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# A TRAJECTORY BASED APPROACH FOR ROBUST STABILITY PROPERTIES OF INFINITE-DIMENSIONAL SYSTEMS

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Abstract: In this paper, we present a trajectory based approach to discuss the robustness of parameterized families of systems with disturbances with respect to arbitrary closed sets in a normed space. Input-to-state stability (ISS) properties are discussed. Our results are applicable to infinite-dimensional systems with disturbances.

Keywords: Robustness, Stability Properties, Infinite-Dimensional Systems.

## 1. INTRODUCTION

Robustness with respect to disturbances is one of the central issues in control. The notion of input-to-state stability provides a very useful framework for robust stability analysis, as well as the controller design, of nonlinear systems, which are governed by ordinary differential equations, see for instance (Sontag and Wang, 1995; Christogists and Teel, 1996; Nešić *et al.*, 1999; Nešić and Teel, 2001; Teel *et al.*, 2003; Sontag, 2005).

In practice, many systems are infinite-dimensional as they are governed by partial differential equations or time delay equations (Curtain and Zwart, 1995; Christofides, 2002; Chen *et al.*, 2003). In order to achieve a practical theory for control design and analysis for a class of infinite-dimensional systems, a natural way is to generalize the results/concepts from classes of finite-dimensional systems. For example, $\mathcal{H}_\infty$ techniques have been extended from finite-dimensional systems to infinite-dimensional systems (Curtain and Zwart, 1995). This paper extends the results in (Moreau *et al.*, 2001) to a more general setting: both the solutions of the system and disturbances are in normed spaces, which may be infinite-dimensional. In (Moreau *et al.*, 2001), it was shown that when an auxiliary system is $(\beta, \gamma)$-ISS, the solutions of parameterized family of systems is *semiglobal practical* $(\beta, \gamma)$-ISS if these solutions can be made arbitrarily close to the solutions of the auxiliary system. In this paper, we first show that this result is also applicable to infinite-dimensional systems. Such an extension, though straightforward, provides a unified and general framework for investigating robust stability properties of parameterized family of infinite-dimensional/finite-dimensional systems via their simpler auxiliary systems and thus is applicable to a range of different situations. There are many possible applications, due to space limitation, we just list several of them.

(1) Obviously, this framework can be directly applied to robustness analysis of finite-dimensional systems. For example, robustness of finite-dimensional systems via singular perturbation and averaging theories discussed in (Teel *et al.*, 2003) is a special case of this framework.

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(2) This framework can be used to analyze robustness of infinite-dimensional systems with respect to disturbances when averaging and singular perturbation theories are employed in infinite-dimensional space (Hale and Lunel, 1990; Bentsman and Hong, 1993; Lehman *et al.*, 1994; Christogists and Daoutidis, 1998*a*; Christogists and Daoutidis, 1998*b*). In this paper, an illustrative example is provided to show how to use this framework to analyze the stability properties of the moving average along the a trajectory of a nonlinear parabolic system with Neumann boundary conditions from its finite-dimensional average (Bentsman and Hong, 1993).
(3) It will also provide a useful tool to investigate the stability properties of sampled-data infinite-dimensional systems from a much simpler finite-dimensional approximation.

Our second result focuses on a special ISS property: input-to-state exponential stability (ISES) with a linear/nonlinear disturbance gain. Exponential stability properties are most frequently used in the control of infinite-dimensional systems (Curtain and Zwart, 1995; Oostveen, 1999; Logemann *et al.*, 2003). It is always desirable that the solutions of parameterized family of systems are ISES. Our second result states that when an auxiliary system is ISES and a new "strong closeness of solutions" condition holds, the solutions of parameterized family of systems are also ISES, *instead of practically ISES*.

This paper is organized as follows. Preliminaries are presented in Section 2. Main results are stated in Section 3 followed by an example in Section 4.

## 2. PRELIMINARIES

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ if it is zero at zero, continuous and nondecreasing. It is of class $\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if for each fixed value of $t \geq 0$ the function $\beta(\cdot, t)$ is a class-$\mathcal{K}$ function and for each fixed value of $s > 0$ it is decreasing to zero. $X$ and $W$ are normed spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_W$ respectively. For any $z \in \mathbb{R}^p$, $|z|$ is the Euclidean norm of $z$. Given a measurable function $w : \mathbb{R}_{\geq 0} \to \mathcal{F} \subset W$, we define its infinity norm $\|w\|_\infty := \operatorname{ess\,sup}_{t\geq 0} \|w(t)\|_W$. If we have $\|w\|_\infty < \infty$, then we write $w \in \mathcal{L}_\infty$. Given an arbitrary nonempty closed subset of $X$, i.e., $\mathcal{A} \subset X$, we define the distance of a point $x \in X$ to the subset $\mathcal{A}$ as $\|x\|_\mathcal{A} := \inf_{s\in\mathcal{A}} \|x - s\|_X$.

## 3. MAIN RESULTS

### *3.1 Semiglobal practical input-to-state stability*

In this part, semiglobal practical input-to-state stability is first discussed. We consider two systems that are defined in a normed space $X$: one is parameterized by a set of parameters while the other is generated by a simple auxiliary system that is independent of the parameters.

For the sake of generality, the results in this section are stated for general flows parameterized by the initial state $x_\circ \in X$, disturbance $w(t) \in \mathcal{F}, \forall t$ and a true parameter $\theta \in \Theta$ (a subset of $\mathbb{R}^p$ such that the origin either belongs to the boundary of the set or to the set itself). Such a flow, which is a subset of functions defined on intervals of the form $[0, T)$ where $T \in [0, \infty]$ ($T$ does not need to be the same for each element of the flow) taking values in $X$, is denoted $\mathcal{S}(x_\circ, w(\cdot), \theta)$. We use the following definitions:

*Definition 1.* A parameterized set $\mathcal{S}$ of functions defined on intervals of the form $[0, T)$ with $T \in [0, \infty]$ is said to be a *flow* if for all $x_\circ \in X$, the following holds

1. $x(\cdot) \in \mathcal{S}(x_\circ, w(\cdot), \theta)$ implies $x(0) = x_\circ$;
2. $x(\cdot) \in \mathcal{S}(x_\circ, w(\cdot), \theta)$ and $t_1 \in \text{domain}(x(\cdot))$ imply $x(t_1 + \cdot) \in \mathcal{S}(x(t_1), w(t_1 + \cdot), \theta)$. ◦

*Definition 2.* Let $\mathcal{A} \subset X$ be a nonempty closed set, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$. The flow $\mathcal{S}$ is said to be *semiglobally practically* $(\beta, \gamma)$*-ISS with respect to* $\mathcal{A}$ if for each pair of strictly positive real numbers $(\delta, r)$, there exists a strictly positive real number $\theta^*$ such that for all $\theta \in \Theta$ with $|\theta| < \theta^*$, for all $\|w\|_\infty \leq r$ and $x_\circ \in X$ with $\|x_\circ\|_\mathcal{A} \leq r$, and for all $t \in [0, \infty)$ each element $x(\cdot)$ of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ exists and satisfies $\|x(t)\|_\mathcal{A} \leq \max\{\beta(\|x_\circ\|_\mathcal{A}, t), \gamma(\|w\|_\infty)\} + \delta$. ◦

*Definition 3.* Let $\mathcal{A} \subset X$ be a nonempty closed subset, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$. The flow $\mathcal{S}$ is said to be *semiglobally practically* $(\beta, \gamma)$*-ISS with respect to* $\mathcal{A}$ *on finite time intervals* if for each triple of strictly positive real numbers $(T, \delta, r)$, there exists a strictly positive real number $\theta^*$ such that for all $\theta \in \Theta$ with $|\theta| < \theta^*$, for all $\|w\|_\infty \leq r$, and $x_\circ \in X$ with $\|x_\circ\|_\mathcal{A} \leq r$, and for all $t \in [0, T]$ each element $x(\cdot)$ of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ exists and satisfies $\|x(t)\|_\mathcal{A} \leq \max\{\beta(\|x_\circ\|_\mathcal{A}, t), \gamma(\|w\|_\infty)\} + \delta$. ◦

With the above definitions, the following proposition indicates that semiglobal practical ISS with disturbance gain $\gamma$ on compact time intervals implies semiglobal practical ISS with disturbance gain $\gamma$ on the semi-infinite interval.

*Proposition 1.* The flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ is semiglobally practically $(\beta, \gamma)$-ISS with respect to $\mathcal{A}$ on finite time intervals if and only if it is semiglobally practically $(\beta, \gamma)$-ISS with respect to $\mathcal{A}$. ◦

**Proof:** The proof is identical to the proof showed in (Moreau *et al.*, 2001, Theorem 1) and it is omitted. ■

Our first result states that if the elements of the parameterized flow can be made arbitrarily *weakly* close (see, Definition 5) on arbitrary compact time intervals to the elements of another flow that is $(\beta,\gamma)$-ISS then the parameterized flow is semiglobally practically $(\beta,\gamma)$-ISS. The utility of this result is that the weak closeness of solutions result may be evident from the regularity hypotheses of the system generating the flow and the ISS property for the auxiliary flow may be easy to check while the (semiglobal practical) ISS property for the parameterized flow may be difficult to check directly.

The auxiliary flow is denoted by $\mathcal{S}_2(y_\circ, w(\cdot))$, where $y_\circ \in X$ and $\|w\|_\infty \leq r$. We will assume that this flow has the following property:

*Definition 4.* Let $\mathcal{A} \subset X$ be a nonempty closed subset, let $\beta \in \mathcal{KL}$ and let $\gamma \in \mathcal{G}$. The flow $\mathcal{S}_2$ is said to be $(\beta,\gamma)$-*ISS with respect to* $\mathcal{A}$ if, all elements $y(\cdot)$ of the flow are defined on $[0,\infty)$ and satisfy the bound

$$\|y(t)\|_{\mathcal{A}} \leq \max\left\{\beta(\|y_\circ\|_{\mathcal{A}}, t), \gamma(\|w\|_\infty)\right\}, \forall t \geq 0,$$

for all $y_\circ \in X$ and $\|w\|_\infty \leq r$. ◦

*Definition 5.* We say that the solutions of the flows $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ and $\mathcal{S}_2(y_\circ, w(\cdot))$ can be made arbitrarily *weakly* close on compact time intervals if the following holds. For each triple $(T, \delta, r)$ of strictly positive real numbers, there exists a strictly positive real number $\theta^*$ such that, for each $\theta \in \Theta$ with $|\theta| \leq \theta^*$, $x_\circ \in X$ with $\|x_\circ\|_{\mathcal{A}} \leq r$, $w \in \mathcal{L}_\infty(r)$, each element $x(\cdot)$ of the flow $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ is defined for all $t \geq 0$ and there exists an $y_\circ \in X$ and an element of $y(\cdot)$ of the flow $\mathcal{S}_2(y_\circ, w(\cdot))$ such that

$$\|y(s) - x(s)\|_X \leq \delta, \tag{1}$$

for all $s \in [0, t]$. ◦

*Theorem 1.* Let $\mathcal{A} \subset X$ be a nonempty closed set. Assume that the following conditions hold:
1. The flow $\mathcal{S}_2(y_\circ, w(\cdot))$ is $(\beta,\gamma)$-ISS with respect to $\mathcal{A}$;
2. The solutions of the flows $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ and $\mathcal{S}_2(y_\circ, w(\cdot))$ can be made arbitrarily *weakly* close on compact time intervals.
Then the flow $\mathcal{S}_1$ is semiglobally practically $(\beta,\gamma)$-ISS with respect to $\mathcal{A}$. ◦

**Proof:** The proof is identical to the proof showed in (Moreau *et al.*, 2001, Corollary 1) and it is omitted. ■

*Remark 1.* Theorem 1 is an extension of (Moreau *et al.*, 2001, Corollary 1) since $X$ and $W$ are arbitrary normed spaces as opposed to Euclidean finite dimensional spaces considered in (Moreau *et al.*, 2001). Obviously, the result in (Moreau *et al.*, 2001, Corollary 1) is a special case of Theorem 1 and belongs to our framework. Such an extension provides a useful tool to analyze the robustness of the systems in normed spaces, including the infinite-dimensional spaces. Similar results appeared in literature, for example, in (Lehman *et al.*, 1994, Theorem 2.1), it was shown that the solutions of parameterized family systems governed by differential delay equations (infinite-dimensional systems disturbances free) are weakly close to their simpler average systems if parameters are tuned properly. It also shown that in such a situation, the parameterized family of systems and their averaged system have the same stability properties.

*Remark 2.* It is natural to extend "strong/weak averaging" techniques (Nešić and Teel, 2001), which discuss the robustness of averaging methods with respect to disturbances for finite-dimensional systems, to address the robustness of averaging methods in infinite-dimensional systems. Our framework will provide a useful tool under such circumstances.

*Remark 3.* Theorem 1 provides a sufficient condition, which is "weak closeness of solutions" on compact time intervals, to obtain the stability properties of the parameterized flow from stability properties of another flow. In practice, finite-dimensional systems are often used to approximate the behavior of the infinite-dimensional systems when the weak closeness of two solutions on compact time interval can be established. For example, by applying (Lax and Richtmyer, 1956, Lax Richtmyer Equivalence Theorem), the solution of the approximation model (a finite-dimensional system) can be made arbitrarily weakly close to the solution of an abstract differential equation (an infinite-dimensional system) on compact time intervals if the numerical approximation is stable [2] . Example 1 also demonstrates how to use our framework to show stability properties of parameterized family of infinite-dimensional systems from their auxiliary systems when the weak closeness of the solutions condition is given.

### *3.2 Input-to-state exponential stability*

Exponential stability is the most desirable type of stability in control of infinite-dimensional systems (Oostveen, 1999). The following definition is standard for systems described by abstract differential equations without disturbances.

*Definition 6.* The flow $\mathcal{S}(x_\circ)$ is said to be *exponentially stable* if there exist positive constants $K \geq 1$ and $\lambda > 0$ such that the following holds

$$\|x(s)\|_X \leq K \exp(-\lambda s) \|x_\circ\|_X ,$$

for all $s \geq 0$. ◦

---

[2] Stability of the numerical method in the numerical analysis literature means that the numerical approximation is uniformly bounded on a compact time interval.

In order to investigate the robustness of infinite-dimensional systems with respect to disturbances, our attention is focused on a special case of $(\beta, \gamma)$-ISS properties: input-to-state exponential stability, where the $\beta$ function takes the form of an exponentially decaying function. We will show in Theorem 2 that when the auxiliary system is ISES, under the condition of "strong closeness of solutions", the parameterized family of systems also have ISES properties. To the best of our knowledge this is the first result that shows parameterized family of systems can achieve ISES from ISES properties of the auxiliary systems under a condition of "strong closeness of solutions". Two corollaries (Corollary 1 and Corollary 2) are also stated for disturbance free systems.

For simplicity, we now consider the stability properties without considering the nonempty closed subset $\mathcal{A}$. The stability properties with respect to $\mathcal{A}$ can be extended easily.

*Definition 7.* Let $\gamma \in \mathcal{G}$. The flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ is said to be *ISES* with disturbance gain $\gamma$ if there exists a strictly positive real number $\theta^*$ such that for all $\theta \in \Theta$ with $|\theta| < \theta^*$, for all $w(t) \in \mathcal{F}$ and $x_\circ \in X$, and for all $t \in [0, \infty)$ each element $x(\cdot)$ of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ exists and there exist positive constants $K, \lambda$ such that the following holds

$$\|x(s)\|_X \leq \max\left\{K \exp(-\lambda s)\, \|x_\circ\|_X, \gamma\left(\|w\|_\infty\right)\right\}$$

for all $s \geq 0$. ◦

*Definition 8.* Let $\gamma \in \mathcal{G}$. We say that the solutions of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ are input-to-state bounded, with disturbance gain $\gamma$ on compact time intervals, uniformly in small $\theta$ if the following holds. For any $t > 0$ there exits $L > 0$ and $\theta^* > 0$ such that for $\theta \in \Theta, |\theta| \leq \theta^*$, for all $w(t) \in \mathcal{F}$ and $x_\circ \in X$, $x(s) \in \mathcal{S}_1(x_\circ, w(\cdot), \theta)$, we have

$$\|x(s)\|_X \leq \max\left\{L\, \|x_\circ\|_X, \gamma\left(\|w\|_\infty\right)\right\},$$

for all $s \in [0, t]$. ◦

An auxiliary flow $\mathcal{S}_2$, which starts from the same initial condition $x_\circ$ of $\mathcal{S}_1$, is denoted as $\mathcal{S}_2(x_\circ, w(\cdot))$, for all $w(t) \in \mathcal{F}$.

*Definition 9.* Let $\gamma \in \mathcal{G}$. We say that the solutions of the flows $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ and $\mathcal{S}_2(y_\circ, w(\cdot))$ can be made arbitrarily *strongly* close on compact time intervals with disturbance gain $\gamma$ if the following holds. For any $t \geq 0$ and any $\delta > 0$, there exist constant $\theta^* > 0$ such that for all $\theta \in \Theta$, $|\theta| \leq \theta^*$, for all $w(t) \in \mathcal{F}$ and $x_\circ \in X$, $x(s) \in \mathcal{S}_1(x_\circ, w(\cdot), \theta)$ and $y(s) \in \mathcal{S}_2(x_\circ, w(\cdot))$, we have that $\|y(s) - x(s)\|_X \leq \max\{\delta\, \|x_\circ\|_X, \gamma(\|w\|_\infty)\}$.

*Remark 4.* Let us compare "weak closeness solutions" (Definition 5) and "strong closeness solutions" (Definition 9). Note that for any disturbance $w \in \mathcal{L}_\infty(r)$ and any $x_o$, weak closeness indicates that the distance between the flow $\mathcal{S}_1$ and the flow $\mathcal{S}_2$ can be made arbitrarily small by choosing appropriate initial condition $y_o$ and the parameter $\theta$. Strong closeness implies that that the distance between two solutions ($\mathcal{S}_1$ and $\mathcal{S}_2$) starting from the same initial condition is a function of the initial condition and the infinity norm of the disturbance. This distance cannot made arbitrarily small by choosing an appropriate parameter $\theta$. In the definition of strong closeness of the solution, it is required that two flows have to start from the same initial condition. There may some cases that the flow $\mathcal{S}_2$ starts from a finite-dimensional subspace of $X$, while the initial value of the flow $\mathcal{S}_1$ is in the infinite-dimensional space $X$. Under such a situation, the strong closeness cannot be ensured. Moreover, when there is no disturbance $w$, strong closeness of solutions is a special case of weak closeness solutions. We will show in Theorem 2 that a stronger stability properties are obtained (ISES, instead of *practical* ISES) under the condition of strong closeness of solutions.

*Remark 5.* Arbitrary "closeness of solutions" (either strong or weak) on compact time intervals plays an important role in obtaining the stability properties of infinite-dimensional systems from their simpler auxiliary systems. As indicated in Remark 3, the solutions of numerical approximation (finite-dimensional) are always shown to be arbitrarily weakly/strongly close to the solutions of an abstract differential/infinite-dimensional equation. Therefore, our result will provide a very useful tool to design a controller/robust controller for classes of infinite-dimensional systems by their finite-dimensional approximations, as long as "closeness of solutions" is ensured.

The second main result is presented as follows.

*Theorem 2.* Suppose that the following conditions hold:

(1). The flow $\mathcal{S}_2(x_\circ, w(\cdot))$ is ISES with disturbance gain $\gamma$;

(2). The solutions of the flows $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ and $\mathcal{S}_2(x_\circ, w(\cdot))$ can be made arbitrarily *strongly* close on compact time intervals with disturbance gain $\gamma_2$.

(3). The solutions of the flow $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ are input-to-state bounded with disturbance gain $\gamma_1$ on compact time intervals, uniformly in small $\theta$;

Then, there exists positive constants $K_1$, $\lambda_1$ and $\gamma_3 \in \mathcal{G}$ such that the following holds.

$$\|x(s)\|_X \leq \max\left\{K_1 \exp(-\lambda_1 s)\, \|x_\circ\|_X, \gamma_3(\|w\|_\infty)\right\}$$

for all $s \geq 0$. ◦

*Proof*: Let $c \in (0,1)$ and $\delta \in (0,c)$ be arbitrary. Let $K$ and $\lambda$ come from the item (1) of the theorem. Let $\Delta t := \frac{1}{\lambda}\ln\left(\frac{K}{c-\delta}\right)$ and let this $\Delta t$ and the given $\delta$ determine $\theta_1^* > 0$ via the item (2) of the theorem. Let this $\Delta t$ generate $L$ and $\theta_2^*$ from the item (3) of the theorem and define $\theta^* = \min\{\theta_1^*, \theta_2^*\}$. Consider now an arbitrary $\theta \in \Theta, |\theta| \le \theta^*$, $x_\circ \in X$ and $x(s) \in \mathcal{S}_1(x_\circ, w(\cdot), \theta)$. Suppose that $x(\Delta t)$ is defined for all $\Delta t \ge 0$. Introduce now a sequence of times $\Delta t_i := i\Delta t$ where $i = 0, 1, 2, \dots$ and consider the solution $x(\Delta t_{i+1}, x_\circ)$. Because of the second property of the flow, we can write that $x(\Delta t_{i+1}) := x(\Delta t_{i+1}, x_\circ, w(\cdot)) = x(\Delta t, x(\Delta t_i), w(\cdot))$. At each $\Delta t_i$, we reinitialize the flow $S_2$ with $x(\Delta t_i)$.

Then, we can write for all $\Delta t_i$, $x(\Delta t_i) \in X$ and $x(s) \in \mathcal{S}_1(x(t_i), w(\cdot), \theta)$:

$$
\begin{aligned}
&\|x(\Delta t_{i+1})\|_X = \|x(\Delta t, x(\Delta t_i), w(\cdot))\|_X \\
&\le \|y(\Delta t, x(\Delta t_i), w(\cdot))\|_X \\
&\quad + \|y(\Delta t, x(\Delta t_i), w(\cdot)) - x(\Delta t, x(\Delta t_i), w(\cdot))\|_X \\
&\le K\exp(-\lambda\Delta t)\|x(\Delta t_i)\|_X + \gamma(\|w\|_\infty) \\
&\quad +\delta\|x(\Delta t_i)\|_X + \gamma_2(\|w\|_\infty) \\
&= (c-\delta)\|x(\Delta t_i)\|_X + \gamma(\|w\|_\infty) \\
&\quad +\delta\|x(\Delta t_i)\|_X + \gamma_2(\|w\|_\infty) \\
&= c\|x(\Delta t_i)\|_X + \tilde{\gamma}(\|w\|_\infty),
\end{aligned}
$$

where $\tilde{\gamma} = \gamma + \gamma_2$. Moreover, we can write that

$$
\begin{aligned}
\|x(\Delta t_i)\|_X &\le c^i\|x_\circ\|_X + \sum_{k=1}^{i} c^{k-1}\tilde{\gamma}(\|w\|_\infty) \\
&\le \exp(-\lambda_1\Delta t_i)\|x_\circ\|_X + \frac{1}{1-c}\tilde{\gamma}(\|w\|_\infty),
\end{aligned}
$$

where $\lambda_1 = \ln(1/c)$. Finally, using the item (3) of the theorem, we can write that for all $s \in [\Delta t_i, \Delta t_{i+1}]$ we have:

$$
\begin{aligned}
&\|x(s)\|_X \le L\|x(\Delta t_i)\|_X + \gamma_1(\|w\|_\infty) \\
&\le L\exp(-\lambda_1\Delta t_i)\|x_\circ\|_X + \gamma_1(\|w\|_\infty) + \tfrac{L}{1-c}\tilde{\gamma}(\|w\|_\infty) \\
&= L\exp(\lambda_1(s-\Delta t_i))\exp(-\lambda_1 s)\|x_\circ\|_X \\
&\quad +\gamma_1(\|w\|_\infty) + \frac{L}{1-c}\tilde{\gamma}(\|w\|_\infty) \\
&\le L\exp(\lambda_1\Delta t)\exp(-\lambda_1 s)\|x_\circ\|_X \\
&\quad +\gamma_1(\|w\|_\infty) + \frac{L}{1-c}\tilde{\gamma}(\|w\|_\infty) \\
&\le \max\{K_1\exp(-\lambda_1 s), \gamma_3(\|w\|_\infty)\}, \qquad (2)
\end{aligned}
$$

which completes the proof with $K_1 := 2L\exp(\lambda_1\Delta t)$ and $\gamma_3(\cdot) := \frac{2L}{1-c}(\gamma(\cdot) + \gamma_2(\cdot)) + 2\gamma_1(\cdot)$. ■

When a disturbance free case is considered, the following two corollaries are obvious. The first one is a special case of Theorem 1

*Corollary 1.* Assume that the following conditions hold:

1. The flow $\mathcal{S}_2(x_\circ)$ is exponentially stable;
2. The solutions of the flows $\mathcal{S}_1(x_\circ, \theta)$ and $\mathcal{S}_2(y_\circ)$ can be made arbitrarily *weakly* close on compact time intervals.

Then $\mathcal{S}_1(x_\circ, \theta)$ is exponentially practically stable. That is, for any $\delta > 0$, $\theta^* > 0$, $K_1 > 0$ and $\lambda_1 > 0$ such that for all $\theta \in \Theta$, $|\theta| \le \theta^*$, all $x_\circ \in X$, $x(s) \in \mathcal{S}_1(x_\circ, \theta)$ and $y(s) \in \mathcal{S}_2(x_\circ)$, the following holds

$$\|x(s)\|_X \le K_1\exp(-\lambda_1 s)\|x_\circ\|_X + \delta.$$

On the other hand, Corollary 2 is a special case of Theorem 2.

*Corollary 2.* Assume that the following conditions hold:

1. The flow $\mathcal{S}_2(x_\circ)$ is exponentially stable;
2. The solutions of the flow $\mathcal{S}_1(x_\circ, \theta)$ are bounded on compact time intervals, uniformly in small $\theta$;
3. The solutions of the flows $\mathcal{S}_1(x_\circ, \theta)$ and $\mathcal{S}_2(x_\circ)$ can be made arbitrarily *strongly* close on compact time intervals.

Then $\mathcal{S}_1(x_\circ, \theta)$ is exponentially stable.

## 4. AN ILLUSTRATIVE EXAMPLE

In this section we will show an example on how to apply Corollary 1 to obtain new results in literature where "weak closeness of solution" was already established in the literature.

*Example 1.* In (Bentsman and Hong, 1993), the following nonlinear parabolic system is considered:

$$u_t = Au_{xx} + Bu_x + C(u) + C_1(f(t), u), \qquad (3)$$

with boundary condition $u_x(0,t) = u_x(1,t) = 0$, $u(x,0) = u_0(x)$ for all $t \ge 0$ and $C_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $C(u) \equiv C(u, \lambda)$ where $\lambda \in \mathbb{R}^m$ is a vibratile parameter. $f(t)$ is a $T$-periodic as $f(t) = \frac{1}{\epsilon}\phi(\frac{t}{\epsilon})$, where $\phi(\cdot)$ is a periodic vector function with the average value that equals to zero.

It is assumed that the system is well-posed in the Sobolev space $H^{n_1}(0,1)$ of vector functions $v(x) = [v_1(x), \cdots, v_n(x)]^T$ with components $v_i(x)$ in $L_2(0,1)$ which have the first distributional derivative. The Sobolev space $H^{n_1}(0,1)$ is a normed space with the following norm:

$$\|v\|_1 \triangleq \left(\int_0^1 v_x^T(x)v_x(x) + v(x)^T v(x)dx\right)^{1/2} \qquad (4)$$

The system (3) is difficult to analyze since it is time-varying. However when $\epsilon$ is sufficiently small, then the trajectories of (3) with a $T$-periodic function $f(t)$ are usually composed of a fast oscillatory part and a slow "evolutionary" part. A time-invariant moving average along the trajectory $u(t) \equiv u(x,t,u_0(x),0)$ of system (3) is defined as follows.

$$\bar{u}(t) \triangleq \frac{1}{T}\int_t^{t+T} u(x,s,u_0(x),0)ds.$$

This moving average is used to approximate the slow part of the system (3). Therefore the behavior of this time-invariant moving average will reveal the global "evolutionary" part of the system (3) without the fast oscillatory component. An ordinary differential equation is considered for the fast part:

$$\dot{\zeta} = C_1(\phi(t),\zeta),$$

with a general solution $\zeta(t) = h(t,q)$ where $q \in \mathbb{R}^n$ is a constant uniquely defined for every pair of initial conditions $(\zeta_0, t_0)$. The average of its solution $h(t,\cdot)$ is defined $\bar{h}(\cdot) \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T h(s,\cdot)ds$, where $\bar{h} : \mathbb{R}^n \to \mathbb{R}^n$. $\bar{h}(\cdot)$ is a time-invariant function.

Substituting the fast part into the system (3) with $u(x,t) = h(t, v(x,t))$ yields

$$v_t = F_1(t,v) + F_2(t,v) + F_3(t,v),$$

with the following parabolic averaged equation [3].

$$z_t = P_1(z) + P_2(z) + P_3(z),$$

with boundary condition $z_x(0,t) = z_x(1,t) = 0$, $z(x,0) = z_0(x), \forall t \geq 0$. Here $z : \mathbb{R}(0,1)\times\mathbb{R}_+ \to \mathbb{R}^n$. We have the following result:

*Corollary 3.* $\bar{u}(t)$ is *practically* exponentially stable provided that the time-invariant map $\bar{h}(z(t))$ is exponentially stable

*Sketch of proof*: Applying Theorem 1 in (Bentsman and Hong, 1993), we can obtain the weak closeness of solutions with respect to the norm defined in (4) between $\bar{u}(t)$ and $\bar{h}(z(t))$ on finite time intervals $[0,\Delta t]$ by tuning $\epsilon$ sufficiently small. The result holds true by applying Corollary 1. ■

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[3] See (Bentsman and Hong, 1993) for more details about $F_1(t,v) - F_3(t,v)$ and $P_1(z) - P_3(z)$.