping the basis vectors, then we define the $R$-matrix:

$$R(x) = P\check{R}(x) \tag{4.6}$$

Let $|0\rangle = (1,0)^T$ and $|1\rangle = (0,1)^T$ denote the standard basis in $\mathbb{C}^2$ and $|i_1 \dots i_N\rangle$, with $i_1, \dots, i_N \in \{0,1\}$, its generalization to the $N$-fold tensor product of $\mathbb{C}^2$. Define similarly the dual basis, then we have:

$$\check{R}(x/y) = \sum_{a,b,c,d=0,1} \Bigg[$$



$$\Bigg] |a,c\rangle\, \langle b,d| \tag{4.7}$$

Let id be the identity matrix in $\mathbb{C}^2$ then $\check{R}$ acts in $\otimes_{i=1}^{N} V_{x_i}$ by:

$$\check{R}_i(x_{i+1}/x_i) := \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-1} \otimes \check{R}(x_{i+1}/x_i) \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{N-i-1} \tag{4.8}$$

and similarly we define $R_i(x_{i+1}/x_i) \in \mathrm{End}\left(\otimes_{i=1}^{N} V_{x_i}\right)$ and the permutation matrix $P_i$. We have the Yang–Baxter equation:

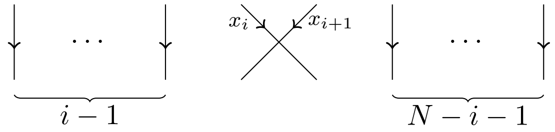
$$\check{R}_i(z/y)\check{R}_{i+1}(z/x)\check{R}_i(y/x) = \check{R}_{i+1}(y/x)\check{R}_i(z/x)\check{R}_{i+1}(z/y) \tag{4.9}$$

and the unitarity relation:

$$\check{R}_i(x/y)\check{R}_i(y/x) = \mathrm{id} \otimes \mathrm{id} \tag{4.10}$$

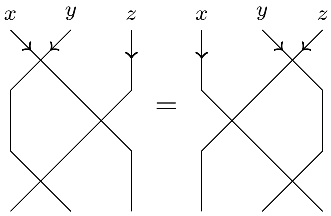
The graphical notation associated to the matrix $\check{R}_i(x_{i+1}/x_i)$ is:

$\check{R}_i(x_{i+1}/x_i)$ :



(4.11)

where the rotation of the cross compared to (4.2) does not present an ambiguity due to the presence of the arrows. The arrows also help us to keep track of the ordering of operators: moving forward w.r.t. the orientation of a line is reading an expression right to left. With this convention the graphical version of the Yang–Baxter equation (4.9) reads:



Using the correspondence of the $\check{R}_i$-matrix in (4.8) and its graphical counterpart in (4.11), we can view the (rotated counterclockwise by $\pi/4$) object in (4.1) as a tensor. In what follows, we define three such tensors: $Z_N(x;y), W_N(x;y)$ and $W_N(x)$, where $W_N(x)$ is a special case of $W_N(x;y)$ which in turn is a projection of $Z_N(x;y)$. The reason for this is that we will be interested in the matrix elements of $W_N(x)$ and will require their properties, which can be determined from the more general objects $Z_N(x;y)$ and $W_N(x;y)$.

**Definition 4.2.** *For two sets of parameters $(x) = (x_1 \dots x_N)$ and $(y) = (y_1 \dots y_N)$ we define the following tensors:*

$$Z_N(x;y) := \check{R}_N(x_N/y_1)\check{R}_{N-1}(x_{N-1}/y_1)\check{R}_{N+1}(x_N/y_2)\cdots\check{R}_N(x_1/y_N) \tag{4.12}$$

$$W_N(x;y) := \left(\left\langle 0^N\right| \otimes \cdot\right) Z_N(x;y) \left(\left|0^N\right\rangle \otimes \cdot\right) \tag{4.13}$$

$$W_N(x) := W(x_1 \dots x_N; qx_1 \dots qx_N) \tag{4.14}$$

The graphical representation for the matrix elements of $Z_N(x;y)$ follows from Definition 4.1:

$$Z_N(x;y) = \sum_{\alpha,\beta,\gamma,\delta} Z_{\alpha,\gamma}^{\beta,\delta}(x;y) \left|\alpha,\gamma\right\rangle \left\langle\beta,\delta\right| \tag{4.15}$$

By setting $W_\gamma^\delta(x;y) := Z_{(0^N),\gamma}^{(0^N),\delta}(x;y)$ and $W_\gamma^\delta(x) := W_\gamma^\delta(\dots x_i \dots; \dots qx_i \dots)$ we also have:

$$W_N(x;y) = \sum_{\gamma,\delta} W_\gamma^\delta(x;y) \left|\gamma\right\rangle \left\langle\delta\right|, \qquad W_N(x) = \sum_{\gamma,\delta} W_\gamma^\delta(x) \left|\gamma\right\rangle \left\langle\delta\right| \tag{4.16}$$

and the graphical representations of $W^{\delta}_{\gamma}(x;y)$ and $W^{\delta}_{\gamma}(x)$ are given by (4.4) but with the specialized labels $\alpha_i = 0$ and $\beta_i = 0$ and parameters $y_i = qx_i$ in the case of $W^{\delta}_{\gamma}(x)$.

**Lemma 4.3.** *For $i = 1 \dots N-1$, the tensor $Z_N(x;y)$ satisfies the following exchange relations:*

$$\check{R}_i(x_{i+1}/x_i) Z_N(x;y) = Z_N(\dots x_{i+1}, x_i \dots ; y) \check{R}_{N+i}(x_{i+1}/x_i) \tag{4.17}$$

$$Z_N(x;y) \check{R}_i(y_i/y_{i+1}) = \check{R}_{N+i}(y_i/y_{i+1}) Z_N(x; \dots y_{i+1}, y_i \dots) \tag{4.18}$$

*The tensor $W_N(x;y)$ satisfies:*

$$W_N(x;y) = W_N(\dots x_{i+1}, x_i \dots ; y) \check{R}_i(x_{i+1}/x_i) \tag{4.19}$$

$$W_N(x;y) = \check{R}_i(y_i/y_{i+1}) W_N(x; \dots y_{i+1}, y_i \dots) \tag{4.20}$$

*and for $W_N(x)$ we have:*

$$\check{R}_i(x_{i+1}/x_i) W_N(x) = W_N(\dots x_{i+1}, x_i \dots) \check{R}_i(x_{i+1}/x_i) \tag{4.21}$$

*Proof.* The two equations (4.17) and (4.18) are well-known exchange relations which are a consequence of the Yang–Baxter equation (4.9) (see e.g. [24]). The exchange relations for $W_N(x;y)$ (4.19) and (4.20) follow from (4.17) and (4.18) by applying the projection in (4.13) and using:

$$\langle 0^N | \check{R}_i(x) = \langle 0^N |, \qquad \check{R}_i(x) | 0^N \rangle = | 0^N \rangle$$

After relabelling the vector spaces $N+i \to i$ we get (4.19) and (4.20). The last equation (4.21) is obtained by equating the right hand sides of (4.19) and (4.20), swapping $x_i \leftrightarrow x_{i+1}$:

$$W_N(x;y) \check{R}_i(x_i/x_{i+1}) = \check{R}_i(y_i/y_{i+1}) W_N(\dots x_{i+1}, x_i \dots ; \dots y_{i+1}, y_i \dots)$$

multiplying both sides by $\check{R}_i(y_{i+1}/y_i)$ on the left and by $\check{R}_i(x_{i+1}/x_i)$ on the right, using the unitarity property (4.10) and then setting $y_i = qx_i$ for all $i$. ☐

### 4.2. The $F$-matrix and transformed tensors.

In this section we introduce the $F$-matrix [33, 2], following the conventions of [7]. Then we will use it to transform the tensors $Z_N(x;y), W_N(x;y)$ and $W_N(x)$ and establish their properties.

**Definition 4.4.** *The 2-site $F$-matrix reads:*

$$F_2(x_1, x_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{(1-t)x_2/x_1}{1-tx_2/x_1} & \frac{1-x_2/x_1}{1-tx_2/x_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.22}$$

*This matrix satisfies the property:*

$$F_2(x_1, x_2) = P F_2(x_2, x_1) \check{R}(x_2/x_1) \tag{4.23}$$

The 2-site $F$-matrix and the property (4.23) are associated to the vector space $V_{x_1} \otimes V_{x_2}$. The $N$-site $F$-matrix and the analogue of (4.23) correspond to $\otimes_{i=1}^{N} V_{x_i}$. This generalization is done with the help of the matrices $\check{R}_\sigma$ and $R_\sigma$ for $\sigma \in \mathcal{S}_N$. Consider $\sigma \in \mathcal{S}_N$ and its decomposition into simple transpositions. For some $i$ and $\sigma' \in \mathcal{S}_N$ we can write:

$$\sigma = s_i \sigma' \tag{4.24}$$

Define recursively the matrices $\check{R}_\sigma$ and $R_\sigma$:

$$\check{R}_{\text{id}} = 1, \qquad \check{R}_\sigma = \check{R}_i(x_{\sigma'(i+1)}/x_{\sigma'(i)}) \check{R}_{\sigma'} \tag{4.25}$$

$$R_{\text{id}} = 1, \qquad R_\sigma = R_{\sigma'(i),\sigma'(i+1)}(x_{\sigma'(i+1)}/x_{\sigma'(i)}) R_{\sigma'} \tag{4.26}$$

where the two-index notation $R_{i,j}(x)$ is defined by setting $R_{i,i+1}(x) = R_i(x)$ and $R_{i,j+1}(x) = P_j R_{i,j}(x) P_j$. Next we consider an explicit decomposition $\sigma = s_{i_k} \cdots s_{i_1}$ and define:

$$P_\sigma = P_{i_1} \cdots P_{i_k} \tag{4.27}$$

With these definitions we have $P_\sigma = R_\sigma|_{x_1 = \cdots = x_N}$ and the relation:

$$\check{R}_\sigma = P_{\sigma^{-1}} R_\sigma \tag{4.28}$$

**Definition 4.5.** *For $k, l \in \{0, 1\}$ and $r = 1 \ldots N$, let $E_r^{(kl)} \in \mathrm{End}\left(\otimes_{i=1}^N V_{x_i}\right)$ be the matrix acting non-trivially on the $r$-th tensor space $V_{x_r}$ by the two by two unit matrix $E^{(kl)}$. The $N$-site $F$-matrix reads:*

$$F_N(x_1 \ldots x_N) = \sum_{\sigma \in \mathcal{S}_N} \sum_{(k_1 \ldots k_N) \in \mathcal{J}_\sigma} \prod_{i=1}^{N} E_{\sigma(i)}^{(k_i k_i)} R_\sigma \tag{4.29}$$

*where the set $\mathcal{J}_\sigma$ is defined by:*

$$\mathcal{J}_\sigma = \Big\{ 0 \le k_1 \le \cdots \le k_N \le 1 : k_i < k_{i+1} \text{ if } \sigma(i) > \sigma(i+1) \Big\} \tag{4.30}$$

*This matrix satisfies the property:*

$$F_N(x_1 \ldots x_N) = P_\sigma F_N(x_{\sigma(1)} \ldots x_{\sigma(N)}) \check{R}_\sigma \tag{4.31}$$

**Lemma 4.6.** *The $F$-matrix has the properties:*

$$\Big\langle 0^{N-k} 1^k \Big| F_N(x) = \Big\langle 0^{N-k} 1^k \Big|, \qquad F_N(x) \Big| 1^k 0^{N-k} \Big\rangle = \prod_{i=1}^{k} \prod_{j=k+1}^{N} \frac{x_i - x_j}{x_i - t x_j} \Big| 1^k 0^{N-k} \Big\rangle \tag{4.32}$$

These are known properties which are a consequence of the explicit form of the $F$-matrix (4.29) (for more details see e.g. [34]).

**Definition 4.7.** *Introduce transformed tensors $\widetilde{Z}_N(x; y)$, $\widetilde{W}_N(x; y)$ and $\widetilde{W}_N(x)$:*

$$\widetilde{Z}_N(x; y) := \left(F_N(x) \otimes F_N(y)\right) Z_N(x; y) \left(F_N^{-1}(y) \otimes F_N^{-1}(x)\right) \tag{4.33}$$

$$\widetilde{W}_N(x; y) := F_N(y) W_N(x; y) F_N^{-1}(x) \tag{4.34}$$

$$\widetilde{W}_N(x) := F_N(x) W_N(x) F_N^{-1}(x) \tag{4.35}$$

The definition of $\widetilde{W}_N(x; y)$ is consistent with (4.13) because of the property (4.32) at $k = 0$, in other words we have:

$$\widetilde{W}_N(x; y) = \left(\langle 0^N | \otimes \cdot\right) \widetilde{Z}_N(x; y) \left(|0^N\rangle \otimes \cdot\right) \tag{4.36}$$

Due to the conjugation by the $F$-matrices the tensors $\widetilde{Z}_N(x; y), \widetilde{W}_N(x; y)$ and $\widetilde{W}_N(x)$ obey new exchange relations.

**Proposition 4.8.** *For $i = 1 \ldots N-1$, the tensor $\widetilde{Z}_N(x; y)$ satisfies the following exchange relations:*

$$P_i \widetilde{Z}(x; y) = \widetilde{Z}(\ldots x_{i+1}, x_i \ldots; y) P_{N+i} \tag{4.37}$$

$$\widetilde{Z}(x; y) P_i = P_{N+i} \widetilde{Z}(x; \ldots y_{i+1}, y_i \ldots) \tag{4.38}$$

*the tensor $\widetilde{W}_N(x; y)$ satisfies:*

$$\widetilde{W}_N(x; y) P_i = \widetilde{W}_N(\ldots x_{i+1}, x_i \ldots; y) \tag{4.39}$$

$$P_i \widetilde{W}_N(x; y) = \widetilde{W}_N(x; \ldots y_{i+1}, y_i \ldots) \tag{4.40}$$

*and $\widetilde{W}_N(x)$ satisfies:*

$$P_i \widetilde{W}_N(x) = \widetilde{W}_N(\ldots x_{i+1}, x_i \ldots) P_i \tag{4.41}$$

*Proof.* Consider two permutations $\sigma, \tau \in \mathcal{S}_N$ and their decompositions into simple transpositions, denote $x_\sigma = (x_{\sigma(1)} \dots x_{\sigma(N)})$ and $y_\tau = (y_{\tau(1)} \dots x_{\tau(N)})$, then we have:

$$\left(\check{R}_\sigma(x) \otimes \check{R}_\tau(y)\right) Z_N(x;y) = Z_N(x_\sigma; y_\tau) \left(\check{R}_\tau(y) \otimes \check{R}_\sigma(x)\right) \tag{4.42}$$

This follows from the recursive definition of $\check{R}_\sigma$ and repetitive application of (4.17) and (4.18). Multiply both sides of (4.42) on the left and on the right by $F$-matrices as follows:

$$(F_N(x_\sigma) \otimes F_N(y_\tau))\,(4.42)\left(F_N^{-1}(y) \otimes F_N^{-1}(x)\right)$$

and use (4.31) and (4.33), the outcome is:

$$(P_{\sigma^{-1}} \otimes P_{\tau^{-1}})\, \widetilde{Z}_N(x;y) = \widetilde{Z}_N(x_\sigma; y_\tau)\,(P_{\tau^{-1}} \otimes P_{\sigma^{-1}}) \tag{4.43}$$

where on the r.h.s. we used (4.31) in a rearranged form:

$$\check{R}_\sigma(x) F_N^{-1}(x) = F_N^{-1}(x_\sigma) P_{\sigma^{-1}}$$

Clearly (4.43) is equivalent to the pair of equations (4.37) and (4.38). The remaining equations (4.39), (4.40) and (4.41) are special cases of the above due to (4.36) and $\widetilde{W}_N(x) = \widetilde{W}_N(\dots x_i \dots; \dots q x_i \dots)$. ☐

In the remaining part of this subsection we compute the matrix elements of $\widetilde{W}_N(x;y)$. The formula which we obtain is a product consisting of several building blocks of fully factorized terms and a *domain-wall* partition function:

$$D_M(x;y) := W_{(1^M)}^{(1^M)}(x_1 \dots x_M; y_1 \dots y_M). \tag{4.44}$$

The following Lemma is a well-known result about the six vertex domain-wall partition function [28, 27].

**Lemma 4.9.** *$D_M(x;y)$ is the six vertex domain-wall partition function which can be computed using a determinant formula:*

$$D_M(x;y) = \frac{\prod_{i,j=1}^M (x_i - y_j)}{\prod_{1 \le i < j \le M} (x_i - x_j)(y_j - y_i)} \det_{1 \le i,j \le M} \frac{(1-t)x_i}{(x_i - y_j)(y_j - t x_i)} \tag{4.45}$$

**Proposition 4.10.** *Fix $k \in \{0 \dots N\}$ and let $\alpha, \beta \in \{0,1\}^N$ be the labels of the matrix elements $\widetilde{W}_\alpha^\beta(x;y)$ of $\widetilde{W}_N(x;y)$ both having $k$ number of 1's. Let $P, S \subseteq [N]$ denote the positions of 1's in $\alpha$ and $\beta$ respectively, then:*

$$\widetilde{W}_\alpha^\beta(x;y) = D_k(x_S; y_P) \prod_{i \in S} \prod_{j \in S^c} \frac{x_i - t x_j}{x_i - x_j} \prod_{i \in P} \prod_{j \in S^c} \frac{y_i - x_j}{y_i - t x_j} \tag{4.46}$$

*Proof.* The proof is based on the properties (4.39) and (4.40) which imply that for a fixed $k$ it is sufficient to compute any one matrix element $\widetilde{W}_\alpha^\beta(x;y)$. There exists a choice of $\alpha$ and $\beta$ for which the matrix element $\widetilde{W}_\alpha^\beta(x;y)$ can be easily shown to be of the form (4.46). Let us explain this in detail.

Consider two permutations $\sigma, \tau \in \mathcal{S}_N$ such that $\alpha = \sigma^{-1}(0^{N-k}1^k)$ and $\beta = \tau^{-1}(0^{N-k}1^k)$ where the action of permutation on a string is by permuting the entries of the string. We can represent the vectors $\langle\alpha|$ and $|\beta\rangle$ as follows:

$$\langle\alpha| = \left\langle 0^{N-k}1^k \right| P_{\sigma^{-1}}, \qquad |\beta\rangle = P_\tau \left| 0^{N-k}1^k \right\rangle$$

therefore:

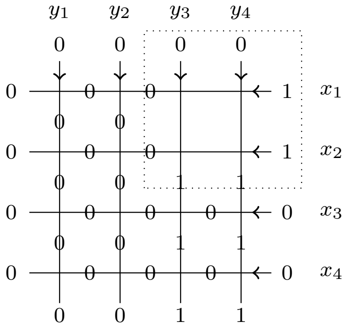
$$\begin{aligned} \widetilde{W}_\alpha^\beta(x;y) = \langle\alpha|\, \widetilde{W}_N(x;y)\, |\beta\rangle &= \left\langle 0^{N-k}1^k \right| P_{\sigma^{-1}} \widetilde{W}_N(x;y) P_\tau \left| 0^{N-k}1^k \right\rangle \\ &= \left\langle 0^{N-k}1^k \right| \widetilde{W}_N(x_\tau; y_\sigma) \left| 0^{N-k}1^k \right\rangle = \widetilde{W}_{(0^{N-k}1^k)}^{(0^{N-k}1^k)}(x_\tau; y_\sigma) \end{aligned}$$

where in the second line we used the properties (4.39) and (4.40) of $\widetilde{W}_\alpha^\beta(x;y)$ (cf. (4.43)). This shows that computing one matrix element of $\widetilde{W}_N(x;y)$ with a fixed number of 1's is sufficient to recover all of them. Consider next the matrix element $\widetilde{W}^{(1^k0^{N-k})}_{(0^{N-k}1^k)}(x;y)$ and write it in terms of $W^{(1^k0^{N-k})}_{(0^{N-k}1^k)}(x;y)$ using (4.34):

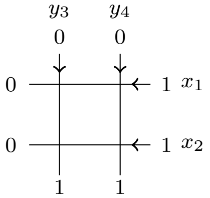
$$\widetilde{W}^{(1^k0^{N-k})}_{(0^{N-k}1^k)}(x;y) = \left\langle 0^{N-k}1^k \right| F_N(y) W_N(x;y) F_N^{-1}(x) \left| 1^k 0^{N-k} \right\rangle$$
$$= \prod_{i=1}^{k} \prod_{j=k+1}^{N} \frac{x_i - tx_j}{x_i - x_j} \times \left\langle 0^{N-k}1^k \right| W_N(x;y) \left| 1^k 0^{N-k} \right\rangle = \prod_{i=1}^{k} \prod_{j=k+1}^{N} \frac{x_i - tx_j}{x_i - x_j} \times W^{(1^k0^{N-k})}_{(0^{N-k}1^k)}(x;y)$$

where in the second line we used the properties (4.32) of the $F$-matrix. The matrix element $W^{(1^k0^{N-k})}_{(0^{N-k}1^k)}(x;y)$ can be evaluated as follows. For demonstration, we choose $N = 4$ and $k = 2$. The graphical depiction of the partition function $W^{(1100)}_{(0011)}(x;y)$ shows several regions which are frozen and one region with domain-wall configurations (enclosed in the dotted square below):

$$W^{(1100)}_{(0011)}(x;y) =$$



$$= \prod_{i=\{3,4\}} \prod_{j=\{3,4\}} \frac{y_i - x_j}{y_i - tx_j} \times$$



$$= \prod_{i\in\{3,4\}} \prod_{j\in\{3,4\}} \frac{y_i - x_j}{y_i - tx_j} D_2(x_1, x_2; y_3, y_4) \tag{4.47}$$

In the first line of the above equation we observed that the left half of the lattice must have only trivial local configurations corresponding to the last vertex in (4.2) whose weight is equal to 1; the bottom right $2 \times 2$ region contains four local configurations all given by the 4-th vertex in (4.2) and therefore they contribute four factors appearing in the second line; finally the top right region, enclosed in the dotted square, equals to the partition function $D_2(x_1, x_2; y_3, y_4)$ (4.44). The labels of $x$'s and $y$'s which appear in (4.47) in $D_2(x_1, x_2; y_3, y_4)$ are determined by the locations of 1's on right and bottom boundaries respectively while the labels of $x$'s and $y$'s in the product appearing in (4.47) are determined by the positions of 0's on the right boundary and positions of 1's on the bottom boundary respectively. Thanks to the property:

$$\widetilde{W}^{\tau^{-1}(0^{N-k}1^k)}_{\sigma^{-1}(0^{N-k}1^k)}(x;y) = \widetilde{W}^{(0^{N-k}1^k)}_{(0^{N-k}1^k)}(x_\tau; y_\sigma)$$

one needs to keep track of the positions of 1's to be able to assign the correct labels to $x$'s and $y$'s. This explains the labelling by the sets $S$ and $P$ given in the statement of the proposition. ☐

## 4.3. **Trace of $W_N(x)$ and the shuffle product.**

In this section, we define the partition function $T_N(x)$ as the trace of the matrix $W_N(x)$. Because of the cyclicity of the trace we will be able to replace $W_N(x)$ with $\widetilde{W}_N(x)$ since they are related by a conjugation (4.35). The trace represents a summation over indices of the matrix $\widetilde{W}_N(x)$, using the exchange relation (4.41) we can replace this summation by a symmetrization over the parameters $x_1 \dots x_N$ acting on specific diagonal elements of $\widetilde{W}_N(x)$. These diagonal elements can be computed via the connection with $\widetilde{W}_N(x;y)$. The result of this computation, summarized in Theorem 4.13, gives a formula for $T_N(x)$ in terms of the shuffle product of $\mathcal{A}^{\circ}$.

**Definition 4.11.** *Let $z_0, z_1$ be two indeterminates. Define the following function:*

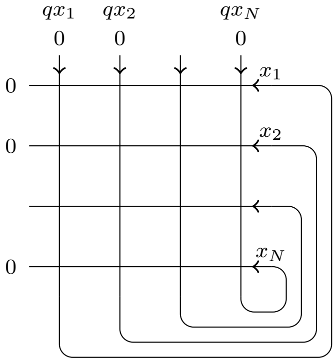
$$T_N(x;z_0,z_1) := \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N} \langle\alpha| W_N(x) |\alpha\rangle \tag{4.48}$$

Note that $W_N(x)$ is block diagonal with blocks labelled by the number of 1's in $\alpha$ and $\beta$ in the matrix elements $W_\alpha^\beta(x)$. This means that $T_N(x;z_0,z_1)$ is the trace of $W_N(x)$ where the monomials in $z_0, z_1$ parameterize different blocks of $W_N(x)$. We can write $T_N(x;z_0,z_1)$ in terms of the matrix elements of $W_N(x)$:

$$T_N(x;z_0,z_1) = \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N} W_\alpha^\alpha(x) \tag{4.49}$$

and using the graphics (4.4) we interpret $T_N(x;z_0,z_1)$ as the following lattice partition function:

$$T_N(x;z_0,z_1) = \tag{4.50}$$



If we rotate this picture by 135 degrees counterclockwise we can match it with the lattice drawn on the cone in (1.1). Therefore $T_N(x;z_0,z_1)$ coincides with the partition function $Z_N$ from the introduction (1.4) where we need to choose $n=1$ and $m=0$.

**Lemma 4.12.** *The function $T_N(x;z_0,z_1)$ is symmetric in $(x_1\dots x_N)$.*

*Proof.* We can rewrite (4.21) in the form:

$$W_N(\dots x_{i+1},x_i\dots) = \check{R}_i(x_{i+1}/x_i)W_N(x)\check{R}_i(x_i/x_{i+1})$$

Using this equation we compute:

$$\begin{aligned}
T_N(\dots x_{i+1},x_i\dots;z_0,z_1) &= \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N}\langle\alpha| W_N(\dots x_{i+1},x_i\dots)|\alpha\rangle\\
&= \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N}\langle\alpha| \check{R}_i(x_{i+1}/x_i)W_N(x)\check{R}_i(x_i/x_{i+1})|\alpha\rangle\\
&= \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N}\langle\alpha| \check{R}_i(x_i/x_{i+1})\check{R}_i(x_{i+1}/x_i)W_N(x)|\alpha\rangle\\
&= \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N}\langle\alpha| W_N(x)|\alpha\rangle = T_N(x;z_0,z_1)
\end{aligned}$$

where we also used the cyclicity of the trace and the unitarity (4.10). □

**Theorem 4.13.** *The function $T_N(x;z_0,z_1)$ can be written as a shuffle product:*

$$T_N(x;z_0,z_1) = \sum_{k=0}^{N} z_0^{N-k} z_1^k \frac{(1-t^{-1})^k}{(1-qt^{-1})^k} H_k(t^{-1}) * E_{N-k}(tq^{-1}) \tag{4.51}$$

*Proof.* Let us use the relation (4.35) between $W_N(x)$ and $\widetilde{W}_N(x)$ in Definition 4.11 of $T_N(x; z_0, z_1)$:

$$T_N(x; z_0, z_1) = \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N} \langle\alpha| F_N^{-1}(x)\widetilde{W}_N(x)F_N(x)|\alpha\rangle$$

Using the cyclicity of the trace we get:

$$T_N(x; z_0, z_1) = \sum_{\alpha\in\{0,1\}^N} z_{\alpha_1}\cdots z_{\alpha_N} \widetilde{W}_\alpha^\alpha(x) \tag{4.52}$$

We can rewrite the summation over $\alpha \in \{0,1\}^N$ as a sum over $k = 0\dots N$ and a sum over permutations of $(0^{N-k}1^k)$:

$$T_N(x; z_0, z_1) = \sum_{k=0}^{N} \frac{1}{k!(N-k)!} z_0^{N-k} z_1^k \sum_{\sigma\in\mathcal{S}_N} \widetilde{W}_{\sigma(0^{N-k}1^k)}^{\sigma(0^{N-k}1^k)}(x) \tag{4.53}$$

The division by the factorials compensates the over-counting when $\sigma$ permutes 0's or 1's. In the above formula the trace from (4.52) is written as a sum over permutations $\sigma$ acting on the indices of the matrix elements of $W_N(x)$, below we will show that this action can be transferred to the action of $\sigma$ which instead permutes the spectral parameters. The diagonal elements $\widetilde{W}_\alpha^\alpha(x)$ in (4.53) can be computed with the help of Proposition 4.10 in which we set $y_i = qx_i$:

$$\widetilde{W}_\alpha^\alpha(x) = D_k(x_S)\prod_{i\in S}\prod_{j\in S^c} \frac{x_i - tx_j}{x_i - x_j}\frac{qx_i - x_j}{qx_i - tx_j} = D_k(x_S)\prod_{i\in S}\prod_{j\in S^c}\zeta(x_i/x_j) \tag{4.54}$$

where $D_k(x) := D_k(x_1\dots x_k; qx_1\dots qx_k)$, the subset $S \subseteq [N]$ records the positions of 1's in $\alpha$, $S^c \subseteq [N]$ is the complement subset to $S$ and thus it records the positions of 0's in $\alpha$. The domain-wall partition function $D_k(x)$ can be computed as a determinant using Lemma 4.9 in the special case $y_i = qx_i$. The determinant formula for $D_k(x)$ matches with the definition of $H_k(t^{-1})$ in (3.13) up to a factor:

$$D_k(x) = \frac{(1-t^{-1})^k}{(1-qt^{-1})^k} H_k(t^{-1}) \tag{4.55}$$

Inserting (4.55) into (4.54) and then inserting the result into (4.53) produces:

$$\begin{aligned} T_N(x; z_0, z_1) = \sum_{k=0}^{N} z_0^{N-k} z_1^k \frac{(1-t^{-1})^k}{(1-qt^{-1})^k} \\ \frac{1}{k!(N-k)!}\sum_{\sigma\in\mathcal{S}_N} H_k(t^{-1}; x_{\sigma(N-k+1)}\dots x_{\sigma(N)}) \prod_{j=1}^{N-k}\prod_{i=N-k+1}^{N}\zeta(x_{\sigma(i)}/x_{\sigma(j)}) \end{aligned} \tag{4.56}$$

After reordering the summation over $\sigma$ we can match the expression in the second line with the shuffle product (3.4) of $H_k(t^{-1})$ and $E_{N-k}(tq^{-1})$ which verifies (4.51). ☐

**Corollary 4.14.** *Consider the generating function of $T_N(x; z_0, z_1)$:*

$$T(v; z_0, z_1) = \sum_{N=0}^{\infty} v^N T_N(x; z_0, z_1) \tag{4.57}$$

*We have:*

$$T(v; z_0, z_1) = \exp_*\left(\sum_{r>0}\frac{1}{r}\frac{1-t^r}{1-q^r}\left(-z_1^r - \frac{q^r - t^r}{1-t^r}z_0^r\right)v^r S_r\right) \tag{4.58}$$

*Proof.* We compute $T(v)$ using (4.51):

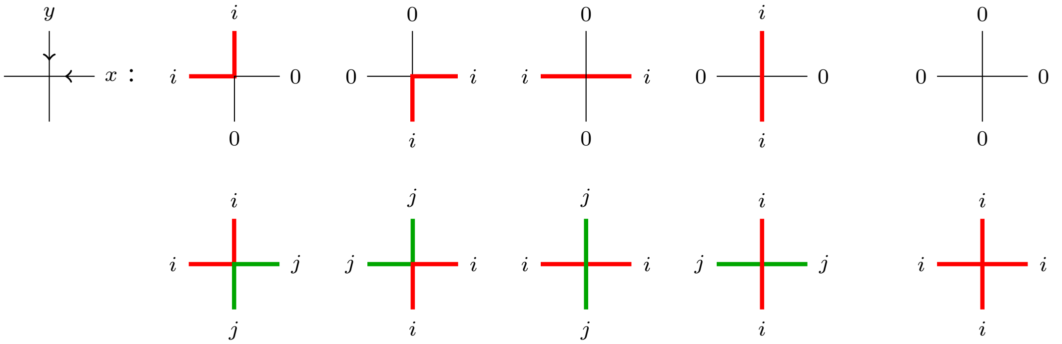
$$\begin{aligned}\sum_{N=0}^{\infty} v^N T_N(x;z_0,z_1) &= \sum_{N=0}^{\infty} v^N \sum_{k=0}^{N} z_0^{N-k} z_1^k \frac{(1-t^{-1})^k}{(1-qt^{-1})^k} H_k(t^{-1}) * E_{N-k}(tq^{-1}) \\ &= \sum_{k=0}^{\infty} (vz_1)^k \frac{(1-t^{-1})^k}{(1-qt^{-1})^k} H_k(t^{-1}) * \sum_{l=0}^{\infty} (vz_0)^l E_l(tq^{-1}) \\ &= \exp_* \left( -\sum_{r>0} \frac{1}{r} \frac{1-t^r}{1-q^r} v^r z_1^r S_r \right) \exp_* \left( -\sum_{r>0} \frac{1}{r} \frac{q^r-t^r}{1-q^r} v^r z_0^r S_r \right) \\ &= \exp_* \left( \sum_{r>0} \frac{1}{r} \frac{1-t^r}{1-q^r} \left( -z_1^r - \frac{q^r-t^r}{1-t^r} z_0^r \right) v^r S_r \right)\end{aligned}$$

where in the second line we reordered the two summations and in the third line we used the exponential generating functions (3.16) and (3.17) for $H_k(t^{-1})$ and $E_k(tq^{-1})$ and finally we combined the two shuffle-exponentials into a single shuffle-exponential. The last step is possible because of the shuffle-commutativity of $H_k(t^{-1})$ and $E_k(tq^{-1})$. ☐

## 5. The $sl(n+1|m)$ vertex model and the shuffle algebra $\mathcal{A}^\circ$

The aim of this section is to prove Theorem 1.1 in the full generality. We will follow the same logic as in Section 4. The main input in Section 4 is the six vertex $R$-matrix. In this section, we will replace it with the $R$-matrix of $U_t(\widehat{sl}(n+1|m)$, a supersymmetric algebra whose representations are given by $\mathbb{Z}_2$ graded vector spaces. The grading splits vectors into two categories which graphically correspond to having bosonic and fermionic lattice paths. In this sense a red path in the six vertex model case is a bosonic path and these paths contribute the $H_k(t^{-1})$ part in the formula (4.51). In this section we will obtain a generalization of the formula (4.51) which, in addition to the bosonic contributions $H_k(t^{-1})$, will have fermionic contributions of $E_k(t^{-1})$.

5.1. **Coloured lattice models.** We use the same diagrammatic formalism as in Section 4. Consider a square lattice as in (4.1) where every edge of a vertex is labelled by $0 \dots n+m$ where 0 denotes the absence of a path, $i > 0$ stands for a coloured path of colour "$i$". If $i \le n$ then the path is called bosonic and if $i > n$ the path is called fermionic. Specifying the boundary conditions in (4.1) means assigning labels to the $4N$ external edges which take values in $\{0 \dots n+m\}$. For colours $i < j$ and "0" being the greatest colour, we list all the possible types of vertices with their corresponding Boltzmann weights:

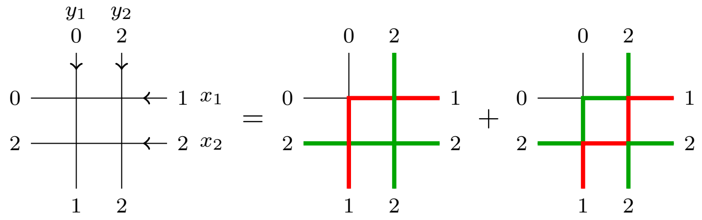


(5.1)

$$\frac{1-t}{1-tx/y} \qquad \frac{(1-t)x/y}{1-tx/y} \qquad \frac{t(1-x/y)}{1-tx/y} \qquad \frac{1-x/y}{1-tx/y} \qquad \begin{cases} 1 & i \le n \\ \dfrac{x/y-t}{1-tx/y} & i > n \end{cases}$$

The weights of all the other vertices are set to 0. Note that the difference between fermionic and bosonic paths shows in the last vertex in (5.1), i.e. when two paths of the same colour touch each other.

Consider an example of (4.1) with $N = 2$ with one bosonic (red) and one fermionic (green) path and a choice of boundary conditions:

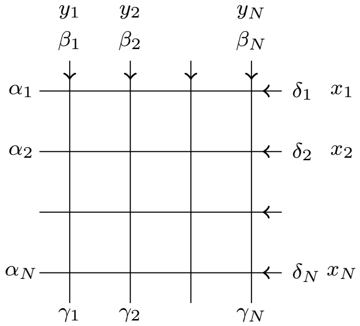


$$= \frac{(1-t)x_1/y_1\,(1-x_2/y_1)\,t\,(1-x_1/y_2)\,(x_2/y_2-t)}{(1-tx_1/y_1)\,(1-tx_2/y_1)\,(1-tx_1/y_2)\,(1-tx_2/y_2)} + \frac{(1-t)^4 x_1/y_1\,x_2/y_1\,x_1/y_2}{(1-tx_1/y_1)\,(1-tx_2/y_1)\,(1-tx_1/y_2)\,(1-tx_2/y_2)}$$

Summing over all possible labels of the internal edges on the l.h.s. of (5.2) gives two possible configurations with non-zero Boltzmann weights which are computed with (5.1) and presented in the second line.

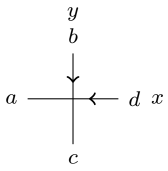
**Definition 5.1.** *Let $\alpha, \beta, \gamma, \delta \in \{0, 1 \ldots n+m\}^N$ (cf. Definition 4.1). The supersymmetric coloured partition function $Z^{\beta,\delta}_{\alpha,\gamma}(x_1 \ldots x_N; y_1 \ldots y_N)$ is defined as the rational function in the spectral parameters equal to the weighted sum over all possible configurations computed with* (5.1) *and the boundary conditions specified below:*

$$Z^{\beta,\delta}_{\alpha,\gamma}(x;y) = \tag{5.3}$$



The supersymmetric $\check{R}$-matrix acts in $V_y \otimes V_x$ with $V_y, V_x \simeq \mathbb{C}^{n+m+1}$. Let $|0\rangle = (1, 0 \ldots 0)^T$ and $|i\rangle = (0 \ldots 0, 1, 0 \ldots 0)^T$, with 1 being on the $i$-th position, denote the standard basis in $\mathbb{C}^{n+m+1}$ and $|i_1 \ldots i_N\rangle$, with $i_1, \ldots, i_N \in \{0, 1 \ldots n+m+1\}$, its generalization to the $N$-fold tensor product of $\mathbb{C}^{n+m+1}$. Define similarly the dual basis, then we have:

$$\check{R}(x/y) = \sum_{a,b,c,d=0,1\ldots n+m+1} \left[ \quad \right] |a,c\rangle\,\langle b,d| \tag{5.4}$$



where the explicit form of the matrix elements is given by the Boltzmann weights (5.1). The permutation matrix $P$ acts in $\mathbb{C}^{n+m+1} \otimes \mathbb{C}^{n+m+1}$. The supersymmetric $R$-matrix is defined as before:

$$R(x) = P\check{R}(x) \tag{5.5}$$

Let id be the identity matrix in $\mathbb{C}^{n+m+1}$ then $\check{R}$, $R$ and $P$ act in $\otimes_{i=1}^{N} V_{x_i}$ by $\check{R}_i(x_{i+1}/x_i)$, $R_i(x_{i+1}/x_i)$ and $P_i$ respectively (see e.g. (4.8)). For these matrices we have the Yang–Baxter equation (4.9) and the same unitarity relation as before (4.10). We will also use the same graphical representation for $\check{R}_i(x_{i+1}/x_i)$ as before.

The tensors $Z_N(x;y), W_N(x;y)$ and $W_N(x)$ in the $sl(n+1|m)$ case are defined algebraically by (4.12), (4.13) and (4.14) respectively and their matrix elements are

$$Z^{\beta,\delta}_{\alpha,\gamma}(x;y), \quad W^{\delta}_{\gamma}(x;y) \quad \text{and } W^{\delta}_{\gamma}(x)$$

with the indices $\alpha,\beta,\gamma,\delta \in \{0,1\dots n+m\}^N$. The graphical interpretation of these matrix elements is given by Definition 5.1 and appropriate specialization of the indices and parameters in the case of $W^{\delta}_{\gamma}(x;y)$ and $W^{\delta}_{\gamma}(x)$. We have the analogue of Lemma 4.3.

**Lemma 5.2.** *For $i = 1\dots N-1$, the tensor $Z_N(x;y)$ satisfies the following exchange relations:*

$$\check{R}_i(x_{i+1}/x_i)Z_N(x;y) = Z_N(\dots x_{i+1}, x_i \dots; y)\check{R}_{N+i}(x_{i+1}/x_i) \tag{5.6}$$

$$Z_N(x;y)\check{R}_i(y_i/y_{i+1}) = \check{R}_{N+i}(y_i/y_{i+1})Z_N(x;\dots y_{i+1}, y_i \dots) \tag{5.7}$$

*The tensor $W_N(x;y)$ satisfies:*

$$W_N(x;y) = W_N(\dots x_{i+1}, x_i \dots; y)\check{R}_i(x_{i+1}/x_i) \tag{5.8}$$

$$W_N(x;y) = \check{R}_i(y_i/y_{i+1})W_N(x;\dots y_{i+1}, y_i \dots) \tag{5.9}$$

*and for $W_N(x)$ we have:*

$$\check{R}_i(x_{i+1}/x_i)W_N(x) = W_N(\dots x_{i+1}, x_i \dots)\check{R}_i(x_{i+1}/x_i) \tag{5.10}$$

This Lemma is based on the same algebraic identities as Lemma 4.3.

5.2. **The $sl(n+1|m)$ $F$-matrix and transformed tensors.** In this section we introduce the supersymmetric $sl(n+1|m)$ $F$-matrix [44]. This matrix satisfies the same properties as the six vertex $F$-matrix therefore we will immediately get the transformed tensors $\widetilde{Z}_N(x;y), \widetilde{W}_N(x;y)$ and $\widetilde{W}_N(x)$ with the desired properties establishes in Section 4.2.

**Definition 5.3.** *For $k,l \in \{0\dots n+m\}$ and $r = 1\dots N$, let $E^{(kl)}_r \in \operatorname{End}\left(\otimes_{i=1}^N V_{x_i}\right)$ be the matrix acting non-trivially on the $r$-th tensor space $V_{x_r}$ by the $(n+m+1)\times(n+m+1)$ unit matrix $E^{(kl)}$. The $N$-site supersymmetric $sl(n+1|m)$ $F$-matrix reads:*

$$F_N(x_1\dots x_N) = \sum_{\sigma\in\mathcal{S}_N}\ \sum_{(k_1\dots k_N)\in\mathcal{J}_\sigma} \left(\prod_{\substack{1\le i<j\le N\\ k_i,k_j>n}} \frac{x_{\sigma(i)}+x_{\sigma(j)}}{x_{\sigma(i)}-tx_{\sigma(j)}}\right) \prod_{i=1}^N E^{(k_ik_i)}_{\sigma(i)} R_\sigma \tag{5.11}$$

*where $R_\sigma$ is defined as before* (4.26) *but based on the $sl(n+1|m)$ $R$-matrix and the set $\mathcal{J}_\sigma$ is defined by:*

$$\mathcal{J}_\sigma = \Big\{0 \le k_1 \le \dots \le k_N \le n+m : k_i < k_{i+1} \text{ if } \sigma(i) > \sigma(i+1)\Big\} \tag{5.12}$$

*The $F$-matrix* (5.11) *satisfies the property:*

$$F_N(x_1\dots x_N) = P_\sigma F_N(x_{\sigma(1)}\dots x_{\sigma(N)})\check{R}_\sigma \tag{5.13}$$

We note that the $F$-matrix in the supersymmetric case is modified compared to (4.29) by the additional rational function in (5.11). Each factor of this rational function is associated to a pair of fermionic labels, therefore if $m = 0$, this factor disappears and one recovers the standard $sl(n+1)$ $F$-matrix.

**Lemma 5.4.** *Fix a composition of non-negative integers $(l_0, l_1\dots l_{n+m})$ such that $N = l_0+\dots+l_{n+m}$. For every such composition we define two ordered compositions $\lambda^-$ and $\lambda^+$:*

$$\lambda^- := (0^{l_0}1^{l_1}\dots(n+m)^{l_{n+m}}), \qquad \lambda^+ := ((n+m)^{l_{n+m}}\dots 1^{l_1}0^{l_0}) \tag{5.14}$$

*The supersymmetric $F$-matrix has the properties:*

$$\left\langle \lambda^- \right| F_N(x) = \left\langle \lambda^- \right| \prod_{\substack{1\leq i<j\leq N\\ \lambda_i^-,\lambda_j^->n}} \frac{x_i+x_j}{x_i-tx_j}, \qquad F_N(x)\left|\lambda^+\right\rangle = f_N(x) \prod_{\substack{1\leq i<j\leq N\\ \lambda_i^+,\lambda_j^+>n}} \frac{x_i+x_j}{x_i-tx_j}\left|\lambda^+\right\rangle \tag{5.15}$$

*where $f_N(x)$ is a rational function define by:*

$$f_N(x) := \prod_{\substack{0\leq i<j\leq N\\ \lambda_i^+\neq\lambda_j^+}} \begin{cases} \dfrac{x_i-x_j}{x_i-tx_j} & \text{if } \lambda_j^+=0\\ t\dfrac{x_i-x_j}{x_j-tx_i} & \text{if } \lambda_i^+,\lambda_j^+>n\\ t\dfrac{x_i-x_j}{x_i-tx_j} & \text{else}\end{cases} \tag{5.16}$$

**Definition 5.5.** *Let $F_N$ be the supersymmetric $F$-matrix* (5.11). *Introduce transformed tensors $\widetilde{Z}_N(x;y)$, $\widetilde{W}_N(x;y)$ and $\widetilde{W}_N(x)$:*

$$\widetilde{Z}_N(x;y) := (F_N(x)\otimes F_N(y))\, Z_N(x;y)\left(F_N^{-1}(y)\otimes F_N^{-1}(x)\right) \tag{5.17}$$

$$\widetilde{W}_N(x;y) := F_N(y)W_N(x;y)F_N^{-1}(x) \tag{5.18}$$

$$\widetilde{W}_N(x) := F_N(x)W_N(x)F_N^{-1}(x) \tag{5.19}$$

**Proposition 5.6.** *For $i=1\dots N-1$, the tensor $\widetilde{Z}_N(x;y)$ satisfies the following exchange relations:*

$$P_i\widetilde{Z}(x;y) = \widetilde{Z}(\dots x_{i+1},x_i\dots;y)P_{N+i} \tag{5.20}$$

$$\widetilde{Z}(x;y)P_i = P_{N+i}\widetilde{Z}(x;\dots y_{i+1},y_i\dots) \tag{5.21}$$

*the tensor $\widetilde{W}_N(x;y)$ satisfies:*

$$\widetilde{W}_N(x;y)P_i = \widetilde{W}_N(\dots x_{i+1},x_i\dots;y) \tag{5.22}$$

$$P_i\widetilde{W}_N(x;y) = \widetilde{W}_N(x;\dots y_{i+1},y_i\dots) \tag{5.23}$$

*and $\widetilde{W}_N(x)$ satisfies:*

$$P_i\widetilde{W}_N(x) = \widetilde{W}_N(\dots x_{i+1},x_i\dots)P_i \tag{5.24}$$

The proof of this proposition is the same as the proof of Proposition 4.8 since the algebraic properties of the more general objects are the same as in the six vertex case. In the remaining part of this subsection we compute the matrix elements of $\widetilde{W}_N(x;y)$ generalizing the result of Proposition 4.10. The formula is again a product consisting of fully factorized terms and domain-wall partition functions:

$$D_M^{(k)}(x;y) := W_{(k^M)}^{(k^M)}(x_1\dots x_M;y_1\dots y_M). \tag{5.25}$$

When the label $k$ is fermionic (i.e. $k>n$) the domain-wall partition function becomes the fermionic version of the six vertex domain-wall partition function and is known to have a factorized formula [21]. We summarize the formulas for $D_M^{(k)}(x;y)$ for different $k$ in the following Lemma.

**Lemma 5.7.** *$D_M^{(k)}(x;y)$ is the domain-wall partition function associated to the label $k$:*

$$D_M^{(k)}(x;y) = \begin{cases} 1 & k=0\\ D_M(x;y) & 0<k\leq n\\ (-1+t)^M\dfrac{\prod_{i=1}^M x_i\prod_{j=i+1}^M(x_j-tx_i)(y_i-ty_j)}{\prod_{i,j=1}^M(tx_i-y_j)} & n<k\leq n+m\end{cases} \tag{5.26}$$

*where $D_M(x;y)$ is defined in* (4.44).

**Proposition 5.8.** *Fix a composition of non-negative integers* $(l_0, l_1 \dots l_{n+m})$ *such that* $N = l_0 + \dots + l_{n+m}$*. Let* $\alpha, \beta$ *be two permutations of* $(0^{l_0} 1^{l_1} \dots (n+m)^{l_{n+m}})$ *and* $P(k), S(k) \subseteq [N]$ *be pairs of compositions which record the positions of* $k$*'s in* $\alpha$ *and* $\beta$ *respectively, then:*

$$\widetilde{W}_{\alpha}^{\beta}(x;y) = \prod_{k=0}^{n+m} D_{l_k}^{(k)}(x_{S(k)}, y_{P(k)}) \times \prod_{k_1=0}^{n+m} \prod_{k_2=k_1+1}^{n+m} \prod_{j\in S(k_1)} \prod_{i\in S(k_2)} \frac{x_i - t x_j}{x_i - x_j} \cdot \prod_{j\in S(k_1)} \prod_{i\in P(k_2)} \frac{y_i - x_j}{y_i - t x_j} \tag{5.27}$$

$$\prod_{k_1=n+1}^{n+m} \prod_{k_2=k_1+1}^{n+m} \prod_{i\in P(k_1)} \prod_{j\in P(k_2)} \frac{y_i + y_j}{y_i - t y_j} \cdot \prod_{i\in S(k_1)} \prod_{j\in S(k_2)} \frac{x_i - t x_j}{x_i + x_j} \times \prod_{k=n+1}^{n+m} \prod_{\substack{i,j\in P(k)\\ i<j}} \frac{y_i + y_j}{y_i - t y_j} \cdot \prod_{\substack{i,j\in S(k)\\ i<j}} \frac{x_i - t x_j}{x_i + x_j}$$

*Proof.* The proof is analogous to the proof of Proposition 4.10. Define two compositions $\lambda^{\pm}$:

$$\lambda^{-} := (0^{l_0} 1^{l_1} \dots (n+m)^{l_{n+m}}), \qquad \lambda^{+} := ((n+m)^{l_{n+m}} \dots 1^{l_1} 0^{l_0})$$

For two permutations $\sigma, \tau \in \mathcal{S}_N$ let $\alpha = \sigma^{-1}(\lambda^-)$ and $\beta = \tau^{-1}(\lambda^-)$. Represent the vectors $\langle \alpha |$ and $| \beta \rangle$ as follows:

$$\langle \alpha | = \left\langle \lambda^- \right| P_{\sigma^{-1}}, \qquad | \beta \rangle = P_{\tau} \left| \lambda^- \right\rangle$$

Different matrix elements $\widetilde{W}_{\alpha}^{\beta}(x;y)$ are related to each other by appropriately permuting the spectral parameters:

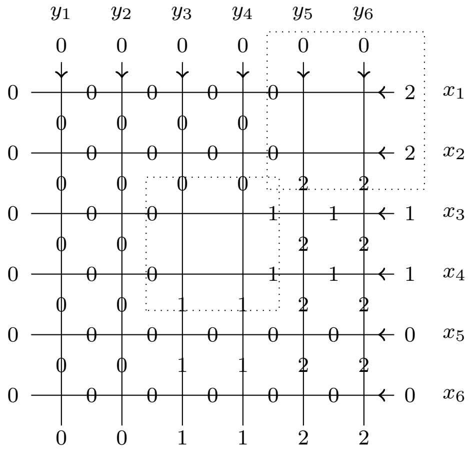
$$\widetilde{W}_{\alpha}^{\beta}(x;y) = \langle \alpha | \widetilde{W}_N(x;y) | \beta \rangle = \left\langle \lambda^- \right| P_{\sigma^{-1}} \widetilde{W}_N(x;y) P_{\tau} \left| \lambda^- \right\rangle = \left\langle \lambda^- \right| \widetilde{W}_N(x_{\tau}; y_{\sigma}) \left| \lambda^- \right\rangle = \widetilde{W}_{\lambda^-}^{\lambda^-}(x_{\tau}; y_{\sigma}) \tag{5.28}$$

where we used the properties (5.22) and (5.23) of $\widetilde{W}_{\alpha}^{\beta}(x;y)$. Consider next the matrix element $\widetilde{W}_{\lambda^-}^{\lambda^+}(x;y)$ and write it in terms of $W_{\lambda^-}^{\lambda^+}(x;y)$ using (5.18):

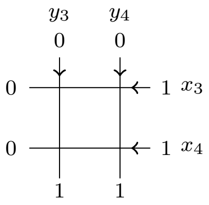
$$\begin{aligned} \widetilde{W}_{\lambda^-}^{\lambda^+}(x;y) &= \left\langle \lambda^- \right| F_N(y) W_N(x;y) F_N^{-1}(x) \left| \lambda^+ \right\rangle \\ &= \frac{1}{f_N(x)} \prod_{\substack{1\le i<j\le N\\ \lambda_i^+, \lambda_j^+ > n}} \frac{1 - t x_j / x_i}{1 + x_j / x_i} \prod_{\substack{1\le i<j\le N\\ \lambda_i^-, \lambda_j^- > n}} \frac{1 + y_j / y_i}{1 - t y_j / y_i} \times W_{\lambda^-}^{\lambda^+}(x;y) \end{aligned} \tag{5.29}$$

where in the second line we used the properties (5.15) of the supersymmetric $F$-matrix. The matrix element $W_{\lambda^-}^{\lambda^+}(x;y)$ can be evaluated in a similar way as the matrix element $W_{(0^{N-k}1^k)}^{(1^k 0^{N-k})}(x;y)$ in the proof of Proposition 4.10. For demonstration we choose $N = 6$, $m = n = 1$ and $l_0 = l_1 = l_2 = 2$. The graphical depiction of the partition function $W_{(001122)}^{(221100)}(x;y)$ shows several regions which are frozen and two regions which contain domain-wall type configurations which we enclosed in the dotted squares for the reference:

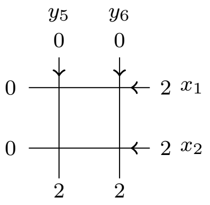
$$W_{(001122)}^{(221100)}(x;y) =$$



$$= \prod_{i\in\{3,4,5,6\}} \prod_{j\in\{5,6\}} \frac{y_i - x_j}{y_i - tx_j} \prod_{i\in\{5,6\}} \prod_{j\in\{3,4\}} t\frac{y_i - x_j}{y_i - tx_j} \times$$

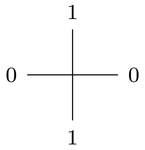


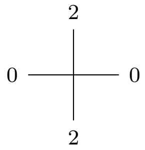
$\times$

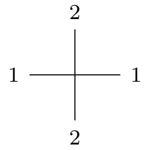


$$= \prod_{i\in\{3,4,5,6\}} \prod_{j\in\{5,6\}} \frac{y_i - x_j}{y_i - tx_j} \prod_{i\in\{5,6\}} \prod_{j\in\{3,4\}} t\frac{y_i - x_j}{y_i - tx_j} \times D^{(1)}_2(x_3, x_4; y_3, y_4) D^{(2)}_2(x_1, x_2; y_5, y_6) \tag{5.30}$$

The partial freezing of the last two columns in the graphical representation given in the first line in (5.30) is due to the fact that the 2's on the right boundary must take either left or down steps and drop to the bottom boundary in the last two columns. This leaves us with domain-wall configurations in the intersections of the first two rows and last two columns. The remaining part of the lattice is of the form (4.47). As a result we need to evaluate two domain-wall partition functions $D^{(1)}_2$ and $D^{(2)}_2$ as indicated in the second line in (5.30) and take into account various factors coming from the vertices:







where the first two are of the fourth type in (5.1) and the third one is of the third type in (5.1) (recall that 0 is considered the highest colour label).

Let us view the lattice configurations in the first line in (5.30) as a block matrix. The example of (5.30) shows that the anti-diagonal blocks must have domain-wall configurations, above the anti-diagonal the configurations are trivial with all edges labelled 0 and below the anti-diagonal we have blocks with vertices of type four and type three in (5.1). The latter blocks can be expressed in terms of $Z^{\alpha,\beta}_{\alpha,\beta}$ with $\alpha = (i^M)$ and $\beta = (j^M)$ for some $i, j$ and evaluated:

$$Z^{(i^M),(j^M)}_{(i^M),(j^M)}(x; y) = t^{\delta_{j>i}M^2} \prod_{a,b=1}^{M} \frac{y_a - x_b}{y_a - tx_b} \tag{5.31}$$

We can write $W^{\lambda^+}_{\lambda^-}(x; y)$ in terms of such block matrix:

$$W^{\lambda^+}_{\lambda^-}(x; y) = \prod_{a,b} \begin{pmatrix} 1 & \dots & 1 & \dots & W_{m+n} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ 1 & \dots & W_k & \dots & W_{k,m+n} \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ W_0 & \dots & W_{0,k} & \dots & W_{0,m+n} \end{pmatrix}_{a,b} \tag{5.32}$$

where we shortened the notations:

$$W_k = D^{(k)}_{\lambda_k}(x^{(k)}, y^{(k)}), \qquad W_{i,j} = Z^{(i^M),(j^M)}_{(i^M),(j^M)}(x^{(i)}, y^{(j)}) \tag{5.33}$$

and $(x^{(k)})$ and $(y^{(k)})$ denote a pair of $\lambda_k$ spectral parameters $x_a$ and $y_b$ with $a$ and $b$ such that $\lambda^+_a = k$ and $\lambda^-_b = k$ for all $a, b$. By evaluating the product in (5.32) we get:

$$W^{\lambda^+}_{\lambda^-}(x; y) = t^{\lambda_0(N-\lambda_0)} \prod_{k=0}^{n+m} D^{(k)}_{\lambda_k}(x^{(k)}, y^{(k)}) \prod_{0\le i<j\le n+m} Z^{(i^M),(j^M)}_{(i^M),(j^M)}(x^{(i)}; y^{(j)}) \tag{5.34}$$

This expression allows us to compute $\widetilde{W}_{\lambda^-}^{\lambda^+}(x;y)$ using (5.29) and by (5.28) we get the final expression (5.27) for all matrix elements $\widetilde{W}_{\alpha}^{\beta}(x;y)$. □

5.3. **Trace of $W_N(x)$ and the shuffle product.** In this section we define the supersymmetric partition function $T_N(x)$ as the trace of the matrix $W_N(x)$. We show that the trace can be rewritten as a symmetrization over the parameters $x_1 \dots x_N$ acting on specific diagonal elements of $\widetilde{W}_N(x)$. These diagonal elements can be computed using Proposition 5.8. The result of this computation, summarized in Theorem 5.11, gives a formula for $T_N(x)$ in terms of the shuffle product of $\mathcal{A}^\circ$.

**Definition 5.9.** *Let $z_0, z_1 \dots z_n$ and $w_1 \dots w_m$ be two sets of indeterminates. Define the following function:*

$$T_N(x;z,w) := \sum_{\alpha\in\{0,1\dots n+m\}^N} \prod_{i=0}^{n} z_{\alpha_i} \prod_{i=1}^{m} (-w_{\alpha_{n+i}}) \langle\alpha| W_N(x) |\alpha\rangle \tag{5.35}$$

The graphical counterpart of (5.35) is given by (4.50) with vertices interpreted as in (5.1). Using this we can see that the function $T_N(x;z,w)$ is equal to the conic partition function $Z_N(x;z,w)$ which was defined in the introduction (1.4) using lattice paths. The matrix $W_N(x)$ is block diagonal with blocks labelled by a collection of non-negative integers $(l_0, l_1 \dots l_{n+m})$ where $l_k$ is the number of $k$'s in $\alpha$ and $\beta$ in the matrix elements $W_\alpha^\beta(x)$. This means that $T_N(x;z,w)$ is the (graded) trace of $W_N(x)$ where the monomials in $z$ and $w$ parameterize different blocks of $W_N(x)$.

**Lemma 5.10.** *The function $T_N(x;z,w)$ is symmetric in $(x_1 \dots x_N)$.*

The proof follows the same lines as the proof of Lemma 4.12. We arrive at the main theorem of this section.

**Theorem 5.11.** *The function $T_N(x;z,w)$ can be written as a shuffle product:*

$$T_N(x;z,w) = \sum_{\substack{l_0\dots l_{n+m}\geq 0\\ l_0+\dots+l_{n+m}=N}} \prod_{i=0}^{n} z_i^{l_i} \prod_{i=1}^{m} (-w_i)^{l_{n+i}} \frac{(1-t^{-1})^{N-l_0}}{(1-qt^{-1})^{N-l_0}}$$
$$E_{l_{n+m}}(t^{-1}) * \dots * E_{l_{n+1}}(t^{-1}) * H_{l_n}(t^{-1}) * \dots * H_{l_1}(t^{-1}) * E_{l_0}(tq^{-1}) \tag{5.36}$$

*Proof.* By the same arguments leading to (4.53) we convert the trace in (5.35) into a summation over the symmetric group:

$$T_N(x;z,w) = \sum_{\substack{l_0\dots l_{n+m}\geq 0\\ l_0+\dots+l_{n+m}=N}} \frac{1}{l_0!\cdots l_{n+m}!} \prod_{i=0}^{n} z_i^{l_i} \prod_{i=1}^{m} (-w_i)^{l_{n+i}} \sum_{\sigma\in\mathcal{S}_N} \widetilde{W}_{\sigma(\lambda^-)}^{\sigma(\lambda^-)}(x) \tag{5.37}$$

where $\lambda^- = (0^{l_0} 1^{l_1} \dots (n+m)^{l_{n+m}})$. Let us show that the summation over $\sigma$ can be written as a shuffle product. The diagonal elements $\widetilde{W}_\alpha^\alpha(x)$ in (5.37) can be computed with the help of Proposition 5.8 in which we set $y_i = qx_i$:

$$\widetilde{W}_\alpha^\alpha(x) = \prod_{k=0}^{n+m} D_{l_k}^{(k)}(x_{S(k)}) \times \prod_{k_1=0}^{n+m} \prod_{k_2=k_1+1}^{n+m} \prod_{j\in S(k_1)} \prod_{i\in S(k_2)} \frac{x_i - tx_j}{x_i - x_j} \frac{qx_i - x_j}{qx_i - tx_j} \tag{5.38}$$

where $D_M^{(k)}(x) := D_M^{(k)}(x_1 \dots x_M; qx_1 \dots qx_M)$ and $S(k) \in [N]$ is the subset that records the positions of $k$'s in $\alpha$. The factors $D_{l_k}^{(k)}(x)$ are the domain-wall partition functions from Lemma 5.7 in the special case

$y_i = qx_i$. We can match these partition functions with the shuffle algebra functions:

$$D_M^{(k)}(x) = \begin{cases} E_M(tq^{-1}) & k = 0 \\ \dfrac{(1-t^{-1})^M}{(1-qt^{-1})^M} H_M(t^{-1}) & 0 < k \leq n \\ \dfrac{(1-t^{-1})^M}{(1-qt^{-1})^M} E_M(t^{-1}) & n < k \leq n+m \end{cases} \tag{5.39}$$

where we remind that $E_M(tq^{-1}) = 1$ but considered to be a function in $(x_1 \ldots x_M)$. By inserting (5.39) into (5.38) and recalling the definition of $\zeta$ in (3.2) we get:

$$\widetilde{W}_\alpha^\alpha(x) = \frac{(1-t^{-1})^{N-l_0}}{(1-qt^{-1})^{N-l_0}} E_{l_0}(tq^{-1}; x_{S(0)}) \prod_{k=1}^{n} H_{l_k}(t^{-1}; x_{S(k)}) \prod_{k=n+1}^{n+m} E_{l_k}(t^{-1}; x_{S(k)}) \tag{5.40}$$
$$\prod_{k_1=0}^{n+m} \prod_{k_2=k_1+1}^{n+m} \prod_{i \in S(k_1)} \prod_{j \in S(k_2)} \zeta(x_j/x_i)$$

Inserting this formula for $\widetilde{W}_\alpha^\alpha(x)$ into (5.37) gives an expression which can be rewritten using the shuffle product (3.4). Note that the positions of the variables $x_j$ and $x_i$ in $\zeta(x_j/x_i)$ decide the order in which the shuffle product is taken: the functions ($H$ or $E$) which depend on $S(k)$ with higher values of $k$ should be placed on the left. This explains the reason for the ordering of the factors in the shuffle product in (5.36). ☐

**Corollary 5.12.** *Consider the generating function of $T_N(x; z, w)$:*

$$T(v; z, w) = \sum_{N=0}^{\infty} v^N T_N(x; z, w) \tag{5.41}$$

*We have:*

$$T(v; z, w) = \exp_* \left( \sum_{k>0} \frac{1}{k} \frac{1-t^k}{1-q^k} \left( \sum_{i=1}^{m} w_i^k - \sum_{i=1}^{n} z_i^k - \frac{q^k - t^k}{1-t^k} z_0^k \right) v^k S_k \right) \tag{5.42}$$

*Proof.* We first note that the elements $E_k(t^{-1})$, $H_k(t^{-1})$ and $E_k(tq^{-1})$ belong to the commutative shuffle algebra $\mathcal{A}^\circ$. The proof of (5.42) follows by writing (5.41) with $T_N$ given by (5.36). Then summing over $N$ produces a shuffle product of $m$ generating functions of $E_k(t^{-1})$, $n$ generating functions of $H_k(t^{-1})$ and a single generating function of $E_k(tq^{-1})$ where all generating functions have different generating parameters. Then we can use the exponential generating functions (3.16) and (3.17) which allows us to rewrite $T(v; z, w)$ in the form given in (5.42). The last step is possible because we are dealing with commutative elements. ☐

The partition function $T_N(v; z, w)$ is equal to $Z_N$ from the introduction (1.4). In Theorem 1.1 we wrote $Z_N$ in terms of yet another conic partition function $L_N$. This result follows from (5.45) below.

**Definition 5.13.** *Consider the matrix $W_N(x)$ associated to $U_t(\widehat{sl}(2|1))$ vertex model, i.e. in the matrix elements $W_\gamma^\delta(x)$, with $\gamma, \delta \in \{0, 1, 2\}^N$, the label 1 denotes the bosonic colour and 2 denotes the fermionic colour. We define $L_N(x)$ to be the following partition function:*

$$L_N(x) := \sum_{\alpha \in \{1,2\}^N} (-1)^{m_2(\alpha)} m_2(\alpha) \langle \alpha | W_N(x) | \alpha \rangle \tag{5.43}$$

*where $m_2(\alpha)$ is the multiplicity of the fermionic index 2 in $\alpha$.*

Using the same methods as in the proof of Theorem 5.11 we can write $L_N(x)$ in terms of a shuffle product of $H_M(t^{-1})$ and $E_M(t^{-1})$ which by (5.39) are equal to $D_M^{(1)}$ and $D_M^{(2)}$ up to a factor. Therefore the partition function $L_N(x)$ can be expressed as a combination of the fermionic and bosonic six vertex domain-wall partition functions:

$$L_N(x) = \sum_{j=1}^{N} (-1)^j j D_{N-j}^{(1)} * D_j^{(2)} \tag{5.44}$$

The following proposition provides a lattice path realization of the shuffle algebra elements $S_N \in \mathcal{A}_N^\circ$.

**Proposition 5.14.** *The partition functions $L_N(x)$ and the elements $S_N \in \mathcal{A}_N^\circ$ are equal up to a factor:*

$$L_N(x) = \frac{1-t^N}{1-q^N} S_N(x) \tag{5.45}$$

*Proof.* Consider $T_N(x; z_0, z_1, w_1)$, i.e. the case $n = m = 1$ of $T_N$. Set $z_0 = 0, z_1 = 1$ and $w_1 = w$. We compute $T_N(x; 0, 1, w)$ using the definition (5.35) of $T_N$:

$$T_N(x; 0, 1, w) = \sum_{\alpha \in \{1,2\}^N} (-w)^{m_2(\alpha)} \langle \alpha | W_N(x) | \alpha \rangle \tag{5.46}$$

By comparing (5.43) with (5.46) we see that the functions $L_N(x)$ and $T_N(x; 0, 1, w)$ are related by:

$$L_N(x)) = \lim_{w \to 1} \frac{\partial}{\partial w} T_N(x; 0, 1, w) \tag{5.47}$$

Using (5.42) we compute the exponential generating function of $T_N(x; 0, 1, w)$:

$$\sum_{N=0}^{\infty} v^N T_N(x; 0, 1, w) = \exp_* \left( \sum_{k>0} \frac{1}{k} \frac{1-t^k}{1-q^k} \left( w^k - 1 \right) v^k S_k \right) \tag{5.48}$$

Next we take the sum over $N$ on both sides in (5.47) with $v^N$ and compute the derivative and the limit on the r.h.s. using (5.48). As a result we get (5.45). ☐

## 6. Skew Macdonald functions and lattice paths

In this section we derive the connection between the trace elements $T_N$ and the skew Macdonald functions $P_{\mu/\nu}$. This connection is based on the mixed Cauchy kernel $K(x; w)$, which is defined by applying the isomorphism $\iota_z$ to the Cauchy kernel $\Pi(z, w)$ (see [15]). Under the evaluation map $\mathrm{ev}_{\mu/\nu}$ the mixed Cauchy kernel $K(x; w)$ reproduces the skew Macdonald functions $P_{\mu/\nu}(w)$. Finally we relate the mixed Cauchy kernel with the generating function $T(v; z, w)$.

6.1. **The mixed Cauchy kernel.** Let us recall the commuting shuffle algebra elements $F_\lambda \in \mathcal{A}^\circ$ and the evaluation map (3.26) from Section 3. The functions $F_\lambda = F_\lambda(x_1 \dots x_k)$, with $\lambda \vdash k$, have an important property:

$$\mathrm{ev}_\mu(F_\lambda) = \delta_{\lambda,\mu} \frac{1}{d_\lambda}$$

Let us compute the evaluations $\mathrm{ev}_{\mu/\nu}$ of $F_\lambda$.

**Lemma 6.1.** *We have the product rule:*

$$F_\lambda * F_\nu = \sum_\mu q^{n(\mu')-n(\lambda')-n(\nu')} \frac{c_\lambda c_\nu}{c_\mu} f_{\lambda,\nu}^\mu F_\mu \tag{6.1}$$

*and evaluations:*

$$\mathrm{ev}_{\mu/\nu}\left(F_\lambda\right) = q^{n(\mu')-n(\lambda')-n(\nu')} \frac{c_\lambda c_\nu}{c_\mu d_{\mu/\nu}} f_{\lambda,\nu}^\mu \tag{6.2}$$

*Proof.* We recall the product rule for Macdonald functions (2.13) :

$$P_\lambda P_\nu = \sum_\mu f^\mu_{\lambda,\nu} P_\mu$$

and apply the isomorphism $\iota^{-1}$ to the l.h.s. and to the r.h.s. of this equation separately:

$$\iota^{-1}\left(P_\lambda P_\nu\right) = \frac{q^{n(\lambda')+n(\nu')}(1-t)^{|\lambda|+|\nu|}}{c_\lambda c_\nu} F_\lambda * F_\nu$$

$$\sum_\mu f^\mu_{\lambda,\nu} \iota^{-1}\left(P_\mu\right) = \sum_\mu \frac{q^{n(\mu')}(1-t)^{|\mu|}}{c_\mu} f^\mu_{\lambda,\nu} F_\mu$$

By comparing the two equations above and noting that $|\lambda| + |\mu| = |\nu|$ by the degree argument we verify (6.1). In order to prove (6.2) we act with (6.1) on $|\varnothing\rangle$ and recall (3.32) that $F_\nu |\varnothing\rangle = |\nu\rangle$:

$$F_\lambda |\nu\rangle = \sum_\mu q^{n(\mu')-n(\lambda')-n(\nu')} \frac{c_\lambda c_\nu}{c_\mu} f^\mu_{\lambda,\nu} |\mu\rangle$$

This equation allows us to compute the matrix elements $\langle\mu| F_\lambda |\nu\rangle$:

$$\langle\mu| F_\lambda |\nu\rangle = q^{n(\mu')-n(\lambda')-n(\nu')} \frac{c_\lambda c_\nu}{c_\mu} f^\mu_{\lambda,\nu}$$

and combining this with (3.27) leads to (6.2). ☐

Next we define the mixed Cauchy kernel $K(x;z)$ which we will expand in $F_\lambda(x)$ and apply to it the evaluation map $\mathrm{ev}_{\mu/\nu}$ using the result (6.2) of the above Lemma.

**Definition 6.2.** *Let* $(z)$ *and* $(w)$ *be two alphabets and* $\Pi(z,w)$ *the Cauchy kernel* (2.15). *Define the mixed Cauchy kernel:*

$$K(x;w) := \iota_z^{-1}\left(\Pi(z,w)\right) \tag{6.3}$$

**Remark 6.3.** *For* $K(x;w)$ *we have the analogues of the exponential formula* (2.15) *as well as bases expansions* (2.17) *for* $\Pi(z,w)$*:*

$$K(x;w) = \exp_*\left(\sum_{r>0} \frac{1}{r}\frac{1-t^r}{1-q^r}\frac{(t-q)^r}{(1-q)^r} p_r(w) S_r\right) \tag{6.4}$$

$$K(x;w) = \sum_\lambda \frac{(1-t^{-1})^{|\lambda|}}{(1-q)^{|\lambda|}} m_\lambda(w) E_\lambda(t^{-1}) \tag{6.5}$$

$$K(x;w) = \sum_\lambda \frac{q^{n(\lambda')}(1-t)^{|\lambda|}}{c'_\lambda} P_\lambda(w) F_\lambda \tag{6.6}$$

*Proof.* The exponential formula (6.4) is obtained by inserting the exponential formula (2.15) for the Cauchy kernel in (6.3) and applying $\iota_z$ to the power sums (3.44). The monomial expansion (6.5) is a consequence of (2.17) and the isomorphism formula (3.45)[4]. The Macdonald expansion (6.6) follows from (2.17), (2.9) and the isomorphism formula (3.41). ☐

**Lemma 6.4.** *Let* $\mu/\nu$ *be a skew partition and* $k = |\mu| - |\nu|$. *The mixed Cauchy kernel* $K(x;w)$ *reproduces the skew Macdonald function* $P_{\mu/\nu}(w)$ *under* $\mathrm{ev}_{\mu/\nu}$*:*

$$P_{\mu/\nu}(w) = \tilde{a}_{\mu,\nu} \mathrm{ev}_{\mu/\nu}\left(K(x;w)\right), \qquad \tilde{a}_{\mu,\nu} := \frac{q^{n(\nu')-n(\mu')} c'_\mu d_{\mu/\nu}}{(1-t)^k c'_\nu} \tag{6.7}$$

---

[4] The monomial expansion of $K(x;z)$ was studied in [15] in relation with the tableaux formula for the Macdonald functions.

*Proof.* We first note that the evaluation map $\mathrm{ev}_{\mu/\nu}$ acts on $F \in \mathcal{A}_j^\circ$ and gives zero unless $j = k$, the number of boxes in $\mu/\nu$. We apply $\mathrm{ev}_{\mu/\nu}$ to $K(x; w)$ written in the form (6.6):

$$\begin{aligned}\mathrm{ev}_{\mu/\nu}\left(K(x;w)\right) =&(1-t)^k \sum_{\lambda\vdash k} \frac{q^{n(\lambda')}}{c'_\lambda} P_\lambda(w) \mathrm{ev}_{\mu/\nu}\left(F_\lambda\right)\\ =&(1-t)^k \frac{q^{n(\mu')-n(\nu')}c_\nu}{c_\mu d_{\mu/\nu}} \sum_{\lambda\vdash k} b_\lambda f^\mu_{\lambda,\nu} P_\lambda(w)\\ =& \frac{(1-t)^k q^{n(\mu')-n(\nu')} c'_\nu}{c'_\mu d_{\mu/\nu}} P_{\mu/\nu}(w)\end{aligned}$$

where in the second line we computed $\mathrm{ev}_{\mu/\nu}(F_\lambda)$ using (6.2) and then used the definition of $b_\lambda$ (2.9). To get the result in the third line we recalled the formula for the skew Macdonald functions (2.14):

$$P_{\mu/\nu} = \sum_\lambda \frac{b_\nu b_\lambda}{b_\mu} f^\mu_{\lambda,\nu} P_\lambda$$

This computation proves (6.7). □

6.2. **The trace elements and the skew Macdonald functions.** The elements of the shuffle algebra $S_r \in \mathcal{A}_r^\circ$ are mapped to the power sums (3.44) under $\iota$, therefore the exponential generating function $T(v; z, w)$ given in (5.42) is related to the Cauchy kernel (2.15) under this isomorphism.

The generating function $T(v; z, w)$ (5.42) is a symmetric function in two alphabets $(w) = (w_1 \dots w_m)$ and $(z) = (z_1 \dots z_n)$ and can be written in terms of the power sums:

$$T(v;z,w) = \exp_*\left(\sum_{r>0} \frac{1}{r}\frac{1-t^r}{1-q^r}\left(p_r(w) - p_r(z) - \frac{q^r-t^r}{1-t^r} z_0^r\right) v^r S_r\right) \tag{6.8}$$

Therefore it is given by the mixed Cauchy kernel $K(x; w)$ to which one needs to apply the following transformation:

$$\pi_{w,z}: \quad p_r(w) \to p_r(w) - p_r(z) - \frac{q^r-t^r}{1-t^r} z_0^r \tag{6.9}$$

The map $\pi_{w,z}$ is a *plethystic substitution* and for a function $f(w)$ from the ring of symmetric functions $\Lambda$ in the alphabet $(w)$ it is common to write:

$$\pi_{w,z}\left(f(w)\right) = f\left[w - z - \frac{q-t}{1-t} z_0\right] \tag{6.10}$$

By comparing (6.8) with (6.4) we find:

$$T(cv; z, w) = \pi_{w,z}\left(K(x;w)\right), \qquad c = \frac{t-q}{1-q} \tag{6.11}$$

This identity together with Lemma 6.4 leads to the following result:

**Theorem 6.5.** *Let $\mu/\nu$ be a skew partition and $N = |\mu| - |\nu|$, we have the identity:*

$$P_{\mu/\nu}\left[w - z - \frac{q-t}{1-t} z_0\right] = a_{\mu,\nu} \mathrm{ev}_{\mu/\nu}\left(T_N(x; z, w)\right), \tag{6.12}$$

*where*

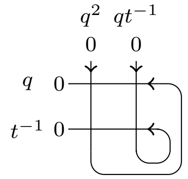
$$a_{\mu,\nu} := \frac{(t-q)^N}{(1-q)^N(1-t)^N} \frac{q^{n(\nu')-n(\mu')} c'_\mu d_{\mu/\nu}}{c'_\nu} \tag{6.13}$$

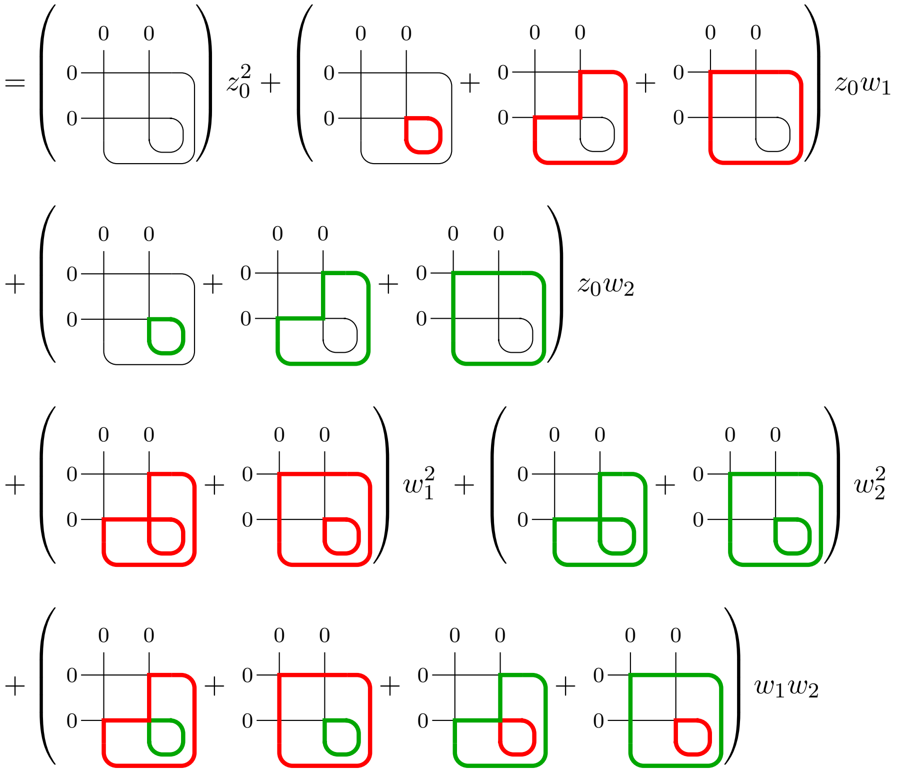
6.3. **Example.** In this section we show how to compute $P_{(2,1)/(1)}(w_1,w_2;z_0)$ using Theorem 6.5. In this case we have $N=2$, $n=0$ and $m=2$, so according to Theorem 6.5 we need to calculate:

$$a^{-1}_{(2,1),(1)}P_{(2,1)/(1)}(w_1,w_2;z_0) = \mathrm{ev}_{(2,1)/(1)}\left(T_2(x_1,x_2;z_0,w_1,w_2)\right) = T_2(q,t^{-1};z_0,w_1,w_2)$$

where $T_2(q,t^{-1};z_0,w_1,w_2)$ is the partition function of lattice path configurations with two fermionic colours and boundary conditions as in (4.50). Let the two colours be red and green where red corresponds to the smaller label and the local configurations are given by the Boltzmann weights of the $R$-matrix of $U_t(\widehat{sl}(1|2))$, given in (5.1). We can find all possible configurations in $T_2(q,t^{-1};z_0,w_1,w_2)$ and write:

$$a^{-1}_{(2,1),(1)}P_{(2,1)/(1)}(w_1,w_2;z_0) = \qquad (6.14)$$





By substituting the Boltzmann weights (5.1) we compute the coefficients in front of each monomial:

$$\begin{aligned}P_{(2,1)/(1)}(w_1,w_2;z_0) =& a_{(2,1),(1)}\Big(z_0^2 + \frac{(1-t)(2+q+t+2qt)}{(1+q)(1+t)(t-q)}(z_0w_1+z_0w_2) \\ &+\frac{(1-t)\left(1-qt^2\right)}{(1+q)(1+t)(q-t)^2}(w_1^2+w_2^2) + \frac{(1-t)^2(2+q+t+2qt)}{(1+q)(1+t)(q-t)^2}w_1w_2\Big)\end{aligned} \qquad (6.15)$$

where the coefficient $a_{(2,1),(1)}$ reads:

$$a_{(2,1),(1)} = \frac{(1+q)(1+t)(q-t)^2}{(1-t)(1-qt^2)} \qquad (6.16)$$

Therefore we computed $P_{(2,1)/(1)}(w_1,w_2;z_0)$. By setting $z_0=0$ in (6.15) we recover the standard skew Macdonald polynomial $P_{(2,1)/(1)}(w_1,w_2)$:

$$P_{(2,1)/(1)}(w_1,w_2)=w_1^2+\frac{(1-t)(2+q+t+2qt)}{1-qt^2}w_1w_2+w_2^2$$

## Appendix A. A shuffle product identity

The determinant formula (3.13) for $H_k(t^{-1})$ is known as the Izergin determinant. This determinant satisfies various summation identities. For our purposes a useful identity is given by a subset formula from [5]:

$$\begin{aligned}&\frac{\prod_{i,j=1}^{k}(x_i-y_j)}{\prod_{1\le i<j\le k}(x_i-x_j)(y_j-y_i)}\det_{1\le i,j\le k}\frac{(1-t)x_i}{(x_i-y_j)(y_j-tx_i)}\\ =&\sum_{r=0}^{k}(-1)^r t^{r(r-1)/2}\sum_{\substack{S\subseteq[k]\\|S|=r}}\prod_{i\in S}\prod_{j\in S^c}\frac{x_j-tx_i}{x_j-x_i}\prod_{i\in S}\prod_{j=1}^{k}\frac{y_j-x_i}{y_j-tx_i}\end{aligned}\tag{A.1}$$

When we set $y_i=qx_i$ the left-hand side above matches with the formula (3.13) for $H_k(t^{-1})$ up to a factor. This leads to a summation formula for $H_k(t^{-1})$:

$$H_k(t^{-1})=\frac{(1-qt^{-1})^k}{(1-t^{-1})^k}\sum_{r=0}^{k}(-1)^r t^{-r(r+1)/2}\sum_{\substack{S\subseteq[k]\\|S|=r}}\prod_{i\in S}\prod_{j\in S^c}\frac{x_i-t^{-1}x_j}{x_i-x_j}\prod_{i\in S}\prod_{j=1}^{k}\frac{x_i-qx_j}{x_i-qt^{-1}x_j}\tag{A.2}$$

The sum over subsets $S$ has the structure of the shuffle product (3.8). To see this we split the product over $j$ in the last factor in (A.2) into two products: $j\in S$ and $j\in S^c$. The second product can be combined with the first factor in (A.2) giving:

$$\begin{aligned}&\sum_{\substack{S\subseteq[k]\\|S|=r}}\prod_{i\in S}\prod_{j\in S^c}\frac{x_i-t^{-1}x_j}{x_i-x_j}\prod_{i\in S}\prod_{j=1}^{k}\frac{x_i-qx_j}{x_i-qt^{-1}x_j}=\sum_{\substack{S\subseteq[k]\\|S|=r}}\prod_{i\in S}\prod_{j\in S^c}\frac{x_i-t^{-1}x_j}{x_i-x_j}\frac{x_i-qx_j}{x_i-qt^{-1}x_j}\prod_{i,j\in S}\frac{x_i-qx_j}{x_i-qt^{-1}x_j}\\ =&\frac{(1-q)^r}{(1-qt^{-1})^r}t^{r(r-1)/2}\sum_{\substack{S\subseteq[k]\\|S|=r}}E_r(q;x_S)\prod_{i\in S}\prod_{j\in S^c}\zeta(x_j/x_i)=\frac{(1-q)^r}{(1-qt^{-1})^r}t^{r(r-1)/2}E_{k-r}(tq^{-1})*E_r(q)\end{aligned}$$

where in the second line we used (3.2), Definition 3.4 of $E_r(q)$ and $E_{k-r}(tq^{-1})$ (which is equal to 1) and the formula for the shuffle product (3.8). As a consequence the element $H_k(t^{-1})$ admits the formula:

$$H_k(t^{-1})=\frac{(1-qt^{-1})^k}{(1-t^{-1})^k}\sum_{r=0}^{k}(-t)^{-r}\frac{(1-q)^r}{(1-qt^{-1})^r}E_{k-r}(tq^{-1})*E_r(q)\tag{A.3}$$

We can derive analogously the identities for $H_k(q)$ and $H_k(tq^{-1})$ and summarize all three formulas by:

$$H_k(q_a)=\sum_{r=0}^{k}q_c^{k-r}\left(\frac{1-q_b}{1-q_bq_c}\right)^{k-r}\left(\frac{1-q_c}{1-q_bq_c}\right)^{r}E_{k-r}(q_b)*E_r(q_c)\tag{A.4}$$

where $(a,b,c)$ is a permutation of $(1,2,3)$.

## Acknowledgments

We would like to thank Jan de Gier, Andrei Neguţ, Michael Wheeler and Paul Zinn-Justin for many interesting discussions. A. G. gratefully acknowledges financial support from the Australian Research Council.

## References

[1] A. Aggarwal, A. Borodin, and M. Wheeler. Colored fermionic vertex models and symmetric functions. *Commun. Amer. Math. Soc.*, 3(08):400–630, 2023. `arXiv:2101.01605`.

[2] T.-D. Albert, H. Boos, R. Flume, and K. Ruhlig. Resolution of the nested hierarchy for rational $sl(n)$ models. *J. Phys. A: Math. Gen.*, 33(28):4963, 2000. `arXiv:nlin/0002027`.

[3] R. Baxter. *Exactly solved models in statistical mechanics.* Elsevier, 2016.

[4] V. Bazhanov and A. Shadrikov. Trigonometric solutions of triangle equations. Simple Lie superalgebras. *Theor. Math. Phys.*, 73(3), 1988.

[5] D. Betea, M. Wheeler, and P. Zinn-Justin. Refined Cauchy/Littlewood identities and six-vertex model partition functions: II. Proofs and new conjectures. *J. Alg. Comb.*, 42:555–603, 2015. `arXiv:1405.7035`.

[6] A. Borodin. On a family of symmetric rational functions. *Adv. Math.*, 306:973–1018, 2017. `arXiv:1410.0976`.

[7] A. Borodin and M. Wheeler. Coloured stochastic vertex models and their spectral theory. *Astérisque*, 437, 2022. `arXiv:1808.01866`.

[8] A. Borodin and M. Wheeler. Nonsymmetric Macdonald polynomials via integrable vertex models. *Trans. Amer. Math. Soc.*, 375(12), 2022. `arXiv:1904.06804`.

[9] B. Brubaker, V. Buciumas, D. Bump, and H. Gustafsson. Colored vertex models and Iwahori Whittaker functions. Preprint, `arXiv:1906.04140`, 2019.

[10] B. Brubaker, D. Bump, and S. Friedberg. Schur polynomials and the Yang-Baxter equation. *Commun. Math. Phys.*, 308(2):281–301, 2011. `arXiv:0912.0911` .

[11] V. Buciumas and T. Scrimshaw. Double Grothendieck polynomials and colored lattice models. *Int. Math. Res. Not. IMRN*, 2020. Art. ID rnaa327. `arXiv:2007.04533`.

[12] L. Cantini, J. de Gier, and M. Wheeler. Matrix product formula for Macdonald polynomials. *J. Phys. A: Math. Theor.*, 48(38):384001, 2015. `arXiv:1505.00287`.

[13] I. Corwin and L. Petrov. Stochastic higher spin vertex models on the line. *Commun. Math. Phys.*, 343:651–700, 2016. `arXiv:1502.07374` .

[14] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, and S. Yanagida. A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials. *J. Math. Phys.*, 50(9):095215, 2009. `arXiv:0904.2291`.

[15] B. Feigin, A. Hoshino, J. Shibahara, J. Shiraishi, and S. Yanagida. Kernel function and quantum algebras. Preprint, `arXiv:1002.2485`, 2010.

[16] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. Quantum toroidal $gl(1)$ and Bethe ansatz. *J. Phys. A: Math. Theor.*, 48(24):244001, 2015. `arXiv:1502.07194`.

[17] B. Feigin, M. Jimbo, and E. Mukhin. Commutative subalgebra of a shuffle algebra associated with quantum toroidal $\mathfrak{gl}_{m|n}$. Preprint, `arXiv:2306.05223`, 2023.

[18] B. Feigin and A. Odesskii. A family of elliptic algebras. *Int. Math. Res. Not.*, 1997(11):531–539, 1997.

[19] B. Feigin and A. Tsymbaliuk. Equivariant $K$-theory of Hilbert schemes via shuffle algebra. *Kyoto J. Math.*, 51(4):831–854, 2011. `arXiv:0904.1679`.

[20] B. Feigin and A. Tsymbaliuk. Bethe subalgebras of $U_q(\widehat{gl}_n)$ via shuffle algebras. *Sel. Math.*, 22:979–1011, 2016.

[21] O. Foda, M. Wheeler, and M. Zuparic. Factorized domain wall partition functions in trigonometric vertex models. *J. Stat. Mech.: Theor. Exp.*, 2007(10):P10016, 2007. `arXiv:0709.4540`.

[22] A. Garbali, J. de Gier, and M. Wheeler. A new generalisation of Macdonald polynomials. *Commun. Math. Phys.*, 352(2):773–804, 2017. `arXiv:1605.07200`.

[23] A. Garbali and M. Wheeler. Modified Macdonald polynomials and integrability. *Commun. Math. Phys.*, 374(3):1809–1876, 2020. `arXiv:1810.12905`.

[24] A. Garbali and P. Zinn-Justin. Shuffle algebras, lattice paths and the commuting scheme. *Contemp. Math. Special Issue: Hypergeometry, Integrability and Lie Theory*, 780, 2022. `arXiv:2110.07155`.

[25] A. Gunna and P. Zinn-Justin. Vertex models for Canonical Grothendieck polynomials and their duals. *Alg. Comb.*, 6(1):109–163, 2023. `arXiv:2009.13172`.

[26] A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, and N. Slavnov. Scalar products of Bethe vectors in the models with $gl(m|n)$ symmetry. *Nucl. Phys. B*, 923:277–311, 2017. `arXiv:1704.08173`.

[27] A. Izergin. Partition function of a six-vertex model in a finite volume. In *Dokl. Akad. Nauk SSSR*, volume 297, pages 331–333, 1987.

[28] V. Korepin. Calculation of norms of Bethe wave functions. *Commun. Math. Phys.*, 86:391–418, 1982.

[29] V. Korepin, N. Bogoliubov, and A. Izergin. *Quantum inverse scattering method and correlation functions*, volume 3. Cambridge University Press, 1997. `arXiv:cond-mat/9301031`.

[30] A. Kuniba, V. Mangazeev, S. Maruyama, and M. Okado. Stochastic $R$ matrix for $U_q(A_n^{(1)})$. *Nucl. Phys. B*, 913:248–277, 2016. `arXiv:1604.08304`.

[31] A. Litvinov and I. Vilkoviskiy. Liouville reflection operator, affine Yangian and Bethe ansatz. *J. High En. Phys.*, 2020(12):1–49, 2020. `arXiv:2007.00535`.

[32] I. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, 1998.

[33] J.-M. Maillet and J. Sanchez de Santos. Drinfel'd twists and algebraic Bethe ansatz. *Amer. Math. Soc. Transl. Ser. 2*, 201, 2000. `arXiv:q-alg/9612012`.

[34] S. McAteer and M. Wheeler. On factorizing F-matrices in $Y(sl_n)$ and $U_q(\widehat{sl_n})$ spin chains. *J. Stat. Mech.*, 2012(04), 2012. `arXiv:1112.0839`.

[35] M. Mucciconi and L. Petrov. Spin q-Whittaker polynomials and deformed quantum Toda. *Commun. Math. Phys.*, 389(3):1331–1416, 2022. `arXiv:2003.14260`.

[36] A. Neguţ. The shuffle algebra revisited. *Int. Math. Res. Not.*, 2014(22):6242–6275, 2014. `arXiv:1209.3349`.

[37] A. Negut. Moduli of flags of sheaves and their $K$-theory. *Alg. Geom.*, 2(1):19–43, 2015. `arXiv:1209.4242`.

[38] A. Neguţ. A tale of two shuffle algebras. Preprint, `arXiv:1908.08395`, 2019.

[39] N. Reshetikhin. Calculation of the norm of Bethe vectors in models with $SU(3)$-symmetry. *J. Sov. Math.*, 46:1694–1706, 1989.

[40] O. Schiffmann and E. Vasserot. The elliptic Hall algebra and the $K$-theory of the Hilbert scheme of $\mathbb{A}^2$. *Duke Math. J.*, 162(2):279–366, 2013. `arXiv:0905.2555`.

[41] N. Slavnov. Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz. *Teor. Mat. Fiz.*, 79(2):232–240, 1989.

[42] M. Wheeler. Scalar products in generalized models with $SU(3)$-symmetry. *Commun. Math. Phys.*, 327:737–777, 2014. `arXiv:1204.2089`.

[43] M. Wheeler and P. Zinn-Justin. Hall polynomials, inverse Kostka polynomials and puzzles. *J. Comb. Theor. Ser. A*, 159:107–163, 2018. `arXiv:1603.01815`.

[44] W.-L. Yang, Y.-Z. Zhang, and S.-Y. Zhao. Drinfeld Twists and Algebraic Bethe Ansatz of the Supersymmetric Model Associated with $U_q(gl(m|n))$. *Commun. Math. Phys.*, 264:87–114, 2006. `arXiv:hep-th/0503003`.

Alexandr Garbali, Ajeeth Gunna, School of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia.

*Email address*: `alexandr.garbali@unimelb.edu.au, agunna@student.unimelb.edu.au`