Intl. Trans. in Op. Res. xx (2021) 1–25

Minimum Steiner trees on a set of concyclic points and their centre

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Received 16 January 2021; received in revised form DD MMMM YYYY; accepted DD MMMM YYYY

Abstract

Consider a configuration of points comprising a point q and a set of concyclic points R that are all a given distance r from q in the Euclidean plane. In this paper we investigate the relationship between the length of a minimum Steiner tree on $R \cup \{q\}$ and a minimum spanning tree on R. We show that if the degree of q in the minimum Steiner tree is 1, then the difference between these two lengths is at least $(2-\sqrt{3})r$, and that this lower bound is tight. This bound can be applied as part of an efficient algorithm to find the solution to the prize collecting Euclidean Steiner tree problem, as outlined in an earlier paper.

Keywords: Steiner tree problem; Prize collecting Steiner tree; Prize collecting Euclidean Steiner tree; Optimization

1. Introduction

Given a finite set of points X lying in the Euclidean plane, the Steiner tree problem asks us to find a connected geometric network $S(X) = (V(S), E(S))$ such that $X \subseteq V(S)$ and the sum of the lengths of the edges $E(S)$ is as small as possible. Any network solving the Steiner tree problem is a tree, and is referred to as a *minimum Steiner tree* (MStT). An MStT on X differs from a minimum spanning tree (MST) on X in that it can include extra vertices not in X, called *Steiner points,* if their inclusion contributes to a reduction in the sum of the lengths of the edges of $S(X)$. Unlike the spanning tree problem in the Euclidean plane (which can be solved in $O(n \log n)$ time), the MStT decision problem is NP-complete (even in a discretised version of the problem which avoids the computational difficulties inherent in computing with irrational numbers), and it follows that the MStT problem is NP-hard; see Garey et al. (1977). For a comprehensive examination of the properties of MStTs refer to Brazil and Zachariasen (2015).

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An MStT is at most as long as an MST, but it can be considerably shorter. One of the enduring problems in the literature is the question of how much shorter it can be, with the *Steiner ratio conjecture* (Gilbert and Pollak (1968)) claiming that the length of an MStT can be no less than $\frac{\sqrt{3}}{2}$ times the length of an MST on the same set of terminals. The Steiner ratio conjecture is widely considered to be true (see Brazil and Zachariasen (2015) pp. 23-24 for a brief history of the conjecture), and has been confirmed for cases with eight or fewer terminals by Kirszenblat (2014). A specific instance that has been solved is that of the Steiner ratio for a set of concyclic terminals. Du et al. (1987) determined the Steiner trees for all cases where the terminals are the vertices of a regular polygon; Weng and Booth (1995) extended this determination to cases that also include a point at the centre. Rubinstein and Thomas (1992b) proved that the Steiner ratio holds for any set of concyclic terminals, and they also proved *Graham's conjecture*: that an MST and MStT coincide for any set of concyclic terminals, provided that no more than one pair of adjacent terminals on the circle is farther than r apart, where r is the radius of the circle (Rubinstein and Thomas (1992a)).

In this paper we are also interested in concyclic terminals, but in our case the configuration includes a point at the centre, and instead of the ratios of lengths, we investigate differences in lengths. More specifically, for any configuration of points comprising a point q and a set of concyclic points R that are all a given distance r from q in the Euclidean plane we determine a lower bound (in terms of r) for the length of an MStT on $R \cup \{q\}$ minus the length of an MST on R. The motivation for investigating this bound comes from its application as part of an efficient algorithm to find an exact solution to the *Prize collecting Euclidean Steiner tree problem*, where it is used as part of a pre-processing procedure to test if q can be ruled out as a possible terminal. This is outlined in an earlier paper (Whittle et al. (2020)). The aforementioned result by Weng and Booth (1995) implies an easy path to finding a lower bound in a constrained case in which concyclic points are the vertices of a regular polygon, however the unconstrained case is more challenging.

In this paper we restrict ourselves to the case where there exists an MStT on $R \cup \{q\}$ with exactly one edge incident to q. Bounds for other cases will be discussed in a future paper.

2. Preliminaries

We first establish some basic notation used throughout this paper. For two distinct given points a and b in the Euclidean plane, we denote the straight line segment between them as ab and the length of that line segment as $|ab|$. For the same two points, we denote the ray from a passing through b as $\overrightarrow{ab}$ and the straight line passing through both points as $\overleftrightarrow{ab}$.

We next review some of the fundamental definitions and properties associated with a minimum Steiner tree (MStT). More details on these can be found in Brazil and Zachariasen (2015).

The given vertices that are interconnected by an MStT are referred to as *terminals*, and the vertices of an MStT that are not terminals are referred to as *Steiner points*. We denote the sum of the lengths of edges of an MStT or MST S by L_S. An MStT is said to be *full* if all terminals have degree 1. We call a pair of degree-1 terminals adjacent to a common Steiner point a *cherry*. We call the underlying graph of an MStT a *Steiner topology*. The difficulties involved in determining the locations of the Steiner points and the topology of the MStT are what make the Steiner tree problem NP-hard.

Well known properties of MStTs include:

- All terminals have degree 1, 2 or 3.
- Each pair of edges incident to a degree 2 or 3 terminal meets at an angle greater than or equal to $\frac{2\pi}{3}$.
- All Steiner points have degree 3 and each pair of edges meets at an angle of $\frac{2\pi}{3}$.
- A full Steiner tree on three or more terminals contains at least two cherries.

Consider a configuration of points $P = R \cup \{q\}$ where $R = \{r_1, r_2, ..., r_m\}$ is a set of m points on a circle with radius r and q is a point at the centre of the circle. Throughout this paper, we normalise the problem, by assuming that $r = 1$, in other words, that the points in R lie on a unit circle, denoted by $\mathbb{D}$. We call the elements of R *Rubin points*[1].

In the context of the sets P and R we have the following notation and definitions:

- $S(P)$ denotes an MStT on P.
- $T(R)$ denotes an MST on R.
- An *R-cherry* is defined to be a cherry in $S(P)$ on two Rubin points.
- R^C is defined to be the closure of the set of all configurations of Rubin points that satisfy the conditions that $S(P)$ is a full MStT and q has degree 1 in $S(P)$.
- R^A is defined to be the set of all configurations of Rubin points in R^C such that no perturbations of the Rubin points (to configurations in R^C) can reduce $L_{S(P)} - L_{T(R)}$.

Note that the set of all configurations of Rubin points that satisfy the conditions that $S(P)$ is a full MStT and q has degree 1 in $S(P)$ is not compact. This is the reason that we investigate configurations of Rubin points in the slightly larger set R^C, so that we can guarantee that configurations of Rubin points that achieve the infimum of $L_{S(P)} - L_{T(R)}$ lie in R^C. It is straightforward to see that as we move configurations of Rubin points to the boundary of R^C, the network $S(P)$ remains an MStT, as otherwise we would have the opportunity to change to a better topology before reaching the boundary (compare for example, Lemma 1.16 and Exercise 1.14 of Brazil and Zachariasen (2015)). However, it may be the case that on the boundary of R^C at one or more terminals the neighbouring Steiner point collapses into that terminal, in which case there may be terminals with degree 2 where the two incident edges meet at an angle of $\frac{2\pi}{3}$. Hence, we have the following lemma:

Lemma 1. *For every configuration of Rubin points in R^A, $S(P)$ is an MStT in which each terminal that is not a degree 1 terminal is a degree 2 terminal where the two incident edges meet at an angle of $\frac{2\pi}{3}$.*

For the remainder of this paper it will be convenient to treat all elements of R^A as full MStTs, where one or more of the edges incident to a terminal may have length 0, but otherwise all properties of full MStTs hold.

An example of a set $P = R \cup \{q\}$ with corresponding MStT on P, $S(P)$, and MST on R, $T(R)$, is shown in Figure 1. Note that Rubin points r_2 and r_3 constitute an R-cherry in the figure.

In Section 8 we prove the main result of this paper, that for all possible instances of P in which q is degree 1 in $S(P)$ we have $\inf(L_{S(P)} - L_{T(R)}) = 2 - \sqrt{3}$.

The key steps in the proof of this result, can be summarised as follows:

- If $m = 1$ it is easy to show that $L_{S(P)} - L_{T(R)} = 1$ (Lemma 2).

[1]Named for one of the co-authors, Hyam Rubinstein, who proposed the use of this construction to aid in finding a solution to the PCEST problem given in Whittle et al. (2020).

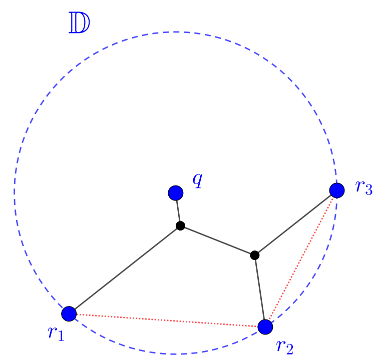

Fig. 1. An example of $S(P)$ (solid lines) and $T(R)$ (dotted lines) in the case that q has degree 1 in $S(P)$.

- If $m = 2$ then $\inf(L_{S(P)} - L_{T(R)}) = 2 - \sqrt{3}$ (Lemma 4).
- If $m > 2$ then again $L_{S(P)} - L_{T(R)} > 2 - \sqrt{3}$. (See Sections 6 and 7).

The bound occurs in the limiting case for $m = 2$.

3. One or two Rubin points in R

Lemma 2. *If R has 1 Rubin point, then $L_{S(P)} - L_{T(R)} = 1$.*

Proof. Observe there is one edge in $S(P)$ with length 1 and $T(R)$ has zero length. □

The following concept from classical Steiner tree theory is used in Lemma 3 and in subsequent lemmas.

Definition 1 (Simpson line $\tilde{S}$ for a in S). Let a denote a degree 1 terminal in an FST S. Let s denote the adjacent Steiner point to a in S. A *Simpson line* $\tilde{S}$ for a in S is a size-2 FST with points $\{a, e\}$ and a single edge ae such that s is a point on ae and $L_{\tilde{S}} = L_S$.

It is well known that for an FST S on a set of three terminals $\{a, b, c\}$, a Simpson line can be found as follows: construct an equilateral triangle $\triangle bce$ such that the vertex e is on the opposite side of $\overleftrightarrow{bc}$ to a; then ae is a Simpson line for the S, and contains the Steiner point of S. Furthermore, e is referred to as the *equilateral point* of the cherry for b and c. A Simpson line for an FST on a larger terminal set can be constructed by iterating this construction, as in the merging phase of the Melzak-Hwang algorithm (Brazil and Zachariasen (2015)), and has the following useful property.

Lemma 3. *For any FST S, a sufficiently small perturbation of the position of some terminal $a \in V(S)$ such that $a \in V(\tilde{S})$ changes the length of S and $\tilde{S}$ equally.*

For a proof of Lemma 3, see Lemma 2.2 in Brazil et al. (2000).

Note that this result does not necessarily apply if S is not full. On the boundary of R^C, for example, S and $\tilde{S}$ still have equal length, but any perturbation of a terminal that takes such a configuration outside R^C will not change the length of S and $\tilde{S}$ equally.

Lemma 4. *If* $m = 2$ *then* $\inf(L_{S(P)} - L_{T(R)}) = 2 - \sqrt{3}$.

Proof. Denote the Rubin points as r_1 and r_2 and the Steiner point as s. Let $\phi := \angle r_1 q r_2$ and let e be the equilateral point for r_1 and r_2 lying outside $\mathbb{D}$, as in Figure 2.

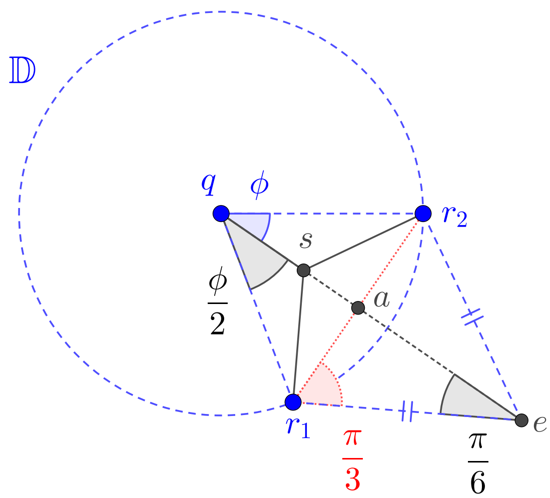

Fig. 2. $S(P)$ (black) and $S(R)$ (red) shown with the equilateral point e and various construction points, angles, edges and line segments as described in the text (Lemma 4).

Note that $T(R)$ consists of a single line segment $r_1 r_2$, while $S(P)$ has a single Steiner point s. The angle conditions at s imply that $0 < \phi < \frac{2\pi}{3}$. We also observe that $L_{S(P)} - L_{T(R)}$ can be parameterised solely in terms of ϕ.

We first calculate $L_{S(P)}$ from the Simpson line qe, where a is the midpoint of $r_1 r_2$:

$$L_{S(P)} = |qe| = |qa| + |ae|. \tag{1}$$

From basic trigonometry:

$$|qa| = \cos\frac{\phi}{2}, \tag{2}$$

$$|ae| = |r_1 e| \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}|r_1 e| = 2\sin\frac{\phi}{2}, \tag{3}$$

where the final equality follows from the fact that $\triangle r_1 r_2 e$ is equilateral.

Substituting from Equations 2 and 3 into 1 and applying standard trigonometric identities, we obtain:

$$L_{S(P)} - L_{T(R)} = 2\sin\left(\frac{\phi}{2} + \frac{\pi}{6}\right) - 2\sin\frac{\phi}{2}.$$

The first derivative of $L_{S(P)} - L_{T(R)}$ is now used to find critical points:

$$\dot{L_{S(P)}} - \dot{L_{T(R)}} = \cos\left(\frac{\phi}{2} + \frac{\pi}{6}\right) - \cos\frac{\phi}{2}.$$

Since cosine is a decreasing function for the domain $(0, \pi)$ it follows that $\cos\left(\frac{\phi}{2} + \frac{\pi}{6}\right) - \cos\frac{\phi}{2}$ is strictly negative for $\phi \in (0, \pi)$. Accordingly, the minimum is found at an end point of the domain:

$$\lim_{\phi \to 2\pi/3} L_{S(P)} - L_{S(R)} = 2 - \sqrt{3},$$

and the statement of the lemma follows. □

Note that the MStTs that achieve the infimum in Lemma 4 correspond to configurations in R^A, but are examples of degenerate full MStTs where the edge between the Steiner point and q has length 0, and hence the corresponding configurations lie on the boundary of R^C.

4. Three or more Rubin points in R

By Lemma 4, we now only need to look for configurations $R \in R^A$ in which $L_{S(P)} - L_{T(R)} \leq 2 - \sqrt{3}$ and $m > 2$ (where m is the number of Rubin points). This section establishes some useful facts about such cases.

Throughout the remainder of this paper we assume that $R \in R^C$, and we use the term *chord* exclusively to refer to a straight line segment between Rubin points that are adjacent on $\mathbb{D}$.

Definition 2 (longest and shorter chords). A *longest chord* is a chord (between adjacent Rubin points on $\mathbb{D}$) such that no other chord is longer. A *shorter chord* is a chord that is strictly shorter than a longest chord.

Lemma 5. *No Rubin point in $T(R)$ has degree greater than 2.*

Proof. Suppose to the contrary, a Rubin point a is adjacent to Rubin points b, c and d in $T(R)$, where a, b, c and d occur in that order on $\mathbb{D}$. Then the quadrilateral $abcd$ is cyclic and hence $\angle abc$ and $\angle cda$ are supplementary angles. Therefore, one of $\angle abc$ and $\angle cda$ is obtuse or both are right angles; so we can assume, without loss of generality, that $\angle abc \geq \frac{\pi}{2}$. This implies that ac is the longest side of $\triangle abc$ and so the edge ac in $T(R)$ can be replaced with bc to create a shorter network, contradicting the minimality of $T(R)$. □

Corollary 1. *The set of edges of $T(R)$ comprises all chords of R, with a longest chord removed.*

We refer to the chord that does not correspond to an edge in $T(R)$ as the *gap* in $T(R)$. Clearly the gap must be a longest chord.

Definition 3 (Bridge in $S(P)$). A bridge in $S(P)$ is a convex path (a path in which every turn is anti-clockwise or every turn is clockwise) between two Rubin points that are adjacent on $\mathbb{D}$.

Definition 4 (Non-bridge in $S(P)$). A non-bridge in $S(P)$ is a path between two Rubin points that are adjacent on $\mathbb{D}$ such that the path contains both clockwise and anti-clockwise turns.

Lemma 6. *For $m > 2$, $S(P)$ contains exactly one non-bridge, and q is adjacent to a Steiner point on the non-bridge.*

Proof. Given two Rubin points r_1 and r_2 adjacent on $\mathbb{D}$, consider the region enclosed by the path in $S(P)$ between r_1 and r_2 and the arc of $\mathbb{D}$ between r_1 and r_2. Since $S(P)$ does not intersect the interior of this arc, it follows that if the path in $S(P)$ between r_1 and r_2 is not convex then there must be a terminal belonging to P in this region. The only possible candidate for such a terminal is q, and the statement of the lemma easily follows. □

Lemma 7. *Every edge of $S(P)$ has length ≤ 1.*

Proof. If $S(P)$ has an edge of length greater than 1, we could remove that edge and then reconnect the tree with a radius of $\mathbb{D}$ from q to a Rubin point in the connected component not containing q, contradicting the minimality of $S(P)$. □

Lemma 8. *The distance between Rubin points corresponding to the ends of a bridge in $S(P)$ is less than $\sqrt{3}$.*

Proof. Let r_1 and r_2 be the two end points of a bridge in $S(P)$, and suppose $|r_1 r_2| \geq \sqrt{3}$. There are four cases to consider:

Case (1)**.** *One Steiner point in the path*: Since q has degree 1, s cannot coincide with q, and hence $\max(|r_1 s|, |r_2 s|) > 1$ contradicting Lemma 7.

Case (2)**.** *Two Steiner points $\{s_1, s_2\}$ in the path*: See Figure 3 Left. The vertex q must be on the opposite side of the edge (s_1, s_2) to vertices r_1 and r_2, otherwise this is not a bridge. Observe that the edge (s_1, s_2) achieves minimum length if it is parallel to $\overleftrightarrow{r_1 r_2}$ and as close as possible to q. Hence $|r_1 r_2| \geq \sqrt{3}$ implies that $|s_1 s_2| > \frac{2}{\sqrt{3}}$ (by basic geometry) contradicting Lemma 7.

Case (3)**.** *Three Steiner points $\{s_1, s_2, s_3\}$ in the path*: See Figure 3 Right. Since $r_1 s_1 || r_2 s_2$, then $|s_1 s_3| \geq \sqrt{3}$ and hence $\max(|s_1 s_2|, |s_3 s_2|) \geq 1$. However, equality cannot occur, since q must lie on the opposite side of either $\overleftrightarrow{s_1 s_2}$ or $\overleftrightarrow{s_2 s_3}$ to $r_1 r_2$; this implies that $\max(|s_1 s_2|, |s_3 s_2|) > 1$ contradicting Lemma 7.

Case (4)**.** *Four or more Steiner points in the path*. Since $|r_1 r_2| \geq \sqrt{3}$, the tangents of $\mathbb{D}$ through r_1 and r_2 meet at an angle less than or equal to $\frac{\pi}{3}$. If there are exactly four Steiner points in the path, where s_1 is the Steiner point adjacent to r_1 and s_4 is the Steiner point adjacent to r_4, then observe that the rays extending through $s_1 r_1$ and $s_4 r_2$ meet at an angle of $\frac{\pi}{3}$. We conclude that either s_1 or s_4 lies outside the boundary of $\mathbb{D}$, contradicting the definition of $S(P)$. A similar contradiction applies if there are more than four Steiner points in the path. □

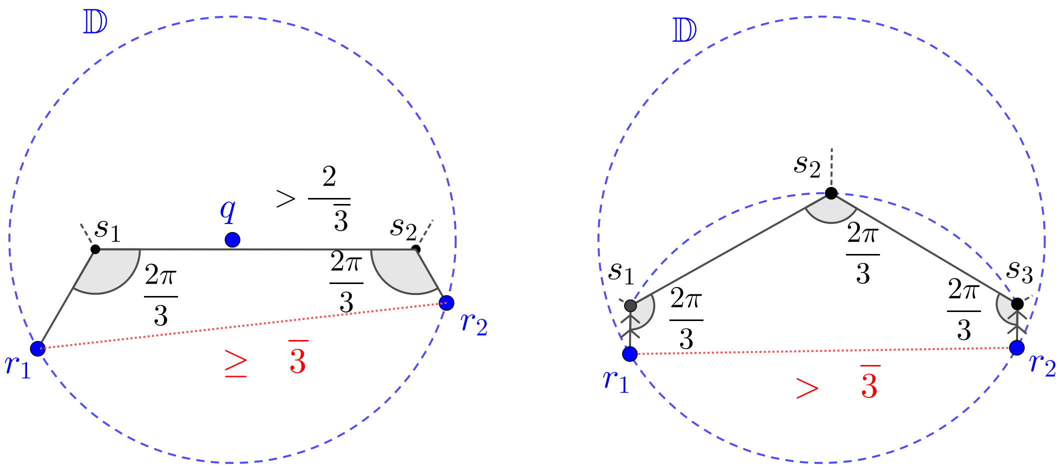

Fig. 3. Left: The case with two Steiner points in the path. Right: The case with three Steiner points in the path (Both Lemma 8).

Lemma 8 easily implies the following result.

Corollary 2. *If $R \in R^C$ then every edge of $T(R)$ has length less than $\sqrt{3}$.*

The proofs of the next two lemmas utilize a *variational approach* developed by Rubinstein and Thomas for their paper on the Steiner ratio conjecture (Rubinstein and Thomas (1991)) and later refined for their paper on the Steiner ratio conjecture for concyclic points (Rubinstein and Thomas (1992b)). In those papers, the method is based on an analysis of the derivative of the length ratio of two networks under certain perturbations of the terminals. Here we apply a similar approach to the length difference of the networks $S(P)$ and $T(R)$.

Lemma 9. *If $R \in R^A$, then every R-cherry of $S(P)$ is adjacent to a longest chord.*

Proof. By Lemma 8, the result is clearly true if $m \leq 3$. So assume that $m > 3$ and $R \in R^A$, and suppose, contrary to the statement of the lemma, that $S(P)$ has an R-cherry (on Rubin points r_2 and r_3, say) that is adjacent to two shorter chords; this is illustrated in Figure 4. We also assume, for the moment, that r_2r_3 is not the unique longest chord, and hence can be assumed to be an edge of $T(R)$.

Let $\tilde{S}(P)$ be a Simpson line of $S(P)$ with endpoints e and e', where e is the equilateral point of the cherry at r_2, r_3 and $\tilde{S}(P)$ contains the Steiner point s adjacent to r_2 and r_3 (as in the figure). We assume (without loss of generality) that q either lies on $\tilde{S}(P)$, or on the same side of $\tilde{S}(P)$ as r_3.

Let b and c denote points on the tangent to $\mathbb{D}$ at r_2 that are as close as possible to r_1 and r_3 respectively. Let d and f denote points on the tangent to $\mathbb{D}$ at r_3 that are as close as possible to r_2 and r_4 respectively. Observe that $\angle cr_2r_3 = \angle r_2r_3d$. We define $\theta_1 := \angle r_1r_2b, \theta_2 := \angle cr_2r_3 = \angle r_2r_3d$ and $\theta_3 := \angle fr_3r_4$.

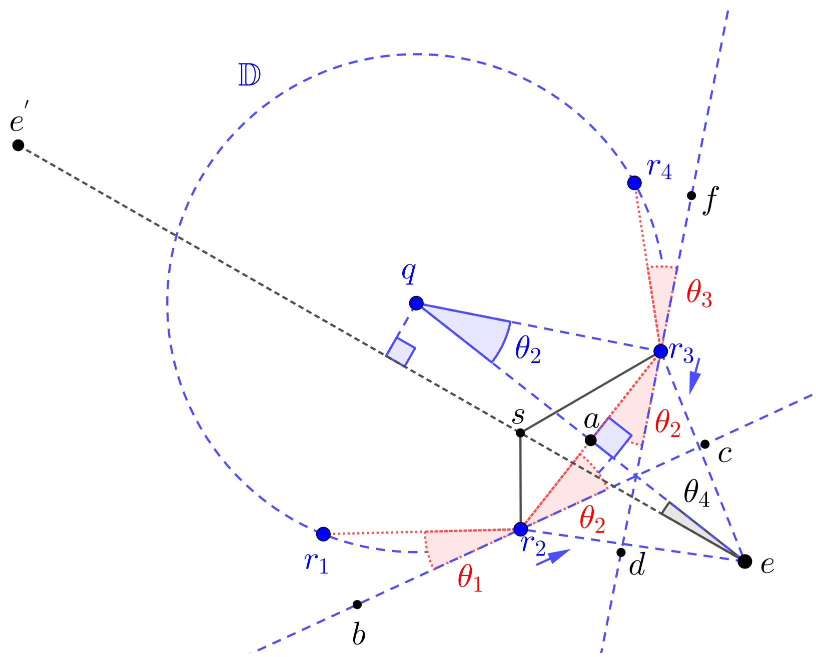

Fig. 4. Illustration of the important elements of $\tilde{S}(P)$ (black dotted line) and $T(R)$ (solid red lines) for Lemma 9.

Consider the effects of perturbing r_2 and r_3 using vectors (tangential to $\mathbb{D}$) v_2 and v_3 respectively, so that e moves at unit speed u directly towards e'. Note that this perturbation only changes the lengths in $S(P)$ of the three edges incident to s; hence under this perturbation R continues to be an element of R^C, even if $S(P)$ contains one or more 0-length edges.

We now decompose the motion of e into $u = u_1 + u_2$, where u_1 is a motion directly towards q and u_2 is perpendicular to u_1. Let $\theta_4 := \angle e'eq$, and note that if $|u| = 1$ then $|u_1| = \cos\theta_4$ and $|u_2| = \sin\theta_4$.

We wish to analyse the rates at which r_1 and r_2 move under each of these two perturbations. For the motion u_1, note that the line segments er_2, er_3 and r_2r_3 all shrink at the same rate. Also observe that r_2 and r_3 move at the same speed, say λ_1. This implies that r_1r_2 shrinks at rate $2\lambda_1\cos\theta_2$; and er_2 shrinks at rate $|u_1|\cos\frac{\pi}{6} + \lambda_1\cos(\frac{\pi}{3} - \theta_2) = \frac{\sqrt{3}}{2}\cos\theta_4 + \lambda_1\cos(\frac{\pi}{3} - \theta_2)$. Hence:

$$2\lambda_1\cos\theta_2 = \frac{\sqrt{3}}{2}\cos\theta_4 + \lambda_1\cos(\frac{\pi}{3} - \theta_2).$$

Applying the addition formula for cosines, we conclude:

$$\lambda_1 = \frac{\cos\theta_4}{2\cos(\theta_2 + \frac{\pi}{6})}.$$

For the motion u_2 we observe that e moves perpendicularly to qe at speed $\sin\theta_4$. Since r_2 and r_3 are equidistant from q this results in each of these Rubin points moving at equal speed, say λ_2, on the respective tangents to $\mathbb{D}$, both moving closer to r_1 and away from r_4. Since e is further from q than r_2

and r_3 are, it follows that $\lambda_2 < \sin\theta_4$.

Combining these two moves, we conclude that r_2 moves via v_2 with speed

$$\frac{\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})} - \lambda_2$$

and r_3 moves via v_3 with speed

$$\frac{\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})} + \lambda_2.$$

It now follows that under this perturbation we obtain

$$\begin{aligned} \dot{L_{S(P)}} \quad - \quad \dot{L_{T(R)}} &= -1 - \cos\theta_1\left(\frac{\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})} - \lambda_2\right) \\ &- \cos\theta_3\left(\frac{\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})} + \lambda_2\right) + \cos\theta_2\left(\frac{2\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})}\right) \end{aligned} \tag{4}$$

where the last three terms are the derivatives of the chord lengths of r_1r_2, r_3r_4 and r_2r_3 respectively.

By Corollary 2 we have that $|r_1r_2|$ and $|r_3r_4|$ are each less than $\sqrt{3}$, and hence that $\theta_1, \theta_3 < \frac{\pi}{3}$. So we obtain

$$\begin{aligned} \dot{L_{S(P)}} - \dot{L_{T(R)}} &< -1 - \frac{1}{2}\left(\frac{\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})} - \lambda_2\right) \\ &\quad - \frac{1}{2}\left(\frac{\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})} + \lambda_2\right) + \frac{\cos\theta_2\cos\theta_4}{\cos(\theta_2+\frac{\pi}{6})} \\ &= -1 - \frac{\cos\theta_4}{2\cos(\theta_2+\frac{\pi}{6})} + \frac{\cos\theta_2\cos\theta_4}{\cos(\theta_2+\frac{\pi}{6})} \\ &= -1 + \frac{\cos\theta_4}{2}\left(\frac{2\cos\theta_2 - 1}{\cos(\theta_2+\frac{\pi}{6})}\right). \end{aligned}$$

If we can show that

$$\frac{2\cos\theta_2 - 1}{\cos(\theta_2+\frac{\pi}{6})} \leq 2 \tag{5}$$

then it follows that $\dot{L_{S(P)}} - \dot{L_{T(R)}} < 0$, giving a contradiction to $R \in R^A$. However, Inequality 5 is equivalent to saying that $(2-\sqrt{3})\cos\theta_2 + \sin\theta_2 \leq 1$, which holds for all $\theta_2 \in [0, \frac{\pi}{3})$, the maximum range of θ_2 since $|r_2r_3| < \sqrt{3}$ (by Corollary 2). Hence the result follows.

Finally, if r_2r_3 is the unique longest chord, then we repeat the same argument but without the term corresponding to r_2r_3 in $\dot{L_{S(P)}} - \dot{L_{T(R)}}$, and again the result follows. □

Lemma 10. *For $R \in R^A$, with $m \geq 4$, let r_1, r_2, r_3, r_4 be four consecutive Rubin points on $\mathbb{D}$, such that r_2 and r_3 belong to an R-cherry of $S(P)$. For each $i \in \{1,2,3\}$ let $\phi_i := \angle r_i q r_{i+1}$. Then $\phi_1+\phi_2+\phi_3 > \pi$.*

Proof. Refer to Figure 5 and consider the same perturbation as in the proof of Lemma 9. We also define the angles $\theta_1, \theta_2, \theta_3$ and θ_4 as in the proof of Lemma 9. As in Lemma 9, we assume that q either lies on

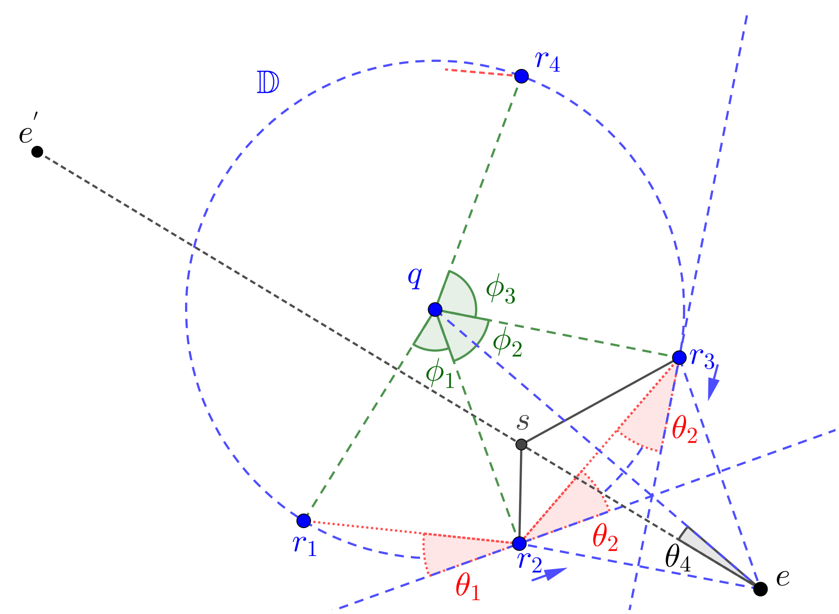

Fig. 5. Illustration of the important elements of $S(P)$ and $T(R)$ for Lemma 10.

$\tilde{S}(P)$, or on the same side of $\tilde{S}(P)$ as r_3. By Lemma 9, at least one of r_1r_2 and r_3r_4 is a longest chord. We will assume for the moment that r_3r_4 is a longest chord (this is the more difficult of the two cases) and that the gap in $T(R)$ corresponds to r_3r_4 both before and after the perturbation.

For this case it is helpful to divide the argument into a series of steps:

Step 1: Arguing by contradiction, assume that $\phi_1 + \phi_2 + \phi_3 \leq \pi$. We have the following constraints: $\theta_1 + \theta_2 \leq \frac{\pi}{3}$, $\theta_4 \leq \frac{\pi}{6}$ and $\theta_2 \leq \frac{\pi}{4}$ (since $\phi_i = 2\theta_i$ for each $i \in \{1, 2, 3\}$). These constraints define a domain $\mathcal{D}$ to which we can restrict our attention.

Step 2: In Equation 4 of Lemma 9 we express the derivative of $L_S - L_T$ in terms of $\theta_1, \theta_2, \theta_3, \theta_4$ and λ_2 in the case where none of the chords r_1r_2, r_2r_3 or r_3r_4 corresponds to the gap. Here the expression will be the same, but without the term corresponding to the change in length of the chord r_3r_4. We also require an explicit expression for λ_2, which is the speed at which r_2 and r_3 rotate around q. This can be computed by multiplying the speed, $\sin\theta_4$, at which e rotates around q by the ratio between the radius of $\mathbb{D}$ and $|qe|$; so it follows that $\lambda_2 = \frac{\sin\theta_4}{|qe|}$. We observe, as in the proof of Lemma 4, that $|qe| = 2\sin(\theta_2 + \frac{\pi}{6})$; hence

$$\lambda_2 = \frac{\sin\theta_4}{2\sin(\theta_2 + \frac{\pi}{6})}$$

and

$$L_{\dot{S(P)}} - L_{\dot{T(R)}} = -1 - \cos\theta_1 \left(\frac{\cos\theta_4}{2\cos(\theta_2 + \frac{\pi}{6})} - \frac{\sin\theta_4}{2\sin(\theta_2 + \frac{\pi}{6})} \right) + \cos\theta_2 \left(\frac{2\cos\theta_4}{2\cos(\theta_2 + \frac{\pi}{6})} \right). \quad (6)$$

Step 3: We now aim to show that $L_{\dot{S(P)}} - L_{\dot{T(R)}} < 0$, giving a contradiction to $R \in R^A$. We do this by showing that $F(\theta_1, \theta_2, \theta_4) := 2\cos(\theta_2 + \frac{\pi}{6})(L_{\dot{S(P)}} - L_{\dot{T(R)}}) < 0$. By Equation 6

$$F(\theta_1, \theta_2, \theta_4) = -2\cos(\theta_2 + \frac{\pi}{6}) + \cos\theta_4(2\cos\theta_2 - \cos\theta_1) + \frac{\sin\theta_4 \cos\theta_1}{\tan(\theta_2 + \frac{\pi}{6})}.$$

Step 4: We focus on the θ_1 variable. In the domain $\mathcal{D}$, we have $0 \leq \theta_1 \leq \frac{\pi}{3} - \theta_2$. Here we show that $\frac{\partial F}{\partial \theta_1} \neq 0$ on the interior of $\mathcal{D}$ and then check the parts of the boundary of $\mathcal{D}$ corresponding to the limiting values of θ_1. Taking the partial derivative:

$$\frac{\partial F}{\partial \theta_1} = \sin\theta_1 \left(\cos\theta_4 - \frac{\sin\theta_4}{\tan(\theta_2 + \frac{\pi}{6})} \right).$$

Hence $\frac{\partial F}{\partial \theta_1} = 0$ implies that either: $\sin\theta_1 = 0$, in which case $\theta_1 = 0$; or $\tan\theta_4 = \tan(\theta_2 + \frac{\pi}{6})$, which implies $\theta_4 = \theta_2 + \frac{\pi}{6}$. The latter situation only occurs in $\mathcal{D}$ if $\theta_2 = 0$ and $\theta_4 = \frac{\pi}{6}$. It follows that $\frac{\partial F}{\partial \theta_1} \neq 0$ on the interior of $\mathcal{D}$.

Step 5: The last two steps involve showing that $F \leq 0$ on the the parts of the boundary of $\mathcal{D}$ corresponding to the extreme values of θ_1. For the boundary where $\theta_1 = \frac{\pi}{3} - \theta_2$ we obtain (via standard trigonometric identities):

$$\begin{aligned} F &= -2\cos(\theta_2 + \frac{\pi}{6}) + \cos\theta_4 \left(2\cos\theta_2 - \cos(\frac{\pi}{3} - \theta_2) \right) + \frac{\sin\theta_4 \cos(\frac{\pi}{3} - \theta_2)}{\tan(\theta_2 + \frac{\pi}{6})} \\ &= -2\cos(\theta_2 + \frac{\pi}{6}) + \cos\theta_4 \left(\sqrt{3}\cos(\theta_2 + \frac{\pi}{6}) \right) + \sin\theta_4 \cos(\theta_2 + \frac{\pi}{6}) \\ &= -2\left(1 - \cos(\theta_4 - \frac{\pi}{6}) \right) \cos(\theta_2 + \frac{\pi}{6}) \ \leq \ 0 \end{aligned}$$

with equality only if $\theta_4 = \frac{\pi}{6}$.

Step 6: Consider the part of the boundary of $\mathcal{D}$ for which $\theta_1 = 0$. For the maximum value of θ_4 on this region we have:

$$F(0, \theta_2, \frac{\pi}{6}) = \sin\theta_2 + \frac{1}{2}\cot(\theta_2 + \frac{\pi}{6}) - \frac{\sqrt{3}}{2}.$$

It is straightforward to check that this function is less than or equal to 0 over the domain $0 \leq \theta_2 \leq \frac{\pi}{4}$. Hence, to show that $F(0, \theta_2, \theta_4) \leq 0$ on the boundary of $\mathcal{D}$ it suffices to show that $\frac{\partial}{\partial \theta_4} F(0, \theta_2, \theta_4) > 0$ over the given domain, where

$$\frac{\partial}{\partial \theta_4} F(0, \theta_2, \theta_4) = \sin\theta_4(1 - 2\cos\theta_2) + \frac{\cos\theta_4}{\tan(\theta_2 + \frac{\pi}{6})}.$$

We argue by contradiction. First note that $\frac{\partial}{\partial\theta_4}F(0,\theta_2,0) > 0$, and suppose there exists a point on the boundary of $\mathcal{D}$ where $\frac{\partial}{\partial\theta_4}F(0,\theta_2,\theta_4) = 0$. This implies that:

$$\tan\theta_4 = \frac{1}{(2\cos\theta_2 - 1)\tan(\theta_2 + \frac{\pi}{6})}. \tag{7}$$

Since $0 \le \theta_4 \le \frac{\pi}{6}$, we have $\tan\theta_4 \le \frac{1}{\sqrt{3}}$. However, it is straightforward to see that the right hand side of Equation 7 is a decreasing function over the domain $0 \le \theta_2 \le \frac{\pi}{4}$, and at $\theta_2 = \frac{\pi}{4}$ we have

$$\frac{1}{(2\cos\frac{\pi}{4} - 1)\tan\frac{5\pi}{12}} = \frac{1}{(\sqrt{2}-1)(2+\sqrt{3})} > \frac{1}{\sqrt{3}}$$

giving the required contradiction. Hence $\frac{\partial}{\partial\theta_4}F(0,\theta_2,\theta_4) > 0$, concluding the argument that $\dot{L_{S(P)}} - \dot{L_{T(R)}} < 0$.

Finally, if r_1r_2 is a longest chord, but r_3r_4 is not, we have a similar, but significantly simpler argument. □

The following two results are immediate consequences of this lemma.

Corollary 3. *If $R \in R^A$ then the length of the longest chord is greater than 1.*

Corollary 4. *If $R \in R^A$ then any $S(P)$ for R has at most three R-cherries. Furthermore, if $S(P)$ has two R-cherries, then those two R-cherries are adjacent (that is, they contain four consecutive Rubin points on $\mathbb{D}$).*

5. More properties of $S(P)$ if $R \in R^A$

In this section we develop a few more useful properties of $S(P)$ relating to R-cherries, the bridges and the non-bridge. Recall that each bridge is a convex path in $S(P)$, whereas the unique non-bridge of R contains both clockwise and anti-clockwise turns. This latter property is due to the central terminal q being adjacent to one of the Steiner points in the non bridge path in $S(P)$, and means that all but one of the Steiner points in the path have turns of the same orientation.

Definition 5 (Internal R-cherry). An *internal R-cherry* is an R-cherry that does not share a Rubin point with a non-bridge in $S(P)$.

Lemma 11. *If $R \in R^A$, then $S(P)$ has no internal R-cherries.*

Proof. Let $R \in R^A$, and contrary to the statement of the lemma suppose there are four adjacent Rubin points r_1, r_2, r_3, r_4 on $\mathbb{D}$ such that r_2 and r_3 correspond to an internal R-cherry of $S(P)$. Let s_1 be the Steiner point of $S(P)$ adjacent to r_2 and r_3. Since the R-cherry is internal, it follows that s_1 is not adjacent to q, and hence must be adjacent to another Steiner point, s_2 (see Figure 6). Again, since the R-cherry is internal, it also follows that s_2 is not adjacent to q. Hence there must be another Steiner point s_3 in $S(P)$ adjacent to s_2, such that q lies in the connected component of $S(P) \setminus \{s_2s_3\}$ containing

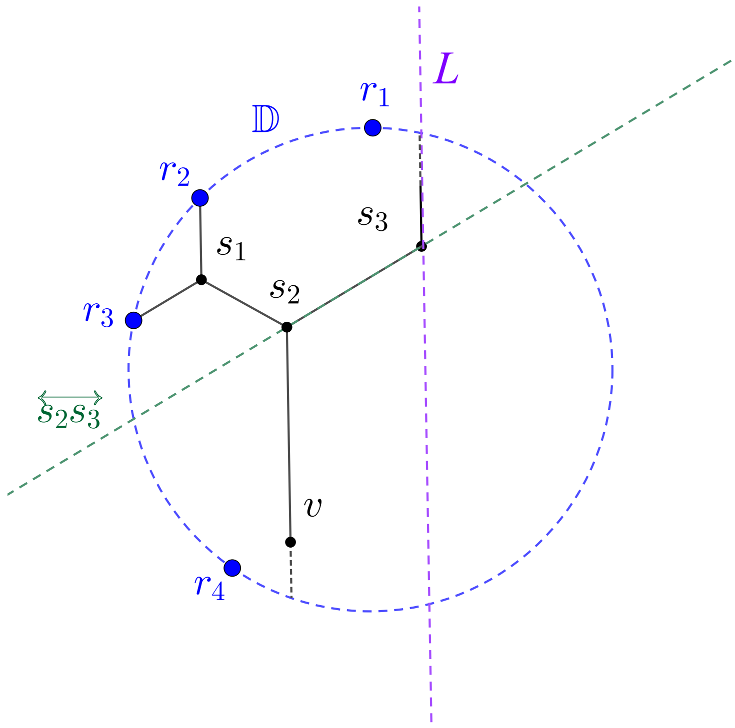

Fig. 6. Part of the structure $S(P)$ when r_2 and r_3 belong to an internal R-cherry (Lemma 11).

s_3. Without loss of generality, we can assume (as in Figure 6) that the path in $S(P)$ from r_2 to s_3 is a convex path with anti-clockwise turns.

Let v be the vertex of $S(P)$ adjacent to s_2, other than s_1 and s_3. Note that v may either be a Steiner point, or the Rubin point r_4. Let L be the line through s_3 parallel to $s_2 v$. It is easy to see, by the angle properties of Steiner points, that r_1, r_2, r_3, r_4 all lie in the same closed half-plane determined by L. There are now three cases to consider, each of which leads to a contradiction:

1. q lies on L or on the opposite side of L to r_2 and r_3. It follows that $\angle r_1 q r_2 + \angle r_2 q r_3 + \angle r_3 q r_4 < \pi$, contradicting Lemma 10.
2. q lies on the same side of L as r_2 and r_3, and on the same side of $\overline{s_2 s_3}$ as v. Then the distance of q from the anti-clockwise convex path in $S(P)$ beginning $s_2, v, \ldots$ and finishing at a Rubin point (possibly v) is less than $|s_2 s_3|$, contradicting the minimality of $S(P)$, since we could remove $s_2 s_3$ and reconnect the tree using a shorter edge.
3. q lies on the same side of L as r_2 and r_3, and on the opposite side of $\overline{s_2 s_3}$ to v. Consider the convex bridge path $p_{2,1}$ from r_2 to r_1, which begins $r_2, s_1, s_2, s_3, \ldots$. This path must contain at least one more Steiner point s_4 (otherwise q necessarily satisfies the conditions of case 1 or case 2). Then it is easy to see that in the final section of $p_{2,1}$ from s_4 to r_1 there exist adjacent vertices w_1 and w_2 such that the Rubin points r_1, r_2, r_3, r_4 all lie in the same closed half-plane determined by $\overline{w_1 w_2}$, but q lies on the opposite side of $\overline{w_1 w_2}$. Hence, as in case 1, we obtain a contradiction to Lemma 10.

□

The following result is an immediate consequence of Lemma 11, noting that a cherry that is not an R-cherry shares a Rubin point with the non-bridge.

Corollary 5. *If $R \in R^A$ then $S(P)$ contains at most two cherries.*

Definition 6 (intermediate bridge). An *intermediate bridge* is a bridge in $S(P)$ that is not adjacent to the non-bridge in $S(P)$.

Lemma 12. *Let r_j and r_k be consecutive Rubin points on $\mathbb{D}$ for which there is a bridge in $S(P)$ containing exactly two Steiner points: s_j adjacent to r_j and s_k adjacent to r_k. Suppose q belongs to the component of $S(P) \setminus \{s_j s_k\}$ containing s_j. Then $|s_k r_k| \leq |s_j r_j|$.*

Proof. Let v_0 denote the vertex adjacent to s_j, other than r_j and s_k; similarly, let v_1 denote the vertex adjacent to s_k, other than r_k and s_j. See Figure 7. Note that $|s_j s_k| \leq |q s_k|$, otherwise (s_j, s_k) can be

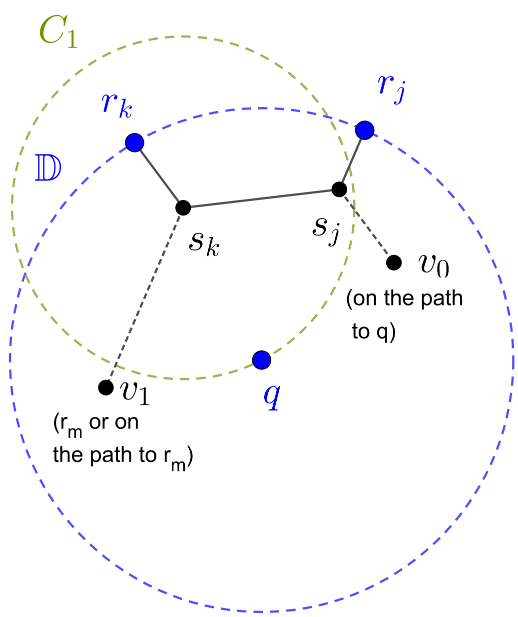

Fig. 7. s_j is inside $\mathbb{D}$ and in the interior of, or on the boundary of C_1 (Lemma 12).

replaced in $S(P)$ with a new edge (q, s_k) giving a shorter tree and contradicting the minimality of $S(P)$.

Consider the case where $|s_k r_k| = |s_j r_j|$, and let $d_{jk} := |s_j s_k|$. Let e be the point where the rays $\overrightarrow{r_j s_j}$ and $\overrightarrow{r_k s_k}$ intersect, and let m be the midpoint of $s_j s_k$, as illustrated in Figure 8. By the angle properties of Steiner trees, $\angle r_j e r_k = \frac{\pi}{3}$, and the triangle $\triangle s_j s_k e$ is equilateral, with side length d_{jk}. By symmetry, the line $\overleftrightarrow{me}$ bisects $\mathbb{D}$, and hence q lies on the ray $\overrightarrow{me}$.

Let C_{jk} be the circumcircle of r_j, r_k, e. Suppose we move e along C_{jk} towards r_j to a new position e', as in Figure 8. Let s'_k be the point on $e' r_k$ a distance d_{jk} from e, let s'_j be the point on $e' r_j$ a distance d_{jk}

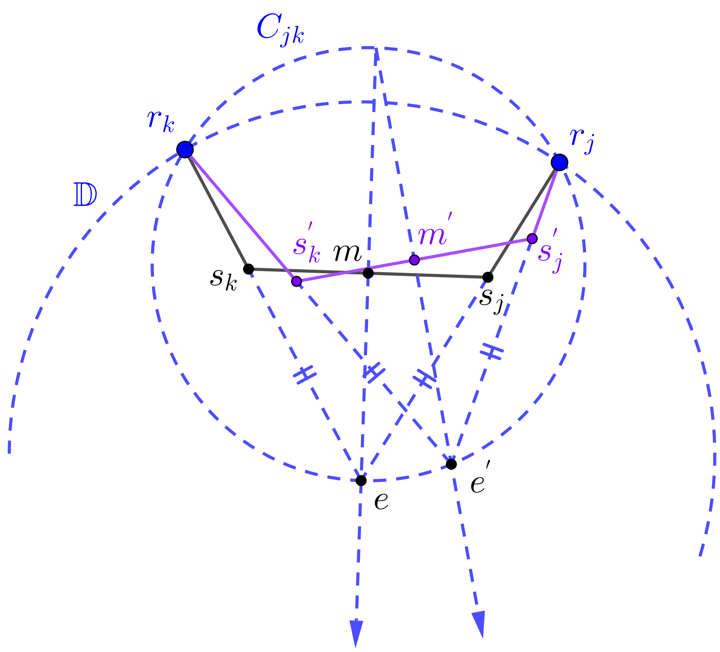

Fig. 8. Construction for showing that if $|s_k r_k| > |s_j r_j|$ then $|qs_k| < |qs_j|$ (Lemma 12).

from e, and let m' be the midpoint of $s_j' s_k'$. Note that after this repositioning of e, $\angle r_j s_j' s_k' = \angle s_j' s_k' r_k = \frac{2\pi}{3}$ (as required for Steiner points) and $|s_k' r_k| > |s_j' r_j|$. Now observe that m is on the same side of the line $\overleftrightarrow{m'e'}$ as s_k, hence $|s_k' m| < |s_j' m|$, which implies that for any point on the ray $\overrightarrow{me}$, s_k' is closer to that point than s_j'. Thus, $|qs_k'| < |qs_j'|$.

Now suppose, contrary to the lemma, that $|s_k r_k| > |s_j r_j|$. The above discussion implies that:

$$|qs_k| < |qs_j|. \tag{8}$$

Let C_1 denote a circle centred on s_k with radius $|qs_k|$. Since $|s_j s_k| \leq |qs_k|$, it follows that s_j is in the interior of, or on the boundary of C_1 (as shown in Figure 7). Let C_2 denote a circle centred on q with radius $|qs_k|$ (see Figure 9 Left). Inequality 8 implies s_j is outside the boundary of C_2.

Let a denote the point at the intersection of C_1 and C_2 closest to s_j. Since C_1 and C_2 have the same radius, it follows that $\triangle qas_k$ is equilateral (as shown in Figure 9 Left) and hence:

$$\angle qs_k a = \frac{\pi}{3}. \tag{9}$$

Since s_k is a Steiner point (at which all incident angles meet at an angle of $\frac{2\pi}{3}$), Equation 9 implies that:

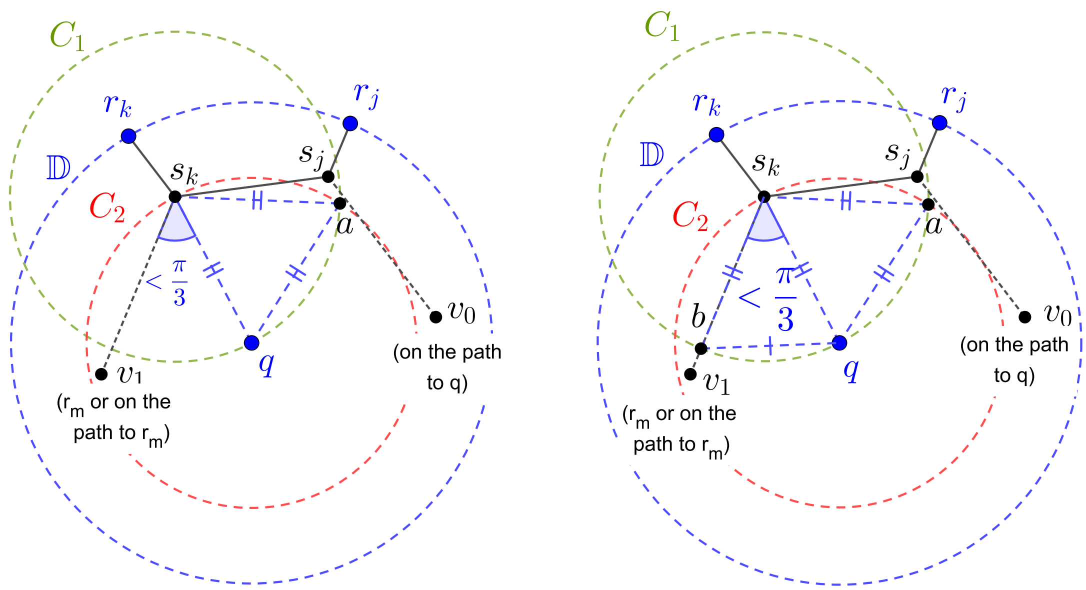

Fig. 9. Left: $\angle v_1 s_k q < \frac{\pi}{3}$ (Lemma 12, Inequality 10). Right: Let b denote a point on (v_1, s_k) such that $|bs_k| = |qs_k|$ (Lemma 12, Case 1).

$$\angle v_1 s_k q = \frac{\pi}{3} - \angle a s_k s_j < \frac{\pi}{3}. \tag{10}$$

There are two cases to consider:

Case (1). $|v_1 s_k| \geq |q s_k|$
Let b denote a point on (v_1, s_k) such that $|bs_k| = |qs_k|$ (Figure 9 Right).

Then $\triangle q s_k b$ is isosceles and given Inequality 10:

$$|qb| < |s_k b|. \tag{11}$$

Consider the following line segment replacement. First remove line segment $s_k b$, separating $S(P)$ into two components, one containing b and the other containing q. Then insert line segment qb. This reconnects the two components. The resulting tree has length $L_{S(P)} - |s_k b| + |qb|$ which, given Inequality 11, contradicts the minimality of $S(P)$.

Case (2). $|v_1 s_k| < |q s_k|$
The case conditions imply that v_1 lies inside C_2 and hence must be a Steiner point. Let b be the point on the ray $\overrightarrow{s_k v_1}$ such that $|s_k b| = |s_k q|$. Let v_2 denote a vertex adjacent to v_1 that is either r_m or on the path from v_1 to r_m. Let c denote the point at the intersection of $\overrightarrow{v_1 v_2}$ and $\overline{qb}$ (Figure 10). Clearly if

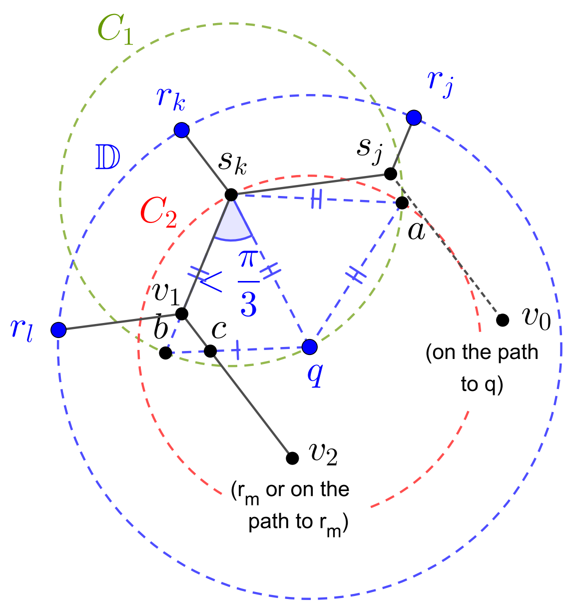

Fig. 10. Showing the location of c (Lemma 12, Case 2).

$|v_1v_2| < |v_1c|$ then $S(P)$ is self-intersecting (since the convex path beginning $v_0, s_j, s_k, v_1, v_2, \ldots$ must eventually terminate at a Rubin point), so we can assume that $|v_1v_2| \geq |v_1c|$.

Observe by Inequality 10 and the fact that $\angle v_2v_1s_k = \frac{2\pi}{3}$ that:

$$|qc| < |s_kv_1|. \tag{12}$$

Consider the following line segment replacement: Remove line segment s_kv_1. This separates $S(P)$ into two components, one containing c and the other containing q. Then insert the line segment qc, reconnecting the two components. The resulting tree has length $L_{S(P)} - |s_kv_1| + |qc|$ which, by Inequality 12, again contradicts the minimality of $S(P)$. □

In the following sections we use these properties to eliminate the various cases that remain as possible candidates for members of R^A with $m \geq 3$.

6. Cases where $S(P)$ has exactly one R-cherry

We first consider some of the consequences of the results from the previous section. In a full Steiner tree S, we refer to the edges not incident to leaves (or equivalently not incident to terminals) of S as *internal edges* of S. It is easy to see that for a full Steiner tree S, the set of internal edges form a subtree S' of S, and that each leaf of S' corresponds to a cherry of S.

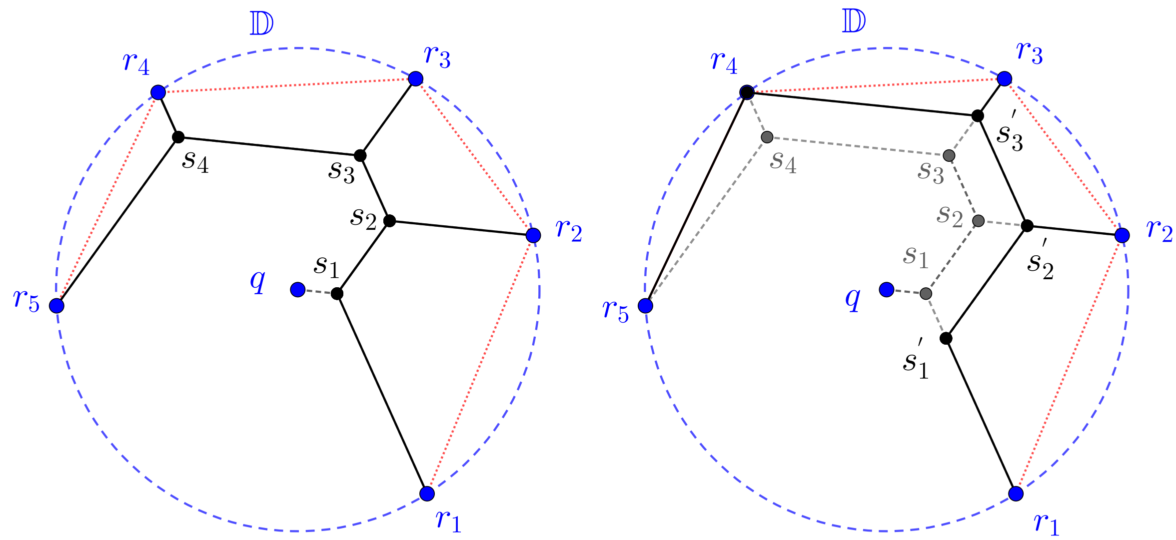

Fig. 11. Left: A Steiner tree $S(P)$ for $m = 5$ containing exactly one R-cherry (Lemma 13). Right: The deformation for Step 2, resulting in the tree $S(P)^2$ (Lemma 13).

Now suppose that $R \in R^A$ and the Steiner tree $S(P)$ contains at least two Steiner points (that is, $m \geq 3$). Then it follows from Corollary 5 that the subtree of internal edges of $S(P)$ is a path.

Suppose, in addition, that $S(P)$ contains only one R-cherry; then there is a second cherry involving q and a Rubin point with Steiner point s_1 (say). Since the R-cherry shares a Rubin point with the non-bridge, it follows that the path of internal edges is part of the non-bridge path of S, and furthermore that the path of internal edges is convex (or else the R-cherry would be internal, contradicting Lemma 11). Hence, if $m = 3$ then up to rotation, reflection and the lengths of edges of $S(P)$, the tree $S(P)$ must look like the Steiner tree in Figure 1. If $m = 4$, then we obtain the same diagram except that we replace the Rubin point on the non-bridge adjacent to s_4, by a new R-cherry. This process can be iterated for $m > 4$. Figure 11 (Left) shows an example with $m = 5$.

Lemma 13. *If $L_{S(P)} - L_{T(R)} \leq 2 - \sqrt{3}$ and $m > 2$, then $S(P)$ contains at least two R-cherries.*

Proof. Here, we illustrate the proof in detail for the case where $m = 5$, and then indicate at the end of the proof how the argument can be generalised to other values of m by adjusting the iterative steps. So, let $m = 5$, and assume contrary to the statement of the lemma that $S(P)$ contains only one R-cherry. By the previous discussion, $S(P)$ can be assumed to have the form illustrated in Figure 11 (Left). The strategy of the proof is to show that there is a series of deformations of $S(P)$ yielding a tree on R with length greater than or equal to $L_{T(R)}$ and such that the deformations monotonically decrease the length of the tree. Furthermore, we will show that the change in length resulting from the deformations must be at least $2 - \sqrt{3}$ and therefore $L_{S(P)} - L_{T(R)} > 2 - \sqrt{3}$, contradicting the assumption of the lemma.

As in Figure 11, let $r_1, \ldots, r_5$ be the five Rubin points, in anti-clockwise order around $\mathbb{D}$. For each $i \in \{1,2,3,4\}$, let s_i be the Steiner point of $S(P)$ adjacent to r_i, such that s_1 is also adjacent to q, and s_4 is also adjacent to r_5. Note that the convex path from q to r_5 has anti-clockwise turns at each Steiner point.

The monotonically length-reducing deformation of $S(P)$ is performed in a series of steps:

Step 1: Remove the terminal q and the edge qs_1 from $S(P)$ and denote the result $S(P)^1$. The tree $S(P)^1$, is a network that interconnects the Rubin points using three standard Steiner points, and an additional vertex (s_1) of degree 2, at which the incident edges meet at an angle of $\frac{2\pi}{3}$.

The remaining steps rely on the observation that for each $i \in \{1,2,3\}$ we have $|s_i r_i| \geq |s_{i+1} r_{i+1}|$, which follows from Lemma 12.

Step 2: Starting with $S(P)^1$, move s_4 to r_4 and, for each $i \in \{1,2,3\}$, move each vertex s_i towards its adjacent Rubin point by a distance of $|s_4 r_4|$, relabelling the vertex s_i', and maintaining the topology of the network (apart from the edge incident with r_4 collapsing into a single point). Denote the resulting tree $S(P)^2$ (Figure 11 Right).

It follows that

$$L_{S(P)^1} - L_{S(P)^2} = \sum_{i=1}^{3}(|s_i s_{i+1}| + |s_i s_i'| - |s_i' s_{i+1}'|) + |s_4 r_4| + |s_4 r_5| - |r_4 r_5|. \tag{13}$$

For both $S(P)^1$ and $S(P)^2$ the angles between incident edges at each internal vertex are all $\frac{2\pi}{3}$, which implies that, for each $i \in \{1,2,3\}$, $s_i' s_{i+1}'$ is parallel to $s_i s_{i+1}$, and hence $|s_i s_{i+1}| + |s_i s_i'| = |s_i' s_{i+1}'|$.

It follows from Equation 13 that

$$L_{S(P)^1} - L_{S(P)^2} = |s_4 r_4| + |s_4 r_5| - |r_4 r_5|. \tag{14}$$

Now, consider a triangle $\triangle abc$ with fixed base ac, such that $\angle abc = \frac{2\pi}{3}$ and such that $|ab| < |bc|$. The ratio

$$\frac{|bc| - |ac|}{|ab|}$$

approaches a minimum as $\triangle abc$ approaches being an isosceles triangle. Hence, by the law of sines:

$$\frac{|bc| - |ac|}{|ab|} \quad > \quad \frac{\sin(\frac{\pi}{6}) - \sin(\frac{2\pi}{3})}{\sin(\frac{\pi}{6})} = 1 - \sqrt{3},$$

which implies that

$$\frac{|ab| + |bc| - |ac|}{|ab|} > 2 - \sqrt{3}. \tag{15}$$

Observe that in the tree $S(P)$ we have $|s_4 r_4| < |s_4 r_5|$, since there is a convex anti-clockwise path from q to r_5, meaning that the line $\overline{s_4 s_3}$ must pass between r_4 and q. Hence, by Equation 14 and Inequality 15:

$$
\begin{aligned}
L_{S(P)^1} - L_{S(P)^2} &= \left(\frac{|s_4 r_4| + |s_4 r_5| - |r_4 r_5|}{|s_4 r_4|} \right) |s_4 r_4| \\
&> (2 - \sqrt{3}) |s_4 r_4|.
\end{aligned}
$$

Step 3: The next step is to iterate the deformation in Step 2 three more times, but each time on a smaller tree, specifically the part of the newly-formed tree on $R\,(S(P)^i)$ that does not include any edges that are chords of $\mathbb{D}$. At each iteration an internal node collapses into a Rubin point, so after the final iteration we are left with a spanning tree of R:

$$W(R) := S(P)^5 = \{r_1 r_2, r_2 r_3, r_3 r_4, r_4 r_5\}.$$

This sequence of trees is illustrated in Figure 12.

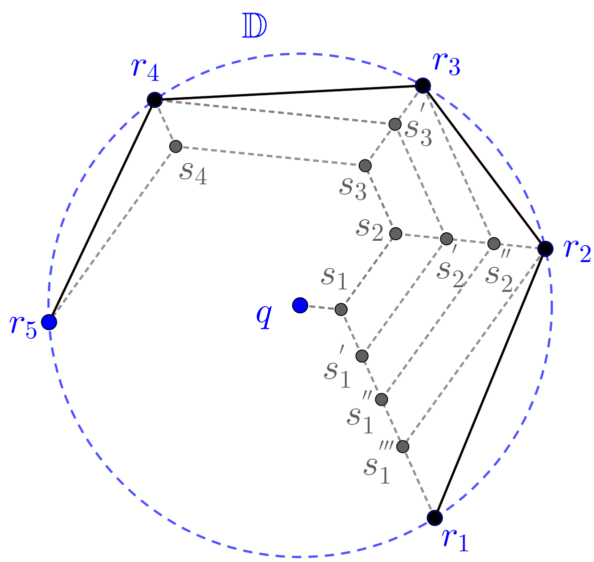

Fig. 12. The resulting trees after the iterations of the deformation in Steps 2 and 3 (Lemma 13).

Following the same argument as in Step 2, and using the notation for the internal vertices for the trees $S(P)^3$, $S(P)^4$ and $S(P)^5$ as indicated in Figure 12, we observe that

$$
\begin{aligned}
L_{S(P)^1} - L_{S(P)^2} &> (2 - \sqrt{3}) |s_4 r_4|, \\
L_{S(P)^2} - L_{S(P)^3} &> (2 - \sqrt{3}) |s_3' r_3|, \\
L_{S(P)^3} - L_{S(P)^4} &> (2 - \sqrt{3}) |s_2'' r_2|, \\
L_{S(P)^4} - L_{W(R)} &> (2 - \sqrt{3}) |s_1''' r_1|.
\end{aligned}
$$

Also note, by the construction, that

$$|s_1r_1| = |s_4r_4| + |s_3'r_3| + |s_2''r_2| + |s_1'''r_1|.$$

Combining these results we conclude that

$$L_{S(P)^1} - L_{W(R)} > (2-\sqrt{3})|s_1r_1|.$$

Now, consider the triangle $\triangle qr_1s_1$, and let $\theta := \angle qr_1s_1$. Since $\angle r_1s_1q = \frac{2\pi}{3}$ and $|qr_1| = 1$, we can rewrite $|qs_1|$ and $|s_1r_1|$ in terms of θ as follows:

$$|qs_1| = \frac{2\sin(\theta)}{\sqrt{3}}; \quad |s_1r_1| = \cos(\theta) - \frac{\sin(\theta)}{\sqrt{3}}.$$

Hence:

$$\begin{aligned} L_{S(P)} - L_{T(R)} &\geq L_{S(P)} - L_{W(R)} \\ &> |qs_1| + (2-\sqrt{3})|s_1r_1|) \\ &= \frac{2\sin(\theta)}{\sqrt{3}} + (2-\sqrt{3})\left(\cos(\theta) - \frac{\sin(\theta)}{\sqrt{3}}\right) \\ &= \sin(\theta) + (2-\sqrt{3})\cos(\theta). \end{aligned}$$

Differentiating this latter function, it is easily checked that in the range $0 \leq \theta \leq \frac{\pi}{3}$ this is an increasing function of θ, and hence has a lower bound when $\theta = 0$ (which implies that $|qs_1| = 0$ and $|s_1r_1| = 1$). Thus

$$L_{S(P)} - L_{T(R)} > 2 - \sqrt{3},$$

giving the required contradiction.

Finally, if $m \neq 5$ we apply essentially the same argument but with either fewer or more iterations in Step 3. □

7. Cases where S(P) has exactly two R-cherries

By Lemma 13 and Corollary 5, the only cases remaining to consider are those where $S(P)$ has exactly two R-cherries. By Lemma 11, each of the two R-cherries in such a network share a terminal with the non-bridge of $S(P)$. An example of such a Steiner tree is illustrated in Figure 13 (Left).

Lemma 14. *If $m > 2$ and $S(P)$ contains two R-cherries, then $L_{S(P)} - L_{T(R)} > 2 - \sqrt{3}$.*

Proof. As in the proof of Lemma 13, we can assume that the Steiner points of $S(P)$ induce a subnetwork of $S(P)$ that is a single path lying on the non-bridge path of $S(P)$. If $m = 5$ it follows that up to rotation, reflection and the lengths of edges, $S(P)$ has the form indicated in Figure 13 (Left). We will outline a proof for this case using an iterative series of deformations, similar to those employed in the

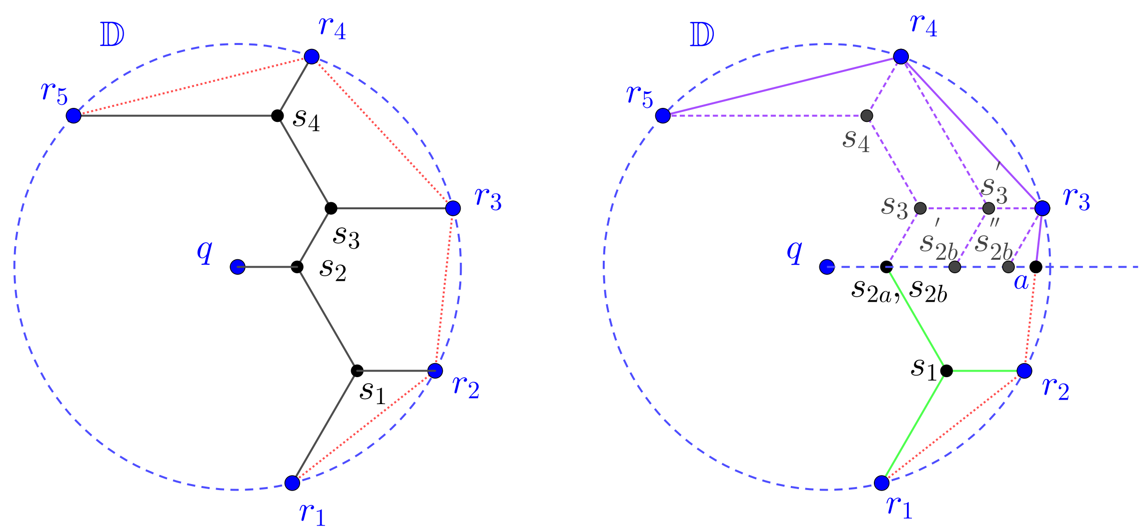

Fig. 13. Left: A Steiner tree $S(P)$ for $m = 5$ containing two R-cherries (Lemma 14). Right: The new edges and vertices resulting from each of the deformations in Steps 2 and 3 (Lemma 14).

proof of Lemma 13. The cases where $m \neq 5$ can be proved by the same method, but using either fewer or more iterations.

Step 1: Remove (q, s_2) from $S(P)$ and denote the result $S(P)^1$. Also, replace the vertex s_2 with coincident but disconnected points s_{2a} and s_{2b}, splitting $S(P)^1$ into two components $S(P)^{1a}$ and $S(P)^{1b}$ accordingly, such that $V(S^{1a}) = \{r_1, r_2, s_1, s_{2a}\}$ and $V(S^{1b}) = \{r_3, r_4, r_5, s_3, s_4, s_{2b}\}$ (using the vertex labels shown in Figure 13 Left). Observe that:

$$L_{S(P)} = |qs_2| + L_{S(P)^{1a}} + L_{S(P)^{1b}}.$$

Let a denote the intersection of $(r_2 r_3)$ and a ray $\overrightarrow{qs_{2a}}$ as shown in Figure 13 (Right).

Step 2: We next perform a deformation on the tree $S(P)^{1b}$, moving s_4 to r_4 and moving s_3 and s_{2b} towards r_3 and a respectively by distance $|s_4 r_4|$. Denote the new positions for s_3 and s_{2b} as s_3' and s_{2b}' respectively (Figure 13 Right), and denote the resulting tree as $S(P)^{2b}$. By the same argument as in Step 2 of the proof of Lemma 13, we have

$$L_{S(P)^{1b}} - L_{S(P)^{2b}} > (2 - \sqrt{3})|s_4 r_4|. \tag{16}$$

Step 3: As in the proof of Lemma 13, we iterate the deformation in Step 2 two more times, but each time on a tree with less Steiner points. The tree $S(P)^{3b}$ is obtained from $S(P)^{2b}$ by moving s_3' to r_3 and moving $s_{2b}^{'}$ towards a by distance $|s_3^{'}r_3|$ to a point $s_{2b}^{''}$ (again see Figure 13 Right). It follows that:

$$L_{S(P)^{2b}} - L_{S(P)^{3b}} > (2-\sqrt{3})|s_3^{'}r_3|. \tag{17}$$

Finally, $S(P)^{4b}$ is obtained from $S(P)^{3b}$ by moving $s_{2b}^{''}$ to a. We observe that the same lower bound easily applies for this final deformation:

$$L_{S(P)^{3b}} - L_{S(P)^{4b}} > (2-\sqrt{3})|s_{2b}^{''}a|. \tag{18}$$

Since, by the construction, $|s_4r_4| = |s_2s_{2b}'|$ and $|s_3'r_3| = |s_{2b}'s_{2b}''|$, it follows from Equations 16, 17 and 18 that

$$L_{S(P)^{1b}} - L_{S(P)^{4b}} > (2-\sqrt{3})|s_2a|. \tag{19}$$

Step 4: Let $S(P)^{3a}$ be the tree with vertices r_1, r_2 and a and edges r_1r_2 and r_2a; then by a similar argument to that in Steps 2 and 3, we obtain:

$$L_{S(P)^{1a}} - L_{S(P)^{3a}} > (2-\sqrt{3})|s_{2a}a|. \tag{20}$$

Let $W(R)$ be the spanning tree on R satisfying $W(R) := S^{3a} \cup S^{4b}$, meaning that $L_{W(R)} = L_{S^{3a}} + L_{S^{4b}}$. Together, Equations 19 and 20 imply that:

$$L_{S(P)^1} - L_{W(R)} > 2(2-\sqrt{3})|s_2a|,$$

and hence

$$L_{S(P)} - L_{W(R)} > |qs_2| + 2(2-\sqrt{3})|s_2a|.$$

Observe that $0 < |r_2r_3| < \sqrt{3}$ (by Corollary 8); it follows that:

$$\frac{1}{2} < |qa| = |qs_2| + |s_2a| < 1.$$

Hence, $L_{S(P)} - L_{W(R)}$ approaches a minimum when $|qa| \to \frac{1}{2}$ and $|qs_2| \to 0$ implying $|s_2a| \to \frac{1}{2}$:

$$L_{S(P)} - L_{T(R)} \geq L_{S(P)} - L_{W(R)} > 2-\sqrt{3}.$$

□

8. Conclusion

The following theorem is the main finding.

Theorem 1. *If q has degree 1 in $S(P)$, then* $\inf L_{S(P)} - L_{T(R)} = 2 - \sqrt{3}$.

Proof. Let m be the number of Rubin points in R. Lemmas 2, 4 prove that if $m \leq 2$ then $L_{S(P)} - L_{T(R)} > 2 - \sqrt{3}$. Furthermore, we observe that $2 - \sqrt{3}$ is the infimum for $L_{S(P)} - L_{T(R)}$ in this case.

When $m > 2$, Lemmas 13 and 14 together show that again $L_{S(P)} - L_{T(R)} > 2 - \sqrt{3}$. It follows that $2 - \sqrt{3}$ is the infimum for $L_{S(P)} - L_{T(R)}$ for all m. □

Acknowledgments

David Whittle is the grateful recipient of the 2015 Gilbert Rigg Scholarship, which supported his graduate research in underground mine plan optimisation. In addition, David was awarded the 2016 John Collier Scholarship to support his research-related travel.

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