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Projective Deformations of Hyperbolic Coxeter 3-Orbifolds

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Abstract By using Klein's model for hyperbolic geometry, hyperbolic structures on orbifolds or manifolds provide examples of real projective structures. By Andreev's theorem, many 3-dimensional reflection orbifolds admit a finite volume hyperbolic structure, and such a hyperbolic structure is unique. However, the induced real projective structure on some such 3-orbifolds deforms into a family of real projective structures that are not induced from hyperbolic structures. In this paper, we find new classes of compact and complete hyperbolic reflection 3-orbifolds with such deformations. We also explain numerical and exact results on projective deformations of some compact hyperbolic cubes and dodecahedra.

Keywords Real projective structure · Orbifold · Moduli space · Coxeter groups · Representations of groups

Mathematics Subject Classification (2000) 57M50 · 57N16 · 53A20 · 53C15

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Introduction

A smooth *n-dimensional orbifold* is a Hausdorff space locally modelled on quotients of open subsets of $\mathbb{R}^n$ by finite groups of diffeomorphisms (see [28, 7, 10] for detailed discussions). In this paper, we deal with *good* orbifolds, which are quotients of a manifold by a discrete group acting properly discontinuously, perhaps with fixed points. The image of the fixed points sets of non-trivial group elements forms the *singular locus* of the orbifold. Isomorphisms of such orbifolds are just diffeomorphisms conjugating the discrete group actions.

Given a Lie group G acting transitively and effectively on an n-dimensional manifold X, Ehresmann introduced the idea of a (G,X)-structure on an n-orbifold as locally modelling the orbifold on open subsets of X modulo finite subgroups of G, with transition maps given by elements of G. We refer to [28, 29, 10] for the details.

When an orbifold M admits such a (G,X)-structure, Thurston [28] showed that there exists a simply connected manifold $\tilde{M}$ and a discrete group Γ of deck transformations so that the quotient orbifold $\tilde{M}/\Gamma$ is isomorphic to M. Then $\tilde{M}$ is said to be a *universal cover* of M and Γ is the *orbifold fundamental group* of M; these are determined uniquely up to diffeomorphism and isomorphism respectively. We write $\pi_1(M) = \Gamma$. (Note that there is also a definition of orbifold fundamental group by Haefliger using paths, see [7].)

Given a (G,X)-structure on an n-orbifold M, we can define an immersion D from the universal cover $\tilde{M}$ to X and a homomorphism

$$h : \pi_1(M) \to G,$$

where $\pi_1(M)$ denotes the orbifold fundamental group of M. Here D is called a *developing map* and h a *holonomy homomorphism* for the (G,X)-structure on M, and D satisfies the equivariance condition

$$D(\gamma \cdot m) = h(\gamma) \cdot D(m), \text{ for all } \gamma \in \pi_1(M), m \in \tilde{M}.$$

Note that (D,h) is determined only up to the following action:

$$(D, h(\cdot)) \mapsto (g \circ D, g \circ h(\cdot) \circ g^{-1}) \tag{1}$$

for $g \in G$. Conversely, the development pair (D,h) determines the (G,X)-structure. (See Thurston [29, chap. 3].)

When M is a closed orbifold, we define $\tilde{\mathfrak{D}}(M)$ to be the space of equivalence classes of development pairs of (G,X)-structures on $\tilde{M}$ modulo isotopies of $\tilde{M}$ commuting with the deck transformation group. Here, the space of development pairs is equipped with the C^1-topology and $\tilde{\mathfrak{D}}(M)$ is endowed with the quotient topology. The *deformation space* $\mathfrak{D}(M)$ of (G,X)-structures on the orbifold M is the quotient space of $\tilde{\mathfrak{D}}(M)$ by the action of G given in equation (1). (See [10], [12].) We can also think of $\mathfrak{D}(M)$ as the space of (G,X)-structures on M up to the equivalence relation given by isotopy in M.

When M is non-compact or has boundary, $\tilde{\mathfrak{D}}(M)$ and $\mathfrak{D}(M)$ are defined as spaces of (G,X)-structures on M up to the equivalence relation given by isotopy and "thickening" of the geometric structure near the ends or boundary of M; see [28, 8]. One of the authors of this paper is writing a more complete version of this theory in [13].

In this paper we study real projective structures and hyperbolic structures. *Real projective geometry* is given by the group $PGL(n+1,\mathbb{R})$ acting by projective transformations on the projective space $\mathbb{R}P^n$. We can represent *hyperbolic geometry* using Klein's projective model: hyperbolic space is an open ball B in $\mathbb{R}P^n$, and the group of hyperbolic isometries

is the subgroup $PO(1,n)$ of $PGL(n+1,\mathbb{R})$ preserving B. Hence hyperbolic orbifolds and manifolds naturally have induced real projective structures.

The study of real projective structures was originally introduced by E. Cartan, and was continued by many people including Chern, Kuiper, Koszul, Milnor, and Benzecri in the late 1950's. In the 1960's, it was unknown whether every real projective structure arises from a hyperbolic structure. In 1967, Kac and Vinberg [31] discovered real projective reflection orbifolds that are not hyperbolic. Sullivan and Thurston [27] produced many examples of real projective manifolds. The theory of real projective structures on 2-dimensional manifolds and orbifolds was developed in Goldman's senior thesis at Princeton from 1977 written under Thurston, and by Goldman and Choi [17], [9], [11] from the 1990's onwards. Earlier in the 1980's, Thurston [28] noticed that there is a projective bending deformation available for a real projective manifold provided it has a totally geodesic submanifold of codimension one. Johnson and Millson [20] used this to construct some deformations of real projective structures on hyperbolic manifolds. Cooper, Long and Thistlethwaite [14,15] investigated whether the closed hyperbolic 3-manifolds of the Hodgson-Weeks census could be deformed into other real projective structures. They show that close to 1 percent of examples deform out of their set of 4500 examples.

We will focus on 3-dimensional reflection orbifolds whose underlying space is homeomorphic to a 3-dimensional convex polyhedron[1], and whose singular locus is its boundary (made up of mirrors). Benoist [5] and Choi [11] investigated these classes first, and Marquis [23] completed the study in some cases.

We will just be concentrating on ones admitting hyperbolic structures and attempt direct computations for some examples as Cooper, Long, and Thistlethwaite have done above. The fundamental group of such an orbifold is a *Coxeter group*, i.e. a group with a set $\{r_1,\dots,r_m\}$ of generators and the following set of defining relations:

$$\begin{aligned} r_i^2 &= 1 \text{ for all } i, \\ (r_i r_j)^{n_{ij}} &= 1 \text{ for some } i \text{ and } j \text{ with } n_{ij} = n_{ji} \geq 2. \end{aligned}$$

Here, r_i represents a reflection in the ith silvered face of P, and $r_i r_j$ represents a rotation of order n_{ij} about an edge where the ith and jth faces meet. The stabilizer of each face is the group $\mathbb{Z}_2$ generated by reflection in the face, and the stabilizer of each edge is the dihedral group $D_{n_{ij}}$ generated by reflections in the adjacent faces.

Let P be a fixed 3-dimensional convex polyhedron, and assign an order $n_e \geq 2$ to each edge e of P. If any vertex of P has more than three edges incident, or has orders of the incident edges not of the form

$$(2,2,k) \text{ with } k \geq 2,\ (2,3,3),\ (2,3,4),\ (2,3,5),$$

(i.e. corresponding to spherical triangular groups), then we remove the vertex. Let $\hat{P}$ denote the differentiable orbifold obtained from P with faces silvered, edge orders n_e, and with vertices removed as above. We say that $\hat{P}$ has a *Coxeter orbifold structure*. For example, let P be a convex hyperbolic polyhedron with dihedral angles submultiples of π; we call P a *Coxeter polyhedron*. Then P will naturally have a Coxeter orbifold structure $\hat{P}$.

Now let $\mathfrak{D}(\hat{P})$ denote the deformation space of real projective structures on the Coxeter 3-orbifold $\hat{P}$. The work of Vinberg [32] implies that each element of $\mathfrak{D}(\hat{P})$ gives a convex projective structure (see Theorem 2 of [12]). That is, the image of the developing map of the orbifold universal cover of $\hat{P}$ is projectively isomorphic to a convex domain in $\mathbb{R}P^3$ and

[1] In this paper a *polyhedron* will be a 3-dimensional polytope with finitely many codimension one faces.

the holonomy is a discrete faithful representation. (For a precise definition of convexity, see [12].)

A point p of $\mathfrak{D}(\hat{P})$ gives a fundamental polyhedron P in $\mathbb{R}P^3$, well defined up to projective automorphisms. We concentrate on the space of $p \in \mathfrak{D}(\hat{P})$ giving a *fixed* fundamental polyhedron P. This space is called the *restricted deformation space* of $\hat{P}$ and denoted by $\mathfrak{D}_P(\hat{P})$. A point t in $\mathfrak{D}_P(\hat{P})$ is said to be *hyperbolic* if it is given by a hyperbolic structure on $\hat{P}$.

Definition 1 Let P be a 3-dimensional hyperbolic Coxeter polyhedron, and let $\hat{P}$ denote its Coxeter orbifold structure. Suppose that t is the corresponding hyperbolic point of $\mathfrak{D}_P(\hat{P})$. We call a neighbourhood of t in $\mathfrak{D}_P(\hat{P})$ the *local restricted deformation space* of P. We say that $\hat{P}$ is *projectively deformable relative to the mirrors*, or simply *deforms rel mirrors*, if the dimension of its local restricted deformation space is positive. Conversely, we say that $\hat{P}$ is *projectively rigid relative to the mirrors*, or *rigid rel mirrors*, if the dimension of its local restricted deformation space is 0.

Choi [12] found a class of Coxeter 3-orbifolds whose restricted deformation spaces are understandable: the orderable Coxeter orbifolds of normal type. A Coxeter orbifold $\hat{P}$ is said to be *orderable* if the faces of P can be ordered so that each face contains at most three edges that are edges of order 2 or edges in a face of higher index. (See §1.4 for the details, and for the definition of normal type.)

In this paper, we will study Coxeter orbifolds that are not orderable. The following theorem describes the local restricted deformation space for a class of Coxeter orbifolds arising from *ideal* hyperbolic polyhedra, i.e. polyhedra with all vertices on the sphere at infinity.

Theorem 1 *Let P be an ideal 3-dimensional hyperbolic polyhedron whose dihedral angles are all equal to $\pi/3$, and suppose that $\hat{P}$ is given its Coxeter orbifold structure. If P is not a tetrahedron, then a neighbourhood of the hyperbolic point in $\mathfrak{D}_P(\hat{P})$ is a smooth 6-dimensional manifold.*

If P is a regular ideal tetrahedron then Theorem 3 of [12], obtained by J.R. Kim in his master's thesis, shows that $\mathfrak{D}_P(\hat{P})$ is a 3-dimensional cell.

The main ideas in the proof of Theorem 1 are as follows. We first show that $\mathfrak{D}_P(\hat{P})$ is isomorphic to the solution set of a system of polynomial equations following ideas of Vinberg [32] and Choi [12]. Since the faces of P are fixed, each projective reflection in a face of the polyhedron is determined by a reflection vector b_i. We then compute the Jacobian matrix of the equations for the b_i at the hyperbolic point. This reveals that the matrix has exactly the same rank as the Jacobian matrix of the equations for the Lorentzian unit normals of a hyperbolic polyhedron with the given dihedral angles. By infinitesimal rigidity of the hyperbolic structure on $\hat{P}$, this matrix is of full rank and has kernel of dimension six; the result then follows from the implicit function theorem. In fact, we can interpret the infinitesimal projective deformations as applying infinitesimal hyperbolic isometries to the reflection vectors.

The other two main results of this paper use various theoretical and computational methods to determine the local restricted deformation spaces of Coxeter orbifolds arising from certain compact cubes and dodecahedra in hyperbolic 3-space. These cubes and dodecahedra were chosen since they are workable using our methods, but not trivially. The results are summarized in the following two theorems; the details are given in §4.4 and §4.5 below.

Theorem 2 *Consider the compact hyperbolic cubes such that each dihedral angle is $\pi/2$ or $\pi/3$. Up to symmetries, there exist 34 cubes satisfying this condition. For the corresponding hyperbolic Coxeter orbifolds, 10 are projectively deformable relative to the mirrors and the remaining 24 are projectively rigid relative to the mirrors. The deformations of three of these orbifolds are not projective bendings.*

Theorem 3 *Consider the compact hyperbolic dodecahedra such that each dihedral angle is $\pi/2$ or $\pi/3$, and each face has at most two dihedral angles equal to $\pi/2$. Up to symmetries, there exist 13 dodecahedra satisfying these conditions. For the corresponding hyperbolic Coxeter orbifolds, only 1 has projective deformations relative to the mirrors and these are not projective bendings; the remaining 12 are projectively rigid relative to the mirrors.*

If a face has more than two edges of order two, then the corresponding reflection is determined. For dodecahedra, we assumed this condition fails for every face and tabulated the results. Without this restriction, the list of the possible dodecahedra would become very large and many of these would prove to be projectively rigid relative to the mirrors by the linear test presented later in §4.2. It is future work to complete the task of fully classifying the Coxeter orbifold structures on dodecahedra and cubes that are projectively deformable relative to the mirrors.

To obtain Theorems 2 and 3, the polyhedra were first enumerated by using a Matlab program to check the conditions of Andreev's theorem. (See cu.m and do.m in [19].) The remaining computations were done by Mathematica. In the case of cubes, we used exact algebraic computations. However, in the case of dodecahedra numerical computations were used. The detailed results of computations by Mathematica can be found at the web page [19].

The remainder of this paper is organized as follows.

Section 1 reviews some well-known facts. In §1.1 we introduce oriented projective structures which are in one-to-one correspondence with real projective structures. In §1.2 we describe Vinberg's results giving the general conditions satisfied by n-dimensional real projective reflection groups. In §1.3 we recall Andreev's theorem characterizing the 3-dimensional hyperbolic polyhedra of finite volume with dihedral angles at most $\pi/2$. In §1.4 we describe the results of Choi [12] on the restricted deformation spaces of 3-dimensional Coxeter orbifolds that are orderable.

Section 2 identifies the restricted deformation space of real projective structures on a Coxeter orbifold $\hat{P}$ with the solution space of a system of polynomial equations. In §2.1 we introduce a space of restricted representations of the orbifold fundamental group $\pi_1(\hat{P})$. In §2.2 we show that this representation space can be identified with the solution space of some polynomial equations as given by Vinberg. In §2.3 we prove that the restricted deformation space can be identified with the set of solutions of Vinberg's equations, when the underlying convex polyhedron P has a discrete projective automorphism group. In §2.4 we recall the description of convex hyperbolic polyhedra by their Gram matrices, and use this to identify the solutions to Vinberg's equations corresponding to a hyperbolic structure.

Section 3 discusses general facts concerning a neighbourhood of a hyperbolic structure in the restricted deformation space of real projective structures on a Coxeter 3-orbifold. The results from §2.3 show that this restricted deformation space is the solution space of a system of polynomial equations. In §3.1 we study the Zariski tangent space of this solution space, and prove some general results on local restricted deformation spaces. In §3.2 we study the Zariski tangent space for the equations defining a hyperbolic structure. In §3.3 we compare the two Zariski tangent spaces at a hyperbolic point, and use Garland-Raghunathan-Weil

infinitesimal rigidity ([16], [35]) to prove Theorem 1. (See Kapovich [22] for a similar work in the conformally flat structures.) In §3.4 we construct families of compact hyperbolic prism orbifolds, with number of faces arbitrarily large, that are deformable relative to the mirrors but non-orderable. In contrast, we note that orderable 3-dimensional compact hyperbolic Coxeter polyhedra are always projectively rigid relative to the mirrors when the number of faces is greater than 7.

Section 4 is concerned with computing the dimension of local restricted deformation spaces for hyperbolic Coxeter orbifolds corresponding to cubes and dodecahedra. We carry out most of the computations using Mathematica. An outline of the computational algorithm is given in §4.1. In §4.2 we provide a simple test for the projective rigidity rel mirrors of 3-dimensional Coxeter polyhedra in real projective space. In §4.3 we describe the notation used in figures and tables in this paper. In §4.4-4.5 we give details of the methods used, and provide detailed tables listing the dimensions of local restricted deformation spaces of cubes and dodecahedra. The results show that computation of the Zariski tangent space is often sufficient; but in other cases, Gröbner bases are used to determine the structure of the local restricted deformation spaces.

Finally in §4.6, we discuss the "projective bending" deformations of Thurston. We show that existence of nontrivial projective bending deformations in the restricted deformation space implies that the polyhedron must be a prism and find a necessary and sufficient condition for a 3-dimensional Coxeter orbifold to have projective bendings relative to the mirrors, where we will exclude triangular prisms for technical reasons.

1 Preliminaries

This section gives the basic background material used in this article.

In §1.1 we give an alternative description of real projective structures that will be more convenient for us in this article, based on the projective sphere $\mathbb{S}^n$ and its group of projective transformations $SL_{\pm}(n+1,\mathbb{R})$. In §1.2 we describe Vinberg's results giving the conditions under which an n-dimensional Coxeter orbifold $\hat{P}$ admits a real projective structure. This is equivalent to showing that the (orbifold) fundamental group $\pi_1(\hat{P})$ is isomorphic to a discrete subgroup of $SL_{\pm}(n+1,\mathbb{R})$. We then concentrate on the case where $\hat{P}$ is a 3-dimensional Coxeter orbifold. In §1.3 we recall Andreev's theorem which explains when $\hat{P}$ admits a finite volume hyperbolic structure. Finally, in §1.4, Choi's results on the restricted deformation spaces of real projective structures on *orderable* Coxeter 3-orbifolds are described.

1.1 Oriented real projective geometry

Instead of working in the n-dimensional real projective space $\mathbb{R}P^n$, it will be more convenient for us to work in the projective sphere $\mathbb{S}^n$, i.e. the set of rays through the origin in $\mathbb{R}^{n+1}$. As a (G,X)-structure, an *oriented projective structure* is a $(SL_{\pm}(n+1,\mathbb{R}),\mathbb{S}^n)$-structure, where

$$SL_{\pm}(n+1,\mathbb{R}) = \{A \in GL(n+1,\mathbb{R}) : \det A = \pm 1\}$$

is the group of projective transformations of $\mathbb{S}^n$. Recall that $\mathbb{S}^n$ double covers $\mathbb{R}P^n$ and $SL_{\pm}(n+1,\mathbb{R})$ double covers $PGL(n+1,\mathbb{R})$. A projective structure on an orbifold corresponds to a unique oriented projective structure and vice versa (see [12] and [29, p. 143]). From now on, by a real projective structure, we always mean an oriented projective structure.

There is a one-to-one correspondence between the set of vector subspaces of $\mathbb{R}^{n+1}$ and the set of great spheres in $\mathbb{S}^n$. In particular, a 1-dimensional subspace corresponds to a pair of antipodal points and an n-dimensional subspace gives a great $(n-1)$-sphere in $\mathbb{S}^n$. Further, a component of the complement of a great $(n-1)$-sphere (i.e. an open hemisphere) can be identified with an affine n-space. We call this an *affine patch*.

In this paper, we define a *convex polytope* P in $\mathbb{S}^n$ to be a precompact convex polytope in an affine patch of $\mathbb{S}^n$. The image of such a polytope under the double covering is called a convex polytope in $\mathbb{R}P^n$. We define $k(P)$ as the dimension of the subgroup of $SL_\pm(n+1,\mathbb{R})$ preserving P. This is the same as the dimension of the group of projective automorphisms of the image of P under the double-covering map.

Hyperbolic geometry arises naturally as a sub-geometry of oriented projective geometry. Let $\langle\cdot,\cdot\rangle$ denote the Lorentzian inner product on $\mathbb{R}^{n+1}$ defined by

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + \ldots + x_{n+1}y_{n+1},$$

and let $B' \subset \mathbb{S}^n$ be the open n-ball consisting of rays through the origin in the cone $\{x \in \mathbb{R}^{n+1} : \langle x,x\rangle < 0, \;\; x_1 > 0\}$. Then we can regard hyperbolic space $\mathbb{H}^n$ as the open ball B', and the group of hyperbolic isometries $\mathrm{Isom}(\mathbb{H}^n)$ is the subgroup $O_0(1,n)$ of $SL_\pm(n+1,\mathbb{R})$ preserving B'.

Radial projection maps B' diffeomorphically to an open n-ball B in the affine hyperplane $x_1 = 1$, and $\mathrm{Isom}(\mathbb{H}^n)$ corresponds to the closed subgroup $PO(1,n) \subset PGL(n+1,\mathbb{R})$ of projective automorphisms of B. This gives the Klein model for hyperbolic geometry.

Alternatively, hyperbolic space $\mathbb{H}^n$ can be embedded in $\mathbb{R}^{n+1}$ as the upper sheet of a hyperboloid

$$\langle x,x\rangle = -1, \;\; x_1 > 0,$$

and $\mathrm{Isom}(\mathbb{H}^n)$ is the subgroup $O_0(1,n)$ of $SL_\pm(n+1,\mathbb{R})$ preserving $\mathbb{H}^n$.

1.2 Vinberg's results

This subsection gives a summary of results from Vinberg's article [32]. An alternative treatment is given in Benoist's notes [6]. Vinberg gave the general conditions under which a Coxeter orbifold admits a real projective structure, and a criterion to decide whether it is a hyperbolic structure or not.

Let V be the $(n+1)$-dimensional real vector space $\mathbb{R}^{n+1}$. A (projective) *reflection* R is an element of order 2 of $SL_\pm(n+1,\mathbb{R})$ which is the identity on a hyperplane U. All reflections are of the form

$$R = Id - \alpha \otimes b$$

for some linear functional $\alpha \in V^*$ and a vector $b \in V$ with $\alpha(b) = 2$. Here, the kernel of α is the subspace U of fixed points of R and b is the *reflection vector*, i.e. an eigenvector corresponding to the eigenvalue -1. A *rotation* is an element of $SL_\pm(n+1,\mathbb{R})$ which is the identity on a subspace of codimension 2 and is conjugate to a matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

in a suitable supplementary basis. The real number $\theta \in [0,\pi]$ is the *angle* of the rotation.

We consider $\mathbb{S}^n$ as the set of rays in $\mathbb{R}^{n+1}$ from the origin. Let P be an n-dimensional convex polytope in $\mathbb{S}^n$ and for each (codimension one) face F_i of P, take a linear functional

α_i for F_i and choose a projective reflection $R_i = Id - \alpha_i \otimes b_i$ with $\alpha_i(b_i) = 2$ which fixes F_i. By making a suitable choice of signs, we will assume that P is defined by the inequalities

$$\alpha_i \leq 0 \qquad i = 1, \ldots, f, \tag{2}$$

where f is the number of faces of P. The group $\Gamma \subset SL_{\pm}(n+1, \mathbb{R})$ generated by all these reflections R_i is called a *linear Coxeter group* if

$$\gamma P^{\circ} \cap P^{\circ} = \varnothing \text{ for every } \gamma \in \Gamma \backslash \{1\},$$

where P° is the interior of P. The $f \times f$ matrix $A = (a_{ij})$, $a_{ij} = \alpha_i(b_j)$, is called the *Cartan matrix* of Γ. Vinberg proved that the following conditions are necessary and sufficient for Γ to be a linear Coxeter group:

(C1) $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.
(C2) $a_{ii} = 2$; and for $i \neq j$, $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4\cos^2 \frac{\pi}{n_{ij}}$, n_{ij} an integer.

In fact, if $a_{ij}a_{ji} = 4\cos^2 \frac{\pi}{n_{ij}}$ then the product R_iR_j is a rotation of angle $2\pi/n_{ij}$ and the group generated by two reflections R_i and R_j is the dihedral group $D_{n_{ij}}$. Note that (C1) and (C2) imply that $a_{ij} = a_{ji} = 0$ if $n_{ij} = 2$; however $a_{ij} \neq a_{ji}$ in general when $n_{ij} > 2$.

For each reflection R_i, α_i and b_i are defined up to transformations

$$\alpha_i \mapsto d_i\alpha_i \text{ and } b_i \mapsto d_i^{-1}b_i \text{ with } d_i > 0.$$

Hence the Cartan matrix of Γ is defined up to conjugation by a diagonal matrix with positive diagonal entries.

For any $x \in P$, let Γ_x denote the subgroup of Γ generated by reflections in those faces of P which contain x. Define $P^f = \{x \in P \,|\, \Gamma_x \text{ is finite}\}$. Then the following statements are true:

1. $C = \cup_{\gamma \in \Gamma} \gamma P$ is convex.
2. Γ is a discrete subgroup of $SL_{\pm}(n+1, \mathbb{R})$ preserving C°.
3. $C^{\circ} \cap P = P^f$, and is homeomorphic to C°/Γ.

Thus C° gives a convex open subset of the projective sphere $\mathbb{S}^n$, and C°/Γ determines a convex real projective structure on the Coxeter orbifold $\hat{P}$ associated with P.

To state the next theorem, we introduce the following notation and definitions: if $X = (X_1, \ldots, X_{n+1}) \in V$, we write $X > 0$ if $X_i > 0$ for every i, and $X \geq 0$ if $X_i \geq 0$ for every i. A matrix A is of *negative type* if there exists $X > 0$ such that $AX < 0$, and if $X \geq 0$ and $AX \geq 0$ imply $X = 0$. A matrix A is *indecomposable* if it cannot be represented as a direct sum of two matrices. Two matrices A and B are said to be *equivalent* if $A = DBD^{-1}$ for a diagonal matrix D having positive entries. A linear Coxeter group Γ is called a *hyperbolic* Coxeter group if Γ is derived from a discrete group generated by reflections in $\mathbb{H}^n$, and no proper plane of $\mathbb{H}^n$ or any point at infinity is Γ-invariant.

Theorem 4 (Vinberg [32]) *A linear Coxeter group Γ is hyperbolic if and only if the Cartan matrix A of Γ is indecomposable, of negative type, and equivalent to a symmetric matrix of signature $(1, n)$.*

1.3 Andreev's theorem

The 3-dimensional Coxeter orbifolds which admit a finite volume hyperbolic structure have been classified by Andreev [1,2].

Let X be an 3-dimensional space of constant curvature, with group of isometries denoted $\mathrm{Isom}(X)$. A convex polyhedron P in X is called a *Coxeter polyhedron* if all the dihedral angles of P are submultiples of π. Let P be a Coxeter polyhedron, and Γ be the group generated by reflections in its faces. Then Γ is a discrete subgroup of $\mathrm{Isom}(X)$, and P is its fundamental polyhedron. Conversely, every discrete subgroup of $\mathrm{Isom}(X)$ generated by reflections can be obtained in this manner.

A nice property of a Coxeter polyhedron is that its dihedral angles are *non-obtuse*, i.e. the dihedral angles do not exceed $\pi/2$. In 1970, E.M. Andreev [1] gave a full description of 3-dimensional compact hyperbolic polyhedra with non-obtuse dihedral angles.

Let C be an abstract 3-dimensional polyhedron and C^* be its dual. A simple closed curve γ is called a *k-circuit* if it consists of k edges of C^*. A circuit γ is *prismatic* if all of the endpoints of the edges of C which γ meets are different.

Suppose that C is *not a tetrahedron* and non-obtuse angles $\theta_{ij} \in (0, \pi/2]$ are given corresponding to each edge $F_{ij} = F_i \cap F_j$ of C, where F_i are the faces of C. Then the following conditions $(A1)$–$(A4)$ are necessary and sufficient for the existence of a compact 3-dimensional hyperbolic polyhedron P which realizes C with dihedral angle θ_{ij} at each edge F_{ij}.

(A1) If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C then

$$\theta_{ij} + \theta_{jk} + \theta_{ki} > \pi.$$

(A2) If F_i, F_j, F_k form a prismatic 3-circuit, then

$$\theta_{ij} + \theta_{jk} + \theta_{ki} < \pi.$$

(A3) If F_i, F_j, F_k, F_l form a prismatic 4-circuit, then

$$\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} < 2\pi.$$

(A4) If C is a triangular prism with triangular faces F_1 and F_2, then

$$\theta_{13} + \theta_{14} + \theta_{15} + \theta_{23} + \theta_{24} + \theta_{25} < 3\pi.$$

Furthermore, this polyhedron is unique up to hyperbolic isometries.

Andreev [2] also showed that the following conditions $(\widetilde{A1})$–$(\widetilde{A6})$ are necessary and sufficient for the existence of a 3-dimensional hyperbolic polyhedron P of finite volume which realizes C with dihedral angle $\theta_{ij} \in (0, \pi/2]$ at each edge F_{ij}.

$(\widetilde{A1})$ If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C then

$$\theta_{ij} + \theta_{jk} + \theta_{ki} \geq \pi.$$

$(\widetilde{A2})$ is the same as (A2).
$(\widetilde{A3})$ is the same as (A3).
$(\widetilde{A4})$ is the same as (A4).
$(\widetilde{A5})$ If $F_{ijkl} = F_i \cap F_j \cap F_k \cap F_l$ is a vertex of C then

$$\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} = 2\pi.$$

($\widetilde{A6}$) If F_i, F_j, F_k are faces with F_i and F_j adjacent, F_j and F_k adjacent, and F_i and F_k are not adjacent but meet in a vertex not in F_j, then

$$\theta_{ij} + \theta_{jk} < \pi.$$

Again, the hyperbolic polyhedron is unique up to hyperbolic isometries.

Note that if the vertices of C are all trivalent then conditions ($\widetilde{A5}$) and ($\widetilde{A6}$) are not needed.

1.4 Orderability results

This subsection describes the main theorem of [12]. As we mentioned in the introduction, if a Coxeter orbifold $\hat{P}$ satisfies the condition of orderability, then we understand the restricted deformation space of real projective structures on $\hat{P}$.

Let P be a fixed 3-dimensional convex polyhedron in $\mathbb{S}^3$ with given edge orders, and let $\hat{P}$ be the corresponding Coxeter orbifold. Denote the numbers of vertices, edges and faces of P by v, e, f respectively. Let e_2 be the number of edges of order 2 in $\hat{P}$. Let $k(P)$ be the dimension of the group of projective automorphisms of P. Then $k(P) = 3$ if P is tetrahedron, $k(P) = 1$ if P is the cone over a polygon other than a triangle, and $k(P) = 0$ otherwise. (See Lemma 3 below.)

The orbifold $\hat{P}$ is called a *normal-type* Coxeter orbifold if it is **not** one of the following types:

- a *cone-type Coxeter orbifold*, whose underlying polyhedron is topologically a cone from a face F to a vertex, and all edges of F have edge order 2,
- a *product-type Coxeter orbifold*, whose underlying polyhedron is topologically a polygon times an interval, and all edges of the top and bottom faces have edge order 2,
- a Coxeter orbifold with finite fundamental group.

Recall that a Coxeter orbifold $\hat{P}$ is said to be *orderable* if the faces of P can be ordered so that each face contains at most three edges that are edges of order 2 or edges in a face of higher index. Then we have:

Theorem 5 ([12]) *Let P be a 3-dimensional convex polyhedron in $\mathbb{S}^3$ and let $\hat{P}$ be given a normal-type Coxeter orbifold structure. Suppose that $\hat{P}$ is orderable. Then the restricted deformation space of projective structures on $\hat{P}$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.*

Examples of orderable orbifolds are obtained if P is any convex polyhedron with all faces triangular. An antiprism (i.e. drum-shaped convex polyhedron with n-gons on the top and bottom joined up by a band of $2n$-triangles) with arbitrary orders given to the edges is orderable, since we can order the top and the bottom faces to have the highest two indices. By Andreev's theorem, an antiprism with all angles $\pi/2$ admits a complete hyperbolic Coxeter orbifold structure (see also Thurston [28]). A triangular prism carries compact hyperbolic Coxeter orbifold structures and these are all orderable.

However the cube and dodecahedron do not carry an orderable Coxeter orbifold structure, since a lowest index face in an orderable orbifold must be triangular.

2 The restricted deformation space of real projective structures

In this section, the restricted deformation space of real projective structures on an n-dimensional Coxeter orbifold $\hat{P}$ is discussed, and identified with a space of representations.

In §2.1 we define a suitable space of restricted representations from $\pi_1(\hat{P})$ into $SL_\pm(n+1,\mathbb{R})$. In §2.2 we show that this restricted representation space can be identified with the solution space of a system of polynomial equations given by Vinberg (Proposition 1). In §2.3 we prove that the restricted deformation space is homeomorphic to the space of restricted representations and to the set of solutions of Vinberg's equations (Theorems 6 and 7), when the underlying convex polyhedron P has a discrete projective automorphism group. In §2.4 we look at the equations satisfied by the Lorentzian unit normals to a hyperbolic polyhedron, and show that a hyperbolic structure on $\hat{P}$ corresponds to the single point in the solution space of §2.2.

2.1 The restricted representation space

Let P be a fixed n-dimensional convex polytope contained in $\mathbb{S}^n$, and $\hat{P}$ an associated Coxeter orbifold. We now identify the deformation space $\mathfrak{D}(\hat{P})$ of real projective structures on $\hat{P}$ with the deformation space of (G,X)-structures on $\hat{P}$, where $G = SL_\pm(n+1,\mathbb{R})$ and $X = \mathbb{S}^n$ is the projective sphere in $V = \mathbb{R}^{n+1}$.

Sending a development pair (D,h) to its holonomy representation h induces a local homeomorphism

$$hol : \tilde{\mathfrak{D}}(\hat{P}) \to \mathrm{Hom}(\pi_1(\hat{P}), G),$$

where $\tilde{\mathfrak{D}}(\hat{P})$ denotes the space of isotopy-equivalence classes of development pairs. (See Theorem 1 of [10] and Proposition 3 of [12], where only a sketch proof is given. One of the authors is writing a more complete account in a generalized setting [13].)

Recall that $\pi_1(\hat{P})$ is a Coxeter group with standard generator r_i corresponding to the ith face of P. To study the restricted deformation space we consider the subset

$$\tilde{\mathfrak{D}}_P(\hat{P}) \subset \tilde{\mathfrak{D}}(\hat{P})$$

giving projective structures *with fundamental polyhedron P*. More precisely, let H_i denote the hyperplane in V containing the ith face of P. Then $\tilde{\mathfrak{D}}_P(\hat{P})$ consists of the isotopy-equivalence classes $[(D,h)]$ of developing pairs (D,h) such that each $h(r_i)$ is a reflection with fixed point set $\mathrm{Fix}(h(r_i)) = H_i$.

Lemma 1 *For every $[(D,h)] \in \tilde{\mathfrak{D}}_P(\hat{P})$ the holonomy h lies in the subset*

$$Hom_P(\pi_1(\hat{P}), G) \subset Hom(\pi_1(\hat{P}), G)$$

consisting of representations such that each $h(r_i)$ is a projective reflection fixing H_i, and $h(r_i r_j)$ is a rotation by $2\pi/n_{ij}$ whenever $F_i \cap F_j$ is a codimension 2 face of P of order n_{ij}.

Proof The definition of the orbifold structure on $\hat{P}$ shows that the local action of r_i and r_j on the universal cover of $\hat{P}$ is given by a standard dihedral group of order $2n_{ij}$, generated by involutions fixing two hypersurfaces meeting transversally at an angle π/n_{ij}. Given a real projective structure on $\hat{P}$, this action is transferred by the developing map into $\mathbb{S}^n$. Hence each $h(r_i)$ is a projective reflection, and $h(r_i)h(r_j)$ is conjugate to a rotation by π/n_{ij}.

We call $\mathrm{Hom}_P(\pi_1(\hat{P}),G)$ the space of *restricted representations* from $\pi_1(\hat{P})$ to G. Lemma 1 shows that *hol* restricts to a map

$$hol_P : \tilde{\mathfrak{D}}_P(\hat{P}) \to \mathrm{Hom}_P(\pi_1(\hat{P}),G).$$

In Theorem 6, we will show that this is a homeomorphism.

2.2 Restricted representations and Vinberg's equations

We now give a very explicit description of the restricted representation space $\mathrm{Hom}_P(\pi_1(\hat{P}),G)$ for the fundamental group of a Coxeter orbifold $\hat{P}$. Let $V=\mathbb{R}^{n+1}$ and let P be a *fixed* convex polytope in $\mathbb{S}^n$.

Assume that P is given by a system of linear inequalities

$$\alpha_i \le 0, \quad i=1,\dots,f,$$

where $\alpha_i \in V^*$ and f is the number of codimension one faces of P. Suppose $b_i \in V$ for $1 \le i \le f$ are reflection vectors with $\alpha_i(b_i)=2$. Let R_i be the reflections defined by $R_i = Id - \alpha_i \otimes b_i$ for $i=1,...,f$, and let $\Gamma \subset SL_{\pm}(n+1,\mathbb{R})$ be the group generated by the R_i. Then the matrix $A=(a_{ij})=(\alpha_i(b_j))$ is the $f\times f$ Cartan matrix of Γ.

We now fix orders n_{ij} for the codimension 2 faces of P and consider the *restricted* deformation space of the corresponding Coxeter orbifold $\hat{P}$. Then the α_i's will be *fixed*, and b_i's are *variables,* so Vinberg's result leads us to solve the following system of polynomial equations:

- For each $i=1,\dots,f$,
$$a_{ii} = \alpha_i(b_i) = 2, \tag{3}$$
- If F_i and F_j are adjacent in P and $n_{ij}>2$,
$$a_{ij}a_{ji} = \alpha_i(b_j)\alpha_j(b_i) = 4\cos^2(\pi/n_{ij}), \tag{4}$$
- If F_i and F_j are adjacent in P and $n_{ij}=2$,
$$a_{ij} = \alpha_i(b_j) = 0 \ \text{ and } \ a_{ji} = \alpha_j(b_i) = 0. \tag{5}$$

(Note the difference between the cases $n_{ij}=2$ and $n_{ij}>2$.)

We call these polynomial equations (3)–(5) the *Vinberg equations*. Let N be the number of Vinberg equations and let $\Phi_{\hat{P}} : V^f = \mathbb{R}^{(n+1)f} \to \mathbb{R}^N$ be the map given by

$$(b_1,\dots,b_f) \mapsto (\Phi_1,\dots,\Phi_N),$$

where $\{\Phi_k\}_{k=1}^N$ is the set of polynomials $a_{ii}-2$, $a_{ij}a_{ji}-4\cos^2(\pi/n_{ij})$, or a_{ij}, a_{ji} as in the above equations (3)–(5). Note that $N=f+e+e_2$, where e is the number of codimension 2 faces of P, and e_2 is the number of codimension 2 faces of order 2.

Proposition 1 *Suppose that the linear functionals α_i defining the faces of P are fixed. Then there is a homeomorphism*

$$\mathcal{H} : \Phi_{\hat{P}}^{-1}(0) \to Hom_P(\pi_1(\hat{P}), SL_{\pm}(n+1,\mathbb{R}))$$

where $\mathcal{H}$ sends $(b_1,\dots,b_f)$ to the homomorphism h with $h(r_i) = Id - \alpha_i \otimes b_i$. The map $\mathcal{H}$ is a polynomial map with a rational inverse $\mathcal{R}$.

Proof Solving the Vinberg equations (3)–(5) is equivalent to finding reflections R_i, $i = 1,...,f$, corresponding to the faces of P, such that R_iR_j is conjugate to a rotation by $2\pi/n_{ij}$ whenever F_i and F_j meet along a codimension 2 face. This follows, for example, from Lemma 1.2 of [6].

Conversely, given a reflection matrix $R_i = Id - \alpha_i \otimes b_i$, the reflection vector b_i is uniquely determined since α_i is fixed. In fact, b_i is the unique eigenvector of R_i with eigenvalue -1 satisfying the normalization condition $\alpha_i(b_i) = 2$. It follows easily that the inverse map $\mathcal{R}$ taking each reflection matrix $h(r_i) = R_i$ to its reflection vector b_i is a rational map.

From now on, the space of representations $\mathrm{Hom}_P(\pi_1(\hat{P}), SL_\pm(n+1,\mathbb{R}))$ will be identified with $\Phi_{\hat{P}}^{-1}(0)$.

2.3 The restricted deformation space

Let P be a convex polytope in $\mathbb{S}^n$, and $\hat{P}$ an associated Coxeter orbifold. In this section, we will show that the restricted space of isotopy classes of real projective structures $\tilde{\mathfrak{D}}_P(\hat{P})$ can identified with the restricted representation space, and with the set of solutions to Vinberg's equations (3)–(5). In the generic case where the group of projective automorphisms of P is discrete, these spaces are also homeomorphic to the restricted deformation space $\mathfrak{D}_P(\hat{P})$ of real projective structures on $\hat{P}$.

Theorem 6 *The maps*

$$\tilde{\mathfrak{D}}_P(\hat{P}) \xrightarrow{hol_P} Hom_P(\pi_1(\hat{P}), G) \xrightarrow{\mathcal{R}} \Phi_{\hat{P}}^{-1}(0)$$

are homeomorphisms.

Proof Given a set of reflection vectors $(b_1,\ldots,b_f) \in \Phi_{\hat{P}}^{-1}(0)$, the work of Vinberg (see [32] or Theorem 1.5 of [6]) shows that

(i) the corresponding reflections $R_i = Id - \alpha_i \otimes b_i$ generate a discrete group Γ isomorphic to $\pi_1(\hat{P})$,
(ii) the images γP for $\gamma \in \Gamma$ tile an convex open subset $\Omega \subset \mathbb{S}^n$, and
(iii) the quotient orbifold Ω/Γ is isomorphic to $\hat{P}$.

Thus we obtain a convex real projective structure on $\hat{P}$ and isotopy class of development pair $[(D,h)] \in \tilde{\mathfrak{D}}_P(\hat{P})$ that maps to $(b_1,\ldots,b_f)$. This gives continuous inverses to the maps in the theorem.

Next we study the restricted deformation space $\mathfrak{D}_P(\hat{P})$. Let

$$G_P = \{g \in G : g(H_i) = H_i \text{ for all } i\}$$

be the subgroup of G that preserves P and each of its faces (and hence preserves each of its vertices). Note that $\dim G_P = k(P)$, where $k(P)$ denotes the dimension of the group of projective automorphisms of P as in [12].

Proposition 2 *The group G_P acts on $\tilde{\mathfrak{D}}_P(\hat{P})$, and the quotient space $\tilde{\mathfrak{D}}_P(\hat{P})/G_P$ is homeomorphic to $\mathfrak{D}_P(\hat{P})$.*

Proof We write $\tilde{\mathfrak{D}} = \tilde{\mathfrak{D}}(\hat{P})$, $\tilde{\mathfrak{D}}_P = \tilde{\mathfrak{D}}_P(\hat{P})$, and $\mathfrak{D}_P = \mathfrak{D}_P(\hat{P})$. Now G acts on $\tilde{\mathfrak{D}}$ by equation (1), and we let $\pi : \tilde{\mathfrak{D}} \to \tilde{\mathfrak{D}}/G = \mathfrak{D}$ be the natural quotient map. Then $\mathfrak{D}_P = \pi(\tilde{\mathfrak{D}}_P) = \hat{\mathfrak{D}}_P/G$, where $\hat{\mathfrak{D}}_P = \pi^{-1}(\mathfrak{D}_P) = G \cdot \tilde{\mathfrak{D}}_P \subset \tilde{\mathfrak{D}}$.

Let $[(D,h)] \in \tilde{\mathfrak{D}}_P$ and $g \in G$. Then $g \cdot [(D,h)] = [(g \circ D, g \circ h \circ g^{-1})]$ and $\mathrm{Fix}(g \circ h(r_i) \circ g^{-1}) = g \cdot \mathrm{Fix}(h(r_i)) = g(H_i)$. Thus

$$g \cdot [(D,h)] \in \tilde{\mathfrak{D}}_P \text{ if and only if } g \in G_P. \tag{6}$$

In particular, it follows that G_P acts on $\tilde{\mathfrak{D}}_P$, and we let $\pi_P : \tilde{\mathfrak{D}}_P \to \tilde{\mathfrak{D}}_P/G_P$ be the natural quotient map.

Now the composition $\tilde{\mathfrak{D}}_P \subset \hat{\mathfrak{D}}_P \to \hat{\mathfrak{D}}_P/G$ is continuous and constant on G_P orbits, so there is an induced continuous map $\tilde{\mathfrak{D}}_P/G_P \to \hat{\mathfrak{D}}_P/G = \mathfrak{D}_P$, taking $G_P \cdot y$ to $G \cdot y$ for $y \in \tilde{\mathfrak{D}}_P$. This is a bijection by observation (6). To finish the proof we show that the inverse is continuous.

First, define $\phi : G \times \tilde{\mathfrak{D}}_P \to \hat{\mathfrak{D}}_P$ by $\phi(g,z) = g \cdot z$. This is a continuous, surjective, open map, hence an identification map. Let $p : G \times \tilde{\mathfrak{D}}_P \to \tilde{\mathfrak{D}}_P$ be the projection onto the second factor. Now if $y, y' \in \phi^{-1}(x)$ then (6) shows that $p(y') = g \cdot p(y)$ for some $g \in G_P$, so $\pi_P \circ p(y) = \pi_P \circ p(y')$. Hence $\pi_P \circ p \circ \phi^{-1}$ is well-defined and gives a continuous function $\hat{\mathfrak{D}}_P \to \tilde{\mathfrak{D}}_P/G_P$. This is constant on G-orbits, so induces a continuous map $\hat{\mathfrak{D}}_P/G \to \tilde{\mathfrak{D}}_P/G_P$. This is the desired inverse. (More explicitly, the inverse is given by $G \cdot x \mapsto (G \cdot x) \cap \tilde{\mathfrak{D}}_P$ for $x \in \hat{\mathfrak{D}}_P$.)

Since the homeomorphism hol_P in Theorem 6 is equivariant with respect to the action of G_P, we also obtain the following.

Corollary 1 *The map hol_P induces a homeomorphism*

$$\mathfrak{D}_P(\hat{P}) \cong \tilde{\mathfrak{D}}_P/G_P \to Hom_P(\pi_1(\hat{P}), G)/G_P,$$

where G_P acts on $Hom_P(\pi_1(\hat{P}), G)$ by conjugation.

In the remainder of this paper we concentrate on the generic case, where $k(P) = 0$. Then we have

Theorem 7 *If $k(P) = 0$ then G_P is a trivial group. Hence we have homeomorphisms*

$$\mathfrak{D}_P(\hat{P}) \cong Hom_P(\pi_1(\hat{P}), G) \cong \Phi_{\hat{P}}^{-1}(0).$$

Proof If $k(P) = 0$ then G_P is a discrete group. Since G_P acts trivially on each face and vertex of P, it follows that the group is a trivial group. The rest follows from Theorem 6 and Proposition 2.

The following result shows that $k(P) = 0$ holds for most convex polyhedra $P \subset \mathbb{S}^3$.

Proposition 3 *Let P be a convex 3-dimensional polyhedron in $\mathbb{S}^3$. Then*

- *$k(P) = 3$ if P is a tetrahedron,*
- *$k(P) = 1$ if P is a convex cone over a polygon other than a triangle, and*
- *$k(P) = 0$ otherwise.*

Thus, $k(P)$ only depends on the combinatorial type of P, and not on the geometric shape of P.

Proof It suffices to consider a projective automorphism that fixes every vertex of P. If there is a face containing four or more vertices, then every point of the face is fixed. (Recall that $n+2$ points in general position form a projective basis for $\mathbb{R}P^n$.) Such an automorphism is represented by a diagonal matrix in $PGL(4,\mathbb{R})$ with diagonal entries $1,1,1,\lambda$. Hence $k(P) = 1$ if P is a cone, and $k(P) = 0$ otherwise. If every face of P is a triangle, then a similar argument shows that $k(P) = 3$ if P is a tetrahedron, and $k(P) = 0$ otherwise. The last sentence follows from these observations.

2.4 The hyperbolic point

Let V be an $(n+1)$-dimensional vector space over $\mathbb{R}$ with coordinates $x_1,\dots,x_{n+1}$, and let P be a Coxeter polytope in Klein's model of n-dimensional hyperbolic space $\mathbb{H}^n$ with faces F_i for $i=1,\dots,f$. Let $\nu_i \in V$ denote the outward unit normal to F_i with respect to the Lorentzian inner product on V, defined by

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + \ldots + x_{n+1}y_{n+1}.$$

Then P is defined by the system of linear inequalities

$$\langle \nu_i, x\rangle \le 0, \quad i=1,\dots,f, \text{ and } x_1 = 1.$$

Now the problem of constructing a hyperbolic polyhedron P with prescribed dihedral angles π/n_{ij} can be expressed as the problem of finding a solution to the following equations:

$$\begin{aligned} \langle \nu_i,\nu_i\rangle &= 1 \text{ for all } i=1,\dots,f, \\ \langle \nu_i,\nu_j\rangle &= -\cos(\pi/n_{ij}) \text{ if faces } F_i \text{ and } F_j \text{ are adjacent in } P. \end{aligned} \tag{7}$$

We call these equations (7) the *hyperbolic equations*.

To compare this with Vinberg's equations, first note that P is defined by the system of linear inequalities

$$\alpha_i \le 0, \quad i=1,\dots,f, \text{ and } x_1 = 1,$$

where the linear functional $\alpha_i \in V^*$ is dual to ν_i under the Lorentzian inner product. In other words, $\alpha_i(x) = \langle \nu_i, x\rangle$.

The hyperbolic reflection in the face F_i is given by

$$R_i(x) = x - 2\langle \nu_i, x\rangle \nu_i = x - \alpha_i(x) b_i$$

where the reflection vector is $b_i = 2\nu_i$. So the reflection point $[b_i] = [\nu_i]$ in $\mathbb{R}P^n$ is the projective dual of the hyperplane containing the face F_i with respect to the sphere at infinity in the Klein model of $\mathbb{H}^n$. There is also a well-known geometric construction corresponding to this kind of duality, see for example [29, p.71].

Thus taking $b_i = 2\nu_i$ gives a point $t = \{t_i\} = \{2\nu_i\}$ in $\Phi_{\hat{P}}^{-1}(0)$ corresponding to the hyperbolic structure on P. This follows since if faces F_i and F_j are adjacent in P then

$$a_{ij} = \alpha_i(2\nu_j) = 2\langle \nu_i, \nu_j\rangle = -2\cos(\pi/n_{ij}),$$

and thus

$$\begin{aligned} a_{ii} &= \alpha_i(\nu_i) = 2\langle \nu_i,\nu_i\rangle = 2 \text{ for all } i=1,\dots,f, \\ a_{ij}a_{ji} &= 4\cos^2(\pi/n_{ij}) \text{ if } F_i \text{ and } F_j \text{ are adjacent in } P \text{ and } n_{ij} > 2, \\ a_{ij} &= 0 \text{ and } a_{ji} = 0 \text{ if } F_i \text{ and } F_j \text{ are adjacent in } P \text{ and } n_{ij} = 2. \end{aligned}$$

Proposition 4 *Let P be a hyperbolic Coxeter polyhedron. Then the space $\Phi_{\hat{P}}^{-1}(0)$ contains a single point corresponding to the hyperbolic structure on $\hat{P}$.*

Proof The hyperbolic reflection in each face of P is determined by the face and a reflection point that is dual to the face in the Klein model. Since P is fixed, the reflection vectors are determined up to scalar multiplication. By the normalization conditions in $\Phi_{\hat{P}}$, we see that the reflection vectors are uniquely determined. Hence the hyperbolic structure on $\hat{P}$ corresponds to a single point.

If we solve the hyperbolic equations (7) directly, then we obtain many algebraic solutions ν_i. However when $n \geq 3$, Mostow-Prasad Rigidity shows that there is only one solution (up to hyperbolic isometries) with geometric meaning. To find this, we need to check that the ν_i's give the desired n-dimensional convex hyperbolic polytope.

Theorem 8 (Vinberg [33]) *Let the Gram matrix G of the set S of vectors $\{\nu_1,\ldots,\nu_f\}$ be an indecomposable matrix (i.e. it cannot be represented as a direct sum of two matrices) with 1's along the diagonal and non-positive entries off it. Assume that S spans V and the cone K defined by the inequalities $\langle x,\nu_i\rangle \leq 0$ $(i=1,\ldots,f)$ intersects the Klein model for $\mathbb{H}^n$. Then G is the Gram matrix of the convex polytope $P = K \cap \mathbb{H}^n$ bounded by the hyperplanes $H_i = \{x \in \mathbb{H}^n \mid \langle x,\nu_i\rangle = 0\}$.*

The following observation will be useful for computational purposes, when we need to select the correct geometric solution from the many algebraic solutions to the hyperbolic equations (7).

Lemma 2 *If the first entry of each ν_i is non-negative, then the cone K defined by the inequalities $\langle x,\nu_i\rangle \leq 0$ $(i=1,\ldots,f)$ intersects $\mathbb{H}^n$.*

Proof $x = (1,0,0,\ldots,0)$ satisfies the inequalities $\langle x,\nu_i\rangle = -\nu_{i1} \leq 0$.

Remark 1 In general, it is difficult to find an exact algebraic solution to the hyperbolic equations (7). However, in 3-dimensions, Roeder's Matlab program [25] can be used to obtain numerical solutions. His construction uses Newton's method and a homotopy to follow the concrete existence proof given by Andreev (as modified in [26]). Heard's program "Orb" [18] can also be used to numerically compute hyperbolic structures on the orientable 3-orbifold obtained by doubling a Coxeter polyhedron along its boundary. In this paper, we will find many exact solutions using Mathematica.

Remark 2 When describing examples in this paper, we will sometimes abuse notation and identify V^* with V as follows: If $\alpha_i \in V^*$ has coordinates $(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{i,n+1})$, and $b_j \in V$ has coordinates $(b_{j1},b_{j2},\ldots,b_{j,n+1})$ then $\alpha_i(b_j) = \alpha_{i1}b_{j1} + \alpha_{i2}b_{j2} + \ldots + \alpha_{i,n+1}b_{j,n+1} = \alpha_i \cdot b_j$, where $\cdot$ denotes the usual Euclidean dot product in $V = \mathbb{R}^{n+1}$. Faces of a polytope will always be specified by the coordinate vectors of the corresponding linear functionals. In particular, for a hyperbolic polytope in $\mathbb{H}^n$, a face with Lorentzian unit normal ν_i corresponds to the linear functional with coordinate vector $\alpha_i = J\nu_i$, where $J \in SL_{\pm}(n+1,\mathbb{R})$ is the diagonal matrix with diagonal entries $-1,1,\ldots,1$.

3 Local restricted deformation spaces of real projective structures near hyperbolic structures

We now concentrate on the case of a 3-dimensional Coxeter orbifold $\hat{P}$. Recall that real projective structures in the restricted deformation space of $\hat{P}$ correspond to solutions to Vinberg's equations (3)–(5). In §3.1 we study the Zariski tangent space to this solution

space, and in §3.2 we study the Zariski tangent space to the solution space of the hyperbolic equations (7). In §3.3 we compare these tangent spaces and combine this with Garland-Raghunathan-Weil infinitesimal rigidity ([16], [35]) to prove Theorem 1. In §3.4 we also use these techniques to study the restricted deformation spaces of 3-dimensional compact hyperbolic prisms.

3.1 The Zariski tangent space to the Vinberg equations

We now study the Zariski tangent space to the solution space of Vinberg's equations, using the notation from §2.2. Let $V = \mathbb{R}^4$, let P be a convex polyhedron in $\mathbb{S}^3$, and let $\hat{P}$ be a corresponding Coxeter orbifold. We assume that P has f faces, and that each linear functional $\alpha_i \in V^*$ is *fixed* for $i = 1, \dots, f$. Then we have variables $b_i \in V$ for $i = 1, \dots, f$, and the equations have the form

- $\Phi_{ii} = \alpha_i(b_i) - 2 = 0$,
- $\Phi_{ij} = \alpha_i(b_j)\alpha_j(b_i) - c_{ij} = 0$ where c_{ij} is a constant if $n_{ij} \neq 2$,
- $\Phi^1_{ij} = \alpha_i(b_j) = 0$ and $\Phi^2_{ij} = \alpha_j(b_i) = 0$ if $n_{ij} = 2$.

Let $\pi_i : V^f \to V$ denote the projection onto the ith factor. Then the derivative of Φ_{ij} at $b = (b_1, \dots, b_f)$, considered as a linear map, is given by:

$$D\Phi_{ij}(\dot{b}) = \alpha_i(\dot{b}_j)\alpha_j(b_i) + \alpha_i(b_j)\alpha_j(\dot{b}_i)$$

for $\dot{b} = (\dot{b}_1, \dot{b}_2, \dots, \dot{b}_f) \in V^f$, or

$$D\Phi_{ij} = a_{ji}(\alpha_i \circ \pi_j) + a_{ij}(\alpha_j \circ \pi_i).$$

Similarly,

$$D\Phi_{ii} = \alpha_i \circ \pi_i,\ D\Phi^1_{ij} = \alpha_i \circ \pi_j \text{ and } D\Phi^2_{ij} = \alpha_j \circ \pi_i.$$

More explicitly, combining Vinberg's equations gives a function $\Phi_{\hat{P}} : V^f = \mathbb{R}^{4f} \to \mathbb{R}^N$ and the rows of the $N \times 4f$ Jacobian matrix $D = [D\Phi_{\hat{P}}]$ are made up of blocks, each consisting of four entries:

$$\begin{aligned}[D\Phi_{ii}] = &\quad (0, \dots, 0, \alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4}, 0, \dots, 0) \\ = &\quad (0, \dots, 0, \underbrace{\alpha_i}_{i\text{th block}}, 0, \dots, 0),\end{aligned}$$

$$[D\Phi_{ij}] = \quad (0, \dots, 0, \underbrace{a_{ij}\alpha_j}_{i\text{th block}}, 0, \dots, 0, \underbrace{a_{ji}\alpha_i}_{j\text{th block}}, 0, \dots, 0), \text{ for } n_{ij} \neq 2,$$

$$[D\Phi^1_{ij}] = \quad (0, \dots, 0, \underbrace{0}_{i\text{th block}}, 0, \dots, 0, \underbrace{\alpha_i}_{j\text{th block}}, 0, \dots, 0) \text{ for } n_{ij} = 2,$$

and

$$[D\Phi^2_{ij}] = \quad (0, \dots, 0, \underbrace{\alpha_j}_{i\text{th block}}, 0, \dots, 0, \underbrace{0}_{j\text{th block}}, 0, \dots, 0) \text{ for } n_{ij} = 2.$$

Note that this Jacobian matrix has two rows for each edge of P with $n_{ij} = 2$, but only one row for each edge with $n_{ij} \geq 3$. (Compare this with §3.3 below.)

Suppose that p is a point of $\Phi_{\hat{P}}^{-1}(0)$. Then the Zariski tangent space at p is the kernel of the Jacobian matrix D evaluated at p. We call this the *infinitesimal restricted deformation space of* P at p because of Theorem 6.

The next result now follows from the implicit function theorem.

Proposition 5 *Let $D = [D\Phi_{\hat{P}}]$ be the Jacobian matrix of Vinberg's equations for $\hat{P}$ at p.*
If $4f - N > 0$ and D has full rank, i.e. rank $D = N$, then $\mathfrak{D}_P(\hat{P})$ is locally a smooth manifold of dimension $4f - N$ near p. So if p is the hyperbolic point, the hyperbolic structure on the Coxeter 3-orbifold $\hat{P}$ deforms relative to the mirrors to a real projective structure which is not a hyperbolic structure.
If $4f - N \leq 0$ and D has full rank, then p is an isolated point in $\mathfrak{D}_P(\hat{P})$. So if p is the hyperbolic point, the hyperbolic structure on $\hat{P}$ is projectively rigid relative to the mirrors in $\mathfrak{D}_P(\hat{P})$.

Note that $4f - N = 4f - (f + e + e_2) = 3f - e - e_2$. The results in [12] are obtained by showing that D has full rank in the *orderable* case.

The following example illustrates the role of orderability; in this case, permutations of rows of D are sufficient to show that D has full rank. This example was originally studied by Benoist [5] and is orderable.

Here and throughout the paper, we use the following notation. Given a diagram of a 3-dimensional hyperbolic polyhedron, if an edge is labelled e_i, then its dihedral angle is π/e_i. Moreover, α_i is the linear functional defining the face F_i.

Example 1 Figure 1 shows a 3-dimensional compact triangular prism P. This satisfies the conditions (A1)–(A4) of Andreev's Theorem so defines a hyperbolic Coxeter orbifold $\hat{P}$.

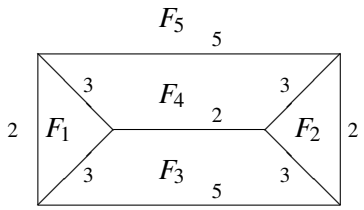


Fig. 1 A compact hyperbolic triangular prism

The following table shows that $\hat{P}$ is orderable:

	faces sharing edges of order 2	faces of higher index not sharing edges of order 2
F_1	F_5	F_3, F_4
F_2	F_5	F_3, F_4
F_3	F_4	F_5
F_4	F_3	F_5
F_5	F_1, F_2	$\varnothing$

The 17×20 Jacobian matrix $D = [D\Phi_{\hat{P}}]$ is shown on the left below. By permuting the rows, we obtain the matrix on the right. (The new ordering corresponds to the entries in the rows

of the table above.)

$$D = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 & 0 \\
0 & 0 & \alpha_3 & 0 & 0 \\
0 & 0 & 0 & \alpha_4 & 0 \\
0 & 0 & 0 & 0 & \alpha_5 \\
a_{13}\alpha_3 & 0 & a_{31}\alpha_1 & 0 & 0 \\
a_{14}\alpha_4 & 0 & 0 & a_{41}\alpha_1 & 0 \\
\alpha_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_1 \\
0 & a_{23}\alpha_3 & a_{32}\alpha_2 & 0 & 0 \\
0 & a_{24}\alpha_4 & 0 & a_{42}\alpha_2 & 0 \\
0 & \alpha_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_2 \\
0 & 0 & \alpha_4 & 0 & 0 \\
0 & 0 & 0 & \alpha_3 & 0 \\
0 & 0 & a_{35}\alpha_5 & 0 & a_{53}\alpha_3 \\
0 & 0 & 0 & a_{45}\alpha_5 & a_{54}\alpha_4
\end{pmatrix} \sim \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 \\
\alpha_5 & 0 & 0 & 0 & 0 \\
a_{13}\alpha_3 & 0 & a_{31}\alpha_1 & 0 & 0 \\
a_{14}\alpha_4 & 0 & 0 & a_{41}\alpha_1 & 0 \\
0 & \alpha_2 & 0 & 0 & 0 \\
0 & \alpha_5 & 0 & 0 & 0 \\
0 & a_{23}\alpha_3 & a_{32}\alpha_2 & 0 & 0 \\
0 & a_{24}\alpha_4 & 0 & a_{42}\alpha_2 & 0 \\
0 & 0 & \alpha_3 & 0 & 0 \\
0 & 0 & \alpha_4 & 0 & 0 \\
0 & 0 & a_{35}\alpha_5 & 0 & a_{53}\alpha_3 \\
0 & 0 & 0 & \alpha_4 & 0 \\
0 & 0 & 0 & \alpha_3 & 0 \\
0 & 0 & 0 & a_{45}\alpha_5 & a_{54}\alpha_4 \\
0 & 0 & 0 & 0 & \alpha_5 \\
0 & 0 & 0 & 0 & \alpha_1 \\
0 & 0 & 0 & 0 & \alpha_2
\end{pmatrix}$$

Note that every coefficient a_{ij} appearing in the Jacobian matrix D corresponds to an edge with $n_{ij} \geq 3$, so satisfies $a_{ij} < 0$. Thus Lemma 3 below implies that the following submatrices have full rank. (Note that the orderability of $\hat{P}$ is used here.)

$$\begin{pmatrix} \alpha_1 \\ \alpha_5 \\ a_{13}\alpha_3 \\ a_{14}\alpha_4 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \alpha_5 \\ a_{23}\alpha_3 \\ a_{24}\alpha_4 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ \alpha_4 \\ a_{35}\alpha_5 \end{pmatrix}, \begin{pmatrix} \alpha_4 \\ \alpha_3 \\ a_{45}\alpha_5 \end{pmatrix}, \begin{pmatrix} \alpha_5 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Therefore, the Jacobian matrix D has full rank $= 17$, and thus $\mathfrak{D}_P(\hat{P})$ is a smooth manifold of dimension 3, since the dimension of the null space of D is always $3 = 20 - 17$.

The following observation will be used again in §4.2.

Lemma 3 *Let P be a 3-dimensional convex polyhedron in $\mathbb{S}^3$ defined by linear inequalities $\alpha_i \leq 0$ where $\alpha_i \in V^*$, and let F_i be the face of P determined by α_i. Suppose that the faces F_{i_1}, F_{i_2}, F_{i_3} are adjacent to the face F_{i_4}. Then the four linear functionals α_{i_1}, α_{i_2}, α_{i_3}, α_{i_4} are linearly independent.*

Proof If not, then the linear functionals α_{i_1}, α_{i_2}, α_{i_3}, α_{i_4} lie in a codimension one subspace of V^* determined by a non-zero vector $b \in V$. In other words,

$$\alpha_{i_1}(b) = \alpha_{i_2}(b) = \alpha_{i_3}(b) = \alpha_{i_4}(b) = 0.$$

Furthermore, we know that three linear inequalities and one equality

$$\alpha_{i_1} \leq 0,\ \alpha_{i_2} \leq 0,\ \alpha_{i_3} \leq 0,\ \alpha_{i_4} = 0$$

give a (2-dimensional) triangle since the three faces F_{i_1}, F_{i_2}, F_{i_3} meet the plane containing the face F_{i_4} in lines. Moreover, these three lines have no common intersection point. This is a contradiction since the three lines meet at the point b by our assumption.

3.2 The Zariski tangent space to the hyperbolic equations

Assume P is a finite volume 3-dimensional hyperbolic polyhedron where the dihedral angle at an edge e_{ij} equals π/n_{ij} for an integer $n_{ij} \geq 2$. Andreev's theorem [2] characterizes such polyhedra.

Constructing such a hyperbolic polyhedron P is the same as solving the system of hyperbolic equations (7) for the unit normals ν_i from §2.4. Equivalently we can write these equations in terms of the reflection vectors $b_i = 2\nu_i$. This gives the following system of $n = f + e$ equations:

$$\Psi_{ii} = \langle b_i, b_i \rangle - 4 = 0 \text{ and } \Psi_{ij} = \langle b_i,\ b_j \rangle + 4\cos(\pi/n_{ij}) = 0.$$

Combining these gives a function $\Psi_{\hat{P}} : V^f = \mathbb{R}^{4f} \to \mathbb{R}^n$ and $\Psi_{\hat{P}}^{-1}(0)$ contains the convex polyhedra in $\mathbb{H}^3$ with the desired dihedral angles. By Andreev's Theorem (or Mostow-Prasad rigidity), there is a unique such polyhedron up to hyperbolic isometries; this corresponds to a 6-dimensional manifold contained in $\Psi_{\hat{P}}^{-1}(0)$.

Now consider the derivative $D\Psi_{\hat{P}}$ at a hyperbolic point t. If $\alpha_i = \langle \nu_i, \cdot \rangle$ are the linear functionals defining the faces of the hyperbolic polyhedron then

$$D\Psi_{ij}(\dot{b}) = \langle \dot{b}_i, b_j \rangle + \langle b_i, \dot{b}_j \rangle = 2\alpha_j(\dot{b}_i) + 2\alpha_i(\dot{b}_j),$$

or

$$D\Psi_{ij} = 2\alpha_j \circ \pi_i + 2\alpha_i \circ \pi_j.$$

When $i = j$ this becomes

$$D\Psi_{ii} = 4\alpha_i \circ \pi_i.$$

Equivalently, the rows of the $n \times 4f$ Jacobian matrix $\hat{D} = [D\Psi_{\hat{P}}]$ are made up of blocks, each consisting of four entries:

$$\begin{aligned} [D\Psi_{ii}] = &\quad (0,\dots,0,4\alpha_{i1},4\alpha_{i2},4\alpha_{i3},4\alpha_{i4},0,\dots,0) \\ = &\quad (0,\dots,0,\ \underbrace{4\alpha_i}_{i\text{th block}},0,\dots,0), \end{aligned}$$

and

$$[D\Psi_{ij}] = \quad (0,\dots,0,\ \underbrace{2\alpha_j}_{i\text{th block}},0,\dots,0,\ \underbrace{2\alpha_i}_{j\text{th block}},0,\dots,0).$$

Then the Zariski tangent space to $\Psi_{\hat{P}}^{-1}(0)$ at t is $\ker D\Psi_{\hat{P}}$.

3.3 The proof of Theorem 1

We now assume that P is a convex ideal polyhedron in $\mathbb{H}^3$ with all edges of order 3. Then all vertices are trivalent, and we have assumed that P is not a tetrahedron. So it follows from Proposition 3 that $k(P) = 0$, and that the results from §2.3 apply.

To prove Theorem 1, we use the results from §3.1 and §3.2 to compare the Jacobian matrices $D = [D\Phi_{\hat{P}}]$ for real projective structures and $\hat{D} = [D\Psi_{\hat{P}}]$ for hyperbolic structures. Since P contains no edges of order 2 we have $N = n$, and each a_{ij} is non-zero. Further, $a_{ij} = a_{ji}$ at a hyperbolic point in $\mathfrak{D}_P(\hat{P})$. Hence, each row of D is a non-zero scalar multiple of a row of $\hat{D}$, so the ranks of D and $\hat{D}$ are equal.

We now use the infinitesimal rigidity of the hyperbolic structure on $\hat{P}$ to compute the rank of $\hat{D}$. The arguments from Proposition 1 show that $\Psi_{\hat{P}}^{-1}(0)$ is locally isomorphic to the algebraic variety $\mathrm{Hom}(\pi_1(\hat{P}), O_0(1,3))$ near a hyperbolic point with holonomy representation h_0. Thus, by the work of Weil [36], the Zariski tangent space to $\Psi_{\hat{P}}^{-1}(0)$ at this point corresponds to the space of 1-cocyles in group cohomology $Z^1(\pi_1(\hat{P}), so(1,3)_{Ad})$, where $\pi_1(\hat{P})$ acts on $so(1,3)$ via the representation $Ad \circ h_0$. (See also [24], [21], [22].)

Let Q be the compact orbifold obtained by truncating the cusps of $\hat{P}$; then Q has a boundary ∂Q consisting of $(3,3,3)$-triangle orbifolds. Now consider the exact sequence

$$H^1(Q, \partial Q; so(1,3)_{Ad}) \to H^1(Q; so(1,3)_{Ad}) \to H^1(\partial Q; so(1,3)_{Ad}).$$

By Garland-Raghunathan-Weil infinitesimal rigidity ([16], [35]) the parabolic group cohomology $PH^1(\pi_1(\hat{P}); so(1,3)_{Ad}) = 0$. This implies that

$$H^1(Q, \partial Q; so(1,3)_{Ad}) = 0.$$

Further, the $(3,3,3)$-triangle group is infinitesimally rigid in $O_0(1,3)$ so

$$H^1(\partial Q; so(1,3)_{Ad}) = 0.$$

Hence

$$H^1(\pi_1(\hat{P}); so(1,3)_{Ad}) \cong H^1(Q; so(1,3)_{Ad}) = 0.$$

It follows that

$$\dim \ker D\Psi_{\hat{P}} = \dim Z^1(\pi_1(\hat{P}), so(1,3)_{Ad}) = \dim so(1,3) = 6.$$

In fact, Weil's argument in [36] shows that a neighbourhood of h_0 in $\mathrm{Hom}(\pi_1(\hat{P}), O_0(1,3))$ coincides with the orbit of h_0 under the group of hyperbolic isometries, and that this is a locally a smooth 6-manifold since the hyperbolic holonomy group $h_0(\pi_1(\hat{P}))$ has trivial centralizer.

Since all vertices of P are trivalent we have $3v = 2e$, and since $v - e + f = 2$ it follows that $4f - N = 3f - e = 6 = 4f - \mathrm{rank}\, D\Psi_{\hat{P}}$. Hence $\mathrm{rank}\, D\Phi_{\hat{P}} = \mathrm{rank}\, D\Psi_{\hat{P}} = N$ and $D\Phi_{\hat{P}}$ has full rank. Therefore, by Proposition 5, a neighbourhood of t in $\mathfrak{D}_P(\hat{P})$ is a smooth 6-dimensional manifold. This completes the proof of Theorem 1.

Remark 3 Looking more closely at the proof of Theorem 1, we see that $\ker D\Phi_{\hat{P}}$ is *equal* to $\ker D\Psi_{\hat{P}}$ and is given by the tangent space to orbit of $b = (b_1, \ldots, b_f)$ under the 6-dimensional group of hyperbolic isometries. This gives a very nice geometric interpretation of the infinitesimal deformations relative to the mirrors provided by Theorem 1. Consider a convex hyperbolic polyhedron P in the Klein model with $P \subset \mathbb{H}^3 \subset \mathbb{R}P^3$. The reflection in each face F_i has a fixed point $[b_i]$ outside the sphere at infinity, corresponding to the reflection vector b_i for the face. Let g_t be a 1-parameter family of isometries of $\mathbb{H}^3$ with $g_0 = identity$. Then $b(t) = (g_t(b_1), \ldots, g_t(b_f))$ is a curve in $V^f = \mathbb{R}^{4f}$ whose derivative at the identity $\dot{b} = (\dot{g}(b_1), \ldots, \dot{g}(b_f))$ is in the kernel of $D\Phi_{\hat{P}}$. Here $\dot{g} \in so(1,3)$ is an infinitesimal isometry of $\mathbb{H}^3$. In other words, all the infinitesimal projective deformations relative to the mirrors are obtained by fixing the polyhedron faces and moving the fixed points of face reflections by infinitesimal hyperbolic isometries. It would be very interesting to extend this observation to give an explicit description of the local projective deformations relative to the mirrors.

Remark 4 This argument extends to convex hyperbolic polyhedra P with trivalent but possibly hyperinfinite vertices, provided *all edges have order at least* 3 and $k(P) = 0$. In general, such a polyhedron is non-compact of infinite volume, but can be truncated along planes orthogonal to the faces at each hyperinfinite vertex to give a compact convex polyhedron. Again infinitesimal rigidity applies since all vertex cross sections give hyperbolic triangle groups, hence are rigid. The argument given above then shows that the restricted deformation space $\mathfrak{D}_P(\hat{P})$ is again locally a smooth 6-dimensional manifold, provided *all edges have order at least* 3. So "almost all" assignments of orders to the edges of P give a hyperbolic Coxeter orbifold $\hat{P}$ that is projectively deformable relative to the mirrors.

3.4 Deformations of prisms

Assume that P is a 3-dimensional compact hyperbolic Coxeter polyhedron, and $\hat{P}$ a corresponding Coxeter orbifold. Theorem 9 below shows that whenever $\hat{P}$ is orderable and the number of faces of P is greater than 7, $\hat{P}$ is projectively rigid relative to the mirrors. Thus, there are only finitely many combinatorial types of convex hyperbolic polyhedra with orderable compact Coxeter 3-orbifold structures that are projectively deformable relative to the mirrors. However when $\hat{P}$ is not orderable, Proposition 6 shows that this is no longer true.

Theorem 9 *Let P be a 3-dimensional compact hyperbolic Coxeter polyhedron. Suppose that $\hat{P}$ is orderable. If the number of faces of P is greater than 7, then it is projectively rigid relative to the mirrors in $\mathfrak{D}_P(\hat{P})$.*

Proof Since P is compact, every vertex is trivalent and is adjacent to an edge of order 2. (See, for example, Andreev's condition (A1) in §1.3.) Also, $v > 10$, since $3v = 2e$ and $v - e + f = 2$. This implies that $e_2 > 5$, and thus $3f - e - e_2 = 6 - e_2 \leq 0$. Hence, there is no local deformations of P relative to the mirrors by Theorem 5

The deformations in the following proposition are actually projective bending deformations as shown in §4.6.

Proposition 6 *For any natural number $f \geq 7$, there exists a 3-dimensional compact hyperbolic prism P with f faces and a corresponding Coxeter orbifold $\hat{P}$ that can be projectively deformed relative to the mirrors.*

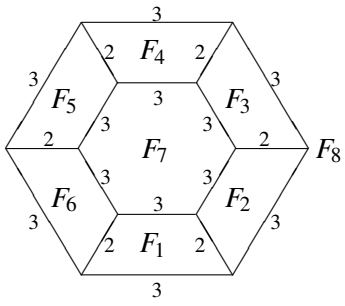


Fig. 2 A hexagonal prism with a rotational symmetry about the axis passing through the centers of faces F_7 and F_8

Proof Let n be a natural number greater than 4. We will construct a prism P with $f = n+2$ faces such that edge orders of the top and bottom n-gons of P are 3 and all the remaining edges orders are 2. (Figure 2 shows the case where $n = 6$.) Note that P is not orderable.

Suppose that $c = \cos(2\pi/n)$ and $s = \sin(2\pi/n)$. Using the rotational symmetry of P, we can find a compact 3-dimensional hyperbolic prism as follows. Let the unit normals ν_i have coordinate vectors:

$$\begin{aligned}
\nu_1 &= \big(\sqrt{\frac{c}{1-c}}, \sqrt{\frac{1}{1-c}}, 0, 0\big),\\
\nu_{k+1} &= L^k\nu_1 \text{ with } k = 1,\dots,n-1,\\
\nu_{n+1} &= \big(\sqrt{\frac{1-c}{4c}}, 0, 0, \sqrt{\frac{1+3c}{4c}}\big),\\
\nu_{n+2} &= \big(\sqrt{\frac{1-c}{4c}}, 0, 0, -\sqrt{\frac{1+3c}{4c}}\big),
\end{aligned}$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & c & -s & 0\\ 0 & s & c & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
&\langle \nu_i, \nu_i\rangle = 1 \text{ for all } i = 1,\dots,n+2,\\
&\langle \nu_{n+1}, \nu_k\rangle = -1/2 \text{ and } \langle \nu_{n+2}, \nu_k\rangle = -1/2 \text{ for all } k = 1,\dots,n,\\
&\langle \nu_1, \nu_2\rangle = \langle \nu_2, \nu_3\rangle = \cdots = \langle \nu_{n-1}, \nu_n\rangle = \langle \nu_n, \nu_1\rangle = 0,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the Lorentzian inner product.

Our aim is to solve Vinberg's equations (3)–(5). Let the linear functionals α_i defining the faces have coordinate vectors $J\nu_i$. We restrict ourselves to solutions of the special form:

$$\begin{aligned}
b_1 &= \big(\sqrt{\frac{4c}{1-c}}, \sqrt{\frac{4}{1-c}}, 0, b_{1,4}\big),\\
b_{k+1} &= L^k b_1 \text{ with } k = 1,\dots,n-1,\\
b_{n+1} &= (b_{n+1,1}, 0, 0, b_{n+1,4}) \text{ with } \alpha_{n+1}(b_{n+1}) = 2,\\
b_{n+2} &= (b_{n+2,1}, 0, 0, b_{n+2,4}) \text{ with } \alpha_{n+2}(b_{n+2}) = 2.
\end{aligned}$$

where $b_{n+1,2} = b_{n+1,3} = b_{n+2,3} = b_{n+2,3} = 0$, and we take $\{b_{1,4}, b_{n+1,4}, b_{n+2,4}\}$ as *three* free variables.

We now solve Vinberg's equations. By computation, we obtain

$$a_{kk} = \alpha_k(b_k) = \alpha_1(b_1) = 2$$

for all $k = 1,\dots,n$, since $(L^k)^T L^k = I$, where M^T denotes the transpose of the matrix M. Then we obtain

$$\begin{aligned}
a_{j,j+1} &= \alpha_j(b_{j+1}) = \alpha_1(b_2) = -2c/(1-c) + 2c/(1-c) = 0,\\
a_{j+1,j} &= \alpha_{j+1}(b_j) = \alpha_2(b_1) = -2c/(1-c) + 2c/(1-c) = 0,
\end{aligned}$$

for all $j = 1, \ldots, n-1$. Similarly, $a_{1,n} = a_{n,1} = 0$. Moreover, if the *two* equations

$$a_{n+1,1}a_{1,n+1} = 1 \text{ and } a_{n+2,1}a_{1,n+2} = 1 \tag{8}$$

are satisfied, then the rotational symmetry implies that *all* of Vinberg's equations (3)–(5) are satisfied, since

$$a_{n+1,k} = \alpha_{n+1}(b_k) = \alpha_{n+1}(b_1) \text{ and } a_{k,n+1} = \alpha_k(b_{n+1}) = \alpha_1(b_{n+1}),$$

for all $k = 1, \ldots, n$. Similarly, $a_{n+2,k} = a_{n+2,1}$ and $a_{k,n+2} = a_{1,n+2}$ for all $k = 1, \ldots, n$.

Since we have three free variables subject to the two equations (8), it follows that the hyperbolic Coxeter orbifold $\hat{P}$ has a one parameter family of real projective structures. (Of course, the dimension of the restricted deformation space might be greater than 1.) This completes the proof.

The deformations keeping the top reflection point and the bottom one in the central axis are determined by the above proof. However, there might be deformations that do not preserve this property. In general, computing the exact dimension seems hard. For regular hexagonal prism, we were able to show that the dimension is exactly 1 but we failed in computing other examples. (See prism.nb and prism6.nb for this.)

4 The numerical and algebraic computations of restricted deformation spaces

The restricted deformation space $\mathfrak{D}_P(\hat{P}) \cong \Phi^{-1}(0)$ of a Coxeter orbifold $\hat{P}$ is defined by Vinberg's system of polynomials $\Phi = \Phi_{\hat{P}}$ and each of these has total degree ≤ 2. It is difficult to understand the general properties of these algebraic varieties. Thus we examine the infinitesimal and local restricted deformation spaces for some interesting examples of Coxeter 3-orbifolds arising from compact hyperbolic cubes and dodecahedra. This work uses a combination of several theoretical and computational methods.

In §4.1 we outline the main algorithm used for our computations. In §4.2 we give a simple linear test for projective rigidity rel mirrors. In §4.3 we describe the notation used in our figures and tables. Finally, in §4.4 and §4.5, we provide detailed tables describing our results on the restricted projective deformation spaces for cubes and dodecahedra, and give detailed descriptions of the methods used.

4.1 The main algorithm for computing local restricted deformations

We use the following steps to compute the local restricted deformation spaces of 3-dimensional compact hyperbolic Coxeter cubes and dodecahedra.

1. We tabulate the 3-dimensional compact hyperbolic cubes (or dodecahedra) satisfying the conditions of Andreev's theorem (A1)-(A4). (See cu.m and do.m in [19].) To obtain manageable finite lists, we restrict the possible edge orders as specified in Theorems 2 and 3. This gives us 34 Coxeter orbifolds (cu1-cu34) based on the cube, and 13 Coxeter orbifolds (do1-do13) based on the dodecahedron.
2. We apply the linear test of rigidity in §4.2 by hand. If the test shows rigidity relative to the mirrors, we stop here and conclude that our orbifold is projectively rigid relative to the mirrors.

3. Next, we explictly construct the 3-dimensional compact hyperbolic Coxeter cubes (or dodecahedra) obtained in step 1. To do this we first choose three faces meeting at a vertex and put the normals to these faces into a standard position. We then use Mathematica to solve the hyperbolic equations (7) for the remaining unit normals. This gives us explicit linear functionals α_i defining the hyperbolic polyhedron.
   For cubes, it is not difficult to find exact *algebraic* values for α_i, since the number of α_i is not large. However, for dodecahedra it is difficult to find these algebraic values. By utilizing a rotational symmetry of do13, we find exact algebraic values of α_i for do13 by hand. We then obtain *numerical* values of α_i for the remaining dodecahedra do1-do12 by deforming the dihedral angles of do13. To obtain the numerical values of α_i, we utilize Mathematica where we can adjust the accuracy to make the errors as small as desired. In fact, we maintain 150 digits of precision in internal computations.
   To check that the α_i obtained here give the desired 3-dimensional compact hyperbolic polyhedron, we use Theorem 8. In fact, the first coordinate of each α_i obtained by Mathematica is non-positive, so we can easily apply Lemma 2
4. We compute the dimension of the Zariski tangent space to $\Phi^{-1}(0)$ for the hyperbolic point, i.e. the dimension of the null space of the Jacobian matrix $D = [D\Phi]$ at the corresponding point. If D is of full rank, step 5 is followed. Otherwise, step 6 is followed.
   For cubes, we use the exact algebraic values obtained in step 3. However, for dodecahedra, we have only numerical values of α_i for the dodecahedra other than do13. To see the accuracy of the numerical calculation of $\operatorname{rank} D$ we use the singular value decomposition of the Jacobian matrix D. In general, the best method for determining the rank of a matrix is to count the number of singular values greater than a judiciously chosen positive number [30]. Note that the singular values of the matrix are non-negative real numbers. We check the minimum of the singular values of D to determine whether D is of full rank or not.
5. If D is of full rank, then the dimension of a neighbourhood of the hyperbolic point is determined by the kernel of D. That is, by Proposition 5, the dimension of the space of infinitesimal restricted deformations is the same as the dimension of the space of local restricted deformations. Therefore, in this case, the algorithm stops and we obtain answers.
6. If D is rank-deficient, we attempt to obtain the Gröbner basis of the ideal I generated by $\{\Phi_k = 0\}_{k=1}^{N}$ with respect to a lexicographic order on the variables. First, we choose new coordinates on $\mathbb{R}^{4f}$ by letting $c_i = b_i - t_i$, where $t = \{t_i\}$ corresponds to our hyperbolic point in $\mathfrak{D}_P(\hat{P})$. In this coordinate system, the hyperbolic point corresponds to the origin 0.
   In general, the entries of α_i are complicated, and thus sometimes it is difficult to calculate the Gröbner basis of the ideal I directly by using Mathematica. Therefore, we express the entries of α_i as elements in a field $\mathbb{Q}(\theta)$ generated by an algebraic number θ, to improve the speed of calculations. In general, the arithmetic within a *fixed* finite extension of $\mathbb{Q}$ is much faster than arithmetic within the field of complex numbers. Using this technique, we obtain a list of the dimensions of local restricted deformation spaces for all the cubes. For do13, using the rotational symmetry, we compute the dimension of its local restricted deformation space. Consequently, we get a list of the dimensions of local restricted deformation spaces for all the dodecahedra.

4.2 A linear test for rigidity

The following *linear test for rigidity* provides a simple, direct proof that seventeen cubes (cu1-cu14, cu16, cu20, cu23) are projectively rigid relative to the mirrors. For other cubes, we go to the next step of the algorithm.

Let P be a 3-dimensional Coxeter polyhedron in $\mathbb{S}^3$. Then there is a simple method to show the rigidity of the corresponding orbifold $\hat{P}$.

1. Find all the faces having more than two edges of order 2. We call them the *rigid faces at level 1*.
2. Relabel all edges of rigid faces at level 1 to become edges of order 2.
3. Again, find all other faces having more than two edges of order 2. We call them *rigid faces at level 2*. Relabel all edges of these faces to become edges of order 2.
4. Continue the process this manner.
5. If every face of P occurs as a rigid face at level k for some $k \geq 1$, then we can conclude $\hat{P}$ is projectively rigid relative to the mirrors.

This test is derived from the following two facts.

- If a face F_i has more than two edges that are of order 2, say $\{F_{ij_1}, \ldots, F_{ij_m}\}$ with $m \geq 3$, then b_i can be eliminated as a variable. This follows since b_i satisfies a system of linear equations
$$\alpha_{j_1}(b_i) = c_{j_1}, \ \ldots \ , \alpha_{j_m}(b_i) = c_{j_m}$$
where $c_{j_1}, \ldots, c_{j_m}$ are constants, and these determine b_i uniquely by Lemma 3
- If b_i is no longer a variable, then $\alpha_i(b_j)\alpha_j(b_i) = 4\cos^2(\pi/n_{ij})$ is a *linear* equation for b_j.

As the number of edges of order 2 increases, this test becomes more effective. In particular, it is often useful if P is a compact 3-dimensional hyperbolic Coxeter polyhedron.

Remark 5 Of course here we are changing the edge orders only temporarily, and when the linear test does not show rigidity we restore the original orders and go to the next step.

Example 2 Figure 3 shows a compact 3-dimensional hyperbolic cube cu23.

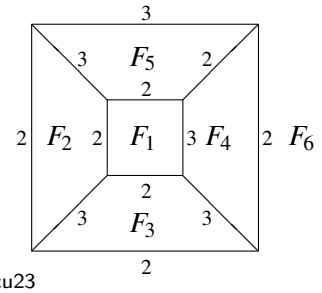


Fig. 3 Labels of faces and edges of cu23

Then the sets of rigid faces at level 1, level 2, and level 3 are $\{F_1, F_6\}$, $\{F_4, F_5\}$, and $\{F_2, F_3\}$, respectively. Hence every face of cu23 occurs as a rigid face at level 1, 2 or 3, and thus cu23 is projectively rigid relative to the mirrors.

4.3 Notations for figures and tables

The following notations will be used in the figures and tables throughout this paper:

- Each e_i is an edge order, corresponding to a dihedral angle π/e_i,
- O = the number of variables − the number of Vinberg equations (3)–(5),
- I = the dimension of infinitesimal restricted deformation space of real projective structures,
- A = the dimension of local restricted deformation space of real projective structures (a bold number means that the result is obtained from the final step of the main algorithm of §4.1),
- L = Is it possible to apply the linear test of rigidity? (yes or no), and the maximum level needed,
- J = Does the calculation of the Jacobian D give a full description of the local restricted deformation space? (yes or no),
- S = the minimum of the singular values of the Jacobian D,
- G = the order of the group of symmetries,
- C = the number of (essential) circuits in the dual graph corresponding to edges of order 3,
- B = the number of totally geodesic 2-dimensional suborbifolds that are not faces of cubes and are embedded (as defined in Remark 8 below).

To compute G, we simply used the maximal symmetry groups for the regular cube and the regular dodecahedron and used a computer program to find which symmetries preserved the assigned edge orders.

The number C was computed by hand. We remark that if $C \neq 0$ then the linear test of rigidity does not apply.

To compute B, we considered the three 4-cycles of faces in the cube, found four vectors corresponding to their reflection points and computed the corresponding determinant. If the determinant is not zero, then there is no corresponding 2-dimensional totally geodesic suborbifold in the cubical orbifold. If the determinant is zero, there is a plane containing the four reflection points and this gives a geodesic suborbifold. If a face parallel to the cycle has all its edge orders equal to 2, then this suborbifold is just a face of the cube. Otherwise, we obtain the desired embedded geodesic suborbifold. (Except for cu29 and cu34, each totally suborbifold was actually the fixed point set of a symmetry of the orbifold.)

4.4 The results for cubes

Let P be a compact hyperbolic cube, all of whose dihedral angles are $\pi/2$ or $\pi/3$. By step 1 in §4.1, the total number of such cubes is 34 (up to symmetries). See Table 1. These orbifolds were tabulated by using Matlab to check the conditions of Andreev's theorem; the Matlab files used are available from the web page (see cu.m in [19].)

By step 2 in §4.1 (the linear test for rigidity), we find that seventeen cubes (cu1-cu14, cu16, cu20, cu23) are projectively rigid relative to the mirrors.

Using steps 3-5 in §4.1, exact algebraic computations of the dimensions of Zariski tangent spaces determine the dimensions of the local restricted deformation spaces for eight cubes (cu15, cu19, cu24, cu25, cu28, cu30-cu32) since we can show that the Jacobian matrices D for these cubes are of full rank. The computations were done with arbitrary precision algorithms in Mathematica, and were also numerically verified using Matlab.

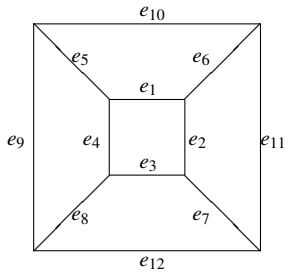


Fig. 4 Labels of edges of cubes

name	$e_1e_2\cdots e_{11}e_{12}$	O	I	A	L	J	G	C	B
cu1	232222232223	-3	0	0	yes, level 2	·	2	0	0
cu2	232222232233	-2	0	0	yes, level 3	·	1	0	0
cu3	232222232322	-3	0	0	yes, level 1	·	6	0	0
cu4	232222232323	-2	0	0	yes, level 2	·	1	0	0
cu5	232222232333	-1	0	0	yes, level 3	·	1	0	0
cu6	232222233322	-2	0	0	yes, level 2	·	2	0	0
cu7	232222233332	-1	0	0	yes, level 3	·	1	0	0
cu8	232222322223	-3	0	0	yes, level 2	·	2	0	0
cu9	232222322332	-2	0	0	yes, level 2	·	1	0	0
cu10	232222323223	-2	0	0	yes, level 3	·	2	0	0
cu11	232222323322	-2	0	0	yes, level 2	·	2	0	0
cu12	232222323323	-1	0	0	yes, level 3	·	1	0	0
cu13	232222323332	-1	0	0	yes, level 2	·	2	0	0
cu14	232222333322	-1	0	0	yes, level 3	·	1	0	0
cu15	232222333332	0	0	0	no	yes	1	1	0
cu16	232223233322	-1	0	0	yes, level 3	·	2	0	0
cu17	232223322323	-1	1	**1**	no	no	2	1	1
cu18	232223323323	0	1	**1**	no	no	4	1	1
cu19	232223333322	0	0	0	no	yes	1	1	0
cu20	232232232233	-1	0	0	yes, level 3	·	2	0	0
cu21	232232232323	-1	1	**1**	no	no	4	1	1
cu22	232232232333	0	1	**1**	no	no	2	1	1
cu23	232232332322	-1	0	0	yes, level 3	·	2	0	0
cu24	232232332323	0	0	0	no	yes	1	1	0
cu25	232232332332	0	0	0	no	yes	1	1	0
cu26	232233332223	0	1	**0**	no	no	2	1	0
cu27	232233332323	1	2	**1**	no	no	2	2	1
cu28	232322232233	-1	0	0	no	yes	2	1	0
cu29	232323232323	0	1	**0**	no	no	4	1	2
cu30	232323323323	1	1	1	no	yes	4	2	1
cu31	232323332323	1	1	1	no	yes	2	2	0
cu32	232323333322	1	1	1	no	yes	2	2	0
cu33	232333332323	2	3	**2**	no	no	8	3	2
cu34	233223233322	0	1	**1**	no	no	12	1	3

Table 1 The list of cubes

We need to look beyond the Jacobian matrix to calculate the dimension of the local restricted deformation spaces for the remaining cubes (cu17, cu18, cu21, cu22, cu26, cu27, cu29, cu33, cu34) as the Jacobian matrices of these cubes are rank-deficient. Using step 6 in §4.1, we instead obtain a Gröbner basis for the ideal generated by Vinberg's equations (3)–(5) using Mathematica. The detailed computations for cu i is saved in a Mathematica

notebook file (cu i.nb) available from the web page [19] for each $i = 15, 17, 18, 19, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34$. (Note that choosing the hyperbolic solutions for the cubes is often a nontrivial process involving geometric considerations, and the choices are explained in the files themselves.)

Example 3 As an example, we use cu21 to illustrate the method for computing the local restricted deformation space. (See Figure 5.) All the local restricted deformation spaces for the cubes except cu27 and cu33 are obtained by a similar method.

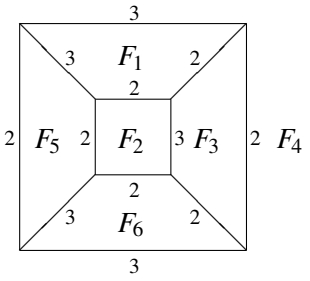


Fig. 5 cu21

First, we note that the set of all rigid faces is $\{F_2, F_3\}$, and these faces are at level 1. So the linear test of rigidity is not applicable.

Second, we find the unit normals ν_i for cu21 as follows. We must solve the system $\{\Psi_k = 0\}_{k=1}^{n}$ of hyperbolic equations (7), where $n = f + e = 18$. We choose the vertex $F_{123} = F_1 \cap F_2 \cap F_3$ whose adjacent three edges have orders $(2,2,3)$. Then, by an isometry of $\mathbb{H}^3$, we can assume that the normals of the adjacent faces are

$$\nu_1 = (0,1,0,0),\ \nu_2 = (0,0,1,0),\ \text{and}\ \nu_3 = (0,0,-\frac{1}{2},\frac{\sqrt{3}}{2}).$$

Hence we satisfy the following six hyperbolic equations:

$$\langle \nu_1, \nu_2 \rangle = \langle \nu_1, \nu_3 \rangle = 0 \ \text{and}\ \langle \nu_2, \nu_3 \rangle = -1/2,$$
$$\langle \nu_1, \nu_1 \rangle = \langle \nu_2, \nu_2 \rangle = \langle \nu_3, \nu_3 \rangle = 1.$$

Since the orders of the two edges F_{14} and F_{34} are 3 and 2 respectively, we let

$$\nu_4 = (x, -\frac{1}{2}, u, \frac{u}{\sqrt{3}}).$$

Similarly, we let

$$\nu_5 = (y, -\frac{1}{2}, 0, \frac{2v}{\sqrt{3}}) \ \text{and}\ \nu_6 = (z, w, 0, 0).$$

Then we satisfy the six hyperbolic equations

$$\langle \nu_1, \nu_4 \rangle = \langle \nu_1, \nu_5 \rangle = -1/2,$$
$$\langle \nu_3, \nu_4 \rangle = \langle \nu_2, \nu_5 \rangle = \langle \nu_2, \nu_6 \rangle = \langle \nu_3, \nu_6 \rangle = 0.$$

Therefore, we must solve the remaining six hyperbolic equations

$$\langle \nu_4, \nu_5 \rangle = 0 \text{ and } \langle \nu_4, \nu_6 \rangle = \langle \nu_5, \nu_6 \rangle = -1/2,$$
$$\langle \nu_4, \nu_4 \rangle = \langle \nu_5, \nu_5 \rangle = \langle \nu_6, \nu_6 \rangle = 1.$$

However, these equations have many solutions. Among them we choose a solution such that non-diagonal entries of the Gram matrix $G = (g_{ij})$ with $g_{ij} = \langle \nu_i, \nu_j \rangle$ are non-positive. In particular, u, v and w are non-positive since $g_{24} = u$, $g_{35} = v$ and $g_{16} = w$. This leaves two solutions:

$$x = y = z = -\frac{\sqrt{5}}{2}, \ u = v = -\sqrt{\frac{3}{2}}, \text{ and } w = -\frac{3}{2} \tag{9}$$

$$x = y = z = \frac{\sqrt{5}}{2}, \ u = v = -\sqrt{\frac{3}{2}}, \text{ and } w = -\frac{3}{2} \tag{10}$$

For solution (9), the cone defined by the inequalities $\langle X, \nu_i \rangle \geq 0$ intersects $\mathbb{H}^3$ since $X = (1,0,0,0)$ satisfies the inequalities $\langle X, \nu_i \rangle = -\nu_{i1} \geq 0$. However, in the case of solution (10), the cone defined by the inequalities $\langle X, \nu_i \rangle \leq 0$ intersects $\mathbb{H}^3$ since $X = (1,0,0,0)$ satisfies the inequalities $\langle X, \nu_i \rangle = -\nu_{i1} \leq 0$. Here, the cone K is defined by the inequalities $\langle X, \nu_i \rangle \leq 0$. Hence only solution (10) is appropriate for our hyperbolic cube. Furthermore, the Gram matrix G is as follows:

$$G = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -\sqrt{\frac{3}{2}} & 0 \\ -\frac{1}{2} & -\sqrt{\frac{3}{2}} & 0 & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\sqrt{\frac{3}{2}} & 0 & 1 & -\frac{1}{2} \\ -\frac{3}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

Hence the solution (10) is the unique solution satisfying the conditions of Theorem 8, and the cone K gives the 3-dimensional compact hyperbolic cube cu21$= K \cap \mathbb{H}^3$.

Remark 6 For each cube, we start by choosing a vertex whose adjacent edges have orders $(2,2,3)$ or $(2,2,2)$. If the edge orders are $2,2,3$ we choose normals ν_1, ν_2, ν_3 for the adjacent three faces as above. This applies to the cases cu21, cu22, cu24, cu25, cu26, cu29, cu30, cu31 and cu32. If the edge orders are $2,2,2$ we let

$$\nu_1 = (0,1,0,0), \ \nu_2 = (0,0,1,0) \text{ and, } \ \nu_3 = (0,0,0,1).$$

This applies to the cases cu15, cu17, cu18, cu19, cu28 and cu34.

Third, using the linear functionals $\alpha_i = J\nu_i$ we form the Jacobian matrix $D = D\Phi$ for Vinberg's equations (3)–(5) at the hyperbolic point. Note that D is a 25×24 matrix. Using Mathematica we find that the rank of D is 23, and so D is rank-deficient. Since the dimension of kernel of the Jacobian matrix D is 1, the dimension of the infinitesimal restricted deformation space of real projective structures is 1.

Finally, to obtain the dimension of the local restricted deformation space of cu21, we compute a Gröbner basis of the ideal $\langle \Phi_1, \ldots, \Phi_N \rangle$ with $N = f + e + e_2 = 25$. Before doing this, we introduce new coordinates on $\mathbb{R}^{4f} = \mathbb{R}^{24}$ by letting $c_i = b_i - t_i$, where $t = \{t_i\}$ corresponds to the hyperbolic point in $\mathfrak{D}_P(\hat{P})$. (Relative to this coordinate system, t is the origin.)

We compute a Gröbner basis of the ideal $\langle \Phi_1, \ldots, \Phi_{25} \rangle$ with respect to the lexicographic order with $c_{41} < c_{42} < c_{43} < c_{44} < c_{51} < c_{52} < c_{53} < c_{54} < c_{61} < c_{62} < c_{63} < c_{64} < c_{11} < c_{12} < c_{13} < c_{14} < c_{21} < c_{22} < c_{23} < c_{24} < c_{31} < c_{32} < c_{33} < c_{34}$. Then the Gröbner basis for cu21 is

$$\begin{aligned}
&\{c_{34}, c_{33}, c_{32}, c_{31}, c_{24}, c_{23}, c_{22}, c_{21}, c_{14}, c_{13}, c_{12}, c_{64}, c_{63},\\
&-\mathbf{c_{11}} + \frac{\mathbf{2c_{62}}}{\sqrt{\mathbf{5}}} + \mathbf{2c_{11}c_{62}}, \sqrt{5}c_{61} + 3c_{62}, c_{54}, c_{53}, -\mathbf{c_{11}} + \frac{\mathbf{2c_{52}}}{\sqrt{\mathbf{5}}} + \mathbf{c_{11}c_{52}},\\
&-\mathbf{c_{52}} + \mathbf{c_{62}} + \mathbf{c_{52}c_{62}}, \sqrt{5}c_{51} + c_{52}, c_{44}, c_{43}, c_{42} - c_{52}, \sqrt{5}c_{41} + c_{52}\}
\end{aligned}$$

Let f_i be the ith polynomial shown in bold letters. The Gröbner basis implies that

- c_{11} is a free variable, and c_{11} determines c_{62} and c_{52},
- c_{62} determines c_{61}, and c_{52} determines c_{51}, c_{42} and c_{41},
- the remaining variables are zero, and
- the equation $f_3 = 0$ is implied by the equations $f_1 = 0$ and $f_2 = 0$.

The last observation follows since the three polynomials in bold letters satisfy the relation

$$c_{62} f_2 + c_{11} f_3 = c_{52} f_1.$$

In other words, if $c_{11} \neq 0$ then $f_3 = (c_{52} f_1 - c_{62} f_2)/c_{11}$. Hence $f_1 = f_2 = 0$ implies $f_3 = 0$. Also, if $c_{11} = 0$ then $f_1 = f_2 = 0$ imply $c_{62} = c_{52} = 0$, and thus $f_3 = 0$. Therefore, this implies that the dimension of the local restricted deformation space is also 1.

Example 4 As another example, we describe the calculation of the local restricted deformation space for cu27, noting the differences to the method used for cu21. (See Figure 6).

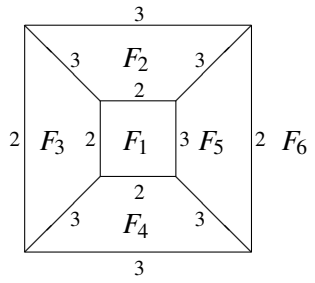


Fig. 6 cu27

Steps 1-2 are similar to those for cu21. (The details are omitted.) To find the exact values of ν_i using Mathematica, we use the reflectional symmetry of cu27, interchanging F_2 and F_4. Hence we let

$$\begin{aligned}
&\nu_1 = (0,0,0,1),\ \nu_2 = (x,-1/2,-u,0),\ \nu_3 = (0,1,0,0),\\
&\nu_4 = (x,-1/2,u,0),\ \nu_5 = (y,v,0,-1/2),\ \nu_6 = (z,0,0,w).
\end{aligned}$$

By similar computations to those for cu21, we obtain the following:

notation	numerical value	a real root of
u	-1.36278	$5+44t^2-144t^4+64t^6=0$
v	-1.45161	$-1-2t+2t^2+2t^3=0$
w	-1.10716	$1-4t+4t^3=0$
x	1.05222	$-1-4t^2+4t^6=0$
y	1.16497	$-1-52t^2-48t^4+64t^6=0$
z	0.47519	$-1+16t^4+16t^6=0$

We again compute the rank of the Jacobian matrix D at the hyperbolic point. Here D is a 23×24 matrix with rank 22. Thus D does not have full rank, and the dimension of kernel of the Jacobian matrix D is 2.

Since the expressions of $\alpha_i = J\nu_i$ for cu27 are complicated, Mathematica is unable to compute a Gröbner basis of the ideal $\langle \Phi_1,\dots,\Phi_{23}\rangle$ in a reasonable time. To make the problem easier for Mathematica to solve, we find an algebraic number θ such that $\{u,v,w,x,y,z\}\subset \mathbb{Q}(\theta)$. Here $\theta\approx -0.395609$ and is a real root of $64-384t^2-208t^4+320t^6-52t^8-24t^{10}+t^{12}=0$, and we convert the above u, v, w, x, y, and z to elements of $\mathbb{Q}(\theta)$ as follows:

notation	as a element of $\mathbb{Q}(\theta)$
u	$\frac{13}{4}\theta+\frac{13}{8}\theta^3-\frac{5}{2}\theta^5+\frac{13}{32}\theta^7+\frac{3}{16}\theta^9-\frac{1}{128}\theta^{11}$
v	$-\frac{47}{32}-\frac{1}{16}\theta^2+\frac{73}{64}\theta^4-\frac{1}{4}\theta^6-\frac{47}{512}\theta^8+\frac{1}{256}\theta^{10}$
w	$-1-\frac{7}{8}\theta^2+\frac{5}{4}\theta^4-\frac{13}{64}\theta^6-\frac{3}{32}\theta^8+\frac{1}{256}\theta^{10}$
x	$-\frac{169}{64}\theta-\frac{45}{128}\theta^3+\frac{195}{128}\theta^5-\frac{89}{256}\theta^7-\frac{129}{1024}\theta^9+\frac{11}{2048}\theta^{11}$
y	$-\frac{11}{4}\theta-\frac{13}{8}\theta^3+\frac{5}{2}\theta^5-\frac{13}{32}\theta^7-\frac{3}{16}\theta^9+\frac{1}{128}\theta^{11}$
z	$-\frac{35}{32}\theta-\frac{55}{64}\theta^3+\frac{9}{8}\theta^5-\frac{7}{64}\theta^7-\frac{37}{512}\theta^9+\frac{3}{1024}\theta^{11}$

Next, we introduce new coordinates on $\mathbb{R}^{4f}=\mathbb{R}^{24}$ by letting $c_i=b_i-t_i$, where $t=\{t_i\}$ correspond to the hyperbolic point in $\mathfrak{D}_P(\hat{P})$. We compute a Gröbner basis of the ideal $\langle \Phi_1,\dots,\Phi_{23}\rangle$ with respect to the lexicographic order with $c_{21}<c_{22}<c_{23}<c_{24}<c_{41}<c_{42}<c_{43}<c_{44}<c_{51}<c_{52}<c_{53}<c_{54}<c_{61}<c_{62}<c_{63}<c_{64}<c_{31}<c_{32}<c_{33}<c_{34}<c_{11}<c_{12}<c_{13}<c_{14}$. Then the Gröbner basis for cu27 is

$$\{c_{14},c_{13},c_{12},c_{11},c_{34},c_{32},c_{31},c_{64},(c_{33}-c_{63})^2,c_{62},c_{61},c_{54},\dots\}.$$

Replacing the polynomial $(c_{33}-c_{63})^2$ by $(c_{33}-c_{63})$ gives a new ideal $\langle c_{33}-c_{63},\Phi_1,\dots,\Phi_{23}\rangle$ with the same underlying solution set. This has the Gröbner basis

$$\begin{aligned}\{\ & c_{14},c_{13},c_{12},c_{11},c_{34},c_{32},c_{31},c_{64},c_{33}-c_{63},c_{62},c_{61},c_{54},c_{33}-c_{53},\\ & c_{52},c_{51},c_{44},-\boldsymbol{\delta c_{33}}+\boldsymbol{\varepsilon c_{43}}+\boldsymbol{c_{33}c_{43}},-\delta c_{42}+c_{43},\mu c_{41}+c_{43},c_{24},\\ & -\varepsilon\boldsymbol{c_{23}}+\delta\boldsymbol{c_{33}}+\boldsymbol{c_{23}c_{33}},-\delta\boldsymbol{c_{23}}+\delta\boldsymbol{c_{43}}+\boldsymbol{2c_{23}c_{43}},\delta c_{22}+c_{23},-\mu c_{21}+c_{23}\},\end{aligned}$$

where

notation	numerical value	a real root of
δ	1.25796	$961-1017t^2+251t^4+5t^6=0$
ε	0.733796	$64-144t^2+44t^4+5t^6=0$
μ	0.597767	$961-2288t^2-1152t^4+80t^6=0$

Let f_i be the ith polynomial shown in bold letters. Then this Gröbner basis implies that

– c_{33} is a free variable, and c_{33} determines c_{63}, c_{53}, c_{43} and c_{23},

– c_{43} determines c_{42} and c_{41}, and c_{23} determines c_{22} and c_{21},
– the remaining variables are zeros, and
– the equation $f_3 = 0$ is implied by the equations $f_1 = 0$ and $f_2 = 0$ where $c_{2,3}f_1 + c_{4,3}f_2 = c_{3,3}f_3$ holds.

Thus, although the dimension of the infinitesimal restricted deformation space is 2, the dimension of the local restricted deformation space is 1. The details of the calculations are available from the webpage [19].

4.5 The results for dodecahedra

Let P be a 3-dimensional compact hyperbolic dodecahedron, all of whose dihedral angles are $\pi/2$ or $\pi/3$. We assume that each face has less than three edges of order 2. Then the total number of such orbifolds is 13 up to symmetries (see table 2). These results were obtained by using Matlab to check the conditions of Andreev's theorem (see do.m in [19]).

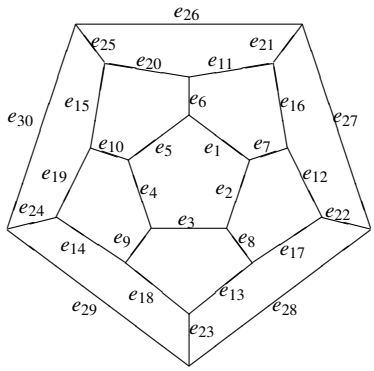


Fig. 7 Labels of edges of a dodecahedron

name	$e_1\, e_2 \cdots e_{29}\, e_{30}$	O	I	A	J	S	G	C
do1	232332323232323333332323233232	-6	0	0	yes	0.17653	2	7
do2	232332333232223333332332233233	-5	0	0	yes	0.13121	2	8
do3	2323233332322233333332333233223	-5	0	0	yes	0.14468	2	8
do4	232332333232223333332333233232	-5	0	0	yes	0.13707	2	8
do5	232332333232323322332333233232	-6	0	0	yes	0.18151	4	7
do6	232332333232333322332332233233	-5	0	0	yes	0.11944	2	8
do7	232333223233323333322223333332	-5	0	0	yes	0.12703	4	8
do8	232333233232223333322332333232	-6	0	0	yes	0.09580	4	7
do9	2323332332322233333322333333232	-5	0	0	yes	0.09365	2	8
do10	232333233232332333322233333322	-5	0	0	yes	0.08277	2	8
do11	232333333222223333333333232323	-4	0	0	yes	0.06115	4	9
do12	232333333222332322233233323323	-6	0	0	yes	0.12412	12	7
do13	2323333332332233233233233323323	-4	1	**1**	no	·	20	9

Table 2 The list of dodecahedra

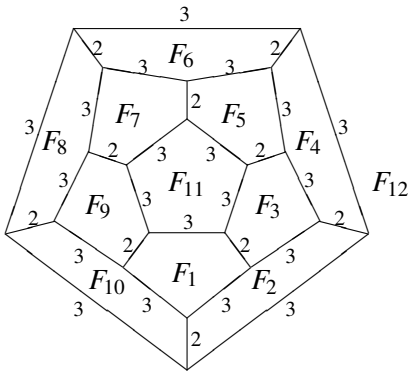


Fig. 8 do13 with five-fold rotational symmetry about the axis passing through the centers of faces F_{11} and F_{12}

Since we assume that each face has less than three edges of order 2, there is no rigid face. Hence we can skip the linear test for rigidity.

As the next step, we want to find unit normals ν_i of the dodecahedra. However, since the number of variables in the hyperbolic equations (7) is large, the exact algebraic solution is hard to obtain for all cases.

Example 5 We first concentrate on do13, shown in Figure 8. This orbifold has rotational symmetry that will allow us to calculate its normals ν_i exactly, and also find the dimension of its local restricted deformation space.

Let $c = \cos(\pi/5)$, $c_3 = \cos(3\pi/5)$, $s = \sin(\pi/5)$, $s_3 = \sin(3\pi/5)$, and

$$d = \sqrt{\frac{1+c}{1-c}}\,\frac{1}{\sqrt{2c+3c^2}+\sqrt{-1+3c^2}}.$$

We obtain the following unit normals for the faces of do13:

$$\begin{aligned}
\nu_1 &= \Big(\sqrt{\frac{c}{4-4c}},\ \frac{1}{\sqrt{2}s},\ 0,\ \sqrt{\frac{2+3c}{4+4c}}\Big),\\
\nu_{k+1} &= L^k\nu_1 \text{ with } k = 1,\ldots,9,\\
\nu_{11} &= \Big(\frac{1}{\sqrt{d^2-1}},\ 0,\ 0,\ \frac{d}{\sqrt{d^2-1}}\Big),\\
\nu_{12} &= \Big(\frac{1}{\sqrt{d^2-1}},\ 0,\ 0,\ -\frac{d}{\sqrt{d^2-1}}\Big).
\end{aligned}$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & c & -s & 0\\ 0 & s & c & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

To check these, we note that direct computation gives:

$$\begin{aligned}
&\langle \nu_i, \nu_i \rangle = 1 \text{ for all } i = 1, \ldots, 12, \\
&\langle \nu_{11}, \nu_{2j-1} \rangle = \langle \nu_{12}, \nu_{2j} \rangle = -1/2 \text{ for all } j = 1, \ldots, 5, \\
&\langle \nu_1, \nu_2 \rangle = \langle \nu_2, \nu_3 \rangle = \cdots = \langle \nu_9, \nu_{10} \rangle = \langle \nu_{10}, \nu_1 \rangle = -1/2, \\
&\langle \nu_1, \nu_3 \rangle = \langle \nu_3, \nu_5 \rangle = \cdots = \langle \nu_7, \nu_9 \rangle = \langle \nu_9, \nu_1 \rangle = 0, \\
&\langle \nu_2, \nu_4 \rangle = \langle \nu_4, \nu_6 \rangle = \cdots = \langle \nu_8, \nu_{10} \rangle = \langle \nu_{10}, \nu_2 \rangle = 0,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the Lorentzian inner product. Further, the cone defined by the inequalities $\langle X, \nu_i \rangle \leq 0$ intersects $\mathbb{H}^3$, since $X = (1,0,0,0)$ satisfies these inequalities.

We now want to solve Vinberg's equations (3)–(5). We were unable to compute a Gröbner basis for the corresponding ideal using Mathematica since the coordinates of the linear functionals $\alpha_i = J\nu_i$ are complicated and the number of variables is very large.

Instead we look for solutions of the special form

$$\begin{aligned}
b_1 &= \big(b_{1,1}, \frac{\sqrt{2}}{s}, 0, b_{1,4}\big) \text{ with } \alpha_1(b_1) = 2, \\
b_2 &= \big(b_{2,1}, \frac{\sqrt{2}c}{s}, \sqrt{2}, b_{2,4}\big) \text{ with } \alpha_2(b_2) = 2, \\
b_{2k+1} &= L^{2k} b_1 \text{ and } b_{2k+2} = L^{2k} b_2 \text{ with } k = 1, \ldots, 4, \\
b_{11} &= (b_{11,1}, 0, 0, b_{11,4}) \text{ with } \alpha_{11}(b_{11}) = 2, \\
b_{12} &= (b_{12,1}, 0, 0, b_{12,4}) \text{ with } \alpha_{12}(b_{12}) = 2.
\end{aligned}$$

Here we assume that $b_{11,2} = b_{11,3} = b_{12,2} = b_{12,3} = 0$ to maintain the rotational symmetry of do13. Therefore, we can choose $\{b_{1,4}, b_{2,4}, b_{11,4}, b_{12,4}\}$ as *four* free variables.

Recall that $\alpha_i = J\nu_i$ for $i = 1, \ldots, 12$, and $a_{ij} = \alpha_i(b_j)$. From direct calculations, we obtain

$$a_{2k+1,2k+1} = a_{1,1} = 2 \quad \text{and} \quad a_{2k+2,2k+2} = a_{2,2} = 2,$$

for all $k = 1, \ldots, 4$. Furthermore,

$$\begin{aligned}
a_{2k-1,2k+1} &= a_{1,3} = a_{1,1} + (a_{1,3} - a_{1,1}) = 2 + \alpha_1(b_3 - b_1) \\
&= 2 + \frac{-1 + \cos(2\pi/5)}{s^2} = 0, \\
a_{2k+1,2k-1} &= a_{3,1} = a_{1,1} + (a_{3,1} - a_{1,1}) = 0, \\
a_{2k,2k+2} &= a_{2,4} = a_{2,2} + (a_{2,4} - a_{2,2}) = 2 + \alpha_2(b_4 - b_2) \\
&= 2 + \frac{c}{\sqrt{2}s} \frac{\sqrt{2}(c_3 - c)}{s} + \frac{s}{\sqrt{2}s} \frac{\sqrt{2}(s_3 - s)}{s} = 0, \\
a_{2k+2,2k} &= a_{4,2} = a_{2,2} + (a_{4,2} - a_{2,2}) = 0,
\end{aligned}$$

for $k = 0, 1, \ldots, 4$, where the indices are taken modulo 10.

If the *three* equations

$$a_{11,1} a_{1,11} = 1, \quad a_{12,2} a_{2,12} = 1, \quad \text{and} \quad a_{1,2} a_{2,1} = 1$$

are satisfied, then so are all the Vinberg equations (3)–(5) since

$$\begin{aligned}
a_{11,2k-1} = a_{11,1} \quad &\text{and} \quad a_{2k-1,11} = a_{1,11}, \\
a_{12,2k} = a_{12,2} \quad &\text{and} \quad a_{2k,12} = a_{2,12}, \\
a_{2k+1,2k} = a_{2k+1,2k+2} = a_{1,10} = a_{1,2} \quad &\text{and} \quad a_{2k,2k+1} = a_{2k+2,2k+1} = a_{10,1} = a_{2,1},
\end{aligned}$$

for $k = 0, 1, \ldots, 4$ and indices modulo 10. It follows that the dimension of the local restricted deformation space is at least 1.

On the other hand, we find that the dimension of the infinitesimal restricted deformation space of do13 is exactly 1 by exact computations using Mathematica. This implies that the dimension of the local restricted deformation space is exactly equal to 1. (See the file do13.nb at [19] for the detailed calculations.)

Remark 7 This work is similar to our work on prisms. Both examples can be understood by letting the reflection points of the top and the bottom faces lie on the axis of rotational symmetry. By choosing the reflection point arbitrarily for the top face, we see that the other faces are orderable up to rotational symmetry. One can also take a quotient orbifold under the rotational symmetry and obtain an "orderable orbifold". Then the other reflection points can be chosen using this ordering. Here, the geometry can be used to show explicitly how the reflection vectors change.

Finally, we describe our method for studying the orbifolds do1-do12; this makes use of numerical computations.

Example 6 We illustrate this for the orbifold do1, shown in Figure 9.

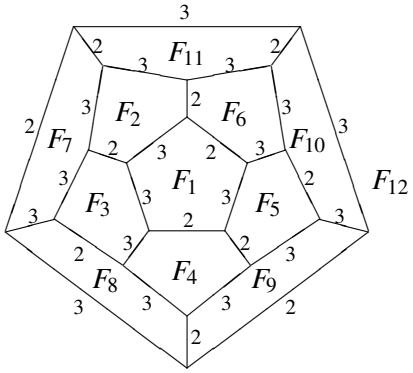


Fig. 9 do1

To find the unit normals ν_i for do1 we make some use of the results for do13. We choose one vertex $F_{ijk} = F_i \cap F_j \cap F_k$ of do1 whose adjacent edges have orders $(2,3,3)$. For example, we choose the vertex F_{123} of do1. Let $c = \cos(\pi/5)$ and $s = \sin(\pi/5)$. We set

$$\nu_1 = \Big(\frac{1}{\sqrt{d^2-1}}, 0, 0, \frac{d}{\sqrt{d^2-1}}\Big) \text{ with } d = \sqrt{\frac{1+c}{1-c}}\,\frac{1}{\sqrt{2c+3c^2}+\sqrt{-1+3c^2}}.$$

$$\nu_2 = \Big(\sqrt{\frac{c}{4-4c}}, \frac{1}{\sqrt{2}s}, 0, \sqrt{\frac{2+3c}{4+4c}}\Big) \text{ and } \nu_3 = L^2\nu_2,$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

From the calculations for do13 we know that

$$\langle \nu_1, \nu_2 \rangle = \langle \nu_1, \nu_3 \rangle = -\frac{1}{2} \text{ and } \langle \nu_2, \nu_3 \rangle = 0.$$

We first fix these normals ν_1, ν_2 and ν_3, and solve the system of hyperbolic equations (7) for do1 using the unit normals for do13 as initial values for a numerical calculation by Mathematica. This gives *numerical* solutions for the unit normals of do1.

We then compute the rank of Jacobian D of do1 by numerical computations in Mathematica, maintaining 150 digits of precision in internal computations. This shows the rank is 48, and thus D is of full rank.

To confirm these numerical computations, we compute the singular value decomposition of D, and list the minimum singular value in table 2. (Note that the singular value decomposition behaves very accurately in numerical matrix computations.)

For all $i = 1, \ldots, 12$, we find that the Jacobian matrix D of do$i$ has rank 48, and thus each do$i$ is projectively rigid relative to the mirrors. The numerical computations using Mathematica are also available from the webpage [19].

4.6 Projective bending deformations and the restricted deformation spaces

In this subsection, we define "bending" deformations and show that some, but not all, of our deformations can be explained as "bendings". First, we give some definitions. Next, we characterize bending in the restricted deformation spaces in Theorem 10. We study these bendings for hyperbolic Coxeter orbifolds in Corollary 2 and then discuss bendings for the restricted deformation spaces of our examples.

Thurston was first to see how to use bendings to deform a projective manifold. See Johnson and Millson [20] for more details (but they only treat manifolds). Notice that each of our orbifolds has a finite cover by a manifold, a so-called *very good cover,* as was first proved by Thurston.

Remark 8 We say that an orbifold S is *embedded* in an another orbifold M if in a very good cover of M, the inverse image of S is also an embedded submanifold. We do not consider bending along "immersed" orbifolds, and we do not allow S to be in the boundary or silvered boundary of the orbifold.

We now give a geometric description of bending for orbifolds: Given a 3-dimensional real projective orbifold M, suppose that it contains an embedded two-sided totally geodesic suborbifold S of codimension one. We suppose that S is not contained in the silvered boundary of M. Then we can split M along S and complete the result to obtain another orbifold M' with totally geodesic boundary components S' and S'' real projectively diffeomorphic to S. Regluing of S' and S'' is possible if there is an open real projective manifold $M'' = M' \cup N' \cup N''$ containing M' and a real projective diffeomorphism $f : N' \to N''$ taking an open collar neighborhood N' of S' in M'' to an open collar neighbourhood N'' of S'' in M'' and sending S' to S''. By identifying N' with N'' in M'' using f, we obtain a new orbifold M''' diffeomorphic to M where M''' is said to be obtained by *bending along* S.

Let $\tilde{f}$ be a lift of f to the universal cover $\tilde{M}''$ of M'' and let $\tilde{N}'$ be a component of the inverse image of N' in $\tilde{M}''$. Then $\tilde{f}$ takes a component $\tilde{S}'$ of the inverse image of S' to a corresponding component $\tilde{S}''$ of the inverse image of S'' and

$$\tilde{f} \circ \gamma' = \gamma'' \circ \tilde{f} : \tilde{N}' \to \tilde{f}(\tilde{N}'), \tag{11}$$

whenever γ' is a deck transformation acting on $\tilde{N}'$ and $\tilde{S}'$, and γ'' is the corresponding deck transformation acting on $\tilde{f}(\tilde{N}')$ and $\tilde{S}''$.

Consider a developing pair (D,h) for M and the induced one (D',h') for M'' obtained by extending (D,h) restricted to the universal cover of $M-S$. Here, $h(\gamma)=h'(\gamma')=h'(\gamma'')$ for each deck transformation γ in $\tilde{M}$ and the deck transformations γ',γ'' in $\tilde{M}''$ corresponding to γ under the splitting. It follows from equation (11) that $\tilde{f}$ corresponds, via the developing map D', to an element $\hat{f}$ in $PGL(4,\mathbb{R})$ centralizing the elements $h(\gamma)$ for $\gamma\in\pi_1(S)$ since $\tilde{f}$ is a real projective map. In fact, any such element gives rise to a suitable f by taking a quotient and vice versa. Note that $\hat{f}$ is usually obtained by a path from the identity, and we require $\hat{f}$ to be in the identity component of $PGL(4,\mathbb{R})$.

Lemma 4 *Let $\hat{P}$ be a 3-dimensional compact Coxeter orbifold obtained from a convex polyhedron P in $\mathbb{R}P^3$ by silvering sides, and assume that every finite index subgroup of $\pi_1(\hat{P})$ has trivial centre. Let S be an embedded totally geodesic 2-dimensional suborbifold meeting at least four sides of P. Then there are non-trivial deformations in $\mathfrak{D}(\hat{P})$ obtained by projective bendings along S.*

Remark 9 The condition that every finite index subgroup of $\pi_1(\hat{P})$ has trivial centre holds, for example, if $\hat{P}$ is a compact hyperbolic orbifold (see Benoist [4]).

Remark 10 Let $\hat{P}$ and S be as in Lemma 4. Then S cannot contain a finite vertex of $\hat{P}$ if S is embedded in the sense of Remark 8 above. The closure of S may contain an ideal vertex of P in the closure of P. However, we will study only the case where $\hat{P}$ is compact. Then S meets any face of P in at most one arc by convexity, and the faces it meets form a cycle. We call this the *cycle of faces* of S. (The underlying space of the 2-orbifold S is always a convex disk in P.)

Proof Let (D,h) be a development pair for $\hat{P}$ with fundamental domain P so that h sends the reflections generating $\pi_1(\hat{P})$ to projective reflections in the sides of P (as in section 2.1). Consider a bending deformation along S in the deformation space $\mathfrak{D}(\hat{P})$ determined by a regluing map $\hat{f}$ as above.

Since $\hat{f}$ commutes with the elements of $h(\pi_1(S))$, it fixes the vertices of the sides of S and acts on the subspaces containing $D(F_1),\dots,D(F_m)$, where $F_1,\dots,F_m$ is the cycle of faces of S and $m\geq 4$. By considering the matrix form of $\hat{f}$, we see that $\hat{f}$ must be of the form

$$\begin{bmatrix} 1/t^3 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix} \text{ with } t>0 \text{ and } t\neq 1, \tag{12}$$

in a coordinate system where the standard coordinate vectors correspond to some of the vertices of S and another point outside the plane containing $D(S)$. ($\hat{f}$ has a unique plane $H_{\hat{f}}$ of fixed points containing the developing image of $\tilde{S}$ and a unique isolated fixed point $p_{\hat{f}}$.) Any such regluing map f gives rise to a real projective structure on M.

A reflection r and a projective automorphism $\hat{f}$ of the form of equation (12) commute if and only if

- either the isolated fixed point p_r of r is contained in the plane $H_{\hat{f}}$ and the plane H_r of fixed points of r intersects $H_{\hat{f}}$ in a line and contains the isolated fixed point $p_{\hat{f}}$ of $\hat{f}$, or
- $H_{\hat{f}}=H_r$ and $p_{\hat{f}}=p_r$.

Since S meets each F_i in a segment for $i = 1,\dots,m$, the subspaces containing $D(F_1),\dots,D(F_m)$ all contain $p_{\hat{f}}$, and the subspace containing $D(S)$ contains the isolated fixed point p_i of the reflection r_i associated with F_i for each $i = 1,\dots,m$ since $\hat{f}$ commutes with each r_i.

By results of Vinberg [32], $\hat{P}$ is properly convex. Since every finite index subgroup of $\pi_1(\hat{P})$ has trivial centre, $h(\pi_1(\hat{P}))$ is irreducible by Corollary 2.13 of Benoist [4]. By Theorem 2.1 of Benoist [3], $h(\hat{P})$ is Zariski dense in some union of components of $PGL(4,\mathbb{R})$ or in $PO(1,3)$ and hence is not contained in a proper parabolic subgroup of $PGL(4,\mathbb{R})$. Thus, $h(\pi_1(\hat{P}))$ is stable by Theorem 1.1 of Johnson and Millson [20].

For each component of $\hat{P}-S$, the centralizer of the holonomy is a subgroup of a finite extension of the one-parameter group $\{\hat{f}^s \mid s>0\}$ since a copy of $\pi_1(S)$ is contained in the fundamental group of each component of $\hat{P}-S$. Each component of $\hat{P}-S$ has a side T not meeting S, since each vertex of P is trivalent and S has at least 4 sides. By convexity of $\hat{P}-S$, the subspace containing $D(T)$ does not contain the isolated fixed point of $\hat{f}$ nor coincide with $H_{\hat{f}}$. Since the holonomy of the fundamental group of each component of $\hat{P}-S$ contains a reflection in T, not commuting with $\hat{f}$, it follows that the holonomy of each component of $\hat{P}-S$ has a zero-dimensional centralizer. By Proposition 5.1 of Johnson and Millson [20] and Corollary 1 of [10], the induced deformation of $\hat{P}$ given by $\hat{f}^s$ is not trivial.

Theorem 10 *Let $\hat{P}$ be a 3-dimensional compact Coxeter orbifold obtained from a convex polyhedron P in $\mathbb{R}P^3$ by silvering sides, and assume that P is not a triangular prism and that every finite index subgroup of $\pi_1(\hat{P})$ has trivial centre. Then there is a nontrivial family of deformations in $\mathfrak{D}_P(\hat{P})$ obtained by projective bendings along an embedded totally geodesic 2-orbifold S with at least four silvered sides if and only if the following conditions hold:*

1. *P is a prism with cycle of faces $F_1,\dots,F_m$ of S contained in subspaces concurrent at a point x, and the top face t and bottom face b are disjoint from S,*
2. *the corresponding reflection points of $F_1,\dots,F_m$ are on a plane L, and*
3. *L and the respective subspaces containing t and b contain a common one-dimensional subspace l.*

Proof We use the notation from the proof of Lemma 4. Assume that there exists a bending deformation along S in the restricted deformation space $\mathfrak{D}_P(\hat{P})$. We showed in the proof of Lemma 4 that in the developing image $D(\tilde{P})$ containing P, there exists a subspace L of dimension 2 so that L contains the respective reflection points $p_1,\dots,p_m$ of the faces $F_1,\dots,F_m$ of P meeting L where $m \geq 4$, and that the regluing map $\hat{f}$ acts as the identity on L. Furthermore, $\hat{f}$ must act on the subspaces $H_1,\dots,H_m$ containing $F_1,\dots,F_m$ for the same reason. This implies that $\hat{f}$ has another fixed point x not on L and that $x \in H_i$ for $i = 1,\dots,m$.

Now, in the bending deformation, $\hat{f}$ is applied to only one component P' of $P-L$ and $\hat{f}(P')$ is reglued to the other component P''. Let $P''' = Cl(\hat{f}(P')) \cup Cl(P'')$ be the union of the closures of $\hat{f}(P')$ and P''. Then P''' is projectively diffeomorphic to P by some $k \in PGL(4,\mathbb{R})$ as our deformation was in the restricted deformation space. Then $k \circ \hat{f}(v') = v'$ for each vertex v' of P in P' and $k(v') = v'$ for each vertex v' of P in P''. Since $k \circ \hat{f}$ acts on $F_1,\dots,F_m$, it fixes x as well.

If P'' has at least four vertices in general position, then k is the identity. Since $\hat{f}$ now fixes each vertex v' of P in P' and the vertices of $L \cap P$ and x, it follows that $\hat{f}$ is the identity, a contradiction. Thus, the vertices in P'' are coplanar and hence P'' contains just one side F'' meeting $F_1,\dots,F_m$.

Since k acts on $F_1,\dots,F_m$, it follows that k fixes x and the vertices of P'', and that the vertices of P'' lie on a plane L''. As $m \geq 4$, it follows that k restricts to the identity map on L''. If P' has at least four vertices in general position, then $k \circ \hat{f}$ is the identity. But since $\hat{f}$

acts by fixing every point of a unique plane L and k acts by fixing every point of a unique plane L'' distinct from L, this is impossible. Thus, the vertices of P' lie on a plane L'.

Now L, L' and L'' are mutually distinct, and $\hat{f}(L')$ and L' are distinct as $\hat{f}$ is not the identity on L'. By choosing $\hat{f}$ close to the identity, we see that L, L', L'' and $\hat{f}(L')$ are mutually distinct.

Let l be the intersection of L' and L. Then we also have $\hat{f}(L') \cap L' = \hat{f}(L') \cap L = l$, since $\hat{f}$ acts on L as the identity. Let l' be the intersection of L' and L'', and l'' the intersection of $\hat{f}(L')$ and L''. Now, $k \circ \hat{f}(L')$ meets L'' at l'' since k fixes every point of L''. Since we have $k \circ \hat{f}(L') = L'$, this implies that $l' = l''$ and that $L', L'', \hat{f}(L')$ meet at $l' = l''$.

Simple geometry shows that L and L'' meet at l also, since otherwise l' and l'' must be different as $\hat{f}(L')$ and L' are distinct. (See Figure 10.) This completes the proof that conditions 1, 2 and 3 must hold.

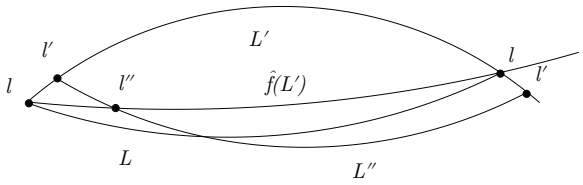


Fig. 10 $L, L', \hat{f}(L')$, and L'' shown as sections of a hyperplane.

For the converse, we use the notation introduced in the proof above. Under the assumptions of the theorem, we see that for any $\hat{f}$ fixing a point x and the points of a plane L, we can find a suitable projective automorphism k fixing x and the points of L'' so that $k \circ \hat{f}(L') = L'$. (k would be of form (12) in some coordinate system.) Thus, the deformation is in $\mathfrak{D}_P(\hat{P})$ and the nontriviality of the deformation follows from Lemma 4.

For triangular prisms, there are always bending deformations unless the edge orders of either the top face or the bottom face are all equal to 2. Also, there is no condition on the subspaces containing the top and bottom faces and the subspace containing the reflection points of the other three sides. Here, the deformation space is of dimension one and consists of projective bendings. (See Example 4.3 of [5] for an alternate view.)

Each of the prisms in the proof of Proposition 6 contains a totally geodesic 2-dimensional suborbifold based on an n-gon in the middle of the prism. The one-dimensional deformation there is clearly of bending type since the proof there shows that the space of deformations where the top reflection point and the bottom one lie on the axis of symmetry is just one dimensional and the bending deformations have this property.

For the dodecahedron example `do13`, the deformations are not projective bendings by Theorem 10 and Remark 9. This completes the proof of Theorem 3.

Corollary 2 *Let $\hat{P}$ be a compact 3-dimensional hyperbolic Coxeter orbifold that is not a triangular prism. Suppose that there exist projective bendings in $\mathfrak{D}_P(\hat{P})$ along a totally geodesic 2-orbifold S meeting a cycle of faces. Then P is a prism with the top face t and the bottom face b not in the cycle and the planes containing faces in the cycle are concurrent. Further, precisely one of the following conditions holds:*

1. *$\hat{P}$ has no edges of order 2 in t and b.*

2. *$\hat{P}$ has exactly two order* 2 *edges in t and b and these occur as opposite edges in a face in the cycle; no other edges on t or b have order* 2.
3. *$\hat{P}$ has exactly four order* 2 *edges in t and b and these occur as opposite edges in two adjacent faces F_i and F_{i+1} in the cycle, with the order* 2 *edges ending at the edge in $F_i \cap F_{i+1}$.*

(*See Figure 11.*)

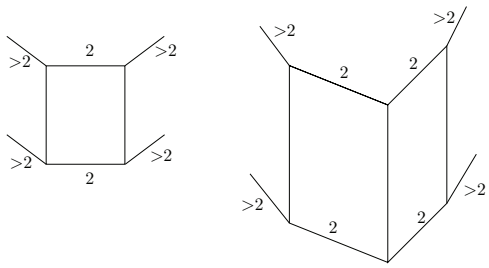


Fig. 11 The second and third possibilities for Corollary 2.

Proof The first conclusion follows from immediately from Theorem 10 and Remark 9. Let L, L' and L'' be the totally geodesic subspaces corresponding to S, t and b respectively, and let $s = L \cap P$. Then, by condition 3 of Theorem 10, L, L' and L'' contain a common subspace of dimension one.

The hyperbolic orbifold $\hat{P}$ develops into a copy of hyperbolic space H lying in the the interior of a conic in $\mathbb{R}P^3$ determined by a Lorentzian inner product B. We will use the hyperbolic metric on H from the Klein model.

Now $l = L \cap L' \cap L''$ is a subspace of dimension one, and l is disjoint from H since P has dihedral angles $\leq \pi/2$ in H. Hence $L \cap H$, $L' \cap H$ and $L'' \cap H$ are mutually disjoint.

Let l' be the dual of l under B; i.e., $B(v,w) = 0$ for every pair of vectors v and w representing points in l and l' respectively. Then $l' \cap H$ is a geodesic in H and is perpendicular to $L \cap H, L' \cap H$ and $L'' \cap H$. This follows since the hyperbolic reflections in the planes $L \cap H, L' \cap H$ and $L'' \cap H$ extend to projective transformations fixing l so must also preserve its dual l'.

The faces in the cycle determined by S are contained in concurrent planes and l' passes through the point of concurrency x as the faces F_i are perpendicular to L and L is perpendicular to l'. Thus, P is contained in a domain D bounded by the union of the lines through x and the faces in the cycle. More precisely,

$$P \text{ is the closure of a component of } D - t - b \qquad (*).$$

Since the dihedral angles are $\leq \pi/2$, the line l' passes through t and b. First, suppose that l' passes through t in its interior. Then l' also passes through b and s in their interiors by $(*)$. Since each face F_j in the cycle is perpendicular to L, we take a perpendicular geodesic l'' in L from $l' \cap s$ to the plane L_j containing the face F_j. Then l' and l'' are contained in a subspace Q of dimension 2 and $Q \cap H$ is totally geodesic. Now l'' together with subsegments of $l' \cap P$, $t \cap Q$, and $L_j \cap Q$ bound a quadrilateral in Q with three $\pi/2$ angles. This implies

that the fourth angle of the quadrilateral is $< \pi/2$ and it follows that all dihedral angles in t are $< \pi/2$. Similarly, the dihedral angles of b are $< \pi/2$, and so the condition 1 holds.

Next, suppose that l' passes through t in the interior of an edge. Then l also passes through the interior of an edge in s and the interior of an edge in b by $(*)$. Moreover $l' \cap P$ is contained in one of the faces F_j in the cycle and does not meet the vertices of F_j. Then the dihedral angles along the edges $F_j \cap t$ and $F_j \cap b$ are $\pi/2$. Further, a similar argument, dropping a perpendicular geodesic from $l' \cap s$ to the remaining faces in the cycle, shows that the other dihedral angles along t and b are $< \pi/2$. This gives the condition 2.

Finally, suppose that l' passes through a vertex of t. Then it also passes through vertices of s and b by $(*)$. In this case condition 3 holds by a similar argument.

We obtain the following results for the orbifolds based on cubes:

- cu31, cu32, and cu34 are the only orbifolds here with deformations relative to the mirrors that are not projective bendings. (These have no bendings since the conditions of Corollary 2 are not met.)
- For cu29 and cu34, there are totally geodesic suborbifolds but these do not yield any projective bendings relative to the mirrors according to Corollary 2.
- cu17, cu18, cu21, cu22, cu27, and cu30 have 1-dimensional deformations relative to the mirrors that are projective bendings by the converse part of Theorem 10, i.e., by checking that the planes containing the top face and the bottom face and the plane including the the reflection points of the face cycle contain a common subspace of dimension 1, or equivalently they have a common perpendicular geodesic in H. These conditions can be easily verified by checking that these orbifolds have order two symmetry about the 2-orbifolds in the middle.
- For cu33, there are projective bendings along two different suborbifolds giving two one-parameter families of bendings by the converse part of Theorem 10. Computations using Mathematica show that the restricted deformation space itself is smooth but that a Gröbner basis for the corresponding ideal has a squared term.

This completes the proof of Theorem 2.

We also did some further experiments increasing the edge orders on cubes. However, the results are not very enlightening so we do not include the details here.

A final comment is as follows:

Corollary 3 *Let $\hat{P}$ be a 3-dimensional Coxeter orbifold. Then if $\hat{P}$ is orderable and is not based on a triangular prism, then the restricted deformation space does not contain projective bendings.*

Proof This follows from Corollary 2 since an orderable orbifold contains a triangular face, so the only orderable prism is the triangular prism.

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