Nonlinear Sampled-Data Systems

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***Abstract*—Sampled-data systems are control systems in which the feedback law is digitally implemented via a computer. They are prevalent nowadays due to the numerous advantages they offer compared to analog control. Nonlinear sampled-data systems arise in this context when either the plant model or the controller are nonlinear. While their linear counterpart is now a mature area, nonlinear sampled-data systems are much harder to deal with and, hence, much less understood. Their inherent complexity leads to a variety of methods for their modeling, analysis and design. A summary of these methods is presented in this article.**

Keywords and phrases: nonlinear, sampled-data, sampler, zero-order-hold, discrete-time

I. Introduction

Definition: A control system in which a continuous-time plant is controlled by a digital computer is referred to as a *sampled-data control system* or simply a *sampled-data system* [4], see Figure 1. *Nonlinear sampled-data systems* arise when either the model of the plant or the controller are nonlinear; otherwise the system is referred to as a linear sampled-data system.

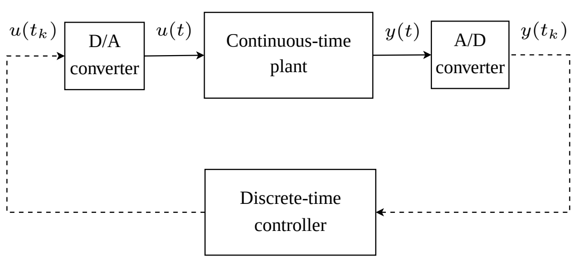

Fig. 1. Sampled-data (control) system

Motivation: Sampled-data control is preferable to continuous-time (analog) control for a range of reasons including reduced cost, reduced wiring, more robust hardware, easier and more flexible programming, and so on. Nowadays, a large majority of controllers are implemented on digital computers and, hence, sampled-data systems are prevalent in practice. On the other hand, nonlinear plant models are necessary in numerous applications when a wide range of operating conditions need to be considered or when truly nonlinear phenomena, such as friction or state/input constraints, are not negligible. Hence, there are many situations where nonlinear plant models are essential, such as vertical take-off and landing of an aircraft, robots, automotive engines, biochemical reactors, to name a few. It has to be noted that the nonlinearity may also come from the controller even when we consider linear plants as it is the case in adaptive control or model predictive control with constraints for example.

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Structure of sampled-data systems: Figure 1 presents a typical structure of a sampled-data system which consists of a continuous-time plant, an analog-to-digital (A/D) converter (i.e. a sampler), a digital-to-analog (D/A) converter (i.e. a hold device) and a discrete-time controller.

The A/D converter takes measurements $y(t_k)$ of a continuous-time output signal $y(t)$, such as temperature or pressure, at sampling time instants $t_k, k = 0, 1, \ldots$ and sends them to the control algorithm. The measurements are obtained with finite precision (i.e. they are quantized); this effect is not considered in this article. The sampling instants t_k are often equidistant, that is $t_k = kT, k = 0, 1, \ldots$, where the distance T between any two consecutive sampling instants is referred to as the *sampling period*. The sampling period is an important degree of freedom in the design of sampled-data systems and it needs to be carefully selected.

The control algorithm is discrete in nature. It takes the sequence of measurements $y(t_k)$ and processes them to produce a sequence of control values $u(t_k)$. The D/A converter converts the sequence of control values $u(t_k)$ into a continuous-time signal $u(t)$ that drives the actuators which control the plant. Typically, a zero-order-hold is used, i.e. $u(t) = u(t_k), \forall t \in [t_k, t_{k+1})$. However, it is possible to use other types of holds.

Note that the system in Figure 1 can be generalized in many ways. An important generalization is *multi-rate sampling* where the output of the system is sampled at one sampling rate while the control inputs are updated at a different sampling rate. Another generalization are *networked control systems* which are discussed in the last section.

II. Modeling

The combination of continuous-time and discrete-time components render the analysis and the design of sampled-data systems challenging. Still, linear systems allow for computationally efficient analysis and design techniques that benefit from the z and δ transforms, as well as convex optimization [4]. Nonlinear sampled-data systems, one the other hand, are much harder to deal with since the aforementioned methods do not apply in this case. This inherent difficulty has lead to a variety of models for different analysis and design methods:

1) Continuous-time models;
2) Discrete-time models;
3) Sampled-data models.

We discuss bellow each of these models, their features and the analysis or design methods that exploit them.

Continuous-time models basically ignore the sampling process and assume that all signals are continuous-time. They are the coarsest approximation of the sampled-data system and they are useful only for very small sampling periods. Nevertheless, they are invaluable and are used as a first step in the controller/observer design in the so called *emulation* design approach.

Discrete-time models only capture the behaviour of the sampled-data system at sampling instants. Indeed, they ignore the inter-sample behaviour of the system and this is their main drawback. There are two ways in which nonlinear discrete-time models arise: (i) from the identification of the plant model using the sampled measurements; (ii) from the discretization of a known continuous-time plant model. For instance, black-box identification methods often lead to nonlinear discrete-time models in input-output form, such as NARMA (nonlinear auto regressive moving average) models [3], [8]. Depending on the approximating functions used, the nonlinearities can be polynomial, neural network type, fuzzy type, and so on. On the other hand, the discretization of the continuous-time plant model requires an exact analytic solution of a set of nonlinear differential equations. When such an analytic solution exists, we can obtain the exact discrete-time models of the system; this is typically assumed for linear plants. Nonlinear sampled-data systems are different from their linear counterparts in that it is typically impossible to obtain the exact discrete-time model and only approximate discrete-time models are available for analysis and design [13], [14].

Sampled-data models capture the true behaviour of the sampled-data system including its inter-sample behaviour. There are several ways in which this can be achieved. One way is to model the piecewise constant signals that arise from zero-order-hold devices as signals with a time-varying delay; this gives rise to time-delay nonlinear models [16]. Another recently proposed approach is to model nonlinear sampled-data systems as hybrid dynamical systems [5]. An extensive analysis and design toolbox has been developed for hybrid dynamical systems and these results can be used for nonlinear sampled-data systems. Another class of models, based on the so called lifting, has been applied for linear systems where the system is represented as a discrete-time system with infinite dimensional input and output spaces. While this approach has been very successful in the linear context [4], it appears that it is not as useful for nonlinear systems due to difficulties arising from harder analysis and prohibitive computational requirements.

III. The Main Issues and Analysis

Controllability/observability: Issues arising due to sampling in linear systems transfer to the nonlinear context although they are less understood in this case. For instance, it is well known that sampling may 'destroy' the controllability and/or observability properties of the system [4]. In other words, if the continuous-time plant model is controllable/observable, then the corresponding exact discrete-time model of the plant may not verify these properties for some sampling periods. A simple test is available for linear systems to avoid this phenomenon but we are not aware of similar results in the nonlinear context.

Finite escape times: A major difference between continuous-time linear and nonlinear systems is that the former have well defined solutions for constant control inputs and arbitrarily long sampling periods. This is not the case, in general, for nonlinear systems as they may exhibit finite escape times. In other words, for a constant input it may happen for some initial conditions of a nonlinear system that solutions blow up within a time that is shorter than the sampling period. As a consequence, for such an initial condition and input the exact discrete-time system cannot be defined. This is a fundamental obstacle to achieving global stability results for nonlinear systems if the sampling period is fixed and independent of the size of the initial state. Nevertheless, it is possible to ensure semi-global stability properties for very general nonlinear systems which means that any compact domain of convergence can be achieved if the sampling period is sufficiently reduced [14].

Model structure is changed: An important issue for nonlinear sampled-data systems is that the sampling modifies the structure of the model. When the continuous-time plant model has a certain structure, such as triangular or affine in the input, the corresponding exact discrete-time model will not inherit it, see [12], [17]. This significantly complicates the design of sampled-data systems via the discrete-time approach since many nonlinear design techniques, like backstepping or forwarding, are heavily reliant on the structure of the model.

Zero dynamics: Probably the most significant aspect of the changed structure are the so called 'sampling zeros'. In linear systems, it is well known that if a continuous-time linear system of relative degree $r \geq 2$ is sampled, then generically for fast sampling the discrete-time models of the plant will have relative degree $r = 1$. In other words, sampling introduces extra zeros in the model which are often unstable and thus render the system non minimum phase. It is well known that the controller design is much harder for non minimum phase systems and, moreover, there are certain fundamental performance limitations in this case. Recently, results that extend the notion of sampling zeros to the nonlinear sampled-data systems have been reported, see the references in [12].

Passivity: Some plant properties like passivity are much more restrictive in discrete-time than in continuous-time. Indeed, it is necessary for a continuous-time plant to have relative degree 1 or 0 to be passive whereas only relative degree 0 discrete-time plants may possess this property. In other words, an exact discrete-time model of a passive continuous-time plant of relative degree 1 will not be passive; that is, sampling typically destroys passivity.

IV. Controller Design

Linearization: The simplest way to design sampled-data nonlinear systems is to linearize the plant at a given operating point. In this case, the nonlinear plant dynamics are approximated by a linear model around a chosen equilibrium and then any of the linear sampled-data techniques can be applied to the linearized model. The obtained solution is then implemented on the true nonlinear plant. The drawback of this technique is that the solution would typically perform well only in the vicinity of the selected equilibrium point.

Nonlinear methods: An alternative is to perform designs that rely on a nonlinear plant model. These approaches can be divided into: feedback linearization, emulation design method, (approximate and exact) discrete-time design method and sampled-data design method.

Feedback linearization: Some classical problems, like feedback linearization, are harder for sampled-data systems than continuous-time ones. It was shown that a class of discrete-time nonlinear systems for which feedback linearization is possible is smaller than the corresponding class of continuous-time systems [6]. This has lead to approximate feedback linearization techniques which consider achieving feedback linearization approximately with an error that can be reduced by reducing the length of the sampling period [2].

Continuous-time design method (Emulation design): Emulation is a design technique consisting of two steps. In the first step, a continuous-time controller or observer is designed for the continuous-time plant while ignoring sampling to achieve appropriate stability, performance and/or robustness guarantees. In the second step, the designed controller/observer is discretized for implementation and the sampling period is reduced sufficiently for the method to work. This method is approximate since the continuous-time plant model approximates well the sampled-data systems only for sufficiently small sampling periods. The discretization can be done using various implicit or explicit Runge-Kutta methods, such as the forward or backward Euler method [12], [17]. The emulation method is probably the best understood of all design methods. It was shown that a range of stability properties that can be cast in terms of dissipation inequalities are preserved in an appropriate sense under the emulation approach [11]. Moreover, non-conservative estimates of the upper bound for the required sampling period in emulation have been reported recently [15].

Exact discrete-time design method: Exact discrete-time design method assumes that an exact discrete-time model of the plant is available to the designer, see [10] and the references cited therein. This approach is reasonable when black box identification techniques are used for modeling. Moreover, in some rare cases it is possible to obtain the exact discrete-time model of the plant by integrating the continuous-time model with fixed inputs (assuming the zero order hold is used). This is the case when the plant dynamics are linear while the control law is nonlinear (e.g. adaptive control) or the plant is linear with state/input constraints, which is a set-up often used in the model predictive control. The literature on exact discrete-time design method is vast and many of the nonlinear continuous-time design techniques, like backtepping, forwarding and passivity based designs, are extended to discrete-time nonlinear systems, see [10], [6]. A drawback of these methods is that they assume a special structure of the discrete-time nonlinear model, such as upper or lower triangular structure, which is typically much more restrictive in discrete-time than in continuous-time due to the loss of structure due to sampling that was discussed earlier.

Approximate discrete-time design method: Due to the nonlinearity, it is impossible in most cases to obtain an exact discrete-time plant model by integrating its continuous-time model equations; instead, a range of approximate discrete-time plant models, such as Runge-Kutta, can be used for controller/observer design. It was recently shown that this design method may lead to disastrous consequences where the controller stabilizes the approximate discrete-time plant model for all (arbitrarily small) sampling periods, while the same controller destabilizes the exact discrete-time plant model for all sampling periods, see [14], [13]. This is true even for linear systems and some commonly used discretization techniques and controller designs. These considerations have lead to the development of a framework for controller design based on approximate discrete-time models [13], [14]. This framework provides checkable conditions on the continuous-time plant model, the approximate discrete-time model and the controller that guarantee that the controllers designed in this manner would stabilize the exact discrete-time model and, hence, the nonlinear sampled-data system for sufficiently small sampling periods. The design is based on families of approximate discrete-time models parameterized with the sampling period and the design objectives are more demanding than in the continuous-time nonlinear systems. Ideas from numerical analysis are adapted to this context. This framework was used to design controllers and observers for classes of nonlinear sampled-data systems where typically Euler approximate discretization is employed to generate the approximate discrete-time model.

Sampled-data design method: Both emulation and discrete-time design methods have their drawbacks. Indeed, the former method ignores the sampling at the design stage, whereas the latter method ignores and may produce unacceptable inter-sampling behaviour. Thus, methods that use a sampled-data model of the plant for design are much more attractive. There are two possible ways in which this can be achieved for nonlinear sampled-data systems.

The first approach consists of representing nonlinear sampled-data systems as systems with time-varying delays [16]. However, controller design tools for such systems need to be further developed.

The second approach involves representing the nonlinear sampled-data system as a hybrid dynamical system. Recent advances on modeling and analysis of hybrid dynamical systems [5] offer great opportunities in this context but the full potential of this approach is still to be exploited. Nonlinear sampled-data systems are just a small subclass of hybrid dynamical systems and developing specific analysis and design tools tailored to this class of systems seems promising.

It should be emphasized that there are many related techniques, such as discrete-time adaptive control and model predictive control, that deal with classes of nonlinear sampled-data systems but are not a part of the main stream nonlinear sampled-data literature.

V. Summary and Future Directions

Summary: Sampled-data control systems are nowadays prevalent and there are many situations where nonlinear models need to be used to deal with wider ranges of operating conditions, more restrictive constraints and enhanced performance specifications. Despite their increasing importance, the design of nonlinear sampled-data systems remains largely unexplored and it is much less developed than its continuous-time counterpart. A variety of models, analysis and design techniques makes nonlinear sampled-data literature very diverse and a comprehensive textbook reference or a unifying approach is still missing. Many open questions remain for nonlinear sampled-data systems, such as results on multi-rate sampling, design techniques based on sampled-data models and other generalizations which are discussed below.

Future directions: In the 1990s, a new generation of digitally controlled systems has evolved from the more classical sampled-data systems which are generally referred to as networked control systems (NCS), see [7] and the references cited therein. These systems exploit digital wired or wireless communication networks within the control loops. Such a set up is introduced to reduce the cost, weight and volume of the engineered systems but its special structure imposes new challenges due to the communication constraints, data packet dropouts, quantization of data, varying sampling periods, time delays etc. At the same time, these systems provide new flexibilities due to the distributed computation within the control system that can be used to improve the performance and mitigate some of the undesirable network effects on the overall system performance. Moreover, embedded microprocessors allow for event-triggered and self-triggered sampling [1] that are still largely unexplored especially for nonlinear systems. Design of NCS was identified as one of the biggest challenges to the control research community in the 21st century and more than a decade of intense research on this topic still has not provided a comprehensive and unifying approach for their analysis and design. Novel results on modeling and Lyapunov stability theory for (nonlinear) hybrid dynamical systems appear to offer the right analysis design tools but they are still to be converted into efficient and easy to use design tools in the control engineers' toolbox.

VI. Recommended Reading:

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