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A Framework for Extremum Seeking Control of Systems with Parameter Uncertainties

Dragan Nešić, Alireza Mohammadi and Chris Manzie

Abstract—**Traditionally, the design of extremum seeking algorithm treats the system as essentially a black-box, which for many applications means disregarding known information about the model structure. In contrast to this approach, there have been recent examples where a known plant structure with uncertain parameters has been used in the online optimization of plant operation. However, the results for these approaches have been restricted to specific classes of plants and optimization algorithms. This paper seeks to provide general results and a framework for the design of extremum seekers applied to systems with parameter uncertainties. General conditions for an optimization method and a parameter estimator are presented so that their combination guarantees convergence of the extremum seeker for both static and dynamic plants. Tuning guidelines for the closed loop scheme are also presented. The generality and flexibility of the proposed framework is demonstrated through a number of parameter estimators and optimization algorithms that can be combined to obtain extremum seeking. Examples of anti-lock braking and model reference adaptive control are used to illustrate the effectiveness of the proposed framework.**

Index Terms—**Extremum seeking, Parameter estimation, Optimization**

I. INTRODUCTION

A STANDING assumption in extremum seeking is that the model of the plant is unknown and that the steady state relationship between reference input signals and plant outputs is such that it contains an extremum [19]. The goal is to tune the system inputs online so that it operates in the vicinity of this extremum in steady state, see [5]. This situation arises in a range of classical, as well as certain emerging engineering applications. Since an extremum seeking controller does not need the exact model of the plant and also can easily deal with multi input systems, it has been successfully used in a range of applications, such as biochemical reactors [6], ABS control in automotive brakes [9], variable cam timing engine operation [24], and mobile sensor networks [22].

There are several main methods for design of extremum seeking controllers. An adaptive control approach for continuous time systems is pursued in [5], [15], [26], [27] whereas a nonlinear programming approach in [29] and simultaneous perturbation stochastic approximation in [25], [24] are proposed for discrete time systems. Results in [29] are significant as they show how to combine a large class of nonlinear programming optimization methods with a gradient estimator in order to achieve extremum seeking. Also in [21], a framework is proposed for design of extremum seeking controllers assuming that the reference-to-output map is not known at all and the estimation of derivatives of this map is done directly. The power of the results in [21], [29] is that they provide a *prescriptive framework* that can be used to design a large class of extremum seeking controllers and show their convergence properties in a unified manner. Within this framework it is shown how to combine the classical nonlinear programming optimization algorithms with various estimation algorithms to obtain a powerful controller design framework for extremum seeking.

Results in [2], [3], [10], [11] are derived under subtly different assumptions. Here the plant is assumed to be parameterized by an unknown parameter and various parameter estimation based techniques are used to achieve extremum seeking. While the parameter is unknown, it is assumed that it is known how the plant model depends on this parameter. This slightly stronger assumption allows a more direct use of classical adaptive control methods in the context of extremum seeking. Nevertheless, the results in [2], [3], [10], [11] are presented for particular classes of plants and particular optimization algorithms are used to achieve extremum seeking. As far as the authors are aware, a prescriptive framework that would allow extremum seeking designers to combine a large class of optimization algorithms with a large class of stable plants (or unstable plants with controller, observer and parameter estimator) that is similar to results in [21], [29] has not been reported in the literature.

It is the purpose of this paper to propose a prescriptive extremum control design framework reminiscent of [21], [29] for methods based on parameter estimation. The framework provides precise conditions under which one can combine a large class of continuous optimization algorithms with a large class of controllers, observers and parameter estimators to achieve convergence of the closed loop trajectories to the desired extremum. Moreover, the results prescribe how the controller parameters need to be tuned in order to achieve convergence to the desired extremum.

It is shown that a large class of optimization algorithms and parameter estimators satisfy the main assumptions and their various combinations can be used to construct various extremum seeking algorithms; this provides a flexible toolbox previously unavailable in the literature. The provided examples

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demonstrate the power of the proposed framework since a large number of new extremum seeking algorithms can be easily obtained following the proposed approach and tuning the parameters in the controller to obtain the desired convergence.

The paper is organized as follows. Section II presents preliminaries. The first main result is presented in Section III for static plants in order to introduce the ideas and assumptions in a simple setting. Subsequently, in Section IV, additional results for stable and unstable dynamic plant models are presented. Section V provides examples of estimation and optimization algorithms that satisfy the assumptions of the framework. Section VI demonstrates the application of the main results through simulation studies and illustrates the generality of the design framework.

II. Preliminaries

The set of real numbers is denoted by $\mathbb{R}$. The continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is nondecreasing and $\alpha(0) = 0$. The continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument, strictly decreasing to zero in its second argument and $\beta(0, t) = 0$ for all $t \geq 0$.

Consider the static mapping $h : \mathcal{B}_\theta \times \mathcal{B}_\xi \to \mathbb{R}$ denoted as

$$y = h(\theta, u) \ , \tag{1}$$

where $\theta \in \mathcal{B}_\theta \subset \mathbb{R}^p$ is a fixed unknown parameter vector, $u \in \mathcal{B}_\xi \subset \mathbb{R}^m$ is the input and $y \in \mathbb{R}$ is the output of the static system. This study is carried out under the following basic assumption on the static map.

Assumption 1: The map $h(\cdot, \cdot)$ is known but the parameter vector θ is unknown. In addition, the map $h(\theta, \cdot)$ is smoothly differentiable sufficiently many times and for any θ there exists an extremum [1].

The vector

$$\mathcal{D}_N(\theta, u) := \begin{pmatrix} h(\theta, u) \\ D^{1,0,\ldots,0}h(\theta, u) \\ \vdots \\ D^{N,\ldots,N}h(\theta, u) \end{pmatrix} ,$$

where $D^{i_1,\ldots,i_m}h(\theta, u) := \frac{\partial^{i_1+\cdots+i_m}h(\theta,u)}{\partial u_1^{i_1}\ldots\partial u_m^{i_m}}$ for $i_m = 0, 1, 2, \ldots, N$ denotes the iterated derivatives of h with respect to its input arguments.

Now consider an optimization scheme of the form:

$$\dot{\xi} = F(\mathcal{D}_N(\theta, \xi), \xi) \tag{2}$$

which is used to generate an extremum seeking scheme. The following assumption is placed on the optimization algorithm.

Assumption 2: For any given (but unknown) θ there exists an equilibrium $\xi^* = \xi^*(\theta)$ of system (2) which corresponds to the extremum of the map $h(\theta, \cdot)$.

Remark 1: Note that sometimes all derivatives in the vector $\mathcal{D}_N$ are not required to generate a certain scheme. For instance, in order to generate the continuous time gradient method, only the first derivative of $h(\theta, u)$ is required. However, some other algorithms require other derivatives and the proofs are provided for the very general case. In addition, (2) can be more general where the model depends on extra states or the right-hand side of (2) is discontinuous. In the latter case, differential equations should be generalized to differential inclusions which requires extra assumptions on the model. However, the further generalization is omitted in this paper to keep the model simple enough to state the main idea of the framework.

[1] Without loss of generality only maxima are considered; minima can easily be treated in the same manner by defining $\tilde{h} = -h$ and then applying the theory to $\tilde{h}$.

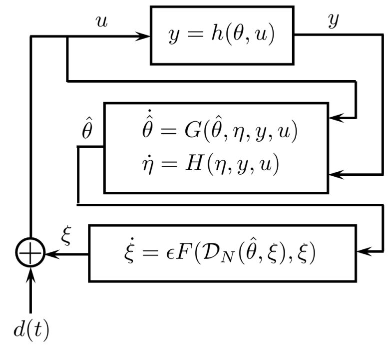

Fig. 1. The proposed framework for static plants

III. Convergence Conditions for Static Plants

The first main result is presented in this section for an extremum seeking scheme applied to a static plant, which leads to a general class of extremum seeking schemes based on estimation of the parameter vector θ. In these schemes, by tuning a parameter in the extremum seeking controller the closed loop system exhibits a time scale separation. Therefore, the reduced system behaves approximately as the given optimization scheme (2) and under appropriate stability properties of the parameter estimator and the optimization scheme, convergence to the extremum of $h(\theta, u)$ can be achieved.

Consider the following class of extremum seeking schemes:

$$y = h(\theta, u) \ , \tag{3}$$
$$u = \xi + d(t) \ , \tag{4}$$
$$\dot{\hat{\theta}} = G(\hat{\theta}, \eta, y, u) \ , \tag{5a}$$
$$\dot{\eta} = H(\eta, y, u) \ , \tag{5b}$$
$$\dot{\xi} = \epsilon \ F(\mathcal{D}_N(\hat{\theta}, \xi), \xi) \ , \qquad \epsilon > 0 \tag{6}$$

where ϵ is a controller tuning parameter (typically, a small positive number) that may be adjusted. Throughout the paper we assume that all functions are sufficiently smooth so that appropriate singular perturbation results can be used.

Equation (3) is the static plant model, (4) is the input into the plant where $d(t)$ is a dither signal that is typically chosen so that appropriate parameter convergence can be achieved and ξ comes from the optimization algorithm (6). The optimization algorithm uses the estimated parameter $\hat{\theta}$ that is obtained from the estimator (5a), (5b). The parameter estimation algorithm (5a) may contain extra states $\eta \in \mathbb{R}^q \times \mathbb{R}^s$ to widen the class of estimators considered; the further possible generalizations are

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explained in Remark 18. Considering η as a non-square matrix is convenient for the later examples, and does not affect any subsequent proofs. Fig. 1 shows the relations between different parts of the closed loop system for a static plant.

Remark 2: By introducing $\tilde{\theta} := \hat{\theta} - \theta$, $\tilde{\eta} := \eta - \eta^*$, $\tilde{\xi} := \xi - \xi^*$ (where (θ, η^*, ξ^*) is the equilibrium of (3)-(6)) and writing the closed loop equations in $\tilde{\theta}, \tilde{\eta}, \tilde{\xi}$ coordinates and in the time scale $\tau := \epsilon(t - t_0)$, the model of the systems (5a), (5b), (6) are obtained in the standard singular perturbation form:

$$\begin{aligned}
\epsilon \frac{d\tilde{\theta}}{d\tau} &= G(\tilde{\theta} + \theta, \tilde{\eta} + \eta^*, h(\theta, \tilde{\xi} + \xi^* + d(t_0 + \frac{\tau}{\epsilon})), \\
&\quad \tilde{\xi} + \xi^* + d(t_0 + \frac{\tau}{\epsilon})) \\
&=: \widetilde{G}(t_0 + \frac{\tau}{\epsilon}, \tilde{\theta}, \tilde{\eta}, \tilde{\xi}) \qquad (7a) \\
\epsilon \frac{d\tilde{\eta}}{d\tau} &= H(\tilde{\eta} + \eta^*, h(\theta, \tilde{\xi} + \xi^* + d(t_0 + \frac{\tau}{\epsilon})), \\
&\quad \tilde{\xi} + \xi^* + d(t_0 + \frac{\tau}{\epsilon})) \\
&=: \widetilde{H}(t_0 + \frac{\tau}{\epsilon}, \tilde{\eta}, \tilde{\xi}) \qquad (7b) \\
\frac{d\tilde{\xi}}{d\tau} &= F(\mathcal{D}_N(\tilde{\theta} + \theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) =: \widetilde{F}(\tilde{\theta}, \tilde{\xi}) \qquad (8)
\end{aligned}$$

While a singular perturbation approach could potentially be used to analyze the system (7a), (7b) and (8) (and therefore (3)-(6)), the standard assumptions in the classical singular perturbation literature on uniform boundedness of the functions $\widetilde{G}$ and $\widetilde{H}$ with small ϵ may not hold because of the dither signal $d\left(t_0 + \frac{\tau}{\epsilon}\right)$. However, the system (3)-(6) is regularly perturbed and the standard singular perturbation results hold under weaker smoothness assumptions than those stated in [14] for the system (7a), (7b) and (8); this was done, for instance, in [30] by dealing directly with a class of systems that subsume (3)-(6).

Next stability assumptions from singular perturbation techniques are used to state the main result for static plants. Consider the following assumption for the parameter estimation algorithm (7a), (7b):

Assumption 3: The origin $\tilde{\theta} = 0$, $\tilde{\eta} = 0$ of the boundary layer (fast) system:

$$\begin{aligned}
\dot{\tilde{\theta}} &= \widetilde{G}(t, \tilde{\theta}, \tilde{\eta}, \tilde{\xi}(t_0)) \qquad (9a) \\
\dot{\tilde{\eta}} &= \widetilde{H}(t, \tilde{\eta}, \tilde{\xi}(t_0)) \qquad (9b)
\end{aligned}$$

is uniformly asymptotically stable (UAS) uniformly in $t \in \mathbb{R}$ and $\tilde{\xi}(t_0)$ with a basin of attraction $\Omega_{\tilde{\theta},\tilde{\eta}}$.

Remark 3: From [14] it follows that when the system is UAS, there exists a $\mathcal{KL}$ function so that the trajectories can be bounded in an appropriate manner.

Then, it is assumed that optimization algorithm (8) satisfies the following hypothesis:

Assumption 4: The origin for the reduced (slow) system:

$$\frac{d\tilde{\xi}}{d\tau} = \widetilde{F}(0, \tilde{\xi}) \qquad (10)$$

is UAS with a basin of attraction $\Omega_{\tilde{\xi}}$.

Now, the stability properties of the overall system (3)-(6) are stated in the following:

Theorem 1: Suppose that Assumptions 1-4 hold. Then, there exist $\beta_1, \beta_2, \beta_3 \in \mathcal{KL}$ and $\Delta > 0$ such that for any strictly positive real number $\nu > 0$, there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ the following holds:

$$\begin{aligned}
|\tilde{\theta}(t)| &\leq \beta_1(|(\tilde{\theta}(t_0), \tilde{\eta}(t_0), \tilde{\xi}(t_0))|, t - t_0) + \nu, \qquad (11) \\
|\tilde{\eta}(t)| &\leq \beta_2(|(\tilde{\eta}(t_0), \tilde{\xi}(t_0))|, t - t_0) + \nu, \qquad (12) \\
|\tilde{\xi}(t)| &\leq \beta_3\big(|\tilde{\xi}(t_0)|, \epsilon(t - t_0)\big) + \nu, \qquad (13)
\end{aligned}$$

for all $(\tilde{\theta}(t_0), \tilde{\eta}(t_0), \tilde{\xi}(t_0)) \in \mathcal{B}_\Delta \subset \Omega_{\tilde{\theta},\tilde{\eta}} \times \Omega_{\tilde{\xi}}$, and $t \geq t_0 \geq 0$. In particular, $\limsup_{t\to\infty} |\tilde{\theta}(t)| < \nu$, $\limsup_{t\to\infty} |\tilde{\eta}(t)| < \nu$ and $\limsup_{t\to\infty} |\tilde{\xi}(t)| < \nu$.

Proof: The proof is omitted since it follows directly from [30]. □

Remark 4: Note that Theorem 1 allows us to combine any continuous optimization method of the form (2) which satisfies Assumptions 2 and 4 with a parameter estimation scheme (5a), (5b) which satisfies Assumption 3 in order to achieve extremum seeking. Hence, results of Theorem 1 provide a prescriptive framework for extremum seeking controller design that combines a large class of optimization methods with a large class of parameter estimation schemes that satisfy conditions of the theorem. Section V provides a range of algorithms for parameter estimator and optimizer and conditions that guarantee satisfaction of the required assumptions.

Remark 5: The conclusions in Theorem 1 are quite intuitive. There exists a ball $\mathcal{B}_\Delta$ of initial conditions such that for any desired accuracy characterized by ν, the parameter ϵ can be adjusted (i.e. reduced) so that for all initial conditions in the ball $\mathcal{B}_\Delta$:

- The parameter estimate $\hat{\theta}$ converges to the $\mathcal{B}_\nu$ ball centered at the true value of the parameter θ in time scale t (see (11));
- The extra state of the parameter estimator η converges to the $\mathcal{B}_\nu$ ball centered at the equilibrium of (5b) η^* in time scale t (see (12));
- The optimizer state ξ converges in the slow time scale ϵt to the $\mathcal{B}_\nu$ ball centered at the optimal value ξ^* (see (13)).

Remark 6: If conditions of Theorem 1 are modified by requiring that the boundary layer and reduced systems are uniformly exponentially stable (UES), then stronger UES stability of the closed loop system can be proved [14]. In other words, under those stated conditions there are positive constants $K_1, K_2, K_3, \lambda_1, \lambda_2, \lambda_3$ such that there exists $\epsilon^* > 0$ so that for all $\epsilon \in (0, \epsilon^*)$:

$$\begin{aligned}
|\tilde{\theta}(t)| &\leq K_1|(\tilde{\theta}(t_0), \tilde{\eta}(t_0), \tilde{\xi}(t_0))| \exp\big(\lambda_1(t - t_0)\big) \qquad (14) \\
|\tilde{\eta}(t)| &\leq K_2|(\tilde{\eta}(t_0), \tilde{\xi}(t_0))| \exp\big(\lambda_2(t - t_0)\big) \qquad (15) \\
|\tilde{\xi}(t)| &\leq K_3|\tilde{\xi}(t_0)| \exp\big(\epsilon\lambda_3(t - t_0)\big)\ , \qquad (16)
\end{aligned}$$

for all $(\tilde{\theta}(t_0), \tilde{\eta}(t_0), \tilde{\xi}(t_0)) \in \Omega_{\tilde{\theta},\tilde{\eta}} \times \Omega_{\tilde{\xi}}$.

Remark 7: Note that the conditions in Theorem 1 can be strengthened so that global asymptotic stability can be assumed for the reduced system and the boundary layer system from Assumptions 3 and 4. In that case, semi-global practical stability of the closed loop system is concluded. However, few optimization algorithms satisfy such strong conditions and so only the local results are stated here.

IV. Convergence Conditions for Dynamic Plants

In this section, extremum seeking is considered for dynamic plants. The extremum seeking controller is parameterized in such a way to exhibit multiple time scales in the closed-loop system which allows singular perturbation theory to be applied. Multi-time-scale singular perturbation results are used to show that if the parameters are tuned appropriately, the system achieves practical asymptotical convergence to the extremum. The following two subsections deal respectively with stable and unstable plants.

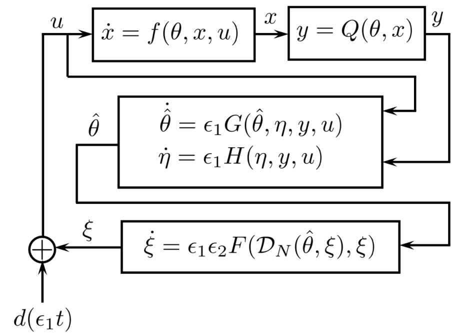

Fig. 2. The proposed framework for dynamic stable plants

A. Stable Plant

The extremum seeking scheme in the previous section can be readily modified to deal with dynamic plants. Consider the following closed loop system with a dynamical plant:

$$\dot{x} = f(\theta, x, u) \ , \tag{17}$$
$$y = Q(\theta, x) \ , \tag{18}$$
$$u = \xi + d(\epsilon_1 t) \ , \tag{19}$$
$$\dot{\hat{\theta}} = \epsilon_1 \ G(\hat{\theta}, \eta, y, u) \ , \tag{20a}$$
$$\dot{\eta} = \epsilon_1 \ H(\eta, y, u) \ , \tag{20b}$$
$$\dot{\xi} = \epsilon_1 \ \epsilon_2 \ F(\mathcal{D}_N(\hat{\theta}, \xi), \xi), \qquad \epsilon_1, \ \epsilon_2 > 0, \tag{21}$$

where $x \in \mathcal{B}_x \subset \mathbb{R}^n$ is the plant state, ϵ_1 and ϵ_2 are tuning parameters of the estimator and optimization scheme, respectively, and all other variables are the same as before (Fig. 2). Note that in this case, with appropriate tuning of ϵ_1 and ϵ_2, there are three time scales, where the plant is the fastest subsystem, $\hat{\theta}$ is the medium system and ξ is the slow system. The scalars $\epsilon_1, \epsilon_2 > 0$ are controller parameters that need to be tuned. The dither signal $d(\epsilon_1 t)$ may be needed to ensure an appropriate persistence of excitation condition that guarantees convergence of $\tilde{\theta}$.

Assumption 5: The following equation:

$$0 = f(\theta, x, u) \ ,$$

has a unique solution $x = \ell(\theta, u)$ and the map:

$$h(\theta, u) := Q(\theta, \ell(\theta, u))$$

has an extremum at $u = \xi^*$.

Remark 8: Assumption 5 says that for every constant u the output y converges to $\ell(\theta, u)$, uniformly in u. Note that this problem formulation is different from classical adaptive control literature where one typically wants to stabilize an unstable plant. In the case considered in this section, the plant is assumed to be stable. If the plant is unstable, then one can use any suitable controller to stabilize it so that the assumption holds.

Suppose that Assumption 1 holds for the map $h(\cdot, \cdot)$. This is a strong assumption since finding the map $\ell(\cdot, \cdot)$ explicitly is hard in general. Moreover, it will be assumed that the optimization scheme of the form (21) and a parameter estimation scheme (20a), (20b) have similar properties to the ones stated in Assumptions 3 and 4, respectively, but with the above constructed map $h(\cdot, \cdot)$.

Writing the closed loop in coordinates $\tilde{x} := x - \ell(\theta, u), \tilde{\theta} := \hat{\theta} - \theta, \tilde{\eta} := \eta - \eta^*, \tilde{\xi} := \xi - \xi^*$ gives

$$\begin{aligned} \dot{\tilde{x}} &= f(\theta, \tilde{x} + \ell(\theta, \tilde{\xi} + \xi^* + d(\epsilon_1 t)), \tilde{\xi} + \xi^* + d(\epsilon_1 t)) \\ &\quad - \epsilon_1 \epsilon_2 \frac{\partial \ell}{\partial u} F(\mathcal{D}_N(\tilde{\theta} + \theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) \\ &=: \widetilde{f}(\epsilon_1 t, \tilde{x}, \tilde{\theta}, \tilde{\xi}, \epsilon_1 \epsilon_2) \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{\tilde{\theta}} &= \epsilon_1 \ G(\tilde{\theta} + \theta, \tilde{\eta} + \eta^*, Q(\theta, \tilde{x} + \ell(\theta, \tilde{\xi} + \xi^* + d(\epsilon_1 t))), \\ &\quad \tilde{\xi} + \xi^* + d\,(t)) \\ &=: \epsilon_1 \ \widetilde{G}(\epsilon_1 t, \tilde{\eta}, \tilde{x}, \tilde{\theta}, \tilde{\xi}) \end{aligned} \tag{23a}$$

$$\begin{aligned} \dot{\tilde{\eta}} &= \epsilon_1 \ H(\tilde{\eta} + \eta^*, Q(\theta, \tilde{x} + \ell(\theta, \tilde{\xi} + \xi^* + d(\epsilon_1 t))), \\ &\quad \tilde{\xi} + \xi^* + d\,(\epsilon_1 t)) \\ &=: \epsilon_1 \ \widetilde{H}(\epsilon_1 t, \tilde{\eta}, \tilde{x}, \tilde{\xi}) \end{aligned} \tag{23b}$$

$$\begin{aligned} \dot{\tilde{\xi}} &= \epsilon_1 \ \epsilon_2 \ F(\mathcal{D}_N(\tilde{\theta} + \theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) \\ &=: \epsilon_1 \ \epsilon_2 \ \widetilde{F}(\tilde{\theta}, \tilde{\xi}) \ . \end{aligned} \tag{24}$$

This system is in singularly perturbed form with tuning controller parameters ϵ_1 and ϵ_2. It has three time scales, the plant is the fastest, then the estimator is the middle and the optimization algorithm is the slowest time scale. The stability analysis is done via three systems, the fast system, the medium system and the slow system that correspond respectively to the plant, the estimator and the optimization algorithm. Stability analysis of this class of systems in standard form was considered in [12], [30] which requires Assumptions 3 and 4 to be replaced by the following three assumptions.

Assumption 6: The origin of the fast system:

$$\begin{aligned} \dot{\tilde{x}} &= \widetilde{f}(0, \tilde{x}, \tilde{\theta}(t_0), \tilde{\xi}(t_0), 0) \\ &= f(\theta, \tilde{x} + \ell(\theta, \tilde{\xi}(t_0) + \xi^*), \tilde{\xi}(t_0) + \xi^*) \end{aligned} \tag{25}$$

is UAS, uniformly in $\tilde{\theta}(t_0)$ and $\tilde{\xi}(t_0)$ with a basin of attraction $\Omega_{\tilde{x}}$.

Assumption 7: The origin $\tilde{\theta} = 0$, $\tilde{\eta} = 0$ of the medium system:

$$\frac{d\tilde{\theta}}{d\tau} = \widetilde{G}(\tau, \tilde{\eta}, 0, \tilde{\theta}, \tilde{\xi}(t_0)) \tag{26a}$$
$$\frac{d\tilde{\eta}}{d\tau} = \widetilde{H}(\tau, \tilde{\eta}, 0, \tilde{\xi}(t_0)) \tag{26b}$$

(in time scale $\tau := \epsilon_1 t$) is UAS, uniformly in $\tilde{\xi}(t_0)$ with a basin of attraction $\Omega_{\tilde{\theta}, \tilde{\eta}}$.

Assumption 8: The origin for the slow system:

$$\frac{d\tilde{\xi}}{d\sigma} = \widetilde{F}(0, \tilde{\xi}) \tag{27}$$

(in the time scale $\sigma := \epsilon_1\epsilon_2 t$) is UAS with a basin of attraction $\Omega_{\tilde{\xi}}$.

Then, the main result is provided in the following:

Theorem 2: Suppose that Assumptions 1, 2 and 5-8 hold. Then, there exist $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathcal{KL}$ and $\Delta > 0$ such that for any strictly positive real number $\nu > 0$, there exists $\epsilon_1^*, \epsilon_2^* > 0$ such that for all $\epsilon_1 \in (0, \epsilon_1^*)$ and $\epsilon_2 \in (0, \epsilon_2^*)$ the following holds:

$$\begin{aligned}
|\tilde{x}(t)| &\leq \beta_1(|(\tilde{x}(t_0), \tilde{\theta}(t_0), \tilde{\xi}(t_0))|, t - t_0) + \nu, & (28)\\
|\tilde{\theta}(t)| &\leq \beta_2(|(\tilde{\theta}(t_0), \tilde{\eta}(t_0), \tilde{\xi}(t_0))|, \epsilon_1(t - t_0)) + \nu, & (29)\\
|\tilde{\eta}(t)| &\leq \beta_3(|(\tilde{\eta}(t_0), \tilde{\xi}(t_0))|, \epsilon_1(t - t_0)) + \nu, & (30)\\
|\tilde{\xi}(t)| &\leq \beta_4\big(|\tilde{\xi}(t_0)|, \epsilon_1\epsilon_2(t - t_0)\big) + \nu, & (31)
\end{aligned}$$

for all $(\tilde{x}(t_0), \tilde{\theta}(t_0), \tilde{\eta}(t_0), \tilde{\xi}(t_0)) \in \mathcal{B}_\Delta \subset \Omega_{\tilde{x}} \times \Omega_{\tilde{\theta},\tilde{\eta}} \times \Omega_{\tilde{\xi}}$ and all $t \geq t_0 \geq 0$. In particular, (28), (29) and (31) imply respectively: $\limsup_{t\to\infty} |\tilde{x}(t)| < \nu$, $\limsup_{t\to\infty} |\tilde{\theta}(t)| < \nu$, $\limsup_{t\to\infty} |\tilde{\eta}(t)| < \nu$ and $\limsup_{t\to\infty} |\tilde{\xi}(t)| < \nu$.

Proof: Stability analysis of this class of systems in standard singular perturbation form was considered in [12]; a similar analysis can be carried out under weaker conditions as in [30] for multiple time scale systems in regular form. These modifications are straightforward and are omitted for space reasons. □

Remark 9: Note that the reason for having a time scale separation between the $\tilde{x}$ and $\tilde{\theta}$ is to be able to use the steady state map $h(\theta, u)$ for parameter estimation. Note that this is not necessary and it is sometimes possible to directly use dynamical equations for the x-subsystem ($\dot{x} = f(\theta, x, u)$) to estimate θ. Note that in this case, there would be two time scales in the overall system rather than three. This was done in the next section for unstable plants.

Remark 10: If Assumptions 6-8 of the Theorem 2 are modified by requiring uniform exponential stability of the slow, medium, and fast systems, then the theorem will be changed by stating uniformly exponential stability for the closed loop system (22)-(24) for all $(\tilde{x}(t_0), \tilde{\theta}(t_0), \tilde{\eta}(t_0), \tilde{\xi}(t_0)) \in \Omega_{\tilde{x}} \times \Omega_{\tilde{\theta},\tilde{\eta}} \times \Omega_{\tilde{\xi}}$ and all $t \geq t_0 \geq 0$, see [8]. Also, by strengthening the stability conditions in assumptions to hold globally, semi-global practical asymptotic stability of the closed loop can be concluded [20].

B. Unstable plant

All of the results in the previous section were stated under the assumption that the plant is stable. This assumption is relaxed in this section and it is assumed that the plant is unstable and needs to be stabilized using the output measurements $\bar{y}$, see [2], [3], [10] for specific plants and optimization schemes. Moreover, results in this section illustrate the point made in Remark 9.

Consider an uncertain dynamic plant:

$$\begin{aligned}
\dot{x} &= f(\theta, x, u) & (32)\\
y &= Q(\theta, x) \ , & (33)
\end{aligned}$$

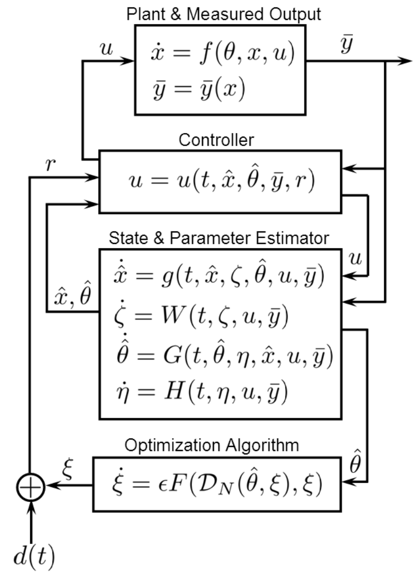

Fig. 3. The proposed framework for dynamic unstable plants

where all variables have the same meaning as in the previous section. Here it may be useful to make a distinction between the *measured output* ($\bar{y}$) that is used to stabilize the plant by designing the observer, controller and parameter estimator and a *performance output* (y) that may be used in extremum seeking.

In this section, it will be assumed that a controller, parameter estimator and state estimator have been designed and they are of the following form:

$$\begin{aligned}
\dot{\hat{x}} &= g(t, \hat{x}, \zeta, \hat{\theta}, u, \bar{y}) & (34a)\\
\dot{\zeta} &= W(t, \zeta, u, \bar{y}) & (34b)\\
\dot{\hat{\theta}} &= G(t, \hat{\theta}, \eta, \hat{x}, u, \bar{y}) & (35a)\\
\dot{\eta} &= H(t, \eta, u, \bar{y}) & (35b)\\
u &= u(t, \hat{x}, \hat{\theta}, \bar{y}, r) \ , & (36)
\end{aligned}$$

where $\hat{x}$ is the estimate of the state, ζ and η are extra states for the state estimator and parameter estimator, respectively, to widen the class of estimators that may be considered, r is the new "reference" input and all other variables were defined in the previous section. The relation between the above equations is shown in Fig. 3.

It is convenient to rewrite the above equations by using $e := \hat{x} - x$, $\tilde{\zeta} := \zeta - \zeta^*$, $\tilde{\theta} := \hat{\theta} - \theta$, $\tilde{\eta} = \eta - \eta^*$ (where $(\hat{x}^*, \zeta^*, \theta, \eta^*)$ is the equilibrium of (34a)-(35b)) which gives:

$$\begin{aligned}
\dot{x} &= \widetilde{f}(t, \theta, x, e, \tilde{\zeta}, \tilde{\theta}, \tilde{\eta}, r) & (37)\\
\dot{e} &= \widetilde{g}(t, \theta, x, e, \tilde{\zeta}, \tilde{\theta}, \tilde{\eta}, r) & (38a)\\
\dot{\tilde{\zeta}} &= \widetilde{W}(t, \theta, x, e, \tilde{\zeta}, \tilde{\theta}, \tilde{\eta}, r) & (38b)\\
\dot{\tilde{\theta}} &= \widetilde{G}(t, \theta, x, e, \tilde{\zeta}, \tilde{\theta}, \tilde{\eta}, r) & (39a)\\
\dot{\tilde{\eta}} &= \widetilde{H}(t, \theta, x, e, \tilde{\zeta}, \tilde{\theta}, \tilde{\eta}, r) \ , & (39b)
\end{aligned}$$

By introducing $X := (x, e, \tilde{\zeta}, \tilde{\theta}, \tilde{\eta})$, (37)-(39b) is rewritten as follows:

$$\dot{X} = N(t, \theta, X, r) \ . \tag{40}$$

Now Assumptions 6-8 are replaced by the following three assumptions.

Assumption 9: There exists ℓ such that

$$N(t,\theta,X,r)=0 \;\Leftrightarrow\; X=\ell(\theta,r):=\begin{pmatrix}\ell_x(\theta,r)\\ \ell_e(\theta,r)\\ \ell_{\tilde\zeta}(\theta,r)\\ \ell_{\tilde\theta}(\theta,r)\\ \ell_{\tilde\eta}(\theta,r)\end{pmatrix}$$

and the map

$$y=h(\theta,r):=Q(\theta,\ell_x(\theta,r))$$

has an extremum (maximum) at $r=\xi^*$.

To state the main result, it is convenient to define $\widetilde{X}:=(\tilde x,e,\tilde\zeta,\tilde\theta,\tilde\eta)$ where $\tilde x:=x-\ell_x(\theta,r)$ to obtain:

$$\dot{\widetilde{X}}=\widetilde{N}(t,\theta,\widetilde{X},r)\ . \tag{41}$$

In order to achieve extremum seeking, the above system is controlled with a slow optimization algorithm:

$$\dot{\tilde\xi}=\epsilon F(\mathcal{D}_N(\tilde\theta+\theta,\tilde\xi+\xi^*),\tilde\xi+\xi^*)=:\epsilon\widetilde{F}(\tilde\theta,\tilde\xi)\ , \tag{42}$$

where $r=\xi+d(t)$, $r=\xi^*$ is the maximum of the map $h(\theta,\cdot)$ and $\tilde\xi:=\xi-\xi^*$.

The relation between plant output, observer, controller, parameter estimator and optimizer is shown in Fig. 3.

Assumption 10: The origin $\widetilde{X}=0$ of the boundary layer (fast) system:

$$\dot{\widetilde{X}}=\widetilde{N}(t,\widetilde{X},\tilde\xi(t_0)) \tag{43}$$

is uniformly asymptotically stable (UAS) uniformly in $t\in\mathbb{R}$ and $\tilde\xi(t_0)$ with a basin of attraction $\Omega_{\widetilde{X}}$.

Assumption 11: The origin for the reduced (slow) system:

$$\frac{d\tilde\xi}{d\tau}=\tilde F(0,\tilde\xi) \tag{44}$$

is UAS with a basin of attraction $\Omega_{\tilde\xi}$.

Then, the following result can be stated:

Theorem 3: Suppose that the Assumptions 1, 2 and 9-11 hold. Then, there exist $\beta_1,\beta_2\in\mathcal{KL}$ and $\Delta>0$ such that for any strictly positive real number $\nu>0$, there exists $\epsilon^*>0$ such that for all $\epsilon\in(0,\epsilon^*)$ the following holds:

$$|\widetilde{X}(t)| \;\le\; \beta_1(|(\widetilde{X}(t_0),\tilde\xi(t_0))|,t-t_0)+\nu, \tag{45}$$
$$|\tilde\xi(t)| \;\le\; \beta_2\big(|\tilde\xi(t_0)|,\epsilon(t-t_0)\big)+\nu, \tag{46}$$

for all $(\widetilde{X}(t_0),\tilde\xi(t_0))\in\mathcal{B}_\Delta\subset\Omega_{\widetilde{X}}\times\Omega_{\tilde\xi}$ and all $t\ge t_0\ge 0$. In particular, (45) and (46) imply respectively: $\limsup_{t\to\infty}|\widetilde{X}(t)|<\nu$ and $\limsup_{t\to\infty}|\tilde\xi(t)|<\nu$.

Proof: This result follows directly from the singular perturbation result in [30]. □

Remark 11: Note that typical adaptive control designs do not produce UAS of the system (41) in general and an appropriate persistence of excitation condition needs to hold in order to get UAS. A range of persistency of excitation conditions that can be used to conclude UAS of (41) can be found in [17]. Note also that while authors of [2], [3], [10], [11] do not state their results at this level of generality, they do require appropriate persistence of excitation that guarantees UAS of (41).

Remark 12: The results in [2], [3], [10], [11] are similar to the main result in Theorem 3 in the sense that in both they assume that the performance function is explicitly known as a function of the system states and uncertain parameters from the dynamic equations. However, the results in [2], [3], [10], [11] are presented for particular classes of plants and particular optimization algorithms are used to achieve extremum seeking.

Remark 13: Time scale separation is only one possible way to ensure convergence of the extremum seeking algorithm. In particular, one could use different conditions based on small gain arguments. Note that (41) and (42) with $\epsilon=1$ is a feedback connection of two systems. If each of these two systems satisfies an input-to-state stability property and a small gain condition holds then it can be concluded that $\tilde\xi$ converges, which implies extremum seeking.

Remark 14: If Assumptions 10-11 are modified by requiring that the the boundary layer and reduced systems are UES, then the closed loop full system (41)-(42) is UES for all $(\widetilde{X}(t_0),\tilde\xi(t_0))\in\Omega_{\widetilde{X}}\times\Omega_{\tilde\xi}$ and all $t\ge t_0\ge 0$.

Remark 15: Note that all of the results can be restated so that instead of uniform asymptotic stability requirements in the assumptions, uniform global asymptotic stability (UGAS) property is used. All of the results of this paper still hold but the conclusions are then stronger as semi-global stability can be achieved. These results are presented in [20].

V. Optimization and Estimation Algorithms

This section provides different optimization algorithms of the form (2) that satisfy Assumptions 4, 8 and 11 in Theorems 1, 2 and 3 and parameter estimators that satisfy Assumptions 3 and 7 in Theorems 1 and 2 respectively. Designers can choose and combine the following parameter estimators and optimization algorithms that are appropriate for their problems. Note that the framework is not restricted to these algorithms and any other algorithms that satisfy the assumptions can be employed.

A range of existing algorithms are taken off-the-shelf from the literature and conditions that guarantee satisfaction of our assumptions are pointed out. In this manner, it is demonstrated that the results of this paper provide a flexible design framework in which a large class of optimizers can be combined with a large class of parameter estimation algorithms to achieve extremum seeking.

Remark 16: Note that in this section, all conditions are provided for the local stability of the optimization and estimation algorithms. By strengthening the conditions, global stability of the algorithms can be concluded. However, to be consistent with the main results, only conditions for local stability are stated.

A. Optimization Algorithms (OA)

While the algorithms provided are taken off-the-shelf from the literature, they demonstrate the utility of the unifying design framework as any of these algorithms can be combined with appropriate parameter estimators to obtain novel extremum seeking algorithms not considered previously in the literature.

It should be noted that the reduced systems in Assumptions 4, 8 and 11 take the same form of optimization algorithms presented in this subsection but they operate on different time scales depending on the problem setting considered. It is not hard to see that modulo the time scale, the dynamics of these systems is exactly the same as that given by (2). Hence, it is assumed that the chosen optimization algorithm is stable in an appropriate sense. In addition, to simplify the notation, the dependence on the unknown parameter is suppressed in this subsection.

OA1. Gradient Descent Algorithm: Consider the continuous-time gradient descent system [1]

$$\dot{\xi} = -\nabla h(\xi), \tag{47}$$

where $\nabla h(\xi)$ denotes the gradient of h at ξ.

Definition 1: A point $z \in \mathbb{R}^n$ is a strict local minimum of $f(x)$ if there exists $\epsilon > 0$ such that $f(x) > f(z)$ for all x such that $0 < |x - z| < \epsilon$.

It was shown in [1] that if equilibrium ξ^* of (47) is a strict local minimum, then ξ^* is asymptotically stable

Proposition 1: If ξ^* is a strict local extremum, then then ξ^* is a uniformly asymptotically stable point of the optimization algorithm (47) and therefore this algorithm satisfies Assumptions 4, 8 and 11.

OA2. Continuous Newton Method: Consider the following continuous Newton method [7]

$$\dot{\xi} = -J_r^{-1}(\xi) M \nabla h(\xi), \tag{48}$$

where J_r represents the Jacobian of $r(\xi) = \nabla h(\xi)$ and M is an arbitrary positive definite matrix.

Condition 1 (Nonsingularity): The Jacobian matrix $J_r(\xi)$ is invertible in some neighborhood $\mathcal{N}(\xi^*)$ of ξ^*.

Proposition 2: [7] Suppose that Condition 1 holds, then for all $\xi(t_0) \in \mathcal{N}(\xi^*)$ the Newton method (48) is exponentially stable and satisfies Assumptions 4, 8 and 11.

OA3. Continuous Jacobian Matrix Transpose: Consider the continuous Jacobian matrix transpose algorithm as follows [7]

$$\dot{\xi} = -M J_r^T(\xi) \nabla h(\xi). \tag{49}$$

Proposition 3: [7] Under Condition 1, the optimization algorithm (49) is asymptotically stable and Assumptions 4, 8 and 11 hold for this algorithm.

OA4. Combination of Newton and Gradient Methods: Consider the following continuous Newton-type differential equation [32]

$$\dot{\xi} = F(\xi), \quad \xi(t_0) = \xi_0, \tag{50}$$

where

$$F(\xi) = \begin{cases} -(\nabla^2 h(\xi))^{-1} \nabla h(\xi), & \text{if } \lambda_{min}(\xi) > \delta_2, \\ -\alpha(\xi)(\nabla^2 h(\xi))^{-1} \nabla h(\xi) & \\ -\beta(\xi) \nabla h(\xi), & \text{if } \delta_1 \le \lambda_{min}(\xi) \le \delta_2, \\ -\nabla h(\xi), & \text{if } \lambda_{min}(\xi) < \delta_1, \end{cases}$$

where λ_{min} denotes the smallest eigenvalue of $\nabla^2 h(\xi)$, $\delta_2 > \delta_1 > 0$ are two predefined positive constants, and $\alpha(\xi)$ and $\beta(\xi)$ are set as

$$\alpha(\xi) = \frac{\lambda_{min}(\xi) - \delta_1}{\delta_2 - \delta_1}, \tag{51}$$

$$\beta(\xi) = 1 - \alpha(\xi) = \frac{\delta_2 - \lambda_{min}(\xi)}{\delta_2 - \delta_1}. \tag{52}$$

Condition 2 (Boundedness): $h(\xi)$ has the following properties:

- $h(\xi)$ is bounded from below by $h^* > -\infty$.
- Let
$$L = \{\xi \in \mathcal{B}_\xi | h(\xi) \le h(\xi_0)\}$$
be the level set of $h(\xi)$, and $L_{h(\xi_0)}$ denote the connected subset of L that contains the point ξ_0, then for any $\xi_0 \in \Omega_\xi$, $L_{h(\xi_0)}$ is bounded.

Proposition 4: [32] Suppose that Condition 2 holds, then optimization algorithm (50) is asymptotically stable and satisfies Assumptions 4, 8 and 11.

OA5. Levenberg-Marquardt Method: Consider a continuous analogue of the Levenberg-Marquardt method [28]

$$\dot{\xi} = -(J_r^T(\xi) J_r(\xi) + \delta I_n)^{-1} J_r^T(\xi) \nabla h(\xi) \tag{53}$$

where δ is a positive number.

Condition 3 (Full Rankness): The Jacobian matrix $J_r(\xi^*)$ is of full rank for an extremum point ξ^* of $h(\xi)$.

Proposition 5: [28] If Condition 3 holds, then ξ^* is an asymptotically stable point of the system (53) and therefore satisfies Assumptions 4, 8 and 11.

OA6. Newton-Raphson-Ben-Israel: Consider a continuous analogue of the Newton-Raphson-Ben-Israel method [28]

$$\dot{\xi} = -J_r^+(\xi) \nabla h(\xi), \tag{54}$$

where $J_r^+(\xi)$ is the Moore-Penrose inverse of $J_r(\xi)$.

Proposition 6: [28] If Condition 3 holds, then ξ^* is an asymptotically stable point of the optimization algorithm (54) and therefore this algorithm satisfies Assumptions 4, 8 and 11.

B. Estimation Algorithms (EA)

In this subsection, different examples of parameter estimation algorithms in the form of (5a), (5b) and (20a), (20b) respectively for Theorems 1 and 2 are presented that satisfy Assumptions 3 and 7 under specified conditions. In these two theorems, parameter estimation algorithms are designed using steady-state mapping between the input and the output. Since the UAS conditions of Assumptions 3 and 7 may be hard to satisfy in general, the estimator design is usually considered for a class of steady-state maps that are linearly parameterized with θ as

$$y = h(\theta, u) = h_1^T(u)\theta + h_0(u). \tag{55}$$

As shown in Fig. 1 and 2, $u(t, \xi(t)) = \xi(t) + d(t)$ and Assumptions 3 and 7 should be verified for $\xi(t) = \xi(t_0)$, hence

$$y(t, \xi(t_0)) = \Phi(t, \xi(t_0))^T \theta(t) + y_0(t, \xi(t_0)) \ , \tag{56}$$

where $\Phi(t, \xi(t_0)) = h_1^T(u(t, \xi(t_0)))$ and $y_0(t, \xi(t_0)) = h_0(u(t, \xi(t_0)))$. Therefore, in the following, different examples of estimation algorithms are presented for the steady-state mapping in the form of (56). In other cases, any parameter estimation algorithm in the form of (5a), (5b) and (20a), (20b) that satisfy the required conditions can be employed. In addition, if the estimator should be designed using plant dynamics instead of steady-state mapping, then Theorem 3 can be used where parameter estimation is in the form of (35a), (35b) and should satisfy Assumption 10.

EA1. Gradient Algorithm: An estimate $\hat{\theta}$ of the unknown parameter vector θ can be obtained via the gradient algorithm [13]:

$$\dot{\hat{\theta}} = \Gamma\Phi(y - \Phi^T\hat{\theta} - y_0) \tag{57}$$

where $\Gamma = \Gamma^T > 0$ is a diagonal matrix of gains.

Condition 4 (Persistency of Excitation): $\Phi(t, \xi(t_0))$ is persistently exciting, i.e., for any $t_0 \geq 0$ there exist $\mu, T > 0$ such that for any fixed $\xi(t_0)$ in a neighborhood of ξ^*,

$$\int_{t_0}^{t_0+T} \Phi(\tau, \xi(t_0))\Phi^T(\tau, \xi(t_0))d\tau \geq \mu I. \tag{58}$$

Proposition 7: [13] If Condition 4 holds, then the estimation algorithm (57) is exponentially stable and satisfies Assumptions 3 and 7 and their respective UES conditions.

EA2. Gradient Algorithm with Integral Cost Function: This parameter estimation algorithm is as follows [13]:

$$\begin{aligned} \dot{\hat{\theta}} &= -\Gamma(R(t)\hat{\theta} + S(t)), \quad \hat{\theta}(0) = \theta_0 && (59)\\ \dot{R} &= -\beta R + \frac{\Phi\Phi^T}{m_s^2}, \quad R(0) = 0, && (60)\\ \dot{S} &= -\beta S - \frac{y\Phi}{m_s^2}, \quad S(0) = 0, && (61) \end{aligned}$$

where $\Gamma = \Gamma^T > 0$ is a design matrix referred to as the adaptive gain and $\beta > 0$ is a design constant acting as a forgetting factor, $m_s^2 = 1 + n_s^2$, and $n_s^2 = \Phi^T\Phi$. Note that by introducing $\eta = [R \;\; S]^T$, estimation algorithm (59)-(61) is in the same form of (5a), (5b) and (20a), (20b).

Remark 17: Note that the asymptotic stability assumption on η can be replaced by a boundedness condition with no consequence for the estimation algorithm. A restatement and proof of Theorems 1 and 2 with only boundedness of η is not undertaken here so as not to cloud the key concept of the presented framework.

Proposition 8: [13] If Condition 4 holds, the estimation algorithm (59)-(61) is exponentially stable and satisfies Assumptions 3 and 7 and their respective UES conditions.

EA3. Pure Least-Squares Algorithm: Consider the following parameter estimation algorithm [13]:

$$\begin{aligned} \dot{\hat{\theta}} &= \eta\Phi(y - \Phi^T\hat{\theta} - y_0), && (62)\\ \dot{\eta} &= -\eta\frac{\Phi\Phi^T}{m_s^2}\eta. && (63) \end{aligned}$$

Proposition 9: [13] If Condition 4 holds for $\frac{\Phi}{m_s}$, then estimation algorithm (62), (63) is asymptotically stable and satisfies Assumptions 3 and 7.

EA4. Recursive Least-Squares Algorithm with Forgetting Factor: An alternative approach to estimate the unknown parameters in (56) is the following continuous-time recursive least-squares (RLS) algorithm [13]:

$$\begin{aligned} \dot{\hat{\theta}} &= \eta\Phi(y - \Phi^T\hat{\theta} - y_0) && (64)\\ \dot{\eta} &= \gamma\eta - \eta\Phi\Phi^T\eta && (65) \end{aligned}$$

where $\gamma \geq 0$ and η is the error covariance matrix.

Proposition 10: [13] If Φ satisfies the persistency of excitation condition (58), then (64) is uniformly exponentially stable and η is bounded.

Remark 18: It should be noted that there exist other estimation algorithms that do not exactly fit the proposed framework, thereby introducing a potential for further generalization. For instance, due to resetting in *Pure Least-Squares Algorithm with Covariance Resetting* [13] and due to the discontinuity of the right-hand side in *Modified Least-Squares Algorithm with Forgetting Factor* [13], these estimators do not satisfy the smoothness condition on the functions required in the main results. Nonetheless, the range of parameter estimators provided in this paper can be combined with optimization algorithms to produce a variety of extremum seeking algorithms.

VI. Applications of the Main Results

In this section, to illustrate the generality of the framework, a given plant with different optimization schemes and parameter estimators from Section V is considered.

A. A class of stable linear systems with output nonlinearities

Below the general result for stable dynamic plants (Theorem 2) is applied to the situation where the dynamic plant (17) and the nonlinear output (18) have special structures. Consider a class of linear systems with linearly parameterized nonlinear outputs

$$\begin{aligned} \dot{x} &= Ax + Bu && (66)\\ y &= \tilde{h}_0(x, u) + \tilde{h}_1(x, u)^T\theta, && (67) \end{aligned}$$

where A is Hurwitz. This class of problems has been investigated in more detail for linear control systems in [3].

By solving

$$0 = Ax + bu \implies x = l(u) := -A^{-1}Bu$$

the steady state map is

$$y = \tilde{h}_0(l(u), u) + \tilde{h}_1(l(u), u)^T\theta = h_0(u) + h_1(u)^T\theta,$$

which is in form of (56). Therefore, any parameter estimation algorithm from subsection V-B satisfying required conditions can be combined with any optimization algorithm presented in subsection V-A satisfying appropriate conditions in order to achieve practical asymptotical convergence to the extremum of the performance function (67).

Example 1: In this example, first a feedback linearizing controller is designed for the Anti-lock Braking System (ABS) problem in order to obtain a stable linear system in the form of (66). Then, a linearly parameterized nonlinear output in the form of (67) is considered for the tire-road friction model. The purpose of the ABS is to regulate the wheel longitudinal

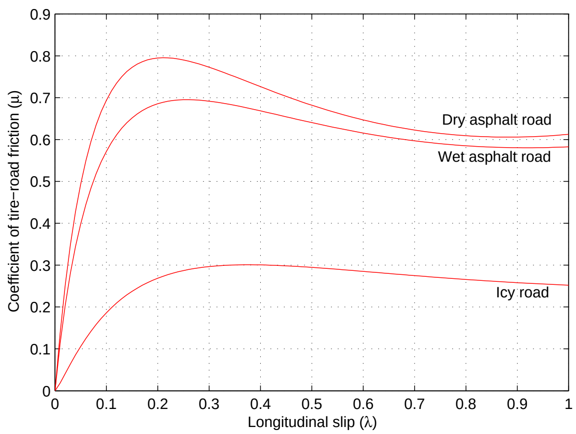

Fig. 4. Friction force coefficient $\mu(\lambda)$ in different road conditions

slip at its optimum point in order to generate the maximum braking force (Fig. 4).

Consider a quarter vehicle model, where the tire dynamics are as follows [5]:

$$m\dot{v} = -N\mu(\lambda) \tag{68}$$
$$I\dot{\omega} = -B\omega + NR\mu(\lambda) - \tau_b \tag{69}$$

Here v is the longitudinal speed of the vehicle, m is its mass, $N = mg$ is the force at the tire, I the wheel inertia, ω the angular velocity, $B\omega$ the bearing friction torque, R the radius of the wheel, τ_b the braking friction torque, $\mu(\lambda)$ the tire-road friction force coefficient and λ is the wheel slip which is defined as

$$\lambda = \frac{v - R\omega}{v}, \quad v \geq R\omega. \tag{70}$$

Using equations (68)-(70) gives

$$\dot{\lambda} = \left(\frac{R\omega}{v^2} + \frac{mR^2}{Iv}\right)\dot{v} + \frac{RB}{Iv}\omega + \frac{R}{Iv}\tau_b. \tag{71}$$

It is assumed that the longitudinal and angular velocities can be measured. Thus, by using the feedback linearizing controller

$$\tau_b = -\frac{cIv}{R}(\lambda - \lambda_0) - B\omega - \frac{I\omega}{v}\dot{v} - mR\dot{v}, \tag{72}$$

where c is a positive constant, the following linear model is obtained

$$\dot{\lambda} = -c(\lambda - \lambda_0), \tag{73}$$

where λ_0 is the control input to the new system which is exponentially stable. The objective here is to achieve maximum friction force by maximizing the friction coefficient $\mu(\lambda)$. Therefore, a parameterized model of the $\mu(\lambda)$ is required that has a maximum μ^* at the optimal slip λ^*. In this paper, the tire-road friction model proposed in [31] is used:

$$\mu(\lambda) = e^{p_1} e^{-p_2\lambda} \lambda^{(p_5\lambda + p_4)} e^{-p_5 v} \tag{74}$$

where p_1, p_2, p_3, p_4, p_5 are unknown parameters to be estimated. It should be noted that although this model depends on v, its optimum value is independent of v and only depends on λ. After applying logarithm to both sides of (74) and rearranging it in vector form, the output is

$$y = h(\theta, \lambda) = ln\mu(\lambda) = \phi^T\theta \tag{75}$$

where $\phi = [1, -\lambda, \lambda ln\lambda, ln\lambda, -v]$ and $\theta = [p_1, p_2, p_3, p_4, p_5]$.

Now, the assumptions of Theorem 2 should be verified. The map $h(\cdot,\cdot)$ in (75) is known and $h(\theta,\cdot)$ is smoothly differentiable. In addition, $h(\theta,\cdot)$ has a global maximum at λ^* which is given by the solution to

$$p_3\lambda^*(ln\lambda^* + 1) = p_2\lambda^* - p_4 \tag{76}$$

which can be seen in Fig. 4, so that Assumption 1 holds.

Next, the continuous Jacobian matrix transpose method (*OA3*) is proposed as the optimization scheme:

$$\dot{\xi} = -\epsilon_1\epsilon_2 M J_r^T(\xi)\nabla h(\xi) = -\epsilon_1\epsilon_2 M\left(\frac{p_3}{\xi} - \frac{p_4}{\xi^2}\right) \times \left(-p_2 + p_3(ln\xi + 1) + p_4(\frac{1}{\xi})\right), \tag{77}$$

To verify Assumptions 2 and 5, consider the system (73) which has an equilibrium at $\lambda = \lambda_0$ and the map $h(\theta, \lambda_0)$ has an extremum at $\lambda_0^* = \lambda^*$ with respect to formulation (76), which corresponds to the equilibrium point of the optimization scheme (77) at ξ^*

$$p_3\xi^*(ln\xi^* + 1) = p_2\xi^* - p_4, \tag{78}$$

if $\xi^* \neq \frac{p_4}{p_3}$ and $\xi^* \neq 0$ so that Assumptions 2 and 5 hold.

To verify UAS of the optimization algorithm, (77) is written in new coordinate $\tilde{\xi} = \xi - \xi^*$ and $\sigma := \epsilon_1\epsilon_2(t - t_0)$:

$$\frac{d\tilde{\xi}}{d\sigma} = M\left(-p_2 + p_3\left(ln(\tilde{\xi} + \xi^*) + 1\right) + p_4\left(\frac{1}{\tilde{\xi} + \xi^*}\right)\right) \times \left(\frac{p_3}{\tilde{\xi} + \xi^*} - \frac{p_4}{(\tilde{\xi} + \xi^*)^2}\right) \tag{79}$$

Since it is assumed that $\xi^* \neq \frac{p_4}{p_3}$ and $\xi^* \neq 0$, the Jacobian matrix $J_r(\xi^*)$ is nonsingular so that Condition 1 holds. Hence, according to Proposition 3, optimization algorithm (79) satisfies Assumption 8.

To verify the last assumption of Theorem 2 regarding recursive least-squares (*EA4*) as estimator, Condition 4 should be hold for ϕ in (75). In the ABS problem, Condition 4 holds using the dither signal $d(t) = 0.013\sin(3t)$ in $\lambda_0 = \xi + d(t)$. Hence, based on Proposition 10 and Remark 17, Assumption 7 holds.

Therefore, all assumptions of Theorem 2 hold and it can be concluded that by tuning the parameters ϵ_1 and ϵ_2 in the extremum seeker, practical asymptotical stability of the closed loop system w.r.t the given bounded region can be achieved.

For simulation purposes, the parameter values used are $m = 400kg$, $B = 0.01$, and $R = 0.3m$. The initial conditions are $v(0) = 15m/s$ and $\omega(0) = 111.11$, which makes $\lambda(0) = 0$, $\hat{\lambda}_0(0) = 0.1$, and $M(0) = 5I_{5\times5}$. The unknown parameters are initialized at $\hat{\theta}(0) = [3.16, 3.3, 2.64, 1.05, 0.01]$, identical to those in [31]. It is assumed that the braking starts on a dry asphalt road and after $5m$ the road becomes icy. The ES schemes estimate values of θ in different road conditions and

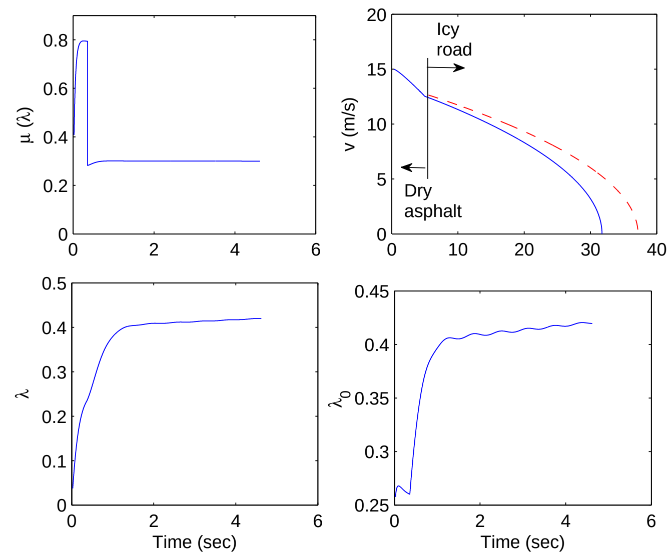

Fig. 5. ABS design using extremum seeking framework. Dashed-line shows the case that extremum seeking scheme is not utilized and λ^* of dry asphalt road in used for the whole road.

then drive λ_0 to its optimal value so that friction coefficient converges to its maximum point at each road condition.

The simulation results in Fig. 5 (solid curves) show that during braking, maximum value of friction coefficient on dry asphalt road ($\mu^* = 0.8$) and icy road ($\mu^* = 0.3$) are reached and the vehicle stopped within the shortest distance ($31.39m$) which is highlighted in Table I. The dashed-curve in Fig. 5 shows the case that λ^* of dry asphalt road in used for the whole road. In this case, the vehicle stops after $37.2m$.

TABLE I
SIMULATED STOPPING DISTANCE UNDER ABS FOR DIFFERENT COMBINATIONS OF OPTIMIZATION AND ESTIMATION ALGORITHMS

	OA1	OA2	OA3	OA4	OA5	OA6
EA1	34.56m	34.40m	34.41m	34.39m	34.43m	34.40m
EA2	34.18m	33.45m	33.30m	33.48m	33.32m	33.45m
EA3	32.19m	31.50m	31.41m	31.52m	31.45m	31.50m
EA4	33.43m	31.46m	***31.39m***	31.43m	31.41m	31.46m

Note that the solution of this problem is not restricted to the above estimator (*EA4*) and optimization algorithm (*OA3*) and any type of methods explained in the previous section can be employed here provided that the required conditions hold. Table I shows that some combinations of optimization algorithms (OA) and estimation algorithms (EA) from the previous section lead to better closed-loop behavior under some conditions, but all are handled by the proposed framework.

B. A class of unstable systems with output nonlinearities

Here, an application of the last main result (Theorem 3) is presented for a class of systems which includes the closed loop system in Model Reference Adaptive Control (MRAC) of linear systems, together with an optimization scheme. The reason for choosing MRAC scheme as an example for application of Theorem 3 is that the plant and the estimator are in the same time scale and a controller also has to be designed to stabilize the nonlinear system. The objective of this example is to manipulate the input of the reference model dynamics in order to achieve the extremum of the performance function appended to the nonlinear system dynamics.

Consider the problem of MRAC with feedback linearizable, linearly parameterized nonlinear system:

$$\dot{x} = f(x) + \sum_{i=1}^{p} q_i(x)\theta_i + g(x)u \tag{80}$$

$$y = Q(\theta, x) \tag{81}$$

where the state $x \in \mathcal{B}_x \subset \mathbb{R}^n$, the performance output $y \in \mathbb{R}$, the input $u \in \mathcal{B}_\xi$, the unknown parameters $\theta \in \mathcal{B}_\theta \subset \mathbb{R}^p$ and $f, g : \mathcal{B}_x \to \mathbb{R}^n$ are smooth functions on $\mathcal{B}_x$.

Since the above system is feedback linearizable, there exists a dipheomorphism T such that the change of variable $z = T(x)$ and a state feedback controller $u = k(x) + \beta(x)v$ transforms (80) and (81) into the form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= \phi_n(z)^T\theta + v \end{aligned} \tag{82}$$

$$y = \bar{Q}(\theta, z) = Q(\theta, T^{-1}(z)) \tag{83}$$

with $\phi_n(\cdot)$ locally Lipschitz. Furthermore, there exists a continuous nondecreasing function $\rho(\cdot)$ such that

$$max\left\{\|\phi(s)\|, \left\|\frac{\partial \phi_n}{\partial s}\right\|\right\} \leq \rho(s) \tag{84}$$

for almost all s. Then (82) can be rewritten in the following compact form

$$\dot{z} = \bar{A}z + b[\phi_n(z)^T\theta + v] \tag{85}$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Now, consider the following reference model for (85) which is described in the controllable canonical form

$$\dot{z}_r = A_r z_r + br \tag{86}$$

where r is an input to closed loop system (85) and A_r is Hurwitz hence there exists $P = P^T > 0$ such that

$$PA_r + A_r^T P = -L, \qquad L = L^T > 0. \tag{87}$$

Assuming that full state z is available for feedback, an adaptive state feedback controller can be designed as follows. Let $e := z - z_r$ and rewrite system (82) as

$$\dot{e} = A_r e + b[\phi_n(z)^T\theta + v - b^T A_r z - r] \tag{88}$$

Next, consider the Lyapunov function candidate

$$V = e^T P e + ||\tilde{\theta}||^2 \quad (89)$$

where $\tilde{\theta} = \hat{\theta} - \theta$, and $\hat{\theta}$ is an estimate of θ to be determined by the parameter adaptive rule. The time derivative of V along the trajectories of (88) is given by

$$\dot{V} = -e^T L e + 2[\phi_n(z)^T \theta + v - b^T A_r z - r]^T b^T P e + 2\tilde{\theta}^T \dot{\tilde{\theta}}$$

Taking

$$v := b^T A_r z - \phi_n(z)^T \hat{\theta} + r \quad (90)$$

the expression for $\dot{V}$ can be rewritten as

$$\dot{V} = -e^T L e + 2\tilde{\theta}^T [\phi_n(z) b^T P e - \dot{\tilde{\theta}}] \quad (91)$$

Therefore, the adaptation rule should be chosen as

$$\dot{\hat{\theta}} = \dot{\tilde{\theta}} = -\phi_n(z) b^T P e \quad (92)$$

to ensure that $\dot{V} \leq 0$. Thus, the system dynamics will be

$$\begin{aligned} \dot{e} &= A_r e + \Phi(t,e)^T \tilde{\theta} && (93) \\ \dot{\tilde{\theta}} &= -\Phi(t,e) P e && (94) \\ \dot{z}_r &= A_r z_r + b r && (95) \\ y &= \bar{Q}(\theta, e + z_r) && (96) \end{aligned}$$

where $\Phi(t,e) := \phi_n(e + z_r) b^T$.

Definition 2: [18] A function $\Phi(\cdot,\cdot)$ is said to be uniformly δ-persistently exciting (uδ-PE) with respect to x_1 if for each $x \in \mathcal{D}_1$ there exist $\delta > 0$, $T > 0$ and $\mu > 0$ s.t. $\forall t \in \mathbb{R}$,

$$|z - x| \leq \delta \Rightarrow \int_t^{t+t} |\Phi(\tau, z)| d\tau \geq \mu,$$

where $x := col[x_1, x_2] \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ and $\mathcal{D}_1 := (\mathbb{R}^{n_1} \setminus \{0\}) \times \mathbb{R}^{n_2}$.

Condition 5: $\Phi(t,e)$ is uδ-PE with respect to $\tilde{\theta}$.

By introducing $X = (e, \tilde{\theta}, z_r)$, equations (93)-(96) can be rewritten in the following compact form

$$\dot{X} = N(t, \theta, X, r) \quad (97)$$

which is in of the form (40). Therefore, in the following, the Assumptions 9-11 of Theorem 3 are verified for (97).

The equilibrium of (97) is given as:

$$N(t, \theta, X, r) = 0 \;\Leftrightarrow\; X = \begin{pmatrix} 0 \\ 0 \\ -A_r^{-1} b r \end{pmatrix} \quad (98)$$

Thus, the steady-state map between the input and the output is

$$y = h(\theta, r) := \bar{Q}(\theta, A_r^{-1} b r) \quad (99)$$

which is assumed to be continuously differentiable with an extremum at $r = \xi^*$. Hence, Assumption 9 holds.

Defining the new coordinates $\widetilde{X} = (e, \tilde{\theta}, \tilde{z}_r)$ where $\tilde{z}_r = z_r + A_r^{-1} b r$ gives

$$\begin{aligned} \dot{\widetilde{X}} &= \begin{pmatrix} A_r e + \Phi(t,e)^T \tilde{\theta} \\ -\Phi(t,e) P e \\ A_r \tilde{z}_r + A_r^{-1} b \dot{r} \end{pmatrix} && (100) \\ y &= \bar{Q}(\theta, e + \tilde{z}_r - A_r^{-1} b r) && (101) \end{aligned}$$

Thus, the boundary layer (fast) system is:

$$\dot{\widetilde{X}} = \begin{pmatrix} A_r e + \Phi(t,e)^T \tilde{\theta} \\ -\Phi(t,e) P e \\ A_r \tilde{z}_r \end{pmatrix}. \quad (102)$$

In order to prove UAS of (102), consider the Lyapunov candidate (89) is augmented with $\tilde{z}_r^2$, i.e.:

$$\widetilde{V} = V + \tilde{z}_r^2.$$

The time derivative of $\tilde{V}$ evaluated along the trajectories of (102) yields

$$\dot{\widetilde{V}} = -(\tilde{z}_r + e)^T L (\tilde{z}_r + e).$$

Hence the system (102) is uniformly stable. Now, to show asymptotic stability of the system (102) an argument similar to the one used in Theorem 1 and Example 2 of [23] is used. Based on this theorem, uδ-PE w.r.t $\tilde{\theta}$ condition on $\Phi(t,e)$ guarantees uniform asymptotic stability of the fast system (102), so Assumption 10 holds.

Now, in order to locate the extremum of the performance function (83), consider an optimization algorithm satisfying appropriate conditions from Section V-A.

Since all of the assumptions of Theorem 3 hold, the full system (82)-(83) is practically asymptotically stable and converges to the extremum of the performance function (83).

Remark 19: Since verification of the uδ-PE condition in general is difficult, Proposition 2 in [23] can be used to relax this condition. Based on this proposition, if there exists bounded function $\bar{\phi}(t,r)$ satisfying PE condition (58), a unitary vector $\zeta \in \mathbb{R}^n$ and $c > 0$, such that

$$||\phi_n(z_r(t,r))^T \zeta|| \geq c ||\bar{\phi}(t,r)^T \zeta|| \quad (103)$$

then $\Phi(t,e)$ uδ-PE w.r.t $\tilde{\theta}$.

Remark 20: The result above is similar to the results presented in [10] and [4]. It yields a generalization of the results in these papers, allowing different types of optimization schemes to achieve extremum of the performance function.

Example 2: Consider the following nonlinear, linearly parameterized system which is similar to the problem addressed in [10]:

$$\begin{aligned} \dot{x}_1 &= x_2 - \theta_1 x_1 \\ \dot{x}_2 &= -\theta_2 x_1^2 + u && (104) \\ y &= 1 - \theta_1 x_1 - \theta_2 x_1^2 && (105) \end{aligned}$$

To show practical asymptotical stability of this system, the conditions mentioned above are verified.

Consider the following change of variables

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 - \theta_1 x_1 && (106) \end{aligned}$$

then z_1 and z_2 and output map satisfy

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\theta_2 z_1^2 - \theta_1 z_2 + u && (107) \\ y &= 1 - \theta_1 z_1 - \theta_2 z_1^2 && (108) \end{aligned}$$

which is in the form of (82)-(83) with $u = v$ and $\phi_n(z)^T = [-z_2 \;\; -z_1^2]$ which is continuously differentiable, hence it is locally Lipschitz. Note that the change of variables $z = T(x)$ is dipheomorphism since the map T is continuously differentiable and there exists continuously differentiable inverse map $T^{-1}(\cdot)$ such that $x = T^{-1}(z)$. The inverse map is as following

$$\begin{aligned} x_1 &= z_1, \\ x_2 &= z_2 + \theta_1 z_1. \end{aligned} \tag{109}$$

Next, consider the following reference model

$$\dot{z_r} = A_r z_r + br, \quad A_r = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{110}$$

where r is an input to the system and A_r is Hurwitz.

The controller (90) for this example is as follows

$$\begin{aligned} v &:= b^T A_r z - \phi_n(z)^T \hat{\theta} + r \\ &= -6z_1 - 5z_2 + \theta_1 z_2 + \theta_2 z_1^2 + r. \end{aligned} \tag{111}$$

Before verifying Assumption 9, the full description of the tracking error and parameter estimate dynamics are established. According to the analysis before this example, the tracking error dynamics (93) and the parameter estimate (94) for this example are

$$\begin{aligned} \dot{e} &= A_r e + b\phi_n^T(e + z_r)\tilde{\theta} \\ \dot{\tilde{\theta}} &= -\phi_n(e + z_r) b^T P e \end{aligned} \begin{matrix} (112) \\ (113) \end{matrix}$$

where $\phi_n(e + z_r) = [-(e_2 + z_{r2}(t, r)), -(e_1 + z_{r1}(t, r))^2]$.

Then, by introducing $X := (e, \tilde{\theta}, z_r)$, the dynamics are given by

$$\dot{X} = N(t, \theta, X, r). \tag{114}$$

To obtain the equilibrium point, set $\dot{X} = 0$ which results in

$$e = 0, \quad \tilde{\theta} = 0, \quad z_r = [r/6 \quad 0]^T.$$

Since $e = 0$, it can be concluded that $z_1 = z_{r1} = r/6$ and $z_2 = z_{r2} = 0$. Thus, the steady-state output map

$$y = h(\theta, r) = 1 - \theta_1(r/6) - \theta_2(r/6)^2 \ , \tag{115}$$

has an extremum at $r = \xi^* = 3\theta_1/\theta_2$, so Assumption 9 holds.

Then, condition (103) is verified for

$$\phi_n(z_r(t, r))^T = [-z_{r2}(t, r) \quad - z_{r1}^2(t, r)]. \tag{116}$$

To this end, it should be shown that there exist a unitary vector ζ, $c > 0$ and persistently exciting function $\bar{\phi}^T$ which satisfy the inequality (103). First, the solution of (110) is obtained as the following, which is required in PE verification of $\phi_n(z_r)$

$$z_{r1} = te^{-(3t^2+5t)} + rt, \qquad z_{r2} = e^{-(3t^2+5t)} + r. \tag{117}$$

Next, $\bar{\phi}^T = [r\sin(t) \quad 0]$ is considered as a function that is persistently exciting. Therefore, with this persistently exciting $\bar{\phi}$ and the unitary vector $\zeta = [1 \ \ 0]^T$, the condition (103) holds as following:

$$||\phi_n(z_r)^T\zeta|| = |e^{-(3t^2+5t)} + r| \geq 0.5 \times |r\sin(t)| = c||\bar{\phi}^T\zeta||.$$

Finally, to verify Assumption 11, the Levenberg-Marquardt method (*OA5*) is used as the optimization scheme:

$$\begin{aligned} \dot{\xi} &= -\epsilon(J_r^T(\xi)J_r(\xi) + \delta I_n)^{-1} J_r^T(\xi)\nabla h(\xi) \\ &= -\epsilon\left(\frac{18\theta_2}{\theta_2^2 + 18\delta}\right) \times \left(\frac{\theta_1}{6} + \frac{\theta_2}{18}\xi\right). \end{aligned} \tag{118}$$

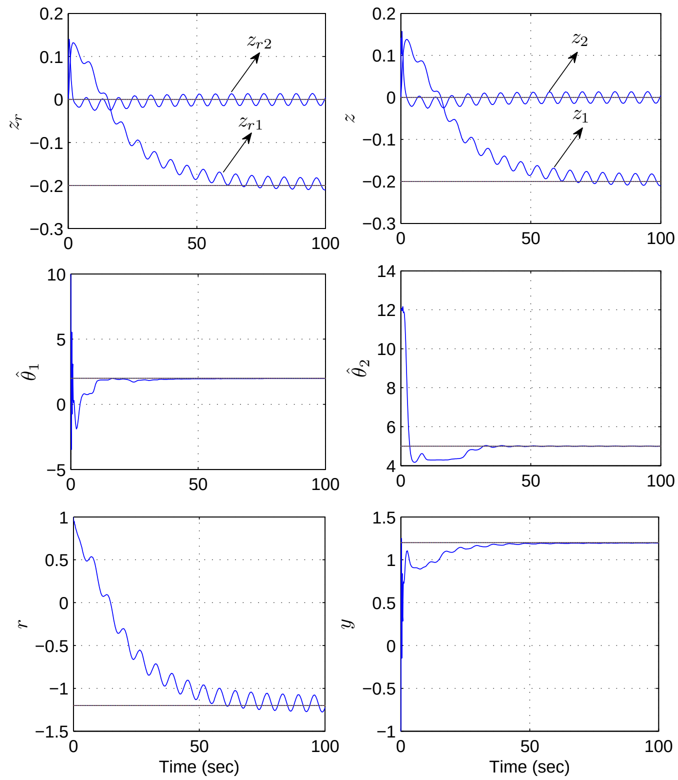

Fig. 6. Reference signal z_r, state z, parameters estimation $\hat{\theta}$, input r and output y

Since $J_r(\xi) = \frac{-\theta_2}{18} \neq 0$, Condition 3 holds and therefore based on Proposition 5, optimization algorithm (118) satisfies Assumption 11.

Since all the assumptions of Theorem 3 hold, it can be concluded that the system (107) is practically asymptotically stable and converges to the extremum of the performance function (108).

Now, the behavior of the extremum seeking controller of the system (107) with performance function (108) is illustrated in simulated conditions. The simulation results are shown in Fig. 6 with the initial conditions $z_{r1}(0) = z_{r2}(0) = z_1(0) = z_2(0) = 0$ and $\hat{\theta}(0) = [10, \ 14]^T$. The controller (111) regulates the state z to the reference state z_r in reference dynamics (110). In (110) the reference input r is produced by optimization scheme (118). As mentioned before, the reference input with a bounded dither signal $(r + \sin(t))$ is appended to provide some richness condition on it. This is necessary to guarantee the convergence of parameter estimate $\hat{\theta}$ to its true value $\theta = [2, \ 5]^T$. It is shown in Fig. 6 that the output converges to its maximum value $y^* = 1.2$ at reference input $r^* = -1.2$.

Fig. 6 shows that the parameter estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ converge faster than optimization signal r due to the time-scale separation with $\epsilon = 0.23$. By increasing the value of ϵ the optimization scheme can converge faster but at the expense of lower accuracy. The tuning parameter ϵ can even be set to 1 which removes any time-scale separation between two systems (plant-estimator and optimizer). However, in such a case, each of the two systems has to satisfy ISS and small gain conditions as discussed in Remark 13.

VII. CONCLUSIONS

A framework is proposed to design extremum seeking controllers for a class of uncertain plants which are parameterized with unknown parameters. This prescriptive framework provides an attractive way to combine a large class of optimization methods with a large class of parameter estimation schemes to achieve convergence of the closed loop trajectories to the desired extremum. The applications of the framework to ABS design and MRAC are used to illustrate the feasibility and flexibility of the proposed framework.

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