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On sampled-data extremum seeking control via stochastic approximation methods

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***Abstract*—This note establishes a link between stochastic approximation and extremum seeking of dynamical nonlinear systems. In particular, it is shown that by applying classes of stochastic approximation methods to dynamical systems via periodic sampled-data control, convergence analysis can be performed using standard tools in stochastic approximation. A tuning parameter within this framework is the period of the synchronised sampler and hold device, which is also the waiting time during which the system dynamics settle to within a controllable neighbourhood of the steady-state input-output behaviour. Semi-global convergence with probability one is demonstrated for three basic classes of stochastic approximation methods: finite-difference, random directions, and simultaneous perturbation. The tradeoff between the speed of convergence and accuracy is also discussed within the context of asymptotic normality of the outputs of these optimisation algorithms.**

***Keywords*—Extremum seeking, sampled-data control, stochastic approximation, recursive optimisation algorithms.**

I. Introduction

Extremum seeking is a non-model based method of locating an optimal operating regime of the steady-state input-output map of a dynamical system [1], [2]. Two categories of extremum seeking controllers can be found in the literature. The first of which is continuous-time controllers which exploit dither/excitation signals to probe the local behaviour of the system to be optimised and continuously transition the system input to one that results in an optimum. See [1], [3], [4] for such methods that employ periodic dithers and [5], [6] for stochastic dithers. The convergence proofs of the former rely on averaging and singular perturbation techniques [7], [8], while the latter on stochastic averaging [6]. On the contrary, discrete-time extremum seeking controllers are examined in [9] within a sampled-data framework. The convergence proof therein is established using Lyapunov arguments. An alternative proof is given in [10] using trajectory-based techniques. In the same paper, the sampled-data framework of extremum seeking is further examined to accommodate global nonconvex optimisation methods, such as those described in [11]. These works demonstrate that a wide range of optimisation algorithms in the literature can be applied to extremum seeking of dynamic plants. This paper attempts to accommodate a special type of stochastic optimisation methods — stochastic approximation — that is designed to handle situations where there are function measurement errors.

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Stochastic approximation methods [12]–[14] are a family of well-studied iterative gradient-based optimisation algorithms that find applications in a broad range of areas, such as adaptive control and neural networks [15]. In contrast to the standard optimisation algorithms such as the steepest descent or Newton methods [16] which exploit direct gradient information, stochastic approximation methods operate based on *approximation* to the gradient constructed from noisy measurements of the objective/cost function. For the former, knowledge of the underlying system input-output relationships are often needed to calculate the gradient using for example, the chain rule. This is not necessary for stochastic approximation, which makes it well-suited for non-model based extremum seeking control.

This paper adapts within a periodic sampled-data framework three classes of discrete-time multivariate stochastic approximation algorithms for extremum seeking control of dynamical systems which can be of infinite dimension. Namely, the Kiefer-Wolfowitz-Blum's Finite Difference Stochastic Approximation (FDSA) [17], [18], Random Directions Stochastic Approximation (RDSA) [12], and Simultaneous Perturbation Stochastic Approximation (SPSA) [13], [19]. It is shown that there exists a sufficiently long sampling period under which semi-global convergence with probability one to an extremum of the steady-state input-output relation can be achieved. Using the asymptotic normality results of [19], [20], it is demonstrated that a longer sampling period increases the accuracy, but also the convergence time. Note that the existence of Lyapunov functions satisfying the conditions in [9] is not known for the stochastic approximation methods, and hence the convergence results therein do not directly apply.

A related work [21] considers an extremum seeking method based on the SPSA within a different setup (i.e. not sampled-data and has continuous plant output measurements). There, the steady-state input-output objective function is assumed to evaluate to a constant after some waiting time with respect to a constant input, and the output measurements are corrupted by noise. By contrast, this paper exploits the fact that the state trajectory of an asymptotically stable dynamical system converges to a neighbourhood of its steady-state value after the system's input is held constant for a pre-selected waiting time. The difference between the sampled output value and the steady-state value is modelled as a random process. The SPSA method has also been applied to optimisation of variable cam timing engine operation in [22], alongside several other optimisation algorithms.

The paper evolves along the following lines. First, the next section states the properties of the nonlinear dynamical systems to which the stochastic approximation methods are applied. These methods are introduced in Section III. Section IV depicts the sampled-data framework within which extremum seeking based on stochastic approximation is introduced and analysed. An illustrative simulation example is provided in Section V, followed by some concluding remarks in Section VII.

II. Dynamical systems

The class of nonlinear, possibly infinite-dimensional, systems considered in this paper is introduced in this section. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ (denoted $\gamma \in \mathcal{K}$) if it is continuous, strictly increasing, and $\gamma(0) = 0$. If γ is also unbounded, then $\gamma \in \mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if for each fixed t, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed s, $\beta(s, \cdot)$ is decreasing to zero [7]. The Euclidean norm is denoted $\|\cdot\|_2$.

Let $\mathcal{X}$ be a Banach space whose norm is denoted $\|\cdot\|$. Given any subset $\mathcal{Y}$ of $\mathcal{X}$ and a point $x \in \mathcal{X}$, define the distance of x from $\mathcal{Y}$ as $\|x\|_{\mathcal{Y}} := \inf_{a \in \mathcal{Y}} \|x - a\|$. Also let

$$\mathcal{U}_\epsilon(\mathcal{Y}) := \{x \in \mathcal{X} \mid \|x\|_{\mathcal{Y}} < \epsilon\}.$$

Definition II.1. Let the state of a time-invariant dynamical system be represented by $x : \mathbb{R}_{\geq 0} \to \mathcal{X}$, where $\mathcal{X}$ is a Banach space with norm $\|\cdot\|$. The input to and output of the system are denoted, respectively, by $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $y : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Given any $u \in \mathbb{R}^m$ and $x_0 \in \mathcal{X}$, let $x(\cdot, x_0, u)$ be the state of the dynamical system starting at x_0 with input u.

The following assumption is based chiefly on [9, Assumption 1].

Assumption II.2. Given a system described in Definition II.1, the following hold:

(i) There exists a function $\mathcal{A}$ mapping from $\mathbb{R}^m$ to subsets of $\mathcal{X}$ such that for each constant $u \in \mathbb{R}^m$, $\mathcal{A}(u)$ is a nonempty closed set and a *global attractor*:
   a) Given any $x_0 \in \mathcal{X}$ and $\epsilon > 0$, there exists a sufficiently large $t > 0$ such that $x(t, x_0, u) \in \mathcal{U}_\epsilon(\mathcal{A}(u))$;
   b) If $x(t_0, x_0, u) \in \mathcal{A}(u)$, then $x(t, x_0, u) \in \mathcal{A}(u)$ for all $t \geq t_0$;
   c) There exists no proper subset of $\mathcal{A}(u)$ having the first two properties above.

   Furthermore,
$$\sup_{u \in \mathbb{R}^m} \sup_{x \in \mathcal{A}(u)} \|x\| < \infty. \tag{1}$$

(ii) There exists a locally Lipschitz function $h : \mathcal{X} \to \mathbb{R}$ such that the system output
$$y(t) = h(x(t, x_0, u)) \quad \forall t \geq 0$$
for any constant input $u \in \mathbb{R}^m$ and $x_0 \in \mathcal{X}$. Moreover, $h(x_a) = h(x_b)$ for every $x_a, x_b \in \mathcal{A}(u)$. Since $\mathcal{A}(u)$ is a global attractor and h is locally Lipschitz, for any $u \in \mathbb{R}^m$ and $x_0 \in \mathcal{X}$,
$$\begin{aligned} Q(u) &:= \lim_{t \to \infty} h(x(t, x_0, u)) \\ &= h\left(\lim_{t \to \infty} x(t, x_0, u)\right) \\ &= h(x_l), \quad \text{for some } x_l \in \mathcal{A}(u) \end{aligned}$$
is a well-defined steady-state input-output map that is Lipschitz on $\mathbb{R}^m$.

(iii) Q takes its global minimum value in a nonempty, compact set $\mathcal{C} \subset \mathbb{R}^m$.

(iv) Q is differentiable and its derivatives are equicontinuous and bounded.

(v) Given any $\Delta > 0$, there exists a class-$\mathcal{KL}$ function β such that
$$\|x(t, x_0, u)\|_{\mathcal{A}(u)} \leq \beta(\|x_0\|_{\mathcal{A}(u)}, t)$$
for all $t \geq 0$, $u \in \mathbb{R}^m$, and $\|x_0\|_{\mathcal{A}(u)} \leq \Delta$.

An example of systems satisfying the above assumption is given here. Consider the following dynamical system [1], [7]:

$$\begin{aligned} \dot{x} &= f(x, u) \quad x(0) = x_0; \\ y &= h(x), \end{aligned} \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions in each argument.

Assumption II.3. There exists a locally Lipschitz function $\ell : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$f(\ell(u), u) = 0 \quad \forall u \in \mathbb{R}^m.$$

Furthermore, $x = \ell(u)$ is globally asymptotically stable uniformly in $u \in \mathbb{R}^m$ [7], i.e. there exists a $\beta \in \mathcal{KL}$ such that for any $u \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} \|x(t, x_0, u)\|_{\ell(u)} &= \|x(t, x_0, u) - \ell(u)\|_2 \\ &\leq \beta(\|x_0 - \ell(u)\|_2, t) \quad \forall t \geq 0, \end{aligned}$$

where $x(\cdot, x_0, u)$ denotes the solution to (2) with respect to the initial condition x_0 and input u.

Definition II.4. Let

$$Q(\cdot) := h \circ \ell(\cdot) : \mathbb{R}^m \to \mathbb{R}$$

be the steady-state input-output map of system (2).

Suppose Q achieves its global minimum value on a nonempty compact set $\mathcal{C} \subset \mathbb{R}^m$ and is differentiable with bounded and equicontinuous derivatives. It can be verified easily that the system (2) satisfies Assumption II.2.

III. Stochastic approximation

This section briefly reviews three types of stochastic approximation (SA) algorithms, i.e. the Finite Difference (FDSA), Random Directions (RDSA), and Simultaneous Perturbation (SPSA). The material in this section is largely based on [13], [20]. Let $Q : \mathbb{R}^m \to \mathbb{R}$ be a differentiable objective/cost function and $g(\theta) := \frac{dQ(\theta)}{d\theta}$. Suppose also that the derivatives of Q are equicontinuous and bounded. SA algorithms are iterative procedures which find a minimising θ^*

of Q such that $g(\theta^*) = 0$. They take the following standard form

$$\theta_{k+1} = \theta_k - a_k g_k(\theta_k), \tag{3}$$

where $\{a_k\}$ is a positive gain sequence and $g_k(\theta_k)$ is an approximation to the gradient g of Q at θ_k.

A. FDSA

Let $\{e_i\}_{i=1}^m$ be the canonical basis for $\mathbb{R}^m$, i.e. e_i is a unit vector in the direction of the i^{th} coordinate of $\mathbb{R}^m$. The FDSA [17], [18] utilises a finite-difference equation to approximate each entry of the gradient g. In particular, let k be the current iteration number and a positive scalar c_k be given, the (two-sided) FDSA takes measurements of Q at design levels $\theta_k \pm c_k e_i$:

$$\begin{aligned} y_k^{i+} &= Q(\theta_k + c_k e_i) + \epsilon_k^{i+} \\ y_k^{i-} &= Q(\theta_k - c_k e_i) + \epsilon_k^{i-}, \end{aligned} \tag{4}$$

where ϵ_k^{i+} and ϵ_k^{i-} represent measurement noise terms that satisfy the martingale difference assumption, i.e.

$$E\left\{\epsilon_k^{i+} - \epsilon_k^{i-} \;\middle|\; \theta_i;\ i \le k\right\} = 0 \text{ a.s. } \forall k. \tag{5}$$

Here, E denotes the expectation. A special instance fulfilling (5) is when the noise is a stationary process. See [14] for generalisations to correlated noise. An estimate of the gradient in (3) used by FDSA is thus given as

$$g_k(\theta_k) = \frac{1}{2c_k}\begin{bmatrix} y_k^{1+} - y_k^{1-} \\ \vdots \\ y_k^{m+} - y_k^{m-} \end{bmatrix}. \tag{6}$$

Assumption III.1. The algorithm sequence satisfies:

(i) $a_k, c_k > 0\ \forall k$; $a_k \to 0$, $c_k \to 0$ as $k \to \infty$, $\sum_{k=0}^{\infty} a_k = \infty$, $\sum_{k=0}^{\infty} (\frac{a_k}{c_k})^2 < \infty$; see (3), (4) and (6).

(ii) $\sup_k \|\theta_k\| < \infty$ a.s.

(iii) θ^* is an asymptotically stable solution of the differential equation

$$\frac{dz(t)}{dt} = -g(z). \tag{7}$$

(iv) Let $D(\theta^*) := \{z_0 \in \mathbb{R}^m : \lim_{t\to\infty} z(t, z_0) = \theta^*\}$, where $z(t, z_0)$ denotes the solution to the differential equation (7) with respect to the initial condition z_0, i.e. $D(\theta^*)$ is the domain of attraction. There exists a compact $S \subset D(\theta^*)$ such that $\theta_k \in S$ infinitely often for almost all sample points.

Proposition III.2 ([18])**.** *Suppose Assumption III.1 holds. Then the output of the FDSA, $\theta_k \to \theta^*$ with probability one.*

Notice that the FDSA requires $2m$ measurements for every iteration of the algorithm. For systems with a large number of inputs, this can thus be very computationally costly and lead to slower convergence speed. This fact has motivated the developments of the RDSA and SPSA as alternatives which use fewer measurements. More specifically, only a pair of measurements are used per iteration of the algorithms.

B. RDSA

Let $\{d_k\}$ be a sequence of independent random vectors, each distributed uniformly over the surface of the m-dimensional sphere of radius m [20] (other probability distributions can also be used). The gradient in (3) is approximated by the RDSA as

$$g_k(\theta_k) = \frac{1}{2c_k} d_k (y_k^+ - y_k^-),$$

where $y_k^{\pm} = Q(\theta_k \pm c_k d_k) + \epsilon_k^{\pm}$, and the measurement noise satisfies

$$E\left\{\epsilon_k^{+} - \epsilon_k^{-} \;\middle|\; \theta_i;\ d_i;\ \epsilon_{i-1}^{\pm};\ i \le k\right\} = 0 \text{ a.s. } \forall k.$$

Assumption III.3. There exist $\alpha_1 > 0$ such that $E(d_k d_k^T) = I$ and $E[(d_k^i Q(\theta_k \pm c_k d_k))^2] \le \alpha_1$, for all k and $i = 1, 2, \ldots, m$, where the superscript T denotes the matrix transpose and I the identity matrix.

Proposition III.4 ([12], [20])**.** *Suppose Assumptions III.1 and III.3 hold and there exists $\alpha_0 > 0$ such that $E[(\epsilon_k^{\pm})^2] \le \alpha_0$ for all k. Then the output of the RDSA, $\theta_k \to \theta^*$ with probability one.*

In general, the RDSA does not have superior efficiency performance to the FDSA because the number of iterations may increase enough to nullify the decrease in the number of measurements per iteration [12], [19]. The SPSA, on the other hand, is the preferable algorithm for systems with multiple inputs on both theoretical and numerical basis [13], [20].

C. SPSA

Let $\Delta_k \in \mathbb{R}^m$ be a vector of m mutually independent zero-mean random variables $[\Delta_k^1, \Delta_k^2, \ldots, \Delta_k^m]$ and $\{\Delta_k\}$ a mutually independent sequence with Δ_k independent of $\theta_0, \theta_1, \ldots, \theta_k$. Δ_k^i can be symmetrically Bernoulli distributed for example, as taken in the numerical studies of [19].

In the SPSA, the gradient in the optimisation procedure (3) is approximated by

$$g_k(\theta_k) = \frac{1}{2c_k}\begin{bmatrix} \frac{y_k^+ - y_k^-}{\Delta_k^1} \\ \vdots \\ \frac{y_k^+ - y_k^-}{\Delta_k^m}, \end{bmatrix} \tag{8}$$

where $y_k^{\pm} = Q(\theta_k \pm c_k \Delta_k) + \epsilon_k^{\pm}$, and the measurement noise terms satisfy

$$E\left\{\epsilon_k^{+} - \epsilon_k^{-} \;\middle|\; \Delta_k;\ \theta_i;\ i \le k\right\} = 0 \text{ a.s. } \forall k.$$

Assumption III.5. There exist $\alpha_1, \alpha_2 > 0$ such that $E(Q(\theta_k \pm c_k \Delta_k))^2) \le \alpha_1$ and $E(\Delta_k^i)^{-2} \le \alpha_2$ for all k and $i = 1, 2, \ldots, m$.

Proposition III.6 ([19])**.** *Suppose Assumptions III.1 and III.5 hold and there exists $\alpha_0 > 0$ such that $E[(\epsilon_k^{\pm})^2] \le \alpha_0$ for all k. Then the output of the SPSA, $\theta_k \to \theta^*$ with probability one.*

Remark III.7. In [13], [19], it is shown that averaging over several gradient approximations (8) using conditionally (on θ_k) independent simultaneous perturbations at each iteration may provide benefits in terms of accuracy. Specifically, the gradient in (3) can be replaced by

$$g_k(\theta_k) = \frac{1}{q}\sum_{i=1}^{q} g_k^{(i)}(\theta_k), \quad q \geq 1,$$

where each $g_k^{(i)}(\theta_k)$ is generated as in (8) based on a different pair of measurements with simultaneous perturbations $\Delta_k^{(i)}$, which are independent conditionally on θ_k. The same idea can also benefit the RDSA; see [20]. Henceforth, the averaged RDSA or SPSA are denoted as RDSA-q and SPSA-q, respectively.

IV. SAMPLED-DATA EXTREMUM SEEKING CONTROL

A. A sampled-data framework

The sampled-data extremum seeking framework of [9] is detailed in this section. A semi-global convergence proof is provided for applying the stochastic approximation (SA) methods in the last section to dynamical systems defined in Section II within this framework. Note that the results in [9] are not applicable here because they rely on existence of Lyapunov functions for the SA methods.

Let $\{u_k\}_{k=0}^{\infty}$ be a sequence of vectors in $\mathbb{R}^m$ and define the zero-order hold (ZOH) operation

$$u(t) := u_k \quad \text{for all } t \in [kT, (k+1)T) \tag{9}$$

and $k = 0, 1, 2, \ldots$, where $T > 0$ denotes the sampling period or waiting time. Furthermore, let the state and output of a dynamical system in Definition II.1 with respect to the input u be respectively x and y and define the ideal periodic sampling operation $x_k := x(kT)$;

$$y_k := y(kT) \quad \text{for all } k = 1, 2, \ldots. \tag{10}$$

Figure 1 shows an extremum seeking scheme based on a sampled-data control law with period T. The following lemma on dynamical systems is needed to establish the main result of this section. The proof is based on ideas from [23, Prop. 1], where finite-dimensional state-space systems with asymptotically stable equilibrium points are considered. Note that infinite-dimensional systems with general attractors are accommodated here.

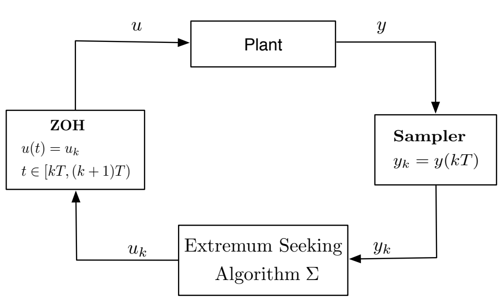

Fig. 1. Sampled-data extremum seeking control.

Lemma IV.1. *Given any dynamical system described in Definition II.1 that satisfies Assumption II.2, $\Delta > 0$, and $\nu > 0$, there exists a $T > 0$ such that for any $\{u_k\}_{k=0}^{\infty} \subset \mathbb{R}^m$ and $\|x_0\|_{\mathcal{A}(u_0)} \leq \Delta$,*

$$|y_k - Q(u_{k-1})| \leq \nu \quad \textit{for all } k = 1, 2, \ldots,$$

where x_0 denotes the initial condition of the system and y_k is as in (10) with y being the output of the system for the input u given by (9).

Proof: Let Δ and ν be given as in the lemma statement and L_h be the Lipschitz constant of h on the compact ball

$$\{x \in \mathcal{X} \mid \|x\| \leq \mathcal{A}_{max} + 1\},$$

where $\mathcal{A}_{max} := \sup_{u\in\mathbb{R}^m} \sup_{x\in\mathcal{A}(u)} \|x\|$, which is finite by Definition II.1. Choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$ so that

$$\epsilon_1(\Delta + 2\mathcal{A}_{max} + 1) + \epsilon_2 \leq \min\left\{\frac{\nu}{L_h}, 1\right\}.$$

By Property (v) of Definition II.1, it follows that there exists a $T > 0$ such that for any $\|x_0\|_{\mathcal{A}(u_0)} \leq \Delta + 2\mathcal{A}_{max} + 1$,

$$\begin{aligned}\|x_1\|_{\mathcal{A}(u_0)} &= \|x(T, x_0, u_0)\|_{\mathcal{A}(u_0)} \\ &\leq \epsilon_1\|x_0\|_{\mathcal{A}(u_0)} + \epsilon_2 \\ &\leq \epsilon_1\Delta + \epsilon_2 \\ &\leq 1,\end{aligned}$$

whereby $\|x_1\| \leq \mathcal{A}_{max} + 1$. We show below that $\|x_k\| \leq \mathcal{A}_{max} + 1$ for all $k = 1, 2, \ldots$ following an inductive argument. Suppose this is true for a $k \in \mathbb{N}$, which implies $\|x_k\|_{\mathcal{A}(u_k)} \leq 2\mathcal{A}_{max} + 1$. Then, by time-invariance of the dynamical system,

$$\begin{aligned}\|x_{k+1}\|_{\mathcal{A}(u_k)} &= \|x(T, x_k, u_k)\|_{\mathcal{A}(u_k)} \\ &\leq \epsilon_1\|x_k\|_{\mathcal{A}(u_k)} + \epsilon_2 \\ &\leq \epsilon_1(2\mathcal{A}_{max} + 1) + \epsilon_2 \\ &\leq 1,\end{aligned}$$

whereby $\|x_{k+1}\| \leq \mathcal{A}_{max} + 1$. Consequently,

$$\begin{aligned}|y_k - Q(u_{k-1})| &= \inf_{x_l\in\mathcal{A}(u_{k-1})} |h(x_k) - h(x_l)| \\ &\leq L_h \inf_{x_l\in\mathcal{A}(u_{k-1})} \|x_k - x_l\| \\ &= L_h\|x_k\|_{\mathcal{A}(u_{k-1})} \\ &\leq L_h(\epsilon_1\|x_{k-1}\|_{\mathcal{A}(u_{k-1})} + \epsilon_2) \\ &\leq L_h(\epsilon_1(2\mathcal{A}_{max} + 1) + \epsilon_2) \\ &\leq \nu,\end{aligned}$$

where the first equality follows from Property (ii) of Definition II.1 and L_h is defined at the beginning of the proof. ■

Given any $\Delta > 0$ and $\nu > 0$, let the sampling period $T > 0$ be chosen such that the conclusion of Lemma IV.1 holds. Define

$$\epsilon_k := y_k - Q(u_{k-1}),$$

and note that $|\epsilon_k| \leq \nu$ for all k. The following standing assumption will be used in the sequel for convergence analysis.

Assumption IV.2. The measurement difference sequence $\{\epsilon_k\}$ defines a random process.

B. Extremum seeking

Consider the feedback configuration in Figure 1 of a dynamical plant in Definition II.1 satisfying Assumption II.2 and an extremum seeking algorithm taken to be the FDSA, RDSA, or SPSA. These are interconnected through a T-periodic sampler (10) and a synchronised zero-order hold (9).

In particular, during the j^{th} SA algorithmic iteration, the output of the extremum seeker is defined in the following manner:

1) for the FDSA,

$$u_{2m(j-1)+2(i-1)} = \theta_j + c_j e_i, \quad i = 1, \dots, m$$
$$u_{2m(j-1)+2(i-1)+1} = \theta_j - c_j e_i, \quad i = 1, \dots, m;$$

2) for the RDSA,

$$u_{2(j-1)} = \theta_j + c_j d_j,$$
$$u_{2(j-1)+1} = \theta_j - c_j d_j;$$

3) and for the SPSA,

$$u_{2(j-1)} = \theta_j + c_j \Delta_j,$$
$$u_{2(j-1)+1} = \theta_j - c_j \Delta_j.$$

The above procedure defines an input sequence $\{u_k\}_{k=0}^{\infty}$ to the plant. The following main results are in order.

Theorem IV.3. *Consider the feedback interconnection of Figure 1 with the extremum seeking controller being the FDSA method in Section III-A. Given any $\Delta > 0$ and $\nu > 0$, obtain a sampling period $T > 0$ from Lemma IV.1. Suppose Assumptions IV.2 and III.1 hold, and*

$$E\{\epsilon_{k+1} - \epsilon_k \mid u_i;\ i \le k\} = 0 \text{ a.s. } \forall k.$$

Then for any $\|x_0\|_{\mathcal{A}(u_0)} \le \Delta$, $u_k \to u^*$ with probability one, where u^* satisfies $\frac{dQ}{du}(u^*) = 0$ and $Q : \mathbb{R}^m \to \mathbb{R}$ is the steady-state input-output map of the plant, as in Assumption II.2.

Proof: This follows directly from Proposition III.2. ■

Theorem IV.4. *Consider the feedback interconnection of Figure 1 with the extremum seeking controller being the RDSA method in Section III-B. Given any $\Delta > 0$ and $\nu > 0$, obtain a sampling period $T > 0$ by applying Lemma IV.1. Suppose Assumptions IV.2 and III.3 hold, and*

$$E\{\epsilon_{k+1} - \epsilon_k \mid u_i;\ d_\ell;\ \epsilon_{i-1};\ i \le k, \ell \le j\} = 0 \text{ a.s. } \forall k, j.$$

Then for any $\|x_0\|_{\mathcal{A}(u_0)} \le \Delta$, $u_k \to u^*$ with probability one, where u^* satisfies $\frac{dQ}{du}(u^*) = 0$.

Proof: Note that by Lemma IV.1, $|\epsilon_k|^2 \le \nu^2$ and hence $E(|\epsilon_k|^2) \le \nu^2$ for all k. The claim then follows from Proposition III.4. ■

Theorem IV.5. *Consider the feedback interconnection of Figure 1 with the extremum seeking controller being the SPSA method in Section III-C. Given any $\Delta > 0$ and $\nu > 0$, let $T > 0$ be the sampling period from applying Lemma IV.1. Suppose Assumptions IV.2 and III.5 hold, and*

$$E\{\epsilon_{k+1} - \epsilon_k \mid u_i;\ \Delta_j;\ i \le k\} = 0 \text{ a.s. } \forall k, j.$$

Then for any $\|x_0\|_{\mathcal{A}(u_0)} \le \Delta$, $u_k \to u^*$ with probability one, where u^* satisfies $\frac{dQ}{du}(u^*) = 0$.

Proof: By Lemma IV.1, $|\epsilon_k|^2 \le \nu^2$. Therefore, $E(|\epsilon_k|^2) \le \nu^2$ for all k. The claim then follows from Proposition III.6. ■

C. Accuracy vs. speed of convergence

Under some slightly strengthened versions of Assumptions III.3 and III.5 and additional mild assumptions, it can be shown that for all of the FDSA, RDSA, and SPSA, the mean square error converges to the form

$$k^\beta E\|u_k - u^*\|_2^2 \to \mathrm{tr} PMP^T + \mu^T \mu \text{ as } k \to \infty,$$

for some $\beta > 0$, $\mu \in \mathbb{R}^m$ and $P \in \mathbb{R}^{m\times m}$, where M is a diagonal matrix whose entries are proportional to

$$\lim_{k\to\infty} E((\epsilon_{k+1} - \epsilon_k)^2 \mid u_i,\ i \le k);$$

see [19], [20]. This follows from the asymptotic normality of u_k established in [19], [20]. If RDSA-q or SPSA-q are used, the expression becomes

$$k^\beta E\|u_k - u^*\|_2^2 \to \frac{1}{q}\mathrm{tr} PMP^T + \mu^T \mu \text{ as } k \to \infty.$$

It is true from Lemma IV.1 that

$$E((\epsilon_{k+1} - \epsilon_k)^2 \mid u_i,\ i \le k) \le 4\nu^2$$

for all k. It is therefore possible to increase accuracy, i.e. reduce ν and $\lim_{k\to\infty} E\|u_k - u^*\|^2$ by elongating the sampling period $T > 0$. Nevertheless, a larger sampling period corresponds to longer waiting time per algorithm iteration, and hence slower speed of convergence. This, however, may be compensated by the decrease in the number of algorithm iterations because of the use of more accurate measurements, as demonstrated by the first numerical example in the ensuing section.

V. Example

Consider the following one-dimensional nonlinear system with a single input:

$$\dot{x} = -x^3 + u^2, \quad x(0) = 2; \qquad y = x^3. \tag{11}$$

Note that for any fixed $u \in \mathbb{R}$, the origin is a globally asymptotically stable equilibrium; see Section II. It is apparent that the steady-state input-output map is $Q(u) = u^2$, of which its unique global minimum is 0. The input is started at $u = 100$ and the FDSA is employed for minimum-seeking of the plant using the sampled-data control law detailed in Section IV.

The gradient approximation step gain c_k is taken to be $1/k^{0.01}$, while the iteration step gain a_k is $1/k$. Note that the combinations of a_k and c_k satisfy Assumption III.1 required for convergence. The algorithm is terminated once an input of magnitude less than 0.2 is found. Using a sampling period of $T = 0.2$s, it takes 1.2s to locate an input estimate $u = -0.0642$; see Figure 2. By increasing the sampling period to $T = 0.4$s, 4.8s is needed to find the input $u = 0.1736$; see Figure 3. If $T = 0.1$s is used instead, 19.4s is required to locate the input $u = 0.1995$ because of the relatively larger variance of the measurement noise (cf. Figure 4); see Section IV-C for a discussion of the tradeoff between accuracy and speed of convergence.

For the case $T = 0.1$s, by adjusting a_k to be $1/k^{0.6}$ and keeping c_k the same, the FDSA takes 2.6s to locate the input $u = 0.194$ (cf. Figure 5). This shows that apart from the waiting time/sampling period T, the parameters in the

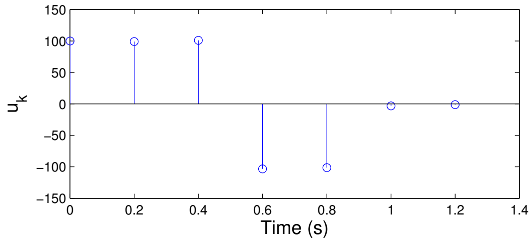

Fig. 2. The output sequence of the FDSA for $T = 0.2$s; $a_k = 1/k$.

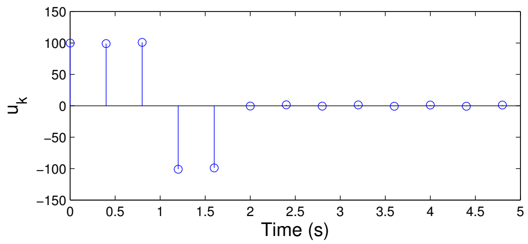

Fig. 3. The output sequence of the FDSA for $T = 0.4$s; $a_k = 1/k$.

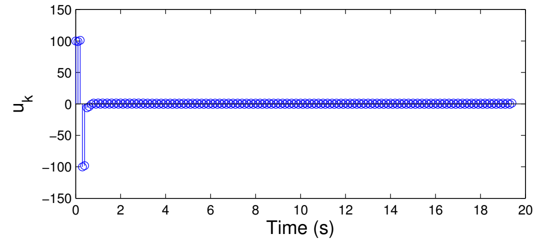

Fig. 4. The output sequence of the FDSA for $T = 0.1$s; $a_k = 1/k$.

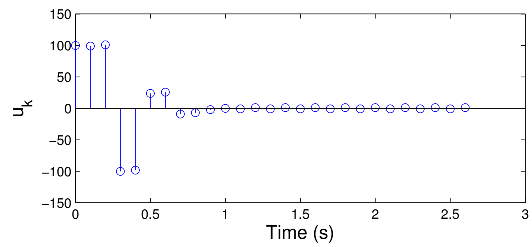

Fig. 5. The output sequence of the FDSA for $T = 0.1$s; $a_k = 1/k^{0.6}$.

stochastic algorithm itself also affects the accuracy/speed of convergence.

Finally, it is remarked that numerous examples of multi-input static-systems with noisy measurements which investigate the enhanced efficiency of the SPSA over the FDSA can be found in [13], [19], [20].

VI. Acknowledgement

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VII. Conclusions

This paper applies a number of stochastic approximation methods to implementing extremum seeking of nonlinear dynamical systems. Future research directions involve investigating generalisation of these to multi-unit and multi-agent settings. A unified framework for different classes of stochastic optimisation methods may also be examined.

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