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RANEY DISTRIBUTIONS AND RANDOM MATRIX THEORY

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Abstract. Recent works have shown that the family of probability distributions with moments given by the Fuss-Catalan numbers permit a simple parameterized form for their density. We extend this result to the Raney distribution which by definition has its moments given by a generalization of the Fuss-Catalan numbers. Such computations begin with an algebraic equation satisfied by the Stieltjes transform, which we show can be derived from the linear differential equation satisfied by the characteristic polynomial of random matrix realizations of the Raney distribution. For the Fuss-Catalan distribution, an equilibrium problem characterizing the density is identified. The Stieltjes transform for the limiting spectral density of the singular values squared of the matrix product formed from q inverse standard Gaussian matrices, and s standard Gaussian matrices, is shown to satisfy a variant of the algebraic equation relating to the Raney distribution. Supported on $[0,\infty)$, we show that it too permits a simple functional form upon the introduction of an appropriate choice of parameterization. As an application, the leading asymptotic form of the density as the endpoints of the support are approached is computed, and is shown to have some universal features.

1. Introduction

For given $s \in \mathbb{N}$ the Fuss-Catalan numbers, also known as the generalized Catalan numbers, are the integer sequence

$$C_s(k) = \frac{1}{sk+1}\binom{sk+k}{k}, \qquad k = 0, 1, 2, \dots \tag{1.1}$$

As is well known (see e.g. [45]) this sequence in the case $s = 1$ — traditionally referred to as the Catalan numbers — first appeared in the work of Euler on counting the number of triangulations of a convex polygon consisting of $k+2$ sides. Note that each elementary triangle in the triangulation must contain at least one side of the $(k+2)$-gon. The other two sides of the elementary triangle therefore naturally

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partition the counting problem into two independent counting problems of the same type but involving polygons of smaller number of sides. Specifically, on one side the counting problem for the triangulation of a (k_1+1)-gon is encountered, and on the other side one has the counting problem for the triangulation of a (k_2+1)-gon, where $k_1+k_2=k-1$. The Catalan numbers therefore satisfy the fundamental recurrence

$$C_1(k)=\sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2=k-1}} C_1(k_1)C_1(k_2) \tag{1.2}$$

valid for $k\geq 1$.

Fuss (see e.g. [45]) generalized the triangulation problem of Euler to counting dissections of a convex $(sk+2)$-gon using $(s+2)$-gons. Here any particular $(s+2)$-gon in the dissection partitions the $(sk+2)$-gon into $(s+1)$ disjoint regions, implying the generalization of (1.2)

$$C_s(k)=\sum_{\substack{k_1,k_2,\ldots,k_{s+1}\geq 0\\ k_1+k_2+\cdots+k_{s+1}=k-1}} C_s(k_1)C_s(k_2)\cdots C_s(k_{s+1}), \tag{1.3}$$

again valid for $k\geq 1$.

Recently, the Fuss-Catalan numbers (1.1) have appeared in several different contexts, for instance, product of random matrices [1, 8, 53], random quantum states [21], free probability and quantum groups [8, 49]. More precisely, the sequence of Fuss-Catalan numbers is the moments of some probability density π_s, which is the limit spectral distribution of the squared singular values of random matrices with the product structures X_1^s (powers of a single matrix) and $X_1\cdots X_s$ (products of independent matrices). The $N\times N$ matrices $X_1,\ldots,X_s$ are each to contain independent, identically distributed zero mean, unit standard deviation random variables, and the squared singular values are to be divided by N before the limit is taken. It is known that the explicit form of π_s can be described in terms of multivariate integral representations [48], in terms of Meijer G-functions [58] or by using the parameterization of the argument. This latter advance is due to Biane [10, n.b. Biane's contribution is in the Appendix of this work, and in particular he is not listed as one of the author's], Haagerup and Möller [37], and Neuschel [54], and its further development forms one of the themes of our paper.

We will consider probability densities with moments given by a family of integer sequences generalizing the Fuss-Catalan sequence (1.1). Thus for $p > 1, 0 < r \leq p$ we introduce the integer sequence

$$R_{p,r}(k) = \frac{r}{pk+r}\binom{pk+r}{k}, \quad k = 0, 1, 2, \ldots. \tag{1.4}$$

Following [58] we refer to $R_{p,r}(k)$ as the k-th Raney number. To tie these numbers in with (1.1) at a combinatorial level requires making note of a combinatorial interpretation of the latter different from that given above in terms of dissections of $(sk+2)$-gons. Thus suppose there are sk numbers $+1$ and k number $-s$ in a sequence. How many ways can the sequence members be arranged so that the partial sum of sequence members 1 up to ℓ is always non-negative for each ℓ? This is a version of the so-called ballot problem (see e.g. [60]). By noting that the final member of the sequence must always be a $-s$, we see that the sequence with the $-s$ removed can be decomposed into $s+1$ sequences of the desired type, each separated by a $+1$. Hence the recurrence (1.3) holds, telling us that the answer to this counting problem is $C_s(k)$.

As a generalization, suppose there are extra r ($r > 0$) $+1$'s and it is required that the partial sums be strictly positive. With $r = 1$ this is equivalent to the ballot problem as reviewed above, since the extra $+1$ must appear at the beginning of a valid ballot sequence. For $r \geq 2$ new ideas are needed [59, 36]. The answer is the Raney number $R_{s+1,r}(k)$. Note from the above discussion that $R_{s+1,1}(k) = C_s(k)$, which can indeed be checked from the explicit forms (1.1) and (1.4).

For the general case of $p > 1, 0 < r \leq p$, Młotkowski [49] has proved that $R_{p,r}(k)$ is the k-th moment of some probability measure $\mu_{p,r}$ (so-called Raney distributions) with compact support contained in $[0, \infty)$. In particular, explicit densities $W_{p,r}(x)$ associated with $\mu_{p,r}$ are given in [58] for integer $p > 1$ and more generally in [50] for rational $p > 1$, both in terms of Meijer G-functions. We will show in this work that the parameterization method of Biane [10], Haagerup and Möller [37], and Neuschel [54] can be generalized to any real $p > 1, 0 < r \leq p$ and further gives explicit densities. Application of this method to a class of probability measures supported on $[0, \infty)$ not possessing finite moments, but nonetheless being intimately related to the sequence (1.4), will be given as will independent derivations of some key polynomial equations determining the measures.

The format of the remainder of our paper is to first review Neuschel's derivation of the explicit parametrization of the Fuss-Catalan distributions, i.e. the cases $p > 1$ and $r = 1$ of the Raney distributions. In the course of this review we will see a fundamental relationship between the case $r = 1$ and the general $0 < r \leq p$ case which enables us to use the results of [54] to obtain a parametrization of the Raney distributions. As an application we give the leading term of the asymptotic expansion of the density as an endpoint of the support is approached. We then turn our attention to realizations of the Fuss-Catalan and Raney distributions in terms of spectral densities of random matrices. In addition to listing known examples, we add a few new cases. We furthermore show how the polynomial equation for the resolvent can be derived from the differential equation for the characteristic equation. For the Fuss-Catalan distribution, an equilibrium problem characterizing the density is identified involving the logarithmic potential with image charges along rays, by making use of recent results of Claeys and Romano [20]. In the case of the spectral density for the squared singular values of a product of q inverse standard Gaussian matrices, and s standard Gaussian matrices, we show how to adapt Neuschel's method to specify a parametrization of the spectral variable which allows a simple closed form for the density. This reclaims a recent result of Haagerup and Möller [37] and furthermore allows this result to be extended. We use this to obtain the leading asymptotic form at the endpoints of the support.

2. Parameterization of the Raney distribution

The Stieltjes transform of the measure $\mu_{p,r}$, also referred to as the resolvent or Green's function, is defined by

$$G_{p,r}(z) = \int_0^{K_p} \frac{1}{z-x}\, d\mu_{p,r}(x) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} R_{p,r}(n), \qquad K_p = p^p |p-1|^{1-p}, \quad (2.1)$$

where the explicit value of the upper terminal in the support K_p corresponds to the reciprocal of the radius of convergence of the series in the last equality, which thus requires that $|z| > K_p$ for its convergence. Crucial to our study is the fact that $w(z) := zG_{p,r}(z)$ satisfies the algebraic equation [50]

$$w^{\frac{p}{r}} - zw^{\frac{1}{r}} + z = 0. \quad (2.2)$$

Before giving its derivation, it is of interest to remark that the equation (2.2) is known in a different area of mathematical physics, namely the theory of anyons. Thus with g denoting the statistical parameter, $0 \le g \le 1$ ($g = 0$, $g = 1$ correspond to Bose and Fermi statistics respectively) it is shown in [63] that the statistical distribution for a single species is given by the average occupation number

$$n_i = \frac{1}{W(e^{\beta(\varepsilon_i - \mu)}) + g}$$

(β is the inverse temperature and μ the chemical potential), where the function $W(x)$ satisfies the functional equation

$$(W(x))^g (1 + W(x))^{1-g} = x,$$

with $x := e^{\beta(\varepsilon - \mu)}$. Introducing the transformation (see e.g. [40]) $w = 1 + 1/W$ the functional equation reads

$$x(w-1) = w^{1-g}$$

which is precisely (2.2) in the case $r = 1$, with $z = x$ and $p = 1 - g$. Another point of interest is that the analytic function defined by the power series on the right hand side of (2.1) has been the subject of a number of earlier studies [39, 40, 6]. In particular, with $\mathcal{B}_p(z) = (1/z)G_{p,1}(1/z)$, it is shown in [39] that

$$\mathcal{B}_p(z) = 1 - \mathcal{B}_{1/p}\Big(-\frac{1}{\sqrt[p]{-z}}\Big). \tag{2.3}$$

To deduce (2.2), one first recalls the Lagrange inversion formula [64]. Thus let $f(z)$ and $\phi(z)$ be analytic in a neighbourhood Ω of a and let t be small enough so that $|t\phi(z)| < |z - a|$, $z \in \Omega$. The Lagrange inversion formula tells us that the equation

$$\zeta = a + t\phi(\zeta) \tag{2.4}$$

has one solution in Ω and furthermore

$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} (f'(a)(\phi(a))^n). \tag{2.5}$$

We observe that in the case $r = 1$ (2.2) can be rearranged to read $w = 1 + (1/z)w^p$ which is of the form (2.4) with $a = 1$, $t = 1/z$, $\phi(\zeta) = \zeta^p$. Choosing $f(\zeta) = \zeta^r$, substitution into (2.5) shows that

$$\begin{aligned} w^r &= 1 + \sum_{n=1}^{\infty} \frac{1}{z^n n!} \frac{d^{n-1}}{da^{n-1}} \Big(r a^{r-1} a^{np} \Big) \Big|_{a=1} \\ &= 1 + r \sum_{n=1}^{\infty} \frac{1}{z^n n!} (np + r - 1)_{n-1} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{z^n} R_{p,r}(n) \ = \ z G_{p,r}(z). \end{aligned}$$

This establishes (2.2) in the case $r = 1$, and moreover shows that

$$(z G_{p,1}(z))^r = z G_{p,r}(z). \tag{2.6}$$

The latter identity together with the validity of (2.2) for $r = 1$ establishes its validity for general r.

Another viewpoint on (2.2) in the case $r = 1$ is that it stems from the recurrence (1.3). Thus multiplying both sides by $1/z^k$ and summing over $k = 1, 2, \dots$ we see

$$\sum_{k=0}^{\infty} C_s(k) z^{-k} - 1 = z^{-1} \Big(\sum_{k=0}^{\infty} C_s(k) z^{-k} \Big)^s.$$

Identifying w with $\sum_{k=0}^{\infty} C_s(k) z^{-k}$, this is (2.2) in the case $r = 1$.

We know from (2.1) that $w(z)$ has a branch cut for z on the real axis between 0 and K_p. Biane [10] and independently Neuschel [54] sort to parametrize the cut by a variable ϕ such that (2.2) in the case $r = 1$ permits a pair of solutions in polar form $w(\phi) = a(\phi) e^{i\phi}$. It was observed that this is possible if one uses the parametrization, a strictly decreasing function (for $0 < c < 1$, the function $\frac{\sin\theta}{\sin(c\theta)}$ is strictly decreasing on $(0, \pi)$)

$$x = \rho(\varphi) = \frac{(\sin p\varphi)^p}{\sin\varphi \, (\sin(p-1)\varphi)^{p-1}}, \qquad 0 < \varphi < \frac{\pi}{p}, \tag{2.7}$$

for then one can immediately verify that the two solutions of (2.2) with $r = 1$ are given by

$$\frac{\sin p\varphi}{\sin(p-1)\varphi} e^{i\varphi} \quad \text{and} \quad \frac{\sin p\varphi}{\sin(p-1)\varphi} e^{-i\varphi}. \tag{2.8}$$

The solutions (2.8) have the property of both converging to the real value $p/(p-1)$ as $\varphi \to 0^+$ (i.e. $x \to K_p^-$) and converging to the real value 0 as $\varphi \to \pi/p$ from below (i.e. $x \to 0^+$). Thus they correspond to the values of $w(z)$ implied by (2.1) in the case that z approaches x, $0 < x < K_p$ from the two sides of the cut. On the other hand, from the inverse formula of the Stieltjes transform, we know that the density function for the measure $\mu_{p,1}$, $W_{p,1}(x)$ say, is given by

$$W_{p,1}(x) = \lim_{\epsilon \to 0^+} \frac{1}{2i\pi} \Big(\frac{w(x-i\epsilon)}{x-i\epsilon} - \frac{w(x+i\epsilon)}{x+i\epsilon} \Big), \quad 0 < x < K_p. \tag{2.9}$$

Consequently, upon making use of (2.8) one has [10, 37, 54]

$$W_{p,1}(\rho(\varphi)) = \frac{1}{\pi\rho(\varphi)} \frac{\sin p\varphi \sin\varphi}{\sin(p-1)\varphi} = \frac{(\sin(p-1)\varphi)^{p-2}(\sin\varphi)^2}{\pi(\sin p\varphi)^{p-1}}. \tag{2.10}$$

Our first new result is the application of the parametrization (2.7) to deduce the explicit form of the density for the Raney distribution in the cases $p \ge r > 0$. For this we observe from (2.6) that with x again parametrized by (2.7), (2.2) permits the solutions

$$\Big(\frac{\sin p\varphi}{\sin(p-1)\varphi} \Big)^r e^{ir\varphi} \quad \text{and} \quad \Big(\frac{\sin p\varphi}{\sin(p-1)\varphi} \Big)^r e^{-ir\varphi}. \tag{2.11}$$

These solutions have the property of each approaching 0 as $x \to 0^+$, and each approaching $p/(p-1)$ as $x \to K_p^-$. These values being real, it follows that as for the case $r = 1$ they correspond to the values of $w(z)$ on either side of the cut. Application of the inverse Stieltjes transform formula then gives the explicit form of the density $W_{p,r}(x)$ of the measure $\mu_{p,r}(x)$ for the Raney distribution, as we now specify.

Proposition 2.1. *Let* $W_{p,r}(x)$ *denote the density supported on* $[0, p^p(p-1)^{1-p}]$ *with k-th moments* $R_{p,r}(k)$ *of* (1.4). *If*

$$x = \rho(\varphi) = \frac{(\sin p\varphi)^p}{\sin\varphi\,(\sin(p-1)\varphi)^{p-1}}, \qquad 0 < \varphi < \frac{\pi}{p},$$

then

$$W_{p,r}(\rho(\varphi)) = \frac{(\sin(p-1)\varphi)^{p-r-1}\sin\varphi \sin r\varphi}{\pi(\sin p\varphi)^{p-r}}, \qquad 0 < \varphi < \frac{\pi}{p}. \tag{2.12}$$

A number of comments are in order.

Remark 2.2. We observe that (2.12) shows why it is necessary to restrict r to $0 < r \le p$: only then will $W_{p,r}(\rho(\varphi))$ be non-negative for all of the support $0 < x < K_p$ [50].

Remark 2.3. It is immediate from (1.4) that $R_{p,p}(k) = R_{p,1}(k+1)$. Consequently, as observed in [58], we must have $W_{p,p}(x) = xW_{p,1}(x)$. Thus functional property is exhibited by Proposition 2.1.

Remark 2.4. If we let $x = y^2$, then we get a symmetric density which have $2k$-moments $R_{p,r}(k)$

$$w_{p,r}(y) = |y| W_{p,r}(y^2), \qquad y \in [-\sqrt{K_p}, \sqrt{K_p}], \tag{2.13}$$

or the standard density with variance 1

$$\widetilde{w}_{p,r}(y) = r|y| W_{p,r}(ry^2), \qquad y \in \big[-\sqrt{K_p/r}, \sqrt{K_p/r}\big]. \tag{2.14}$$

These densities, restricted to $y > 0$, are for $r = 1$ the density for the singular values (rather than the singular values squared) of the random matrices introduced in the second paragraph of the Introduction.

We now turn our attention to an application of Proposition 2.1. We would like to use the explicit form of the density therein to analyze its singularities near the boundary of the support, i.e. the spectrum edges. As to be revised in subsection 3.2 below, the case $p = 2$, $r = 1$ is equivalent to the Marchenko-Pastur law for the scaled density of the eigenvalues of the random matrix product X^*X (X a matrix of standard Gaussians, for example). With the scaling such that the density is supported on $[0, 4]$, it is immediate that

$$W_{2,1}(x) \underset{x \to 0^+}{\sim} \frac{1}{\pi x^{1/2}}, \qquad W_{2,1}(x) \underset{x \to 4^-}{\sim} \frac{1}{2\pi} \sqrt{1 - x/4}. \tag{2.15}$$

The first of these behaviours distinguishes the hard edge in classical random matrix theory (see [28, Ch. 7]), which in turn comes about when the density is strictly zero for $x < 0$, and the joint distribution of eigenvalues can be interpreted as the Boltzmann factor of a one-component log-potential Coulomb gas supported on the half line $x > 0$. Knowledge of this leading singular form of the density allows the leading asymptotic $s \to \infty$ decay of the hard edge scaled (spacing between eigenvalues in the vicinity of $x = 0$ of order unity) gap probabilities for there being k eigenvalues in $(0, s)$ to be computed using the Dyson log-gas heuristic [34]. Similarly the second of the behaviours in (2.15) distinguishes the soft edge in classical random matrix theory (see [28, Ch. 7]). This edge of the support is referred to as "soft" due to the eigenvalue density being non-zero in the region $x > 4$ before the large N limit is taken. As at the hard edge, knowledge of the leading asymptotic form of the density at the soft edge can be used in combination with the Dyson log-gas heuristic to obtain predictions for the leading asymptotic $s \to -\infty$ decay of the soft edge scaled (spacing between eigenvalues in the vicinity of the largest eigenvalue of order unity) gap probabilities for there being k eigenvalues in (s,∞).

Crucial to these applications of the asymptotics of the global density to the asymptotics of gap probabilities is the matching of the former with the asymptotics of the microscopic hard and soft edge scaled densities, with the latter expanded into the bulk [35]. For Gaussian Hermitian, and Wishart matrices, with real, complex and real quaternion elements this was shown explicitly in [35, 32]. Thus, in addition to it being an intrinsic property of the densities themselves, there is much interest in isolating the leading singular forms at the boundaries of support for the Raney distribution. For rational p it is known from [50, 58] that as $x \to 0^+$ the density $W_{p,r}(x)$ is proportional to $x^{-(p-r)/p}$ for $r<p$, while for $r=p$ it is proportional to $x^{1/p}$. Proposition 2.1 allows us to give the explicit leading asymptotic form of the density upon the approach of either boundary of its support.

Corollary 2.5. *As $x \to 0^+$, we have*

$$W_{p,r}(x) \sim \begin{cases} \frac{1}{\pi}\sin\frac{r\pi}{p}\, x^{-\frac{p-r}{p}}, & r<p; \\ \frac{1}{\pi}\sin\frac{\pi}{p}\, x^{\frac{1}{p}}, & r=p. \end{cases} \tag{2.16}$$

As $x \to p^p(p-1)^{1-p}$ from below, we have

$$W_{p,r}(x) \sim \frac{\sqrt{2}r}{\pi}\frac{(p-1)^{p-r-3/2}}{p^{p-r+1/2}}\sqrt{1-p^{-p}(p-1)^{p-1}x}. \tag{2.17}$$

Proof. According to (2.7), x approaches the left boundary of support $x=0$ when $\varphi \to \pi/p$, and the precise functional form of this approach is given by

$$x \sim \Big(\frac{\sin p\varphi}{\sin\frac{\pi}{p}}\Big)^{p}. \tag{2.18}$$

Taking the same limit in (2.12) shows that for $r<p$

$$W_{p,r}(x) \sim \frac{1}{\pi}\Big(\frac{\sin\frac{\pi}{p}}{\sin p\varphi}\Big)^{p-r}\sin\frac{r\pi}{p}$$

while for $r=p$

$$W_{p,r}(x) \sim \frac{1}{\pi}\sin p\varphi.$$

Substituting (2.18) the first assertion follows.

The right boundary of support $x = p^p(p-1)^{1-p}$ is approached as $\phi \to 0$. Thus it follows from (2.7) that

$$x = \frac{p^p}{(p-1)^{p-1}}\Big(1 - \frac{p(p-1)}{2}\varphi^2 + o(\varphi^2)\Big). \tag{2.19}$$

Taking this same limit in (2.12) we have

$$W_{p,r}(x) \sim \frac{r(p-1)^{p-r-1}}{\pi p^{p-r}}\varphi. \tag{2.20}$$

Solving (2.19) for φ and substituting in (2.20) gives the second assertion. □

Remark 2.6. In the case $r = 1$ the asymptotic form (2.16) was recently given in [30, eq. (2.16)], by using the same argument as above specialised to Biane, Haagerup and Möller, and Neuschel's result (2.10). Moreover, a matching of this asymptotic form with the asymptotic form of the corresponding hard edge scaled microscopic density was exhibited for some values of s.

Remark 2.7. Applying the same analysis directly to (2.7) shows that as $x \to 0^+$,

$$w(x) \sim (e^{\pm \pi i}x)^{r/p}. \tag{2.21}$$

In the case $r = 1$, $w(x)$ is the function $\mathcal{B}_p(x)$ introduced above (2.3). Substituting (2.21) shows that for $x \to \infty$, to leading order in $1/x$

$$\mathcal{B}_p(x) \sim 1 + \frac{1}{x}.$$

This is consistent with (2.1), upon recalling that $R_{p,1}(0) = 1$, $R_{p,1}(1) = 1$.

Remark 2.8. The functional equation (2.3) can be used to give the complete series expansion of $W_{p,1}(x)$ about $x = 0$. Thus recalling (2.1) it gives

$$zG_{p,1}(z) = 1 - \sum_{n=0}^{\infty}(-\sqrt[p]{-z})^n R_{1/p,1}(n).$$

Recalling $W_{p,1}(x) = xG_{p,1}(x)$ we can now apply (2.9) to conclude

$$W_{p,1}(x) = \frac{1}{\pi x}\sum_{n=0}^{\infty}(-1)^n \sin \pi(n+1)/p \, R_{1/p,1}(n+1)\, x^{(n+1)/p}. \tag{2.22}$$

In fact this equation has been given very recently by Dupic and Castillo [25, displayed eq. above (10)]. This greatly simplifies the small-x expansion of [58, Eq. (22) with $r = 1$] involving the function ${}_pF_{p-1}$.

Remark 2.9. A direct application of Corollary 2.5 provides the asymptotic form of the symmetric densities given in Remark 2.4. For instance, as $y \to 0$ we have

$$\widetilde{w}_{p,r}(y) \sim \begin{cases} \frac{1}{\pi} r^{\frac{r}{p}} \sin\frac{r\pi}{p} \, |y|^{-1+\frac{2r}{p}}, & r < p; \\ \frac{1}{\pi} r^{1+\frac{1}{p}} \sin\frac{\pi}{p} \, |y|^{1+\frac{2}{p}}, & r = p. \end{cases} \tag{2.23}$$

3. Some special cases

There are a number of special cases of the Raney distribution for which the corresponding density, given in Proposition 2.1 in terms of our extension of Biane's and Neuschel's parametrization (2.7), can be written in an explicit algebraic form using the original spectral variable. Here we list these cases. Moreover, for a number of these a realization as the spectral density of a random matrix ensemble is known. We give some new cases, and show in all the examples how the corresponding algebraic equation for the resolvent (2.1) can be deduced from the differential equation satisfied by the corresponding characteristic polynomial. For this purpose, we will see that in many cases the characteristic polynomial can be written in terms of a generalized hypergeometric function, for which the differential equation is well known.

3.1. **Relating the averaged characteristic polynomial to the resolvent.**

The averaged characteristic polynomial for a random matrix ensemble can be written

$$\langle \det(\lambda I - X) \rangle_X = \langle e^{\log \det(\lambda I - X)} \rangle_X = \langle e^{\sum_{j=1}^N \log(\lambda - \lambda_j)} \rangle_X,$$

where $\{\lambda_j\}$ denotes the eigenvalues of X. Now, so-called weak-convergence to the normalised spectral measure $\mu(x)$ tells us that for $\lambda \notin J$, where J denotes the support of $\mu(x)$, and N large

$$\Big\langle \sum_{j=1}^N \log(\lambda - \lambda_j) \Big\rangle_X \sim N \int_J \log(\lambda - x) \, d\mu(x). \tag{3.1}$$

This suggests that for well behaved functions f,

$$\Big\langle f\Big(\sum_{j=1}^N \log(\lambda - \lambda_j) \Big) \Big\rangle_X \sim f\Big(N \int_J \log(\lambda - x) \, d\mu(x) \Big), \tag{3.2}$$

and thus in particular

$$\langle \det(\lambda I - X) \rangle_X \sim e^{N \int_J \log(\lambda - x) d\mu(x)}, \quad \lambda \notin J. \tag{3.3}$$

Further insight into (3.3) follows by recalling that by definition the characteristic function for a linear statistic $A = \sum_{j=1}^N a(\lambda_j)$ is equal to $\langle \prod_{j=1}^N e^{ika(\lambda_j)} \rangle_X$. With $\bar{a} := \int_J a(x)\, d\mu(x)$, it is known that for a large class of random matrix ensembles and linear statistics A

$$\Big\langle \prod_{j=1}^N e^{ika(\lambda_j)} \Big\rangle_X \sim e^{-k^2 \mathrm{Var}\, A/2}, \tag{3.4}$$

where $\mathrm{Var}\, A = O(1)$ (see e.g. [28, 56]). As well as implying (3.3), by choosing $k = -i$, $a(\lambda_j) = \log(\lambda - \lambda_j)$ this implies that the error term in the exponent on the RHS is then $O(1)$.

We recognise (3.4) as an example of a Gaussian fluctuation formula. Its validity for the linear statistic $a(\lambda_j) = \log(\lambda - \lambda_j)$, $\lambda \notin J$, and $X = (Y^\dagger + Y)/2$ where Y is a complex standard Gaussian matrix is known from [14]. Thus for this ensemble at least

$$\lim_{N\to\infty} \frac{1}{N} \frac{d}{d\lambda} \log \langle \det(\lambda I - X) \rangle_X = \int_J \frac{d\mu(x)}{\lambda - x}, \qquad \lambda \notin J. \tag{3.5}$$

Hence we have an asymptotic relationship between the averaged characteristic polynomial and the resolvent. This equation is stated in [11, above (10)] without derivation. As an aside, we make mention of a recent study relating the zeros of the averaged characteristic polynomial to the spectral density [38], and the work [15] relating to Gaussian fluctuation formulas for biorthogonal ensembles.

The statement (3.3) is weaker than (3.4) with $a(\lambda_j) = \log(\lambda - \lambda_j)$, $\lambda \notin J$, and is expected to hold true for a large class of matrix ensembles X, and thus so is (3.5). We will assume its validity below, and in particular its corollary (3.5), even though to our knowledge it is yet to be formally proved for $X = Y^\dagger Y$, where Y is a product of standard complex random matrices or their inverses, which are examples to be discussed in §3.3 and §5.

3.2. **$p = 2$, $r = 1$.** As is well known [49, 58], this case corresponds to both the Marchenko-Pastur law for the global density of the squared singular values of a single random matrix

$$W_{2,1}(x) = \frac{1}{2\pi} \sqrt{\frac{4 - x}{x}}, \qquad 0 < x \leq 4, \tag{3.6}$$

as well as the Wigner semi-circle law for the eigenvalues of a single real symmetric or complex Hermitian random matrix

$$\widetilde{w}_{2,1}(y) = \frac{1}{2\pi}\sqrt{4-y^2}, \qquad -2 \le y \le 2. \tag{3.7}$$

We would like to relate the algebraic equation satisfied by the resolvent for the Wigner semi-circle law, $\tilde{G}_{2,1}(z)$ say, to the differential equation satisfied by the characteristic polynomial. First we note that changing variables $z \mapsto z^2$ in (2.1) shows that this resolvent is related to the resolvent $G_{2,1}(z)$ for the Marchenko-Pastur law by $\tilde{G}_{2,1}(z) = zG_{2,1}(z^2)$. Recalling (2.2), we read off the well known fact (see e.g. [56]) that $\tilde{G}_{2,1}(z)$ satisfies the quadratic equation

$$\tilde{G}_{2,1}^2 - z\tilde{G}_{2,1} + 1 = 0. \tag{3.8}$$

Now it is well known (see e.g. [56]) that the Wigner semi-circle law is the limiting spectral density for real symmetric matrices, or complex Hermitian matrices, with elements on and above the diagonal independently distributed with zero mean and variance $1/2N$. On the other hand, it is similarly well known (see e.g. [31]) that the averaged characteristic polynomial for such matrices is proportional to the Hermite polynomial $H_N(\sqrt{N/2}x)$. Furthermore, it is a classical result that this polynomial satisfies the second order differential equation

$$\frac{2}{N}u'' - 2xu' + 2Nu = 0. \tag{3.9}$$

According to (3.5), $\tilde{G}_{2,1} = \lim_{N\to\infty} \frac{1}{N}\frac{d}{dx}\log u$. Manipulating (3.9) to be an equation in u'/u and expanding for large N, (3.8) is reclaimed.

3.3. $p = 3$, $r = 1$. The probability density $W_{3,1}(x)$ having moments given by the Raney numbers (1.4) in the case $p = 3$, $r = 1$, or equivalently the Fuss-Catalan numbers (1.3) with $s = 2$, first appeared in the work of Penson and Solomon [57] on quantum mechanical coherent states. In that work it was shown

$$W_{3,1}(x) = \frac{1}{2^{4/3}\pi}\frac{(3\sqrt{3}+\sqrt{27-4x})^{1/3} - (3\sqrt{3}-\sqrt{27-4x})^{1/3}}{x^{2/3}} \tag{3.10}$$

for $0 < x \le \frac{27}{4}$, or equivalently

$$\widetilde{w}_{3,1}(y) = \frac{1}{2^{4/3}\pi}\frac{(3\sqrt{3}+\sqrt{27-4y^2})^{1/3} - (3\sqrt{3}-\sqrt{27-4y^2})^{1/3}}{|y|^{1/3}} \tag{3.11}$$

for $-\sqrt{27/4} \le y \le \sqrt{27/4}$, $y \neq 0$. Subsequently (3.11) appeared as the limiting spectral density for Hermitian random matrices

$$iX^TJX, \qquad J = \mathbb{I}_N \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.12}$$

where X is a $2N \times 2N$ standard real Gaussian matrix. This ensemble was introduced in [47] in the context of a study into a random matrix model for disordered bosons, and later in a more mathematical context in [24]. The eigenvalues of (3.12) occur in the pairs $\pm w_j$ $(j = 1, \dots, N)$. It was shown in [47] that the limiting density of the scaled eigenvalues $\pm x_j := \pm w_j/\sqrt{2}N$ is equal to (3.11).

Here we take up of the task of computing the algebraic equation for the corresponding resolvent according to the method just given for the resolvent of the Wigner semi-circle law. For this purpose, we first identify the characteristic polynomial for the matrices (3.12) in terms of a particular generalized hypergeometric function ${}_1F_2$.

Proposition 3.1. *We have*

$$\langle \det(\lambda \mathbb{I}_{2N} - iX^TJX) \rangle = (-2)^{-N}(2N)! \ {}_1F_2\left({-N \atop 1/2, 1} \Big| \frac{\lambda^2}{2} \right). \tag{3.13}$$

Proof. We use ideas involving averages over the orthogonal group contained in [33], and further developed in [29]. For Y a square matrix, denote by $e_k(Y)$ the k-th elementary symmetric function (polynomial) in the eigenvalues, $\{\lambda_j\}_{j=1,\dots,N}$ of Y so that

$$e_k(Y) = \sum_{1 \le j_1 < \cdots < j_k \le N} \prod_{l=1}^{k} \lambda_{j_l}. \tag{3.14}$$

We then have

$$\det(\lambda \mathbb{I}_{2N} - iX^TJX) = \sum_{p=0}^{2N} \lambda^{2N-p} (-1)^p e_p(iX^TJX). \tag{3.15}$$

Thus our task is to compute the matrix averages $\langle e_p(iX^TJX) \rangle_X$, where X is drawn from the set of $2N \times 2N$ real standard Gaussian random matrices.

Using the formula [33, (3.2)] we have the simplification

$$\langle e_p(iX^TJX) \rangle_X = \frac{e_p(iJ)}{e_p(\mathbb{I}_{2N})} \langle e_p(X^TX) \rangle_X. \tag{3.16}$$

As a consequence of the eigenvalues of iJ being equal to ± 1, each with multiplicity N, we have $\sum_{p=0}^{2N} z^p e_p(iJ) = (1-z^2)^N$, and so

$$e_p(iJ) = \begin{cases} 0, & p \text{ odd} \\ (-1)^{p/2}\binom{N}{p/2}, & p \text{ even.} \end{cases} \tag{3.17}$$

Furthermore, we read off from [33, (3.9)] that

$$\frac{1}{e_p(\mathbb{I}_{2N})}\langle e_p(X^T X)\rangle_X = 2^{p/2}\prod_{j=1}^{p}(N-(j-1)/2). \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.16), then substituting the result in (3.15) with p replaced by $2N-p$, we see after minor manipulation that

$$\langle \det(\lambda \mathbb{I}_{2N} - iX^T JX)\rangle = (-2)^{-N}(2N)!\sum_{p=0}^{N}\frac{(-N)_p}{p!(1/2)_p(1)_p}(\lambda^2/2)^p.$$

This is precisely (3.13). □

To make use of Proposition 3.1, we require the standard fact that the generalized hypergeometric function ${}_pF_q\left({a_1,\dots,a_p \atop b_1,\dots,b_q}\middle| x\right)$ satisfies the differential equation

$$x\prod_{n=1}^{p}\Big(x\frac{d}{dx}+a_n\Big)f = x\frac{d}{dx}\prod_{n=1}^{q}\Big(x\frac{d}{dx}+b_n-1\Big)f. \tag{3.19}$$

The Green's function corresponding to (3.11), $\tilde{G}_{3,1}(z)$ say, is related to the Green's function for (3.10) by $\tilde{G}_{3,1}(z) = zG_{3,1}(z^2)$, and thus recalling (2.2) must satisfy

$$z\tilde{G}_{3,1}^3 - z\tilde{G}_{3,1} + 1 = 0. \tag{3.20}$$

We know the Green's function is related to the characteristic polynomial by (3.5). Denoting the characteristic polynomial by $p(\lambda)$, it follows from Proposition 3.1 and (3.19) that

$$\frac{\lambda^2}{2}\Big(\frac{\lambda}{2}\frac{d}{d\lambda}-N\Big)p = \frac{\lambda}{2}\frac{d}{d\lambda}\Big(\frac{\lambda}{2}\frac{d}{d\lambda}\Big)\Big(\frac{\lambda}{2}\frac{d}{d\lambda}-\frac{1}{2}\Big)p. \tag{3.21}$$

By expressing higher derivatives of p in terms of the logarithmic derivative, and recalling from (3.5) that the latter is proportional to N for large N, we see that to leading order in N

$$\frac{p^{(k)}}{p} \sim \Big(\frac{p'}{p}\Big)^k \tag{3.22}$$

(in the case $k=2$ this equation has already been used in going from (3.9) to (3.8)). Using this fact, now with $k=3$, and furthermore replacing λ by $\sqrt{2}Nz$ and recalling (3.5) with N by $2N$ we see that (3.21) reduces to (3.20) in the $N \to \infty$ limit.

If we consider (3.10) rather than (3.11) there is another random matrix interpretation to the Raney distribution with parameters $p = 3$, $r = 1$. This has already been mentioned in the third paragraph of the Introduction: it gives the limit spectral distribution of the squared singular values of the random matrix power X^2, or random matrix product $X_1 X_2$, with X, X_1, X_2 standard real Gaussian random matrices (for example) [58, 65]. This in turn is a special case of the result [58] that the Raney distribution with parameters $p = s+1$, $r = 1$ gives the limit spectral distribution of the squared singular values of the random matrix power X^s, or random matrix product $X_1 \cdots X_s$. For this latter problem, we can also deduce the polynomial equation (2.1) satisfied by the resolvent from knowledge of the differential equation for the corresponding averaged characteristic polynomial, as done in the above calculations.

Let X_i, $i = 1, \dots, s$ be independent standard complex Gaussian $N \times N$ matrices. The averaged characteristic polynomial has been shown in [4, 5] to be equal to

$$(-1)^N (N!)^s \, {}_1F_s\Big({-N \atop 1, \dots, 1} \Big| \lambda \Big). \tag{3.23}$$

We thus read off from (3.19) that the characteristic polynomial satisfies the differential equation

$$\lambda\Big(\lambda \frac{d}{d\lambda} - N\Big) p = \lambda \frac{d}{d\lambda} \prod_{n=1}^{s} \lambda \frac{d}{d\lambda} p. \tag{3.24}$$

Changing variables $\lambda = N^s z$, replacing $p'(z)/p(z)$ by $NG(z)$ as is consistent with (3.5), and expanding for large N using (3.22) we deduce that

$$z(zG(z) - 1) = (zG(z))^{s+1}. \tag{3.25}$$

With $zG(z) = w(z)$ this is precisely (2.1) in the case $p = s+1, r = 1$.

3.4. $p = 3$**,** $r = 2$**.** Penson and Życzkowski [58] have shown that with these parameters, the density for the Raney density takes on the explicit form

$$W_{3,2}(x) = \frac{1}{2^{5/3} 3^{1/2} \pi} \frac{(3\sqrt{3} + \sqrt{27 - 4x})^{2/3} - (3\sqrt{3} - \sqrt{27 - 4x})^{2/3}}{x^{1/3}} \tag{3.26}$$

for $0 < x \le \frac{27}{4}$, or equivalently

$$\widetilde{w}_{3,2}(y) = \frac{|y|^{1/3}}{2\sqrt{3}\pi} \Big((3\sqrt{3} + \sqrt{27 - 8y^2})^{2/3} - (3\sqrt{3} - \sqrt{27 - 8y^2})^{2/3} \Big) \tag{3.27}$$

for $-\sqrt{27/8} \le y \le \sqrt{27/8}$. They do not give any random matrix realization. However, it is possible to find (3.27) in the thesis of Nadal [52, eq. (6.118)], where it appears as the global density for the Pearcey process. The Pearcey process corresponds to the eigenvalue density for the critical case of the Gaussian unitary ensemble with two sources, symmetrically place about the origin [16]. Explicitly, the random matrices for this ensemble are of the form $X + X_0$, where X is a member of the Gaussian unitary ensemble (matrices $(Y + Y^\dagger)/2$ where Y is a standard complex Gaussian matrix), and X_0 is a diagonal matrix with half the diagonal entries equal to a and the other half equal to $-a$. The parameter a is be related to N in such a way that for large N the spectrum vanishes as a power law as the origin is approached — this is the critical case described by the Pearcey process [13].

In this setting one encounters in [13] the cubic equation

$$\xi^3 - x\xi^2 + x = 0. \tag{3.28}$$

Introducing the variable $h := 1 - \xi/x$ this reads

$$h^3 - 2h^2 + h - 1/x^2 = 0. \tag{3.29}$$

We would like to relate (3.28) to an equation satisfied by $G_{3,2}(z)$. First, we note from (2.2) that $w(z) := zG_{3,1}(z)$ satisfies the cubic equation $w^3 - zw + z = 0$. Taking z to the RHS and squaring gives $w^6 - 2zw^4 + z^2w^2 = z^2$. On the other hand, (2.6) gives that $w^2 = zG_{3,2}(z)$ and so

$$(zG_{3,2})^3 - 2z(zG_{3,2})^2 + z^2(zG_{3,2}) - z^2 = 0. \tag{3.30}$$

We recognize this equation as identical to (3.29) with $G_{3,2} = h$ and $z = x^2$. Introducing $\tilde{G}_{3,2}(z) = zG_{3,2}(z^2)$, in keeping in going from (3.26) to (3.27), (3.30) reads

$$(\tilde{G}_{3,2})^3 - 2z(\tilde{G}_{3,2})^2 + z^2\tilde{G}_{3,2} - z = 0. \tag{3.31}$$

In the remainder of this section, we will show how the cubic equation (3.31) can be deduced from the differential equation satisfied by the characteristic polynomial for the Gaussian unitary ensemble with two sources.

The eigenvalue distribution for the Gaussian unitary ensemble with a source is an example of a biorthogonal ensemble, with the corresponding biorthogonal system being that which specifies the so-called type II multiple Hermite polynomials. General theory [12, 26] tells us that the characteristic polynomial must then be given in terms of the latter. In the case of interest, where the source is specifed by a diagonal matrix X_0 with diagonal entries taking on only two possible values, the multiple Hermite polynomials are indexed by two non-negative integers m and n say. It is known that these polynomials satisfy a third order linear differential equation [41, 23, 27], which with source parameters denoted by c_1 and c_2 we read off from [27] to be given by

$$\begin{aligned} p'''(x) + (c_1 + c_2 - 4x)p''(x) &+ (c_1(c_2 - 2x) + 2(m + n - 1 - c_2 x + 2x^2))p'(x) \\ &+ 2(c_1 m + c_2 n - 2(m + n)x)p(x) = 0. \end{aligned} \tag{3.32}$$

More specifically, we want X_0 to consist of N values a and N values $-a$. The characteristic polynomial will then be proportional to the type II multiple Hermite polynomial with indices $m = n = N$. Specifying (3.32) as such, choosing $c_1 = -c_2 = 2\sqrt{N}a$ and scaling $x \mapsto \sqrt{N}x$, we obtain that the characteristic polynomial satisfies the differential equation

$$\frac{1}{N^2}p'''(x) - \frac{4x}{N}p''(x) + 4\Big(-a(a+x) + (1 - \frac{1}{2N} + x^2)\Big)p'(x) - 8Nxp(x) = 0. \tag{3.33}$$

To be consistent with (3.5) we replace $p'(x)/p(x)$ with $2Ng(x)$ (the factor of 2 comes about because there are a total of $2N$ eigenvalues), then expand for large N using (3.22). This gives that $g(x)$ satisfies the cubic equation

$$g^3 - 2xg^2 + (1 - a^2 + x^2)g - x = 0. \tag{3.34}$$

Setting $a = 1$, which corresponds to the critical case, we see that this is precisely the equation (3.31).

Remark 3.2. Replacing g in (3.34) by ξ according to $\xi = x - g$ we see that ξ satisfies

$$\xi^3 - 2x\xi^2 - (a^2 - 1)\xi + xa^2 = 0. \tag{3.35}$$

This equation appears in [13] as the generalization of (3.28) for general (non-critical) value of the rescaled source parameter a.

Remark 3.3. Complex chiral matrices have the structure

$$\begin{bmatrix} 0_N & X \\ X^* & 0_N \end{bmatrix}, \tag{3.36}$$

where X is a complex standard Gaussian matrix. The eigenvalues of such matrices occur in pairs $\pm\lambda_j$ ($\lambda_j > 0, j = 1, \dots, N$). A source is obtained by adding to (3.36) a matrix of the same structure as (3.36) but with X replaced by $X_0 = c\mathbb{I}_N$, $c > 0$. Furthermore the matrix X is scaled by a time variable t, and so in particular it vanishes for $t = 0$. The distribution of the squared eigenvalues for this model, or equivalently the distribution of the eigenvalues of the shifted mean Wishart matrices

$$(X + X_0)^*(X + X_0), \tag{3.37}$$

is identical to that for N non-intersecting squared Bessel processes all starting from $x = c$ at time $t = 0$ [42]. By scaling c with N, there is a well defined global density, and there is a critical value of this parameter for which the density touches the origin for the first time.

Moreover, the (squared) eigenvalue distribution is an example of a biorthogonal ensemble, and the corresponding characteristic polynomial is an example of a type II multiple Laguerre polynomial. The latter satisfies a third order differential equation [22]. In fact the equation satisfied by the large N form of the logarithmic derivative — which we know from (3.5) gives the resolvent — has been deduced from this [44] and is given by

$$z = \frac{1}{w(1-w)^2}.$$

This is just a rewrite of (3.30) with $w = G_{3,2}$.

Remark 3.4. Let X_0 be defined as in (3.37), let X be a standard complex Gaussian matrix, and similarly let $Y_1, \dots, Y_s$ be standard complex Gaussian matrices, all of size $N \times N$. What can be said about the scaled global density of the squared singular values of the product $(X + X_0)Y_1 \cdots Y_s$ in the case that the parameter c in X_0 is scaled to correspond to the critical case? As just discussed, the scaled density for the squared singular values of $X + X_0$ in the critical case is the Raney distribution $W_{3,2}$, while as rederived in the above subsection, this scaled density for the product $Y_1 \cdots Y_s$ is the Raney distribution $W_{s+1,1}$. In the language of free probability (see e.g. [8]) we are seeking the value of the free multiplication convolution $W_{3,2} \boxtimes W_{s+1,1}$. It is known [49, eq. (4.14)] that in general

$$W_{p,r} \boxtimes W_{s+1,1} = W_{p+rs,r}, \tag{3.38}$$

and so $W_{3,2} \boxtimes W_{s+1,1} = W_{3+2s,2}$. It is worth stressing that the leading form for $W_{3+2s,2}$ near zero, being given by

$$\frac{1}{\pi} \sin \frac{2\pi}{3+2s}\, x^{-1+\frac{1}{1+s+1/2}},$$

is different from the Fuss-Catalan distribution, which may indicate some new universality phenomenon at the hard edge.

3.5. $p = 3/2$**,** $r = 1/2$**.** We see from (2.2) that this choice of parameters, as with the previous two considered above, gives a cubic equation for the resolvent. In keeping with this is the explicit form of the density [50]

$$W_{3/2,1/2}(x) = \frac{1}{2^{5/3}3^{1/2}\pi} \frac{(3\sqrt{3}+\sqrt{27-4x^2})^{2/3} - (3\sqrt{3}-\sqrt{27-4x^2})^{2/3}}{x^{2/3}} \tag{3.39}$$

for $0 < x \le \sqrt{27/4}$ or equivalently

$$\widetilde{w}_{3/2,1/2}(y) = \frac{1}{4\sqrt{3}\pi} \frac{(3\sqrt{3}+\sqrt{27-2y^4})^{2/3} - (3\sqrt{3}-\sqrt{27-2y^4})^{2/3}}{|y|^{1/3}} \tag{3.40}$$

for $-\sqrt[4]{27/2} \le y \le \sqrt[4]{27/2}, y \ne 0$. Furthermore, it was observed in [50] that (3.39) occurs in random matrix theory as the so called Bures distribution [61]. This is realized as the global eigenvalue density of the random matrices $(1+U)XX^*(1+U^*)$, where X is a complex standard Gaussian matrix, and U is a random unitary matrix with Haar measure [55]. We would like to use this realization to deduce first the explicit form of the averaged characteristic polynomial, and then from that the cubic equation for the resolvent.

Proposition 3.5. *For X and U members of the ensembles as specified, we have*

$$\langle \det(\lambda \mathbb{I}_N - (1+U)XX^*(1+U^*))\rangle_{X,U} = (-1)^N (N+1)!\, {}_2F_2\left({-N, N+2 \atop 1, 3/2} \Big| 4\lambda \right). \tag{3.41}$$

.

Proof. We proceed as in the proof of Proposition 3.1. Thus we write

$$\det(\lambda \mathbb{I}_N - (1+U)XX^*(1+U^*)) = \sum_{p=0}^{N} \lambda^{N-p}(-1)^p e_p((1+U)XX^*(1+U^*)), \tag{3.42}$$

showing that our task is to compute the matrix averages $\langle e_p((1+U)XX^*(1+U^*))\rangle_{X,U}$, where X is drawn from the set of $N \times N$ complex standard Gaussian random matrices, and U from the set of unitary matrices with Haar measure.

Analogous to (3.16) we have

$$\langle e_p((1+U)XX^*(1+U^*))\rangle_{X,U} = \frac{\langle e_p((1+U)(1+U^*)\rangle_U}{e_p(\mathbb{I}_N)} \langle e_p(X^TX)\rangle_X. \tag{3.43}$$

Furthermore, we read off from [33, (3.13)] that

$$\frac{1}{e_p(\mathbb{I}_N)} \langle e_p(X^TX)\rangle_X = \prod_{j=1}^{p} (N-(j-1)). \tag{3.44}$$

To compute the average over U, we first note that

$$\sum_{p=0}^{N} \zeta^p \langle e_p((1+U)(1+U^*)\rangle_U = \Big\langle \det\Big(\mathbb{I}_N + \zeta(1+U)(1+U^*)\Big)\Big\rangle_U$$

In terms of the eigenvalues of U, $\{z_j\}_{j=1,\dots,N}$ say with $|z_j| = 1$, the determinant factorizes to read

$$\det\Big(\mathbb{I}_N + \zeta(1+U)(1+U^*)\Big) = \prod_{j=1}^{N}\Big(1+2\zeta+\zeta(z_j+\frac{1}{z_j})\Big).$$

Recalling now that any average over the unitary group involving a product of the individual eigenvalues can be written as a Toeplitz determinant (see e.g. [28, eq. (5.76)]), and noting that in the present case the elements in position (jk) of the latter are given by

$$\mathrm{CT}\Big(1+2\zeta+\zeta(z+\frac{1}{z})\Big)z^{j-k} = \begin{cases} \zeta, & |j-k|=1 \\ 1+2\zeta, & j=k \\ 0, & \text{otherwise} \end{cases},$$

where CT denotes the constant term in the Laurent expansion with respect to z, we see that our task is reduced to one of evaluating a symmetric tridiagonal Toeplitz determinant which is constant down both the diagonal and its two neighbours.

We can verify that with B the symmetric $N \times N$ matrix with entries x in positions with $|j-k| = 1$ and entries 0 elsewhere

$$\det(\mathbb{I}_N + B) = \sum_{p=0}^{[N/2]} \binom{N-p}{p} (-1)^p x^{2p}.$$

This allows us to deduce that

$$\Big\langle \det\Big(\mathbb{I}_N + \zeta(1+U)(1+U^*)\Big)\Big\rangle_U = \sum_{p=0}^{[N/2]} \binom{N-p}{p}(-1)^p \zeta^{2p}(1+2\zeta)^{N-2p}.$$

Extracting the coefficient of ζ^p from this gives a sum over binomial coefficients which can be evaluated, and we conclude

$$\langle e_p((1+U)XX^*(1+U^*))\rangle_{X,U} = \frac{(2N-p+1)!}{(2N-2p+1)!p!}. \tag{3.45}$$

Substituting (3.44) and (3.45) in (3.42) we obtain

$$\Big\langle \det(\lambda \mathbb{I}_N - (1+U)XX^*(1+U^*))\Big\rangle_{X,U} = (-1)^N N! \sum_{q=0}^{N} \frac{(N+q+1)!}{q!(2q+1)!(N-q)!}(-\lambda)^q. \tag{3.46}$$

Noting that

$$\frac{N!}{(N-q)!} = (-1)^q(-N)_q, \quad (2q+1)! = 2^{2q}(1)_q(3/2)_q, \quad (N+q+1)! = (N+1)!(N+2)_q$$

we see that (3.46) is equivalent to (3.41). □

Replacing λ by $Nx/4$, it follows from (3.41) and (3.19) that to leading order in N

$$x\Big(\Big(x\frac{d}{dx}\Big)^2 - N^2\Big)p = \Big(x\frac{d}{dx}\Big)^3 p.$$

Now making use of (3.22) in the cases $k = 2, 3$ and recalling (3.5) it follows that the limiting resolvent, g say, satisfies the cubic equation

$$x((xg)^2 - 1) = (xg)^3,$$

which with $xg = w$ is precisely (2.2) in the case $p = 3/2$, $r = 1/2$.

Remark 3.6. The eigenvalues of the random matrix $(1+U)XX^*(1+U^*)$ are equal to the squared singular values of $(1+U)X$. What can we say about the squared singular values of $(1+U)XX_1\cdots X_s$, where X and each X_i is a standard complex Gaussian matrix, and U is a random unitary matrix chosen with Haar measure? We are asking for $W_{3/2,1/2} \boxtimes W_{s+1,1}$. Applying (3.38) tells us that this is equal to $W_{(3+s)/2,1/2}$. In particular, we have [50]

$$W_{2,1/2}(x) = \frac{\sqrt{2-\sqrt{x}}}{2\pi x^{3/4}}, \qquad 0 < x < 4. \tag{3.47}$$

4. Equilibrium problem

In Section 3.4 a realization of the Raney distribution with parameters $p = 3$, $r = 2$ as the eigenvalue density for the Pearcey process was given. It was commented that the corresponding symmetric density $\tilde{w}_{3,2}(y)$ as specified by (3.27) can be found in the thesis of Nadal [52]. In the latter it is furthermore shown that $\tilde{w}_{3,2}(y)$ is the density $\rho_{(1)}(x)$ which minimizes the energy functional

$$\begin{aligned}\tilde{E}_{3,2}[\rho_{(1)}(x)] = &\int_0^L (x + 1/\sqrt{2})^2 \rho_{(1)}(x)\, dx \\ &- \frac{1}{2}\int_0^L dx \int_0^L dx'\, \rho_{(1)}(x)\rho_{(1)}(x') \log\Big(|x - x'||x^2 - (x')^2|\Big), \quad (4.1)\end{aligned}$$

subject to the normalization $\int_0^L \rho_{(1)}(x)\, dx = 1/2$. The functional (4.1) is very revealing when compared against the energy functional known for the Raney distribution with $p = 3$, $r = 1$ using squared variables, which we know has density (3.11). It is shown in [47] that $\tilde{w}_{3,1}(y)$ is the density $\rho_{(1)}(x)$ which minimizes the energy functional

$$\begin{aligned}\tilde{E}_{3,1}[\rho_{(1)}(x)] = &\int_0^L x\rho_{(1)}(x)\, dx \\ &- \frac{1}{2}\int_0^L dx \int_0^L dx'\, \rho_{(1)}(x)\rho_{(1)}(x') \log\Big(|x - x'||x^2 - (x')^2|\Big), \quad (4.2)\end{aligned}$$

subject to the normalization $\int_0^L \rho_{(1)}(x)\, dx = 1/2$.

We thus see that both (4.1) and (4.2) are special cases of the family of energy functionals

$$\begin{aligned}\tilde{E}(V;\theta)[\rho_{(1)}(x)] = &\int_0^L V(x)\rho_{(1)}(x)\, dx \\ &- \frac{1}{2}\int_0^L dx \int_0^L dx'\, \rho_{(1)}(x)\rho_{(1)}(x') \log\Big(|x - x'||x^\theta - (x')^\theta|\Big),\end{aligned} \quad (4.3)$$

for $\theta = 2$ and particular V. In the case $\theta \in \mathbb{Z}^+$ by writing $|x^\theta - (x')^\theta| = \prod_{k=0}^{\theta-1} |x - \omega^k x'|$ where $\omega = e^{2\pi i/\theta}$ we can interpret this as the Boltzmann factor for a log-potential Coulomb gas on the half line at inverse temperature $\beta = 2$, with image particles of charge $+1/2$ along rays in the direction of ω^k for $k = 1, \dots, \theta - 1$ (see [28, §3.1.4] for a related interpretation of an energy functional). The significance of (4.3) is that in the case that $V(x)$ is such that $\rho_{(1)}(x)$ is supported on $(0, L)$ (one-cut assumption with hard edge) or (L_0, L) with $0 < L_0 < L$ the equilibrium problem for (4.3) has been recently solved by Claeys and Romano [20] for all $\theta \geq 1$. Examination of their results reveals an intimate relationship with the Raney distribution, and in fact allows us to specify the equilibrium problem for all Raney distributions with $r = 1$ (Fuss-Catalan case) and also to provide a qualitative understanding of the cases $r \in \mathbb{Z}^+$ for $r < p$.

First, some notation is needed. With $\theta \geq 1$ the same parameter as in (4.2), define

$$J(s) = (s+1)\Big(\frac{s+1}{s}\Big)^{1/\theta}\theta. \tag{4.4}$$

As done in [20], it is easy to verify that there are two complex conjugate curves $\gamma_\pm$, in the upper and lower half plane respectively, which join $s = -1$ and $s = 1/\theta$ and are such that $J(s)$ is real for $s \in \gamma_\pm$. Define $I_\pm(x) \in \gamma_\pm$ by the requirement that

$$J(I_\pm(x)) = x. \tag{4.5}$$

And with $\rho_{(1)}(x)$ the density minimizing (4.3), and $H_\theta = \{z \in \mathbb{C} : -\pi/\theta < \arg z < \pi/\theta\}$ define

$$\tilde{g}(z) = \int_0^L \log(z^\theta - y^\theta)\rho_{(1)}(y)\,dy, \qquad z \in H_\theta \backslash [0,\infty) \tag{4.6}$$

and

$$\tilde{G}(z) = \frac{d}{dz}\tilde{g}(z). \tag{4.7}$$

Proposition 4.1. [20, §4.5.1] *Let $V(x) = x$ and set $L = (1+\theta)^{1+1/\theta}$. Then the Green's function (4.7), as it approaches the interval $[0, L]$ from the upper plane (+) or the lower plane (−) is given by*

$$x\tilde{G}_\pm(x) = \theta(I_\mp(x) + 1). \tag{4.8}$$

Remark 4.2. In [20], Proposition 4.1 is stated with the restriction that $\theta > 1$. However, we suspect that this is not necessary.

Recalling (4.4) and (4.5) we thus have

$$(x\tilde{G}_\pm(x)/\theta)^{1+1/\theta}\frac{1}{(x\tilde{G}_\pm(x)/\theta - 1)^{1/\theta}} = x. \tag{4.9}$$

Let us now replace x by $x^{1/\theta}$ and write $x^{1/\theta}\tilde{G}_\pm(x^{1/\theta})/\theta = w$. Minor manipulation of (4.9) then gives

$$w^{\theta+1} = x(w-1), \tag{4.10}$$

which we recognize as being identical to the equation (2.2) in the case $\theta + 1 = p$, $r = 1$, uniquely determining the Fuss-Catalan distribution. As a consequence, we have identified an equilibrium problem for this distribution.

Corollary 4.3. *The energy functional*

$$\begin{aligned} E_\theta[\rho_{(1)}(y)] =& \theta \int_0^L y^{1/\theta} \rho_{(1)}(y)\,dy \\ & - \frac{1}{2}\int_0^L dy \int_0^L dy'\, \rho_{(1)}(y)\rho_{(1)}(y') \log\Big(|y^{1/\theta} - (y')^{1/\theta}||y-y'|\Big), \end{aligned} \tag{4.11}$$

subject to the normalization $\int_0^L \rho_{(1)}(y)\,dy = 1$ *is minimized by the density function for the Raney distribution* $p = \theta+1$, $r = 1$, *or equivalently Fuss-Catalan distribution with* $s = \theta$.

Remark 4.4. Changing variables $y = x^\theta$, the point process corresponding to (4.11) has its joint probability density function proportional to

$$\prod_{j=1}^N e^{-x_j} \prod_{1\le j<k\le N} |x_j - x_k||x_j^\theta - x_k^\theta|.$$

This was introduced into random matrix theory by Muttalib [51] (for a realization as in terms of a Wishart matrix formed out of triangular matrices, see the very recent work [18]), and the correlations at the hard edge subsequently computed by Borodin [17]. Specifically, with Wright's generalized Bessel function defined as

$$J_{a,b}(x) = \sum_{m=0}^\infty \frac{(-x)^m}{m!\Gamma(a+bm)},$$

it was shown in [17] that

$$\rho_{(1)}^{\rm h}(x) = \theta \int_0^1 J_{1/\theta,1/\theta}(xt) J_{1,\theta}((xt)^\theta)\,dt,$$

where the superscript "h" denotes the hard edge limit. In keeping with Remark 2.6, we expect a matching of the corresponding large x form with (2.16) in the case $r = 1$,

$$\rho_{(1)}^{\rm h}(x)\,dx\Big|_{y=x^\theta} \underset{x\to\infty}{\sim} \frac{1}{\pi}\sin\Big(\frac{\pi}{\theta+1}\Big) y^{-1+1/(\theta+1)}\,dy. \tag{4.12}$$

We next turn our attention to the case $r = 2$ of the Raney distributions. In the case $\theta = 2$, and using squared variables, we know that the corresponding energy functional is given by (4.1). In [20, §4.5.2] it is argued (without giving a rigorous proof) that for the energy functional (4.3) with potential $V(x) = (x - c)^2$, there is a value of c for which the lower support of the equilibrium density goes from being positive to zero (or equivalently soft to hard edge), and for this value of c the density is proportional to $x^{(\theta-1)/(\theta+1)}$. Changing variables $y = (\theta x)^\theta$ as done in arriving at (4.11), the density becomes proportional to $y^{-(\theta-1)/(\theta+1)}$ which is in agreement with (2.16) in the case $p = \theta + 1$, $r = 2$.

Remark 4.5. Another way for the exponent $(\theta - 1)/(\theta + 1)$ to change sign is to replace θ by $1/\theta$. With $0 < \theta < 1$, the sign changed exponent for the singular behaviour of a density specifed by an energy functional can be found in the works of Zinn-Justin [66] and Kostov [43], relating to the matrix model formalism of the six-vertex model on a random lattice. With $\omega = e^{-i\pi\theta/2}$, $0 < \theta < 1$, the energy functional is

$$E_\theta[\rho_{(1)}(y)] = \int_0^L (y-c)^2 \rho_{(1)}(y)\, dy - \frac{1}{2}\int_0^L dy \int_0^L dy'\, \rho_{(1)}(y)\rho_{(1)}(y') \log\left|\frac{y - y'}{\omega y + \omega^{-1} y'}\right|,$$

where c is to be tuned so that the lower boundary of support goes from being positive to zero. In [66], the n-th moment of $\rho_{(1)}(y)$, after suitable scaling of y, is computed to be equal to

$$2\frac{\Gamma(1 + (1+\theta)n/2)\Gamma(1 + (1-\theta)n/2)}{\Gamma(3 + n)}$$

(cf. (1.4)).

Remark 4.6. This (heuristic) understanding of the Raney distribution for $r = 2$ suggests that a realization of the general $r \in \mathbb{Z}^+$ case, $r < p$, results from the equilibrium problem (4.3) with $V(x)$ given by a certain "tuned" degree r polynomial in the soft-to-hard transition (for "tuned" polynomial potentials in the case of one-cut matrix models generalizing the Gaussian ensembles — so called critical unitary random matrix ensembles — see [19, Eq. (1.8)]). One would expect that it is also necessary to change variables $y = x^\theta$, and identify $p = \theta + 1$ as required for $r = 1$ and $r = 2$.

Remark 4.7. The equilibrium problem for the Raney distribution with parameters $p = 3/2$, $r = 1/2$ is known from [61]. Thus one knows that the corresponding density $\rho_{(1)}(x)$ minimizes the energy functional

$$\begin{aligned} E_{3/2,1/2}[\rho_{(1)}(x)] =& \theta \int_0^L x\rho_{(1)}(x)\,dx \\ & - \frac{1}{2}\int_0^L dx \int_0^L dx'\, \rho_{(1)}(x)\rho_{(1)}(x') \log\Big(\frac{|x-x'|}{|x+x'|}|x^\theta - (x')^\theta|\Big), \end{aligned} \tag{4.13}$$

with $\theta = 1$ and subject to the normalization $\int_0^L \rho_{(1)}(x)\,dx = 1$. An obvious question is to ask if this same energy functional characterizes the Raney distribution with parameters $p = \theta/2 + 1$, $r = 1/2$, and if changing the linear potential x in the first term to a specially tuned degree k polynomial allows for r to be varied from 1/2. The case $\theta = 2$ of (4.13) is the well known energy functional for the Marchenko-Pastur law (3.6), i.e. the case $\theta = 1$ of (4.11). On the other hand the density for the Raney distribution with parameters $p = 2$, $r = 1/2$ is given by (3.47). This agrees with (3.6) upon the change of variables $y = x^\theta$, so as found for (4.3), this change of variables will also be required as a final step.

5. A FURTHER CLASS OF POLYNOMIAL EQUATIONS

In the case $r = 1$, $p > 1$ and integer, the general algebraic equation (2.2) specializes to the polynomial equation $w^p - zw + z = 0$. Here we will show that a random matrix structure recently considered in [30], involving the product of standard complex Gaussian matrices and their inverses, leads to a natural generalization of this equation for the resolvent.

For $s, q \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, let $X_1, \ldots, X_s, \widetilde{X}_1, \ldots, \widetilde{X}_q$ be independent standard complex Gaussian matrices. Moreover, let X_j and $\widetilde{X}_k$ be of dimension $N_j \times N_{j-1}$ ($j = 1, 2, \ldots, s$) and $\widetilde{N}_k \times \widetilde{N}_{k-1}$ ($k = 1, 2, \ldots, q$) respectively. Require that $N_0 = \min\{N_0, \ldots, N_s\}$, $\widetilde{N}_0 = \min\{\widetilde{N}_0, \ldots, \widetilde{N}_q\}$, and furthermore that $N_0 = \widetilde{N}_0 = N$. With these specifications, introduce the products

$$Y_s = X_s X_{s-1} \cdots X_1, \qquad \widetilde{Y}_q = \widetilde{X}_q \widetilde{X}_{q-1} \cdots \widetilde{X}_1,$$

and use these to define product Wishart-type matrices involving inverse Gaussians according to

$$A_{s,q} = \big(\widetilde{Y}_q^* \widetilde{Y}_q\big)^{-1/2} \big(Y_s^* Y_s\big) \big(\widetilde{Y}_q^* \widetilde{Y}_q\big)^{-1/2}. \tag{5.1}$$

Another viewpoint is to replace the rectangular matrix $\widetilde{X}_k$ by an $N \times N$ complex Gaussian matrix $\widetilde{X}_k$ with distribution proportional to

$$\big(\det \widetilde{X}_k^* \widetilde{X}_k\big)^{\mu_k} e^{-\mathrm{tr} \widetilde{X}_k^* \widetilde{X}_k},$$

where $\mu_k = \widetilde{N}_k - N$. Then $\tilde{Y}_q$ is a product of square matrices, and as in [30] we can modify the definition (5.1) to a more familiar Wishart form $((Y_s \tilde{Y}_q)^{-1})^* (Y_s \tilde{Y}_q^{-1})$.

For $q > 0$, the averaged characteristic polynomial of $A_{s,q}$ is not well defined, since for example the averaged determinant diverges. However, as noted in [30], the eigenvalues of $A_{s,q}$ are the same as the eigenvalues in the generalized eigenvalue problem $Y_s^* Y_s \vec{v} = \lambda \widetilde{Y}_q^* \widetilde{Y}_q \vec{v}$. The characteristic polynomial for the latter is

$$P_N^{(s,q)}(z) := \big\langle \det \big(z \widetilde{Y}_q^* \widetilde{Y}_q - Y_s^* Y_s\big)\big\rangle. \tag{5.2}$$

In [30, Prop. 2] this has been evaluated, telling us that

$$P_N^{(s,q)}(z) = (-1)^N \prod_{l=1}^{s} (\nu_l + 1)_N \, {}_{q+1}F_s\Big({-\widetilde{N}_0, -\widetilde{N}_1, \dots, -\widetilde{N}_q \atop \nu_1 + 1, \dots, \nu_s + 1} ; (-1)^q z\Big), \tag{5.3}$$

where $\nu_l = N_l - N$ $(l = 1, 2, \dots, s)$. Introducing the rescaled polynomial

$$f(z) = P_N^{(s,q)}\Big(\frac{N_1 \cdots N_s}{\widetilde{N}_1 \cdots \widetilde{N}_q} z\Big),$$

and recalling (3.19) we see that f satisfies the differential equation

$$\frac{(-1)^q}{\widetilde{N}_1 \cdots \widetilde{N}_q} \prod_{j=0}^{q} \big(z \frac{d}{dz} - \widetilde{N}_j\big) f = \frac{1}{N_1 \cdots N_s} \frac{d}{dz} \prod_{l=1}^{s} \big(z \frac{d}{dz} + \nu_l\big) f. \tag{5.4}$$

As was the theme in Section 3, we want to introduce the logarithmic derivative $g(z) = f'(z)/f(z)$, which we know from (3.5) is proportional to N for N large, and to take the $N \to \infty$ limit. For this purpose, let us expand upon the reasoning which leads to (3.22). With the aim being to express $(z \frac{d}{dz})^k f$ in terms of $g, g', \dots, g^{(k-1)}$ and f, we will give a recursive definition of a polynomial $Q_{k-1}(g)$ in $g, g', \dots, g^{(k-1)}$ with coefficients in $\mathbb{C}_{k-1}[z] = \langle 1, z, \dots, z^{k-1} \rangle_{\mathbb{C}}$.

First, let $Q_0 = 0$, then we have

$$\big(z \frac{d}{dz}\big) f = (zg + zQ_0) f. \tag{5.5}$$

Secondly, suppose that Q_{k-1} is well defined such that

$$(z\frac{d}{dz})^k f = ((zg)^k + zQ_{k-1})f, \tag{5.6}$$

and set

$$Q_k = Q_{k-1} + z(g + \frac{d}{dz})Q_{k-1} + k(zg)^{k-1}(g + zg'). \tag{5.7}$$

It is easy to verify that

$$(z\frac{d}{dz})^{k+1} f = ((zg)^{k+1} + zQ_k)f. \tag{5.8}$$

Moreover, by induction we know from (5.7) that the term of highest degree of Q_k in $g, g', \ldots, g^{(k-1)}$ is

$$\frac{k(k+1)}{2}(zg)^{k-1}(g + zg'), \tag{5.9}$$

which implies the degree of Q_k is k.

Let $e_k(\nu_1, \ldots, \nu_s)$ be the k-th elementary symmetric polynomial in $\nu_1, \ldots, \nu_s$, as is consistent with (3.14). With (5.6) and (5.8) in mind, expanding both sides of (5.4) we have

$$\begin{aligned}&\frac{(-1)^q}{\widetilde{N}_1 \cdots \widetilde{N}_q} \sum_{j=0}^{q+1} \big((zg)^j + zQ_{j-1}(g)\big) e_{q+1-j}(-\widetilde{N}_0, \ldots, -\widetilde{N}_q) f \\ &\qquad = \frac{1}{N_1 \cdots N_s} \sum_{k=0}^{s} \big((zg)^k g + Q_k(g)\big) e_{s-k}(\nu_1, \ldots, \nu_s) f. \end{aligned} \tag{5.10}$$

Here $Q_{-1} = 0$ by convention. Let $G_N(z) = \frac{1}{N}\frac{d}{dz}\log f(z)$, so that $g = NG_N$. Substituting it in the above and dividing both sides by N show

$$\begin{aligned}&-\prod_{j=0}^{q}\big(1 - \frac{N}{\widetilde{N}_j} zG_N\big) + \frac{(-1)^q}{\widetilde{N}_0 \cdots \widetilde{N}_q} \sum_{j=0}^{q+1} zQ_{j-1}(NG_N)\, e_{q+1-j}(-\widetilde{N}_0, \ldots, -\widetilde{N}_q) \\ &= G_N \prod_{k=1}^{s}\big(\frac{N}{N_k} zG_N + \frac{\nu_k}{N_k}\big) + \frac{1}{NN_1 \cdots N_s} \sum_{k=0}^{s} Q_k(NG_N)\, e_{s-k}(\nu_1, \ldots, \nu_s). \end{aligned} \tag{5.11}$$

Since the degree of Q_k is k, we can immediately deduce the following proposition.

Proposition 5.1. *Let* $P_N^{(s,q)}(z)$ *be given in* (5.2). *Write*

$$G_N(z) = \frac{1}{N}\frac{d}{dz}\log P_N^{(s,q)}\Big(\frac{N_1 \cdots N_s}{\widetilde{N}_1 \cdots \widetilde{N}_q} z\Big),$$

and

$$G(z) = \lim_{N\to\infty} G_N(z).$$

Assume that as $N \to \infty$

$$\frac{N}{\widetilde{N}_j} \to u_j \in (0,1] \text{ and } \frac{N}{N_k} \to v_k \in (0,1] \text{ for } j = 0, \dots, q,\ k = 1, \dots, s, \tag{5.12}$$

where $u_0 = 1$*. Then* $G(z)$ *satisfies the polynomial equation*

$$-\prod_{j=0}^{q} \big(1 - u_j z G(z)\big) = G(z) \prod_{k=1}^{s} \big(v_k z G(z) + 1 - v_k\big). \tag{5.13}$$

Remark 5.2. The equation (5.13) has been obtained for products of random matrices with general independent entries when $q = 0$ [2, 3], and also for products of Gaussian random matrices involving inverses when all the u_j's and v_k's are equal to 1 [30, 37], where the tools from free probability theory were used. Actually, the case that all the u_j's are equal to 1 is special. Thus if $u_j < 1$ for all $j = 1, \dots, q$, then the equation (5.13) has one analytic solution at infinity such that $zG(z) \to 1$ as $z \to \infty$ and thus the corresponding density has compact support. Specially, for $q = s = 1$, i.e., the F matrix case in statistics [7, 30], an explicit density has been given in Theorem 4.10, [7] and its limit of u_1 and v_1 approaching 1 is exactly the case of $s = q = 1$ given in [30]. However, if $q > 0$ and at least one of $u_1, \dots, u_q$ equals 1, (5.13) has no analytic solution at infinity. Actually, we know from free probability theory that in this case the support of the density is unbounded. To be precise, let $f_s(x)$ be the Fuss-Catalan distribution of degree s with support on $[0, K_s]$, and let $\tilde{f}_q(x)$ be the distribution of q inverse product of random matrices, then $\tilde{f}_q(x) = (1/x^2) f_q(1/x)$ with support on $[1/K_q, \infty)$. Use of tools from free probability shows that $f_{s,q}(x)$ is the multiplicative free convolution of $f_s(x)$ and $\tilde{f}_q(x)$, hence for $q > 0$ the support of $f_{s,q}(x)$ is not compact. Furthermore, it is known from [37, Prop. 6] that the density is continuous.

Remark 5.3. In the case $q = 0$, with each $N_j - N = \nu_j$ fixed in the limit $N \to \infty$, differential equations have been shown to also characterize different observable quantities, namely the gap probabilities at the hard edge. Moreover, these differential equations are nonlinear, and in fact related to isomonodromy preserving deformations of linear systems [62]

Remark 5.4. With $q = 0$ and each $v_k < 1$ the explicit large-z expansion of $zG(z)$ as implied by (5.13) has recently been given by Lenczewski and Salapata [46] as a series in so called mutivariate Fuss-Narayana polynomials.

6. FURTHER DEVELOPMENT OF THE PARAMETERIZATION METHOD

We start from the polynomial equation (5.13) satisfied by the Stieltjes transform of the inverse product (5.1) and use the parameterization method of the spectral variable, independently due to Biane and Neuschel, to give an explicit form of the limiting density in a special case when all the u_j's ($j = 1, \dots, q$) and v_k's ($k = 1, \dots, s$) are equal to 1. Recently, this same task has been undertaken by Haagerup and Möller [37] using the strategy of Biane [10]. We remark too that in the case of $q = s$ the exact form of the density has been computed in [9] and independently in [30] without the use of a parametrized spectral variable, giving

$$f_{s,s}(x) = \frac{1}{\pi} \frac{x^{-s/(s+1)} \sin \pi/(s+1)}{1 + 2x^{1/(s+1)} \cos \pi/(s+1) + x^{2/(s+1)}}, \qquad 0 < x < \infty. \tag{6.1}$$

As an application, the $x \to 0^+$ leading asymptotics of the density can be read off to be equal to $\frac{1}{\pi} x^{-\frac{s}{1+s}} \sin \frac{1}{1+s}\pi$. It was noticed in [30] that this is the same form as that for $q = 0$, as deduced from the result (2.10) (take $r = 1$, $p = s + 1$ in the first case of (2.16)), from which it was conjectured that this should be a universal feature valid for general s, q but independent of q. As an application of our explicit determination of the density for general $s \neq q$, we are able to obtain the corresponding $x \to 0^+$ leading asymptotic form, and so give an affirmative answer to this conjecture.

6.1. **Explicit densities.**

6.1.1. *General procedure.* Assume that the Stieltjes transform

$$G(z) = \int_0^\infty \frac{1}{z - x} f_{s,q}(x) dx$$

of the density $f_{s,q}(x)$ satisfies the equation

$$(zG)^{1+s} + z(1 - zG)^{1+q} = 0. \tag{6.2}$$

This equation is a limit case of (5.13) when all u_i and v_j approach 1 from the below. Then the support changes from the compact case to the noncompact one (more precisely, as remarked above this happens as soon was one of the u_j becomes equal to 1). Let $w(z) = zG(z)$. We will try to find two special solutions of the algebraic equation

$$w^{1+s} + z(1 - w)^{1+q} = 0, \tag{6.3}$$

one as the cut $0 \le z < \infty$ is approached from $\mathrm{Im}(z) > 0$, the other as it is approached from $\mathrm{Im}(z) < 0$, from which the density immediately follows.

Note that (6.3) can be rewritten as

$$w^{\frac{1+s}{1+q}} + (-(-z)^{\frac{1}{1+q}})(1-w) = 0. \tag{6.4}$$

If we treat w as a function of the new variable $\hat{z} = -(-z)^{1/(1+q)}$, then (6.3) is just the equation (2.2) with $p = (1+s)/(1+q)$, $r = 1$ and z is substituted by $\hat{z}$. We know that in the variable $\hat{z}$ this has the solution about infinity given by the series in (2.1). But this is not analytic about infinity in the variable z when $q > 0$, and so the support must be unbounded.

Following Neuschel's strategy, and augmenting this by explicit knowledge of the structure of the parameterization (2.8), we begin by seeking a complex conjugate pair of solutions of (6.3) in the parameterized polar coordinates form

$$w_{\pm} = \frac{\sin(a\varphi + \varphi + b)}{\sin(a\varphi)}\, e^{\pm i(\varphi+b)}, \tag{6.5}$$

where a and b are to be determined. Note that this has the property that $w_{\pm} = 1$ for $\varphi = -b$. Simple manipulation then gives

$$1 - w_{\pm} = -\frac{\sin(\varphi + b)}{\sin(a\varphi)}\, e^{\pm i(a\varphi+\varphi+b)}. \tag{6.6}$$

Substituting (6.5) and (6.6) in (6.3) one establishes the corresponding parameterization of z,

$$z = \frac{(\sin(a\varphi+\varphi+b))^{1+s}}{(\sin(\varphi+b))^{1+q}(\sin(a\varphi))^{s-q}}\, e^{\mp i(a(1+q)\varphi + q\pi - (s-q)(\varphi+b))}. \tag{6.7}$$

To ensure that z lies in one cut of the real axis we suppose the phase satisfies

$$a(1+q)\varphi + q\pi - (s-q)(\varphi+b) \equiv 2k\pi$$

for some suitably chosen $k \in \mathbb{Z}$. Therefore we get

$$a(1+q) - (s-q) = 0, \qquad q\pi - (s-q)b = 2k\pi,$$

and thus

$$a = \frac{s-q}{1+q}, \qquad b = \frac{2k-q}{q-s}\pi \ (s \neq q). \tag{6.8}$$

Supposing that $s \neq q$, it follows from this working that if we use the parameterization

$$x = \rho(\varphi) = \frac{(\sin(\frac{1+s}{1+q}\varphi + \frac{2k-q}{q-s}\pi))^{1+s}}{(\sin(\varphi + \frac{2k-q}{q-s}\pi))^{1+q}\,(\sin(\frac{s-q}{1+q}\varphi))^{s-q}}, \tag{6.9}$$

then the complex conjugate pair of solutions of (6.3) is given by

$$\frac{\sin(\frac{1+s}{1+q}\varphi + \frac{2k-q}{q-s}\pi)}{\sin(\frac{s-q}{1+q}\varphi)}\, e^{\pm i(\varphi + \frac{2k-q}{q-s}\pi)}. \tag{6.10}$$

According to the inverse formula of the Stieltjes transform the density function, $f_{s,q}(x)$ say, is given by

$$f_{s,q}(x) = \lim_{\epsilon\to 0+} \frac{1}{2i\pi}\Big(\frac{w(x-i\epsilon)}{x-i\epsilon} - \frac{w(x+i\epsilon)}{x+i\epsilon}\Big),$$

where w is one of the above two solutions. Note that the imaginary parts of $w(z)$ and z have opposite sign; we choose $w = w_-$ for $s > q$ while $w = w_+$ for $s < q$. We then have for $s \neq q$

$$\begin{aligned} f_{s,q}(\rho(\varphi)) &= \frac{1}{\pi\rho(\varphi)} \frac{\sin(\frac{1+s}{1+q}\varphi + \frac{2k-q}{q-s}\pi)}{\operatorname{sign}(s-q)\sin(\frac{s-q}{1+q}\varphi)}\, \sin(\varphi + \frac{2k-q}{q-s}\pi) \\ &= \frac{1}{\pi} \frac{(\sin(\varphi + \frac{2k-q}{q-s}\pi))^{2+q}}{\operatorname{sign}(s-q)(\sin(\frac{1+s}{1+q}\varphi + \frac{2k-q}{q-s}\pi))^{s}}\, (\sin(\frac{s-q}{1+q}\varphi))^{s-q-1}. \end{aligned} \tag{6.11}$$

The remaining tasks are the determination of k as well as the range of φ.

6.1.2. *Case* $s > q$. To ensure that the right-hand sides of (6.9) and (6.11) are nonnegative we must choose an appropriate k and restrict the range of φ. First, a restriction following from the periodicity of the sine functions is $0 < \varphi + \frac{2k-q}{q-s}\pi < \pi$. Second, the nonnegativity of the density suggests that both $\frac{1+s}{1+q}\varphi + \frac{2k-q}{q-s}\pi$ and $\frac{s-q}{1+q}\varphi$ should belong to $(2l\pi, (2l+1)\pi)$ for some $l \in \mathbb{Z}$. So we can choose $k = q$ for convenience and thus get the range

$$\frac{q}{s-q}\pi < \varphi < \frac{s}{1+s}\frac{1+q}{s-q}\pi. \tag{6.12}$$

The final form now follows. This is stated in Proposition 6.1 below, where for convenience φ has been replaced by $\varphi + \frac{q}{s-q}\pi$.

6.1.3. *Case $s < q$.* In this case we rewrite (6.9) as

$$x = \rho(\varphi) = \frac{(\sin(\frac{q-2k}{q-s}\pi - \frac{1+s}{1+q}\varphi))^{1+s}}{(\sin(\frac{q-2k}{q-s}\pi - \varphi))^{1+q}\,(\sin(\frac{q-s}{1+q}\varphi))^{s-q}}, \tag{6.13}$$

and choose $k = 0$. To ensure that the angles of the sine functions above lie in the interval $(0, \pi)$ we must restrict φ to the range

$$\frac{s}{1+s}\frac{1+q}{q-s}\pi < \varphi < \frac{q}{q-s}\pi. \tag{6.14}$$

The sought form of the parameterization is specified in Proposition 6.1, where for convenience φ has been replaced by $\frac{q\pi}{q-s} - \varphi$, along with the case $s > q$ of the previous subsection, and the case $s = q$ which can be checked separately (cf.(6.1)).

Proposition 6.1. *Assume $s, q \geq 0$ and $(s, q) \neq (0, 0)$. If we use the parameterization (a strictly decreasing function)*

$$x = \rho(\varphi) = \frac{(\sin(\frac{1+s}{1+q}\varphi + \frac{q\pi}{1+q}))^{1+s}}{(\sin\varphi)^{1+q}\,(\sin(\frac{s-q}{1+q}\varphi + \frac{q\pi}{1+q}))^{s-q}}, \qquad 0 < \varphi < \frac{\pi}{1+s}, \tag{6.15}$$

then

$$f_{s,q}(\rho(\varphi)) = \frac{1}{\pi}\frac{(\sin(\frac{s-q}{1+q}\varphi + \frac{q\pi}{1+q}))^{s-q-1}}{(\sin(\frac{1+s}{1+q}\varphi + \frac{q\pi}{1+q}))^{s}}\,(\sin\varphi)^{2+q}. \tag{6.16}$$

A direct application of Proposition 6.1 gives the explicit leading asymptotic form of the density upon the approach of either boundary of its support.

Corollary 6.2. *Let $s, q > 0$. We have, for $x \to 0^+$*

$$f_{s,q}(x) \sim \frac{1}{\pi}\sin\frac{\pi}{1+s}\,x^{-\frac{s}{1+s}}, \tag{6.17}$$

while for $x \to \infty$

$$f_{s,q}(x) \sim \frac{1}{\pi}\sin\frac{\pi}{1+q}\,x^{-\frac{2+q}{1+q}}. \tag{6.18}$$

Proof. Noting that $x \to 0$ as $\varphi \to \frac{\pi}{1+s}$, and $x \to \infty$ as $\varphi \to 0$, a simple computation completes the proof. □

Actually, for general $s, q \geq 0$ we have all leading asymptotics as follows: (i) for $s > 0$ and $q \geq 0$, the leading form is $\frac{1}{\pi}\sin(\pi/(1+s))x^{-s/(1+s)}$ as $x \to 0^+$; (ii) for $q > 0$ and $s \geq 0$, the leading form is $\frac{1}{\pi}\sin(\pi/(1+q))x^{-(2+q/)(1+q)}$ as $x \to \infty$. This latter behavior is consistent with the fact that all moments diverge. We remark too, as proved respectively from the Stieltjes transform and the S- transform in [30] and [37], that there is a duality relation between the densities, being unchanged by the mappings

$$s \longleftrightarrow q, \qquad x f_{s,q}(x) \longrightarrow x f_{q,s}(x), \qquad x \longrightarrow \frac{1}{x}. \tag{6.19}$$

Remark 6.3. Recently, Haagerup and Möller have proved the same results as in Proposition 6.1, see [37, Theorem 6]. They obtained the parametrization representation by studying the free multiplicative convolution and the S-transform, while our starting point is the Stieltjes transform and the related equation (6.2). We will also give direct expression of densities in terms of spectral variables in two special cases in the subsequent subsection.

Remark 6.4. It is possible to extend the $x \to 0^+$ expansion (6.17) to give an expansion analogous to (2.22). Writing (6.3) as $(1-Y)^{1+s} = -zY^{1+q}$ where $Y = 1-w$ and taking the $(s+1)$ root shows

$$Y^{\frac{1+q}{1+s}} + (-(-z)^{-\frac{1}{1+s}})(1-Y) = 0$$

(cf. (6.4)). We recognise this as identical to (2.2) upon the identifications $p = (1+q)/(1+s)$, $r = 1$, $w = Y$ and $z \mapsto -(-z)^{-\frac{1}{1+s}}$, and thus we have the expansion

$$w = 1 - \sum_{n=0}^{\infty} (-\sqrt[s+1]{-z})^n R_{(q+1)/(s+1),1}(n). \tag{6.20}$$

Applying (2.9) then gives

$$f_{s,q}(x) = \frac{1}{\pi x} \sum_{n=0}^{\infty} (-1)^n \sin \pi(n+1)/p \, R_{(q+1)/(s+1),1}(n+1) \, x^{(n+1)/(s+1)}, \tag{6.21}$$

where as in obtaining (2.22) we have made the specific choice of branch $(-1)^{-1/(s+1)} = e^{-\pi i/(s+1)}$. With $q = 0$, $p = s+1$, this reproduces (2.22). Furthermore with $s = q$ it agrees with the small-x expansion of (6.1).

6.2. **Two special cases.** In this subsection we discuss the special case $1+s = 2(1+q)$ or $1+q = 2(1+s)$, and give an explicit form of the density in the original spectral variable, analogous to the expression (6.1) in the case $1+s = 1+q$. Inspection of (6.3) shows that these two special cases give a quadratic equation in w, which permits further analysis.

Consider first the case $1+s=2(1+q)$ or equivalently $s=1+2q$. The quadratic equation then reads

$$w^2+(-z)^{1/(1+q)}w-(-z)^{1/(1+q)}=0, \tag{6.22}$$

and we read off for the roots

$$w_\pm=\frac{1}{2}\big(-(-z)^{1/(1+q)}\pm\sqrt{(-z)^{2/(1+q)}+4(-z)^{1/(1+q)}}\big). \tag{6.23}$$

Here the square root is specified as the one with the positive imaginary part. Note that we require $w(z)\to 1$ as $z\to-\infty$, so we choose w_+ as the solution corresponding to the Stieltjes transform. From this we compute the density

$$\begin{aligned}xf_{s,q}(x)&=\frac{1}{\pi}\lim_{\epsilon\to 0^+}\operatorname{Im}w_+(x-i\epsilon)\\&=\frac{1}{2\pi}\operatorname{Im}\big\{-(xe^{i\pi})^{1/(1+q)}+\sqrt{(xe^{i\pi})^{2/(1+q)}+4(xe^{i\pi})^{1/(1+q)}}\big\}.\end{aligned}$$

To take the imaginary part, we notice that if we set for $q>0$ (the case $q=0$ is just the Marchenko-Pastur law)

$$1+4x^{-1/(1+q)}e^{-i\pi/(1+q)}=Re^{-i\theta},\qquad 0<\theta<\pi/(1+q),$$

where the positive number R satisfies

$$R^2=1+16x^{-2/(1+q)}+8x^{-1/(1+q)}\cos\tfrac{\pi}{1+q}, \tag{6.24}$$

then

$$\begin{aligned}xf_{s,q}(x)&=\frac{1}{2\pi}x^{1/(1+q)}\big(-\sin\tfrac{\pi}{1+q}+\sqrt{R}\sin(\tfrac{\pi}{1+q}-\tfrac{\theta}{2})\big)\\&=\frac{1}{2\pi}x^{1/(1+q)}\Big(-\sin\tfrac{\pi}{1+q}+\sqrt{\tfrac{R+1}{2}+2x^{-1/(1+q)}\cos\tfrac{\pi}{1+q}}\,\sin\tfrac{\pi}{1+q}\\&\quad-\sqrt{\tfrac{R-1}{2}-2x^{-1/(1+q)}\cos\tfrac{\pi}{1+q}}\,\cos\tfrac{\pi}{1+q}\Big).\end{aligned} \tag{6.25}$$

The case $1+q=2(1+s)$ or equivalently $q=1+2s$ is similar. We obtain for $s>0$

$$\begin{aligned}xf_{s,q}(x)&=\frac{1}{2\pi}x^{-1/(1+s)}\Big(-\sin\tfrac{\pi}{1+s}+\sqrt{\tfrac{\tilde{R}+1}{2}+2x^{1/(1+s)}\cos\tfrac{\pi}{1+s}}\,\sin\tfrac{\pi}{1+s}\\&\quad-\sqrt{\tfrac{\tilde{R}-1}{2}-2x^{1/(1+s)}\cos\tfrac{\pi}{1+s}}\,\cos\tfrac{\pi}{1+s}\Big),\end{aligned} \tag{6.26}$$

where the positive number $\tilde{R}$ satisfies

$$\tilde{R}^2=1+16x^{2/(1+s)}+8x^{1/(1+s)}\cos\tfrac{\pi}{1+s}. \tag{6.27}$$

Remark 6.5. Careful computations using (6.25) and (6.26) give the same asymptotic behaviours of densities for $x \to 0^+$ and $x \to \infty$ obtained in the previous subsections.

Remark 6.6. It is of interest to note how the above working relates to the parameterization approach. In the latter, with w parameterized according to (6.5), the variable $(-z)^{1/(1+q)}$ is written $(-z)^{1/(1+q)} = w^2/(1-w)$ as is consistent with (6.22). The terms inside the square root of (6.23) can then be written as a perfect square, thus eliminating the square root and providing simplification.

6.3. **Densities from mixed equations.** Our extension of the parameterization method is also applicable to the more general equation with any $s, q \geq 0$

$$w^{\frac{1+s}{r}} + z(1 - w^{\frac{1}{r}})^{1+q} = 0, \qquad 0 < r \leq 1 + s, \tag{6.28}$$

which is a mixed case of equations (2.2) and (6.3). Here the Stieltjes transform

$$G(z) = \int_0^\infty \frac{1}{z - x} f_{s,q,r}(x) dx$$

of the density $f_{s,q,r}(x)$ satisfies (6.28) with $w = zG(z)$.

6.3.1. $s = q$. In this case we get from (6.28) that

$$w = \left(\frac{(-z)^{1/(1+s)}}{1 + (-z)^{1/(1+s)}} \right)^r. \tag{6.29}$$

Thus,

$$\begin{aligned} x f_{s,s,r}(x) &= \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \big(\operatorname{Im} w(x - i\epsilon) - \operatorname{Im} w(x - i\epsilon) \big) \\ &= \frac{x^{r/(1+s)}}{2\pi i} \frac{\big(x^{1/(1+s)} + e^{i\pi/(1+s)}\big)^r - \big(x^{1/(1+s)} + e^{-i\pi/(1+s)}\big)^r}{\big(1 + 2x^{1/(1+s)} \cos \frac{\pi}{1+s} + x^{2/(1+s)}\big)^r}. \end{aligned} \tag{6.30}$$

Furthermore, for $x > 0$, if we let

$$x^{1/(1+s)} + e^{i\pi/(1+s)} = Re^{i\varphi}, \qquad 0 < \varphi < \tfrac{\pi}{1+s},$$

where

$$R = \sqrt{1 + 2x^{1/(1+s)} \cos \tfrac{\pi}{1+s} + x^{2/(1+s)}}, \tag{6.31}$$

then we have

$$x f_{s,s,r}(x) = \frac{1}{\pi} \frac{x^{r/(1+s)}}{R^r} \sin(r\varphi). \tag{6.32}$$

6.3.2. $s \neq q$. As in the subsection 6.1, if we use the parameterization

$$x = \rho(\varphi) = \frac{(\sin(\frac{1+s}{1+q}\varphi + \frac{2k-q}{q-s}\pi))^{1+s}}{(\sin(\varphi + \frac{2k-q}{q-s}\pi))^{1+q}\,(\sin(\frac{s-q}{1+q}\varphi))^{s-q}}, \tag{6.33}$$

then the complex conjugate pair of solutions of (6.28) is given by

$$w_{\pm} = \Big(\frac{\sin(\frac{1+s}{1+q}\varphi + \frac{2k-q}{q-s}\pi)}{\sin(\frac{s-q}{1+q}\varphi)}\, e^{\pm i(\varphi + \frac{2k-q}{q-s}\pi)}\Big)^{r}. \tag{6.34}$$

Choose $k = q, w = w_-$ for $s > q$ while $k = 0, w = w_+$ for $s < q$, after similar discussion in the subsection 6.1 we have the following proposition including (6.32).

Proposition 6.7. *Assume $s, q \geq 0$, $(s,q) \neq (0,0)$ and $0 < r \leq 1+s$. If we use the parameterization*

$$x = \rho(\varphi) = \frac{(\sin(\frac{1+s}{1+q}\varphi + \frac{q\pi}{1+q}))^{1+s}}{(\sin\varphi)^{1+q}\,(\sin(\frac{s-q}{1+q}\varphi + \frac{q\pi}{1+q}))^{s-q}}, \qquad 0 < \varphi < \frac{\pi}{1+s}, \tag{6.35}$$

then

$$f_{s,q,r}(\rho(\varphi)) = \frac{1}{\pi}\frac{(\sin(\frac{s-q}{1+q}\varphi + \frac{q\pi}{1+q}))^{s-q-r}}{(\sin(\frac{1+s}{1+q}\varphi + \frac{q\pi}{1+q}))^{1+s-r}}\,(\sin\varphi)^{1+q}\sin(r\varphi). \tag{6.36}$$

A corollary immediately follows from Proposition 6.7.

Corollary 6.8. *Let $s, q > 0$. We have, for $x \to 0^+$*

$$f_{s,q,r}(x) \sim \begin{cases} \frac{1}{\pi}\sin\frac{\pi}{1+s}\, x^{\frac{1}{1+s}}, & r = 1+s; \\ \frac{1}{\pi}\sin\frac{r\pi}{1+s}\, x^{-1+\frac{r}{1+s}}, & r < 1+s, \end{cases} \tag{6.37}$$

while for $x \to \infty$

$$f_{s,q,r}(x) \sim \frac{r}{\pi}\sin\frac{\pi}{1+q}\, x^{-\frac{2+q}{1+q}}. \tag{6.38}$$

Remark 6.9. The equation (6.28) with some special r may occur in the products and inverses of random matrices, as in Remarks 3.4 and 3.6.

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