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Contracting Mechanisms for Stable Sourcing Networks

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Problem definition: We study profit allocation for a sourcing network in which a buyer sources from a set of differentiated suppliers with limited capacity under uncertain demand for the final product. While the buyer takes the lead in forming the sourcing network and designing the contract mechanism, due to their substantial bargaining power, the suppliers take the lead in determining the terms of the contract. **Academic/Practical Relevance**: We identify contracting mechanisms that will ensure the stability of the sourcing network in the long term, where a stable sourcing network requires an effective profit allocation scheme that motivates all members to join and stay in the network. **Methodology:** We apply methods from game theory to model the network and analyze the Nash equilibrium of a non-cooperative game under a proposed contracting mechanism. We then use a cooperative game model to study the stability of the resulting equilibrium. **Results:** We show that the optimal network profit, as a set function of the set of suppliers, is submodular, which allows us to demonstrate that the core of the cooperative game is not empty. We also establish a set of conditions that are equivalent to, but much simpler than, the original conditions for the core. We use these results to demonstrate that the proposed fixed-fee contracting mechanism can implement a stable network in the competitive setting by achieving a profit allocation that is in the core of the cooperative game. We also demonstrate that the grand coalition is stable in a farsighted sense under the Shapley value allocation. **Managerial Implications:** Under the fixed fee mechanism, the buyer's decisions maximize the network profit and each supplier earns a profit equal to its marginal contribution. When the aggregate capacity of the supplier network is high relative to demand, or demand is more likely to be small, the fixed fee mechanism is likely to outperform the Shapley value allocation from the perspective of the buyer.

Key words: Capacity Planning and Investment, Game Theory, Incentives and Contracting, Supply Chain Management

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1. Introduction

Sourcing networks, consisting of a buyer along with multiple independent suppliers, are common in many industries, including transportation, manufacturing and services. Buyers source from multiple suppliers because doing so has a number of benefits, such as creating competition between the suppliers, which can help reduce the buyer's costs. Multi-sourcing also helps guarantee access to sufficient supply in settings where the individual suppliers have limited capacity (Kumar and Subrahmanya 2010). Thus, multi-sourcing can help a buyer meet all demand, particularly in highly volatile environments or when there is uncertainty regarding the market size. See Kumar and Subrahmanya (2010), Li (2013) and Li (2020) for discussions of other reasons why buyers choose to multi-source.

In this paper, we consider the contract design and order allocation problem for a sourcing network in which the buyer must contract with a set of differentiated suppliers who provide critical production capacity, or a key input of production, when there is uncertainty regarding demand for the end-product produced and/or sold by the buyer. While the capacity or inputs procured by the buyer from the suppliers are identical (or equivalent), the suppliers differ in terms of their costs and capacities. The suppliers are assumed to have reasonable knowledge of each other's costs and capacities, as would be the case for a commodity-type product, or when the industry is well-established with mature technologies and publicly-traded companies. Finally, a critical aspect of our model setting is that, although the multiple suppliers compete for a share of the buyer's procurement quantity, those suppliers have sufficient power in the relationship with the buyer that they act as Stackelberg leaders when negotiating a contractual agreement. Given this problem setting, we first characterize the system-optimal order allocation policy for a given set of suppliers. We then design procurement mechanisms, which determine the payment by the buyer to each participating supplier, under which the total network profit is maximized and the profit allocation across the suppliers ensures the stability of the supplier network.

While a number of industries fit the general problem setting described above, here we focus on one particular industry that is both critical to the global economy and under-studied in the operations and supply chain management literature, i.e., the iron ore industry. Iron ore is the key raw material used in the production of steel. A recent overview of the global iron ore industry and market can be found in Deloitte CIS Research (2020). As noted by Holmes and Lu (2015), the production of iron ore has increased rapidly since 2001 in response to growing demand for the steel used in construction, shipbuilding, automotive manufacturing, heavy industry, etc.[1] On the supply side, Australia and Brazil are currently the largest exporters of iron ore in the global market, and the industry is dominated by a handful of large multinational companies[2]. According

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[1] As these authors note "Asian countries continue to drive the expansion of the international iron ore industry." For example, in 2019, China was by far the largest importer of iron ore, representing 66.8% of all iron ore imports, followed by Japan and South Korea; see `https://oec.world/en/profile/hs92/iron-ore`.

[2] During 2019, Australia and Brazil represented 54.3% and 18.5% of the global iron ore market, respectively; see `https://oec.world/en/profile/hs92/iron-ore`. The major mining companies include Vale in Brazil and Rio Tinto, BHP Billiton and Fortescue Metals Group in Australia; see Investopedia (2020).

to Floren et al. (2019), the “corporate concentration for iron ore mining is much higher than that for steel production,” which “reduces the bargaining power of steel companies.” As a result, “iron ore miners in general have more bargaining power than the often smaller steel companies.”

The cost per unit (typically measured in dollars per metric ton) for the iron ore produced by the major mining companies varies according to a number of factors, including the specific type of ore extracted at a given location, and the costs associated with transporting the ore to the market, which depend primarily on distance to market and fuel costs. As a commodity item, information on unit costs tends to be publicly available, from a variety of sources, including the mining companies' own publications and news releases. See Investopedia (2020)[3]. Capacities (defined as production quantities plus reserves) are also publicly known for the major mining companies and are tracked by entities such as the U.S. Geological Survey (2020).

In this paper, we focus on the problem faced by purchasers of iron ore, such as steel manufacturers, who typically obtain ore from a relatively small set of mining companies. Given the current oversupply in the industry (Lau 2021), obtaining sufficient ore to meet demand has not been a challenge, despite the significant increase in demand in recent years. Instead, the key challenge faced by these purchasers in recent years has been the use of short-term contracts, which has led to significant price volatility[4]. Such price volatility was not always an issue for the industry. As discussed in Knowledge@Wharton (2009), historically the industry relied on a benchmark pricing system in which prices were negotiated between the buyers and suppliers on an annual basis. Such long-term agreements were beneficial for both parties, providing welcome stability and allowing them to “set production schedules, estimate earnings, and plan staffing and other costs” (Knowledge@Wharton 2009). For example, during 2012, about one-half of ore purchases took place under long-term contracts (Reserve Bank of Australia 2012). However, this system broke down about a decade ago, resulting in shorter-term contracts[5].

There is evidence that major consumers and producers of iron ore have an interest in, and willingness to, investigate new contractual forms that will ensure more stable and long-term relationships between producers and consumers (Floren et al. 2019). For example, in response to the move to shorter-term contracts, the Chinese government has begun to take an active role in negotiating contracts with the mining companies, reflecting concerns regarding the power of the large

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[3] Figure 1.51 in Holmes and Lu (2015) depicts the unit costs during 2013-14, which ranged from \$30-\$120 per ton.

[4] For example, “[p]rices peaked at \$187 per metric ton in February 2011, then plunged to about \$41 per ton in December 2015; see Investopedia (2020). Also, Figure 1.50 in Holmes and Lu (2015) provides a graphical depiction of the fluctuations in ore prices between 2010-2014.

[5] As noted in Chen et al. (2013), starting in 2010, “contracts between major iron ore producers and Chinese steel makers have proceeded with a quarterly system.”

mining firms and the volatility of short-term agreements[6]. In addition, in recent years, some steel manufacturers have sought long-term contracts with mines to ensure the supply of specific types of high-quality ore (Xu and Serapio 2018). On the supply side, some of the large mining companies have also indicated a willingness to consider new approaches (Newman 2020)[7].

Motivated by these observations, we investigate the contract design problem for a buyer who sources from a set of differentiated suppliers with limited capacity under uncertain demand for the buyer's final product. Since the buyer's need for the product drives the sourcing and procurement decisions, she takes the lead in forming the sourcing network and designing the contract mechanism with the suppliers. However, due to their limited number and substantial bargaining power, the suppliers take the lead in determining the terms of the contract, i.e., the price parameters. We seek to identify contracting mechanisms that will ensure the stability of the sourcing network in the long term, where a stable sourcing network requires an effective profit allocation scheme that motivates all members to join and stay in the network. From the perspective of the suppliers, staying in the network implies that they are committing to make their capacity available to the buyer, should it be needed to fill demand for the final product. More specifically, we address the following research questions:

- How should the buyer design a contracting mechanism to achieve a stable sourcing network with multiple suppliers, limited supplier capacity and uncertain end-product demand, given that the suppliers hold significant power?
- How do the profit allocations between the buyer and suppliers differ under two alternative mechanisms that can ensure stability, i.e., a fixed fee mechanism and a mechanism based on the Shapley value allocation? Which mechanism performs best from the perspective of the buyer?

1.1. Overview of Results and Contributions

To address these research questions, we first propose an easily implementable competitive mechanism that can be used to achieve a stable sourcing network for the problem setting studied in this paper. Under this mechanism, each supplier quotes a fixed upfront fee to the buyer. Given this fee, the buyer can access the supplier's capacity at cost. This *fixed fee mechanism* is similar to capacity reservation contracts, in which the manufacturer pays in advance to reserve the supplier's capacity (Hazra and Mahadevan 2009, Yang et al. 2018). To study the performance of this mechanism, we

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[6] For example, as noted in Knowledge@Wharton (2009), "[a]lthough it does not want to abandon the traditional price negotiations, CISA [the China iron and steel association] still wants more bargaining power as the world's largest buyer of iron ore. . . . [T]he association will try to establish a new pricing model for iron ore imports in order to negotiate on China's own terms, rather than 'blindly' following agreements made between foreign steelmakers and leading miners BHP, Rio Tinto and Vale."

[7] For example, "Rio Tinto has said it is willing to work with the China iron and steel association (Cisa) to improve iron ore pricing mechanisms... The pledge came after the Chinese association urged it to further develop price mechanisms amid another bout of market volatility" (Newman 2020).

develop a non-cooperative game model of our sourcing network. We use that model to demonstrate that, under this mechanism, the network profit is maximized at equilibrium and each supplier earns a profit equal to its marginal contribution to the network.

Then, to demonstrate that the fixed fee mechanism induces a profit allocation that will ensure the stability of the sourcing network, we develop and analyze a cooperative game framework. Because we focus on sourcing networks, our game has a special structure in which the buyer is a necessary player, i.e., if the buyer is not included in the network, the network profit is zero. Since the buyer is a necessary player, she takes a lead role in creating the network and designing the contracting mechanism, despite the relative bargaining power of the suppliers. We use the cooperative game model and the concept of the core to demonstrate that the profit allocation induced by the fixed fee mechanism achieves myopic stability (Osborne and Rubinstein 1994). Thus, the fixed fee mechanism offers an easily implementable way for the buyer to create a stable sourcing network. In terms of theoretical results, we also show that the optimal network profit exhibits both supermodularity and submodularity, which differentiates our model from the commonly studied convex games (Shapley 1971). Further, we demonstrate that the core for the sourcing game is non-empty and establish a set of equivalent, but simpler, conditions which allocations in the core must satisfy, and which can be checked in polynomial time.

Although the fixed fee mechanism is stable in the myopic sense, in general, it may not be stable in the farsighted sense. Therefore, as a benchmark, we characterize the Shapley value allocation (Shapley 1953) and demonstrate that the grand coalition, i.e., the sourcing network consisting of all suppliers, is stable in a farsighted sense under that allocation. However, since the sourcing game is not convex, the Shapley value allocation may not be in the core, implying that it is not stable in the myopic sense. Further, implementation of the Shapley value allocation requires cooperation between the buyer and suppliers, which is not required for the fixed fee mechanism. Therefore, to understand which mechanism will be preferred by the buyer, we compare the profit earned by the buyer under each mechanism. Specifically, we use numerical results to study the equilibrium profit allocation under the fixed fee mechanism and the Shapley value allocation. We find that the buyer's profit allocation is increasing in the capacity of the suppliers and decreasing in the variation in their unit costs under both mechanisms. Further, when the aggregate capacity of the suppliers is high relative to demand, or demand is more likely to be small, the fixed fee mechanism will generally be preferred by the buyer over the Shapley value allocation.

1.2. Literature Review

There are several streams of literature related to our work. First, there is a body of literature in economics developing theories for special classes of cooperative games where the core is not empty

(Osborne and Rubinstein 1994). Since the seminal paper by Shapley (1971), this literature has focused on convex games where the characteristic function is *supermodular*. In operations management, cooperative game theory has seen far less application than non-cooperative game theory; see Cachon and Netessine (2004) and Nagarajan and Sošić (2008) for surveys. Hartman et al. (2000) study inventory pooling where multiple retailers split the profit that arises from risk pooling. They prove the existence of a non-empty core for symmetric and/or normal demand distributions. Müller et al. (2002) relax these restrictions and show that the core is always non-empty. Beyond inventory pooling, cooperative game theory has been applied to other operational problems, including lot-sizing (Chen and Zhang 2009), inventory transshipment (Anupindi et al. 2001, Kemahlioglu-Ziya and Bartholdi 2011, Fang and Cho 2014), joint replenishment (Anily and Haviv 2007, Zhang 2009, He et al. 2012), logistics cooperation (Özener and Ergun 2008), service operations (Anily and Haviv 2010), and supply chain scheduling (Hall and Liu 2008, 2010, Aydinliyim and Vairaktarakis 2013).

Our paper differs from this literature in several ways. From a modeling standpoint, our sourcing problem leads to a new form of game: the players in most of the studies above are horizontally placed, while in our two-echelon supply chain the buyer and the suppliers have different roles to play. Kemahlioglu-Ziya and Bartholdi (2011) also consider a cooperative game in which the players span two echelons, i.e., they study inventory pooling given a single supplier and multiple retailers. In contrast, we consider a supply chain with multiple suppliers and one buyer. As a result, our game exhibits a dual characteristic of supermodularity and submodularity. This has important implications because it allows us to significantly simplify the conditions for core allocations. Further, Kemahlioglu-Ziya and Bartholdi (2011) only consider the Shapley value allocation and show that it ensures farsighted stability of the grand coalition. In contrast, we propose a fixed fee mechanism that delivers a profit allocation in the core. However, inspired by Kemahlioglu-Ziya and Bartholdi (2011), as well as others, e.g., Sošić (2006) and Nagarajan and Bassok (2008), we also study the Shapley value allocation and prove that it is stable in a farsighted sense for our game. Thus, we extend the results from the literature related to the farsighted stability of the Shapley value to a sourcing problem in a many-to-one supply chain.

Second, this paper establishes a link between non-cooperative and cooperative games for sourcing networks. In this respect, our work is similar to Özen et al. (2008), where the authors first consider a cooperative inventory game and then introduce a non-cooperative game. Unlike in the inventory game, here we study a sourcing network problem, and our cooperative game involves a necessary player (i.e., the buyer). We also show that the optimal network profit is a submodular function of supplier indices, which is not found in Özen et al. (2008).

Third, this work investigates the performance of a proposed competitive mechanism. Thus, the literature on supplier competition is relevant. Considering the types of contracts used by suppliers, existing studies have focused on wholesale price contracts (Babich et al. 2007, Cachon and Kok 2010), option contracts (Wu and Kleindorfer 2005, Martinez-de Albeniz and Simchi-Levi 2009, Anderson et al. 2017) and quantity discount contracts (Cachon and Kok 2010). We contribute to this literature by studying strategic alliances between buyers and suppliers. In doing so, we apply the cooperative game framework to the capacity sourcing problem and show how the proposed fixed fee mechanism achieves a stable network, which has not been considered in this literature.

Finally, our paper relates to the literature on capacity allocation; see Hall and Liu (2008) for an excellent survey. Cachon and Lariviere (1999) consider a supply chain in which a supplier sells capacity to multiple retailers using the fixed and turn-and-earn allocation rules. Liu (2012) focuses on the uniform allocation mechanism when there is demand competition between two retailers and shows that the retailers' ordering decisions exhibit a gaming effect. Cho and Tang (2014) extend that work to consider a general number of retailers and identify the exact conditions under which the gaming effect occurs. However, these papers study the problem of a single supplier allocating its limited capacity, while we consider a buyer who allocates demand across multiple suppliers.

2. Model Description and Terminology

We consider a sourcing network consisting of a buyer ("she") and a group of suppliers ("he"). We use $M := \{1, \ldots, m\}$ to denote the set of suppliers, where $m \geq 2$. Each supplier is indexed by i, for $i = 1, \ldots, m$. The buyer must fill a random demand for the final product, which may come from the market or from the buyer's customer(s). In either case, the buyer will fill this demand by contracting with the suppliers to procure a key input or production capacity. We use D to denote the random demand, which follows distribution $F(d)$ over the support $[\underline{d}, \bar{d}]$. The buyer collects revenue of r per unit of the final product, which is assumed to be fixed and exogenously specified.

The suppliers have limited capacity, and we allow for heterogeneous capacities and unit costs across the suppliers. We let $c_i \geq 0$ and $K_i > 0$ denote supplier i's unit cost and capacity, respectively. Without loss of generality, we assume that the indices of the suppliers are ordered so that $c_1 \leq c_2 \leq \cdots \leq c_m$. We assume that the total capacity across the suppliers, excluding the highest cost supplier, satisfies $\sum_{i=1}^{m-1} K_i < \bar{d}$ and that $Pr\left(D > \sum_{i=1}^{m-1} K_i\right) > 0$. If these conditions do not hold, then the highest cost supplier, i.e., supplier m, will never be utilized by the buyer and can, thus, be removed from consideration[8].

In addition to the suppliers, the buyer can also purchase from an uncapacitated backup source, but at a higher cost. We let p_o denote the unit cost at the backup source, and we assume $p_o > c_i$

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[8] More broadly, in our model setting, any arbitrary portfolio of suppliers can be rank-ordered according to their costs, and then that set of suppliers can be trimmed down to the first m suppliers, where m is defined such that the capacity of the first $m-1$ meets the two conditions specified here.

for all $i \in M$, which reflects the cost advantages of the suppliers over the backup source[9]. We also assume $p_o < r$. If this condition does not hold, the buyer would earn a negative profit from using the backup source. In other words, if $p_o \geq r$, the backup source will never be used and can be removed from consideration. If the buyer relies only on the backup source, her expected profit is $(r - p_o)\mathbb{E}[D]$. This is the lowest profit available to the buyer, and this profit can be improved by using the suppliers due to their lower costs.

In summary, we assume that the unit cost paid by the buyer to the backup source is greater than the unit cost of the highest cost supplier and less than the unit selling price for the final product, i.e., we assume $c_m < p_o < r$, and that the backup source has an unlimited supply. Clearly, the latter assumption will not be applicable in all settings. Further, the former assumption may limit the applicability of our model in some settings. In practice, when faced with a large order from an important customer, buyers sometimes take a long-term perspective and fill the customer's order, even if doing so leads to a loss for that particular order, in anticipation of earning future profits from the long-term relationship with that customer. Although this behavior is not captured in our model, it could be considered in future research through the inclusion of a penalty cost associated with unfilled or partially-filled orders.

We study the profit allocation problem arising from the buyer's capacity sourcing problem. The sequence of events is as follows: (1) the buyer selects a subset of suppliers from which she will source and contracts with those suppliers; (2) random demand is realized; and (3) the buyer decides how to allocate demand across the selected suppliers. We first study this sourcing problem within a competitive game framework and propose a fixed fee contracting mechanism that will maximize the network profit at the equilibrium. We then study a cooperative game version of this sourcing network. While the cooperative game does not apply to the practical settings that motivate this work, the analysis of the cooperative game is useful because it allows us to demonstrate that the competitive equilibrium generated by our fixed fee mechanism is stable in the myopic sense.

Finally, for reference, in the appendix we formally define two concepts from combinatorial optimization that will be used in our analysis, i.e., a *rank function* and a *polymatroid.*

3. A Competitive Sourcing Game

In this section, we develop a competitive game model of the sourcing network and propose a fixed fee mechanism which the buyer can use to contract with the suppliers and achieve the optimal network profit. In this model, each supplier quotes (or charges) a fixed upfront fee $f_i \geq 0$ which the

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[9] For example, the backup source could be a supplier located in a different geographical region than the buyer. While this supplier has plentiful supply, the buyer does not usually contract with him due to the transportation cost required to ship from the supplier's location.

buyer must pay on top of the supplier's unit cost. A contract of this form is useful in industries that experience significant market uncertainty and demand variation, such as the iron ore industry. When a supplier contracts with a buyer to provide capacity in advance of demand realization, doing so creates risk for the supplier that some (or all) of their capacity may go unused if the realized demand is low. To compensate for this risk, suppliers may ask the buyer to pay in advance to reserve capacity. For more discussion of capacity reservation, see Hazra and Mahadevan (2009).

Under our proposed contract, if the buyer wants to include a supplier in her network, she needs to pay the supplier the upfront fee as a way of reserving capacity. The buyer then has the right to access the supplier's capacity at the supplier's unit cost. Thus, if the buyer allocates $x_i > 0$ demand to supplier i, the total payment to that supplier will be $f_i + c_i x_i$. Because the buyer must select the suppliers prior to seeing the realized demand, it is possible that the buyer will select supplier i for the sourcing network and pay the supplier f_i, but will not need to allocate any work to that supplier, e.g., if the realized demand is quite low. In that case, the total payment to supplier i is just f_i.

The model induced by this proposed fixed fee mechanism is a Stackelberg game in which each supplier moves first (quoting the fixed fee), and the buyer follows by determining the optimal subset of suppliers to include in her network (before demand is realized) and then determines her allocation policy (after demand is realized). In analyzing this game, we assume that each supplier knows his competitors' unit costs and capacities. This assumption fits situations in which the industry is relatively well-established, the item to be procured is a commodity, or where the technology/equipment, labor requirements and inputs of production are fairly standard. This assumption may also hold when there are a few relatively large suppliers competing to earn orders.

In the remainder of this section, we characterize the equilibrium for this game. We first determine the optimal demand allocation, i.e., the allocation that maximizes the network profit, for a given realized demand and a given set of suppliers. We then analyze the buyer's problem under the fixed fee mechanism and characterize the Nash equilibria for the suppliers. We will demonstrate that each supplier's equilibrium fee will be equal to his marginal contribution to the network, and that the buyer's equilibrium allocation decision will match the socially optimal allocation decision.

3.1. Optimal Demand Allocation for Sourcing Network

To analyze the optimal allocation, we consider a general problem in which an arbitrary subset $S \subseteq M$ of suppliers has been selected by the buyer to provide capacity. We use d to denote the realized value of D and x_i to denote the quantity allocated to supplier i, for $i \in S$. Given d, the optimal demand allocation problem is formulated as follows:

$$\max_{\mathbf{x}\in\mathbb{R}_+^{|S|}} \left\{ rd - \sum_{i\in S} c_i x_i - \left(d - \sum_{i\in S} x_i \right) p_o \right\}, \tag{1}$$

$$\text{s.t.} \qquad x_i \le K_i, \quad \forall i \in S, \tag{2}$$

$$\sum_{i \in S} x_i \le d, \tag{3}$$

where $\mathbf{x}$ denotes a vector of demand allocations to suppliers in S. The objective is to maximize the total profit of the network. The first constraint prevents any supplier's allocation from exceeding his capacity, while the second ensures that the total allocation will not be more than demand.

We first establish a set of conditions that are equivalent to the constraints in the above problem.

Lemma 1. *Constraints (2) and (3) are equivalent to the constraints below:*

$$\sum_{i \in A} x_i \le \min\left(d, \sum_{i \in A} K_i\right), \qquad \forall A \subseteq S. \tag{4}$$

The conditions in (4) state that the aggregate allocation to any subset of suppliers cannot exceed the minimum of the aggregate capacity and the realized demand. This equivalence result enables us to explore the properties of the feasible region. To this end, we define the following notion:

$$z(A) := \min\left(d, \sum_{i \in A} K_i\right), \quad \forall A \subseteq S, \tag{5}$$

and

$$\mathbb{P}(S,z) := \left\{ \mathbf{x} \in \mathbb{R}_+^{|S|} : \sum_{i \in A} x_i \le z(A), \forall A \subseteq S \right\}. \tag{6}$$

The following lemma gives the properties of $z(A)$ and $\mathbb{P}(S,z)$.

Lemma 2. *Function $z(A)$ is a rank function for $A \subseteq S$. Thus, $\mathbb{P}(S,z)$ is a polymatroid.*

We can now reformulate the optimization problem in (1)-(3) to the following:

$$\max_{\mathbf{x} \in \mathbb{P}(S,z)} \left\{ (r - p_o)d + \sum_{i \in S} (p_o - c_i)x_i \right\}, \tag{7}$$

where $\mathbb{P}(S,z)$ is the feasible set. The first term of the objective function is the profit when only the backup source is used and is independent of the allocation decision. The second term is the additional profit that can be earned by using the suppliers in S. The optimization problem is essentially to maximize this additional profit. The second term of the objective is a linear separable function of the allocations, and the feasible set is a polymatroid. The existing literature has established the optimal solution for this class of problem. The following lemma[10], attributed to Edmonds (2003) and He et al. (2012), characterizes the optimal allocation policy for the problem in (7).

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[10] Of course, it is possible to establish the result in Lemma 3 by solving the original problem directly. The idea of casting our problem as a polymatroid optimization problem is to show that the optimal objective value as a set function is submodular, which we will show in Theorem 1.

Lemma 3 **(Edmonds (2003), The Greedy Algorithm; He et al. (2012), Lemma 1)**. *The allocation* $\mathbf{x}^*(S,d)$ *is an optimal solution to the optimization problem in (7), where* $\mathbf{x}^*(S,d) = (x_i^*(S,d) : i = 1, 2, \cdots, |S|)$ *is defined as follows:*

$$x_i^*(S,d) = \begin{cases} z(\{i\}) & \text{for } i = 1, \\ z(\{1,\cdots,i\}) - z(\{1,\cdots,i-1\}) & \text{for } i = 2,\cdots,|S|. \end{cases}$$

Further, if $c_i \neq c_j$ *for any* $i, j \in S$ *and* $i \neq j$, $\mathbf{x}^*(S,d)$ *is the unique optimal solution.*

Thus, the total demand should be allocated to the cheapest supplier first until his capacity is exhausted or all demand is filled. The total allocated quantity when suppliers in S are available is

$$\sum_{i=1}^{|S|} x_i^*(S,d) = z(\{1,\cdots,|S|\}) = \min\left(d, \sum_{i\in S} K_i\right).$$

The assumptions that $\sum_{i=1}^{m-1} K_i < \bar{d}$ and $Pr\left(D > \sum_{i=1}^{m-1} K_i\right) > 0$ guarantee a strictly positive allocation for all suppliers when all suppliers are included in the sourcing network and the realized demand is equal to the maximum possible value, i.e., $x_i^*(M,\bar{d}) > 0$ for all $i = 1,\ldots,m$. When the realized demand is less than the maximum value, some suppliers may receive no allocation.

Let $\pi(S,d)$ be the optimal objective function for the optimization problem in (7), that is,

$$\pi(S,d) := \max_{\mathbf{x}\in\mathbb{P}(S,z)} \left\{ (r-p_o)d + \sum_{i\in S}(p_o - c_i)x_i \right\}. \tag{8}$$

Taking expectations over D yields $\Pi(S) := \mathbb{E}[\pi(S,D)]$. We call $\Pi(S)$ *the optimal network profit* when the suppliers in S are available. The following theorem examines the properties of $\Pi(S)$.

Theorem 1 **(Submodularity of Optimal Network Profit)**. *The optimal network profit,* $\Pi(S)$*, is submodular in* S *for* $S \subseteq M$*, that is,*

$$\Pi(A\cup\{i\}) - \Pi(A) \geq \Pi(B\cup\{i\}) - \Pi(B), \quad \forall A \subseteq B \subseteq M\setminus\{i\}. \tag{9}$$

The submodularity of $\Pi(S)$ implies that each supplier's marginal (expected) value to an existing set of suppliers decreases as the set grows. Since the suppliers' capacities are substitutable, the potential use of a new supplier's capacity is reduced when there are more suppliers available.

In general, it is a challenging task to establish the submodularity of an optimal objective value as a set function. However, by showing that $\mathbb{P}(z,S)$ is a polymatroid, we are able to apply a result from the literature (He et al. 2012) to prove the submodularity of $\Pi(S)$. Note that Schulz and Uhan (2010) also show that the optimal objective function is supermodular when minimizing a linear separable function over a polymatroid. An immediate result from Theorem 1 is the following:

Corollary 1. *For any* $S \subseteq M$*, we have* $\Pi(M) - \Pi(S) \geq \sum_{i\in M\setminus S}(\Pi(M) - \Pi(M\setminus\{i\}))$.

The corollary states that the marginal contribution of a given subset of suppliers is greater than the sum of marginal contributions from each supplier in the subset.

3.2. The Buyer's Supplier Selection Problem

We next consider the buyer's problem of selecting the optimal subset of suppliers to contract with to maximize her expected profit, given a set of fees $\{f_i : i \in M\}$ quoted by the suppliers, and given that the buyer allocates demand across the selected suppliers in order to maximize her own profit. While supplier selection takes place prior to realization of demand, the demand allocation decision takes place after the fixed fees have been paid to the suppliers in S and demand has been realized. Since the fees are sunk, the buyer only needs to consider the unit costs of the suppliers in S when making the allocation decision. Thus, the buyer's allocation problem is the same as in (1)-(3), the optimal allocation is $\mathbf{x}^*(S,d)$, as given in Lemma 3, and the buyer's optimal profit (excluding the fees) is $\Pi(S)$, as defined in (8). After accounting for the fees, the buyer's net expected profit is:

$$\Pi_B(S) = \Pi(S) - \sum_{i \in S} f_i, \tag{10}$$

where the subscript B denotes the buyer. This profit has the following property:

Lemma 4. *The buyer's profit, $\Pi_B(S)$, is submodular in $S \subseteq M$.*

Given $\Pi_B(S)$, the buyer's supplier selection problem is $\Pi_B^*(N) := \max_{S \subseteq N} \Pi_B(S)$. The submodularity of $\Pi_B(S)$ implies that the marginal contribution of a supplier to the buyer's expected profit decreases as the set of selected suppliers expands. In addition, $\Pi_B(S)$ is not monotone: choosing more suppliers does not necessarily lead to a greater profit for the buyer because she may not need the extra capacity, but will still need to pay fees to those additional suppliers. Thus, the buyer's problem is to maximize a non-monotone submodular function, which is NP-hard (Feige et al. 2011). While this prevents us from obtaining an analytical solution, we can still characterize the equilibrium for this game, as shown in the next subsection.

Depending on the suppliers' fees, there may be multiple optimal solutions to the buyer's problem. Therefore, in order to be able to characterize the suppliers' best response strategies, we must specify a tie-breaking rule. Our non-cooperative game is a Stackelberg game in which the suppliers are leaders and the buyer is the follower. There are several possible ways of breaking ties in such a problem. We adopt an optimistic approach by assuming that the follower (i.e., the buyer) is willing to support the leaders (i.e., the suppliers). We thus have the following assumption:

Assumption 1 **(Tie-Breaking Rule)**. *If two subsets of suppliers yield the same expected profit, the buyer chooses the subset with the most suppliers.*

This tie-breaking rule has also been used in Cachon and Kok (2010) and Anderson et al. (2017).

3.3. The Suppliers' Problem and Equilibrium Analysis

We next characterize the best response strategy for each supplier, i.e., given the other suppliers' fixed fees, $\{f_j : j \neq i, j \in M\}$, we consider supplier i's best response, f_i. When all suppliers except for i are available, we use $\Pi^*_B(M \setminus \{i\})$ to denote the buyer's optimal profit, where

$$\Pi^*_B(M \setminus \{i\}) := \max_{S \subseteq M \setminus \{i\}} \Pi(S) - \sum_{j \in S} f_j.$$

Let $\mathbf{f} = (f_1, \cdots, f_{i-1}, 0, f_{i+1}, \cdots, f_m)$, i.e., $\mathbf{f}$ is the vector of fees for all suppliers, with $f_i = 0$. Consider the buyer's optimal choice of suppliers and the corresponding optimal allocation when presented with $\mathbf{f}$. For a given subset of suppliers S and their fees $\{f_j : j \in S\}$, the optimal allocation is $\mathbf{x}^*(S, d)$, while the optimal network profit is $\Pi(S)$. Next, we define:

$$\hat{\Pi}^*_B(M) := \max_{S \subseteq M} \Pi(S) - \sum_{j \in S, j \neq i} f_j,$$

which is the buyer's optimal profit when $f_i = 0$, given the other suppliers charge f_j, for all $j \neq i$.

Intuitively, supplier i will set the fee f_i as high as possible, while still ensuring that he is selected by the buyer. The following theorem formalizes the optimal strategy of supplier i.

Theorem 2 **(Best Response)**. *For a given set of fees $\{f_j : j \neq i, j \in M\}$, supplier i's best response is to set the fee equal to $\hat{f}_i = \hat{\Pi}^*_B(M) - \Pi^*_B(M \setminus \{i\})$.*

Thus, it is optimal for supplier i to set a fee equal to his marginal value to the network. Intuitively, if supplier i sets a higher fee than $\hat{f}_i$, he will not be selected by the buyer. However, if he sets a lower fee, supplier i can increase that fee and still remain in the buyer's optimal set of suppliers.

We can now characterize the equilibrium fees and profits for the suppliers and buyer.

Theorem 3 **(Nash Equilibrium)**. *There exists a Nash equilibrium in which each supplier i's fee is $f^*_i = \Pi(M) - \Pi(M \setminus \{i\})$ and the buyer's optimal allocation is $\mathbf{x}^*(M, d)$. In equilibrium, supplier i earns an expected profit equal to $\pi^*_i = \Pi(M) - \Pi(M \setminus \{i\})$ for $i \in M$, and the buyer earns expected profit equal to $\pi^*_B = \Pi(M) - \sum_{i \in M}(\Pi(M) - \Pi(M \setminus \{i\}))$.*

Theorem 3 shows that the buyer's allocation policy in equilibrium will match the socially optimal policy, i.e., $\mathbf{x}^*(M, d)$. This is an interesting result and implies that the fixed fee mechanism does not result in any efficiency loss. The theorem also shows that each supplier sets his fee, and earns equilibrium profit, equal to his marginal contribution to the entire network. Finally, since the network profit is maximized in equilibrium, the buyer retains the remainder of the optimal network profit and, given the submodularity of $\Pi(S)$, the buyer's profit is non-negative.

The following corollary, which follows from Theorem 3, indicates that, for all suppliers, there is at least one possible realization of demand that will provide a strictly positive demand allocation, which, in turn, implies that all suppliers will earn a strictly positive profit.

Corollary 2. *For all* $i \in M$, *there exists at least one value of demand, denoted by* $\hat{d}_i$, *such that* $x_i^*(M, \hat{d}_i) > 0$, *where* $\hat{d}_i \in [\underline{d}, \bar{d}]$ *and there is a positive probability mass on* $\hat{d}_i$. *Thus,* $f_i^* > 0$.

3.4. Discussion of Equilibrium Results

In this section, we demonstrate the equilibrium characterized in Section 3.3. We then provide some discussion and consider a special case with closed-form results. We first present an example.

Example 1. Consider a set of three suppliers, i.e., $M = \{1, 2, 3\}$. The unit revenue is $r = 10$. For simplicity, we consider a binary demand distribution where the demand is $\underline{d} = 14$ with probability 0.6 and $\bar{d} = 18$ with probability 0.4. The capacities of the suppliers are $K_1 = 8$, $K_2 = 7$ and $K_3 = 5$, and their production costs are $c_1 = 4$, $c_2 = 5$ and $c_3 = 6$. The unit cost for the backup source is $p_o = 9$. Using the results in Lemma 3, we can compute the optimal allocation policies and optimal network profits for different subsets of suppliers. Table 1 summarizes these results. Then, using Theorem 3, we can compute supplier 1's equilibrium fee as follows:

Table 1 The optimal allocation policies and network profits for Example 1

S	$x_1^*(S,\underline{d})$	$x_2^*(S,\underline{d})$	$x_3^*(S,\underline{d})$	$\underline{d} - \sum x_i^*$	$\Pi(S,\underline{d})$	$x_1^*(S,\bar{d})$	$x_2^*(S,\bar{d})$	$x_3^*(S,\bar{d})$	$\bar{d} - \sum x_i^*$	$\Pi(S,\bar{d})$
$\emptyset$	-	-	-	14	14	-	-	-	18	18
$\{1\}$	8	-	-	6	54	8	-	-	10	58
$\{2\}$	-	7	-	7	42	-	7	-	11	46
$\{3\}$	-	-	5	9	29	-	-	5	13	33
$\{1,2\}$	8	6	-	0	78	8	7	-	3	86
$\{1,3\}$	8	-	5	1	69	8	-	5	5	73
$\{2,3\}$	-	7	5	2	57	-	7	5	6	61
$\{1,2,3\}$	8	6	0	0	78	8	7	3	0	95

$$f_1^* = 0.6 \times (\Pi(\{1,2,3\}, \underline{d}) - \Pi(\{2,3\}, \underline{d})) + 0.4 \times \left(\Pi(\{1,2,3\}, \bar{d}) - \Pi(\{2,3\}, \bar{d})\right) = 0.6 \times 21 + 0.4 \times 34 = 26.2.$$

In the same manner, we can compute $f_2^* = 14.2$ and $f_3^* = 3.6$. Therefore, the buyer's profit is

$$\pi_B^* = \left(0.6 \times \Pi(\{1,2,3\}, \underline{d}) + 0.4 \times \Pi(\{1,2,3\}, \bar{d})\right) - (f_1^* + f_2^* + f_3^*) = 40.8. \quad \square$$

We next discuss the implementation of the fixed fee mechanism. First, the buyer identifies a network of suppliers that meets the criteria discussed in Section 2, including the requirement regarding the total capacity of the network relative to the support of the demand distribution. Given this network, the buyer proposes the fixed fee mechanism to the suppliers selected for the network and asks the suppliers to quote their fixed fees for participating in the network. We know from Theorem 3 and Corollary 2 that all suppliers in the network will quote positive fees. The buyer will accept all of the suppliers' bids and contract with each of the suppliers, i.e., the buyer will pay each supplier's fee in order to secure access to their capacity at cost. The buyer's demand will then be realized, and this demand will be allocated across the suppliers in the network using Lemma 3, i.e., the buyer will order x_i^* from supplier i and will pay the supplier $c_i x_i^*$.

Finally, as will be seen in the numerical results presented in Section 4.5, the equilibrium characterized in Theorem 3 can be quite disparate in terms of profit allocation. The buyer may be able to extract a majority of the network profit or just a slim profit, depending on the system parameters. Unfortunately, providing an exact expression for the buyer's profit is not possible in general. However, we can characterize the profit for the following special case in which the aggregate capacity of the suppliers is low relative to demand. Such a case will occur periodically in industries where demand is tied to economic conditions, which are highly cyclical. For example, since the demand for steel depends on the strength of the global economy, during times of strong economic growth, or when disruptions limit supply, demand can exceed the available supply. See, for example, the discussion of the impact of the COVID-19 pandemic on the relative demand and supply of iron ore in Reserve Bank of Australia (2020). Similarly, the semiconductor industry experienced a rapid increase in demand during 2021 as the global economy recovered from the pandemic. As a result, capacity in the industry was insufficient to meet all demand during much of 2021.

Corollary 3. *Suppose* $\sum_{i\in M} K_i \leq \underline{d}$*. In equilibrium, each supplier* i *earns a profit equal to* $\pi_i^* = (p_o - c_i)K_i$ *for* $i \in M$*, and the buyer earns the reservation profit, i.e.,* $\pi_B^* = (r - p_0)\mathbb{E}[D]$.

When the aggregate capacity of the suppliers is insufficient to fill all of the realized demand, there is no competition between suppliers, i.e., all suppliers will get their maximum possible demand allocation. Thus, each supplier i's marginal contribution is $(p_o - c_i)K_i$ and the buyer earns the remaining system profit, which is equal to her reservation profit, $\pi_B^* = (r - p_0)\mathbb{E}[D]$.

4. A Cooperative Sourcing Game

In this section, we model the sourcing network as a cooperative game and use that model, along with concepts from cooperative game theory, to determine whether a stable sourcing network can be formed under the competitive fixed fee mechanism proposed in Section 3. We also use those results to study how the profits should be allocated across the network to achieve stability. In doing so, we consider *myopic stability*, which allows for one-step deviations from the grand coalition. Our analysis uses the solution concept referred to as *the core*, which is a profit allocation under which no coalition has an incentive to deviate. We will demonstrate that, under the fixed fee mechanism, the non-cooperative sourcing network studied in this paper exhibits myopic stability.

We will then consider *farsighted stability*, which is an alternative stability concept that allows for a series of subsequent deviations following an initial deviation from the grand coalition. As a benchmark for comparison with our fixed fee mechanism, which may not possess farsighted stability in some settings, we characterize the Shapley value allocation (Shapley 1953) for our network and show that the grand coalition is stable in the farsighted sense under the Shapley value allocation.

Finally, we compare the Shapley value allocation to the fixed fee mechanism. Since our sourcing game is not convex, the Shapley value allocation may not be in the core, implying that, unlike the fixed fee mechanism, it is not stable in the myopic sense. Further, the Shapley value allocation is a solution concept for cooperative settings and, thus, implementation requires cooperation between the buyer and suppliers, which is not required for the fixed fee mechanism. However, our numerical results comparing the two approaches will indicate that, in some settings, the Shapley value allocation can out-perform the fixed fee mechanism from the perspective of the buyer.

4.1. Formulation of the Cooperative Game

A cooperative game with transferable payoff, denoted by (N, v), is characterized by a finite grand coalition of players N and a characteristic value function, v, that associates with every subset $S \subseteq N$ a real number $v(S)$. In our problem setting, the grand coalition N consists of the buyer and a group of suppliers (i.e., $N = M \cup \{B\}$ where B denotes the buyer). A subset S of the grand coalition will be referred to as a coalition. The characteristic value function, $v(S)$, captures the optimal network profit that can be achieved for a given coalition S. Specifically, we obtain

$$v(S) = \begin{cases} 0, & \text{if } B \notin S, \\ \Pi(S \setminus \{B\}), & \text{if } B \in S, \end{cases}$$

where $\Pi(S \setminus \{B\})$ is given in (8). Clearly, if $B \notin S$, the optimal network profit is zero and, hence, $v(S) = 0$. If $B \in S$, the characteristic value is $v(S) = \Pi(S \setminus \{B\})$, which is the optimal network profit when all suppliers in $S \setminus \{B\}$ are present. Given $v(S)$, it is easy to show that the cooperative game is monotone, i.e., satisfies $v(S) \leq v(T)$ for $S \subseteq T \subseteq M$, using the fact that $\Pi(S) \leq \Pi(T)$.

This sourcing game is different from the typical cooperative game considered in the operations management literature, in which the players are horizontal in terms of how they relate to each other. For example, in inventory management games, a set of newsvendors may place joint orders to benefit from demand pooling (Müller et al. 2002). In such a setting, no given newsvendor is pivotal, i.e., the other newsvendors do not need to work with any given newsvendor to obtain a non-zero value. In our game, however, the buyer plays a special role, i.e., the suppliers must join forces with the buyer to achieve a non-zero value. Thus, the buyer is a *necessary player* for our sourcing game (van der Brink and Gilles 1996). As noted above, because the buyer is a necessary player, with a need to procure the input of production, the buyer takes a lead role in creating the sourcing network and designing the contracting mechanism, despite the relative bargaining power of the suppliers. Cooperative games with necessary players have received little attention in the operations management literature, with the exception of Kemahlioglu-Ziya and Bartholdi (2011). Further, while the economics literature indicates that monotone games with a necessary player will satisfy the *necessary player property*, which states that a necessary player earns at least as much as any other player (van der Brink and Gilles 1996), that property is generally defined in the context of the Shapley value allocation, and may not hold under other allocations.

We next consider the properties of our cooperative sourcing game.

THEOREM 4. *For the cooperative sourcing game* (N, v)*, we obtain the following results:*

(a) For $S \subseteq T \subseteq M$*, we have* $v(S \cup \{B\}) - v(S) \leq v(T \cup \{B\}) - v(T)$*. In other words, since* $v(S) = v(T) = 0$*, we have* $v(S \cup \{B\}) \leq v(T \cup \{B\})$*;*

(b) For $S \subseteq T \subseteq N$*, we have*

(i) If $B \in S$*, then* $B \in T$ *and* $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$ *for* $i \in M$*;*

(ii) If $B \notin S$ *and* $B \in T$*, then* $v(S \cup \{i\}) - v(S) = 0 \leq v(T \cup \{i\}) - v(T)$ *for* $i \in M$*;*

(iii) If $B \notin T$*, then* $B \notin S$ *and* $v(S \cup \{i\}) - v(S) = v(T \cup \{i\}) - v(T) = 0$ *for* $i \in M$*.*

Part (a) of Theorem 4 states that the buyer's marginal contribution increases as the subset of suppliers grows, i.e., the value function is *supermodular* when considering the buyer's contribution. Part (b) considers the marginal contribution of a supplier to an existing set of players, which depends on whether or not that set includes the buyer. When the buyer is not included, i.e., in part (b)(iii) of the theorem, the marginal contribution of a supplier is zero. When the buyer is included in both subsets S and T, i.e., in part (b)(i) of the theorem, the marginal contribution of a supplier decreases as the existing coalition grows. In other words, the value function is *submodular* when considering *each supplier*'s contribution. Overall, Theorem 4 reveals a new feature of our sourcing network game, that is, the dual characteristic of submodularity and supermodularity. This result is due, in part, to the fact that the buyer is a necessary player in the supplier network.

4.2. Myopic Stability: The Core

A common solution concept to study the stability of a cooperative game is the core, a set of feasible payoff profiles $\{a_i : i \in N\}$ for which no coalition deviates from the payoff profiles (Osborne and Rubinstein 1994). In our setting, the feasible payoff refers to the profit allocated to each supplier and to the buyer. Specifically, define the total payoffs allocated to players in S as $a(S) := \sum_{i \in S} a_i$ for all $S \subseteq N$. A profit allocation is in the core if and only if the following conditions hold:

$$a(N) = v(N) \quad \text{and} \quad a(S) \geq v(S) \quad \forall S \subset N. \tag{11}$$

The first condition (referred to as the efficiency condition) states that the entire optimal network profit is allocated to the network members. The second condition (referred to as the coalitional rationality condition) states that no coalition of players can improve its total allocated profit by walking away from the grand coalition. When a set of payoffs exists that satisfies these two sets of conditions, we say the core is non-empty, and the grand coalition is stable in the myopic sense.

It is useful to draw a parallel between our sourcing game and convex games. In convex games, the characteristic function is *supermodular*. In contrast, Theorem 4 indicates that the characteristic function in our game is *submodular*. Thus, our game is not convex and the properties of convex games may not apply. For instance, the greedy algorithm by Shapley (1971) to find the core of a convex game may lead to a payoff profile that lies outside the core for our problem setting.

We next demonstrate that there exists a profit allocation for our game that is in the core:

THEOREM 5 **(The Core)**. *The cooperative sourcing game* (N, v) *has a non-empty core. Moreover, an allocation* $\{a_i : i \in N\}$ *is in the core if and only if the following conditions hold:*

$$\sum_{i\in N} a_i = v(N), \quad \text{and} \quad 0 \le a_i \le v(N) - v(N \setminus \{i\}) \quad \forall i \in N. \tag{12}$$

The first part of the theorem shows there is a non-empty core in our sourcing game, which implies that a stable network is possible with an appropriate profit allocation scheme. The second part of the theorem characterizes a set of conditions that are equivalent to, but much simpler than, the original conditions of the core in (11). In particular, the second set of conditions in (11) says that the allocation to each player cannot exceed that player's marginal contribution to the entire network. From (12), we see that the equivalent conditions are based on each player's unilateral deviation, rather than the deviations of all the possible coalitions. This result is due to the properties of our game characterized in Theorem 4. In the proof of this theorem, we show that if it is in no individual player's interest to separate from the grand coalition, then no coalition of players will have any incentive to split away either. Therefore, the original core conditions in (11) defined on all the subsets with more than one player become redundant, and the number of conditions reduces from $2^{m+1}+1$ to $m+2$. As a consequence of this result, one is able to check whether or not a given profit allocation is a core allocation in polynomial time. This is important because the problem of determining whether or not an allocation lies in the core is generally NP-hard (He et al. 2012).

4.2.1. Connection Between Competitive Equilibrium and the Core Next, we demonstrate that the Nash equilibrium under the fixed fee mechanism, as proposed in Section 3, is in the core of the cooperative game.

THEOREM 6 **(Connection between Core and Equilibrium)**. *The equilibrium profit allocation, as characterized in Theorem 3, lies in the core of the cooperative sourcing game* (N, v).

This result implies that the fixed fee mechanism can provide a method to implement a myopically stable sourcing network. The fixed fee mechanism is appealing for several other reasons. First, it is easy to implement because each supplier is invited to simply quote a fixed fee. Second, the profit split arising from the fixed fee mechanism is driven by the level of competition, as is observed in practice and verified by our numerical results in Section 4.5. Third, the network profit is maximized in equilibrium, so there is no efficiency loss for the sourcing network.

While the equilibrium profit allocation is part of the core, a core allocation does not necessarily form an equilibrium. To understand this, consider the following allocation: $a_B = v(N)$ and $a_i = 0$ for all $i \in M$. Under this allocation, as long as the coalition includes the buyer, we allocate maximum profit $v(N)$ to the coalition; if the coalition does not include the buyer, we allocate zero profits to the coalition. As shown in the proof of Theorem 5, this is a core allocation, but it is not an equilibrium because any supplier has an incentive to unilaterally deviate by charging a positive fee. As long as the supplier sets his fee to be less than his marginal contribution, he will still be selected by the buyer, so this core allocation does not form an equilibrium.

4.3. Farsighted Stability: The Shapley Value

The concept of the core is myopic in the sense that it only looks at one-step deviations of the players, i.e., if there exists a coalition which benefits from a one-step deviation from the grand coalition, then the grand coalition will be considered to be unstable in the myopic sense. This concept implicitly assumes that the new coalition structure after the one-step deviation is final. In practical situations, however, there exists the possibility of a series of subsequent deviations that follow the initial deviation. If, at the end of these deviations, the players who initiated the first deviation end up either (i) in the grand coalition again or (ii) in a different coalition in which they are allocated a lower profit than in the grand coalition, then these players will not choose to deviate in the first place. In other words, the possibility of a reduced profit allocation may deter the initial deviation, and the grand coalition, which appears to be unstable from the myopic sense, will actually be stable in the *farsighted* sense. To consider farsighted stability, we will use the concept of the *largest consistent set* (LCS), which was introduced by Chwe (1994) and allows players to consider a series of deviations by any coalition. The concept of LCS has been applied in the operations management literature, including Granot and Sošić (2005), Sošić (2006), Nagarajan and Sošić (2007), Granot and Yin (2008), Nagarajan and Bassok (2008), Nagarajan and Sošić (2009), Kemahlioglu-Ziya and Bartholdi (2011), Nagarajan et al. (2019) and Tian et al. (2019).

We are interested in whether, for our sourcing game, the grand coalition is stable in a farsighted sense. We will study the Shapley value, which allocates the total profit of a coalition to its members based on each one's marginal contribution. It is regarded as "fair" because it is the only distribution that satisfies the desirable axioms of symmetry, dummy players and additivity (Shapley 1953). Specifically, under the Shapley value allocation, the profit allocated to each player is

$$\phi_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \left(v(S \cup \{i\}) - v(S)\right) \quad \forall i \in N. \tag{13}$$

It has been established that there exists a unique Shapley value for any cooperative game with transfer utilities (Cachon and Netessine 2004). Moreover, for convex games, the Shapley value is in the core (Shapley 1971). However, our game is not convex, so the Shapley value may not be in the core for our sourcing game. We demonstrate this point using the following example.

EXAMPLE 2. Consider two suppliers, i.e., $M = \{1,2\}$. The unit revenue is $r = 10$. Consider deterministic demand with $d = 10$ and assume there is no backup source. The capacities are $K_1 = 9$ and $K_2 = 9$, and the production costs are $c_1 = 9$ and $c_2 = 9.5$. Using the result of Lemma 3 we obtain: $\Pi(\{1,2\}) = 9.5$, $\Pi(\{1\}) = 9$ and $\Pi(\{2\}) = 4.5$. Thus, by definition $v(\{B,1,2\}) = 9.5$, $v(\{B,1\}) = 9$, $v(\{B,2\}) = 4.5$, $v(\{B\}) = 0$ and $v(S) = 0$ for all $S \subseteq M$. Using (13), we find that the Shapley values are: $\phi_1 = 19/6$, $\phi_2 = 11/12$ and $\phi_B = 65/12$. Thus, $\phi_1 + \phi_B = 103/12$, which is less than $v(\{B,1\}) = 9$, so supplier 1 and the buyer would see increased profit by separating from supplier 2 and forming a subcoalition. Thus, the Shapley value violates condition (11) for the core. □

The above example demonstrates the distinction between the Shapley value and the core. On the surface, the Shapley value is calculated on the basis of marginal contributions, while the core is defined on the basis of coalition deviations. In the above example, the Shapley value assigns too much profit to supplier 2. Supplier 2's marginal contribution is $v(\{B,2\}) - v(\{B\}) = 4.5$ when he combines with $\{B\}$ and $v(\{B,1,2\}) - v(\{B,1\}) = 0.5$ when he combines with $\{B,1\}$. Thus, the marginal value of supplier 2 to coalition $\{B,2\}$ is much greater than to the grand coalition. Since the Shapley value is a weighted average of these two marginal values, supplier 2 is allocated a profit higher than his marginal contribution to the overall network (i.e., $\phi_2 = 12/11 > 0.5$). In fact, this profit is so high that the buyer and supplier 1 should separate from the grand coalition. Thus, an allocation based on the Shapley value may not lead to a stable sourcing network in the myopic sense. However, as we will demonstrate below, the Shapley value is stable in a farsighted sense.

Let ϕ_B^T (ϕ_i^T) denote the profit allocated to the buyer (supplier i) when she (he) is a member of coalition T for $T \subseteq N$ and $B \in T$. The following theorem demonstrates that the buyer's profit under the Shapley value allocation increases as the coalition expands and that all players earn a strictly positive profit when the grand coalition is formed. These results allow us to show the key result that the grand coalition is the LCS and, thus, is stable in the farsighted sense.

THEOREM 7. *Under the Shapley value allocations: (i)* $\phi_B^T > \phi_B^{T \setminus \{i\}}$*, for all* $i \in N \setminus T$*; (ii)* $\phi_i^N > 0$ *for all* $i \in N$*; (iii) the grand coalition is the LCS and, thus, is stable in the farsighted sense.*

The concept of LCS is not without limitation. It is criticized for being overly inclusive when it comes to the possible stable outcomes. Further, it does not prescribe which outcome will occur. Instead, it produces a set of outcomes from which deviations are not likely to happen. However, if we treat the grand coalition as the status quo, Theorem 7 suggests that all possible deviations from the grand coalition will be deterred if the players are farsighted, i.e., if the players consider future deviations, they will not make an initial attempt to defect from the grand coalition.

Finally, while there are no closed-form expressions for the Shapley value allocations for our general sourcing game, we next provide simple expressions for the allocations for a special case in which the aggregate capacity across the suppliers is always insufficient to meet demand.

COROLLARY 4. *Suppose* $\sum_{i\in M} K_i \le \underline{d}$. *The Shapley value allocations are* $\phi_i^N = \frac{1}{2}(p_o - c_i)K_i$ *for* $i \in M$ *and* $\phi_B^N = \mathbb{E}[(r-p_o)D] + \frac{1}{2}\sum_{i\in M}(p_o - c_i)K_i$.

This result implies that, when the aggregate capacity across the suppliers is low, each supplier i splits the value they provide relative to the backup source, i.e., $(p_o - c_i)K_i$, with the buyer.

4.3.1. Implementing the Shapley Value Allocation in a Sourcing Network We conclude by considering how the Shapley value allocation can be implemented. By Theorem 7, each supplier earns a positive profit in the grand coalition under the Shapley value allocation. Thus, if the reservation profit for all suppliers is zero, then all suppliers will choose to join the coalition. Further, the grand coalition is stable in the farsighted sense by Theorem 7. For these two reasons, we focus our implementation discussion on the grand coalition.

The buyer selects a subset of suppliers prior to the realization of the random demand, but the actual allocation decision is made after demand is realized. This two-stage decision structure is in line with the existing studies on inventory pooling, e.g., Özen et al. (2008) and Kemahlioglu-Ziya and Bartholdi (2011), and inventory transshipment, e.g., Granot and Sošić (2003) and Sošić (2006). Given this sequence of events, we can implement the Shapley value allocation as follows: The buyer invites the suppliers to join a sourcing network by announcing that, given the realized demand and subset of suppliers $S \subseteq M$, each supplier will be allocated a quantity $x_i^*(S,d)$, as given in Lemma 3, and will be compensated based on the Shapley value allocation rule, $\phi_i(d)$, defined as:

$$\phi_i(d) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}\left(v(S\cup\{i\}|d) - v(S|d)\right), \tag{14}$$

where $v(T|d)$ is the value function for a given coalition, $T \subseteq N$, and realized demand, d, where $v(T|d) = 0$ if $B \notin T$ and $v(T|d) = \pi(T\setminus\{B\}, d)$ if $B \in T$, and $\pi(\cdot, d)$ is given in (8). If we take the expectation of $\phi_i(d)$ over the random demand, it is easy to show that this mechanism provides each supplier with an expected profit that is equal to the Shapley value given in (13).

4.4. Fixed Fee Mechanism vs. Shapley Value Allocation

The Shapley value allocation allocates the total profit based on each supplier's marginal contribution to the network and is generally seen as fair (Shapley 1953). However, as can be seen from (14), the Shapley value allocation would be challenging to implement since $\phi_i(d)$, i.e., the profit allocation announced by the buyer to supplier i given the realized demand d, is a complex function that depends on the realized demand. In contrast, the fixed fee mechanism, which possesses myopic stability but may not have farsighted stability in all settings, is simpler to implement due to the fact that the suppliers' fixed fees and, hence, their allocated profits, are independent of the realized demand, d. Further, unlike the Shapley value allocation, the fixed fee mechanism does not presume cooperation between the buyer and supplier. Given these trade-offs, we next compare the relative performance of these two allocation mechanisms from the perspective of the buyer.

We first provide an analytical comparison for the special case in which capacity is limited. Then, in Section 4.5, we present numerical results regarding the relative performance of these two mechanisms in more general settings.

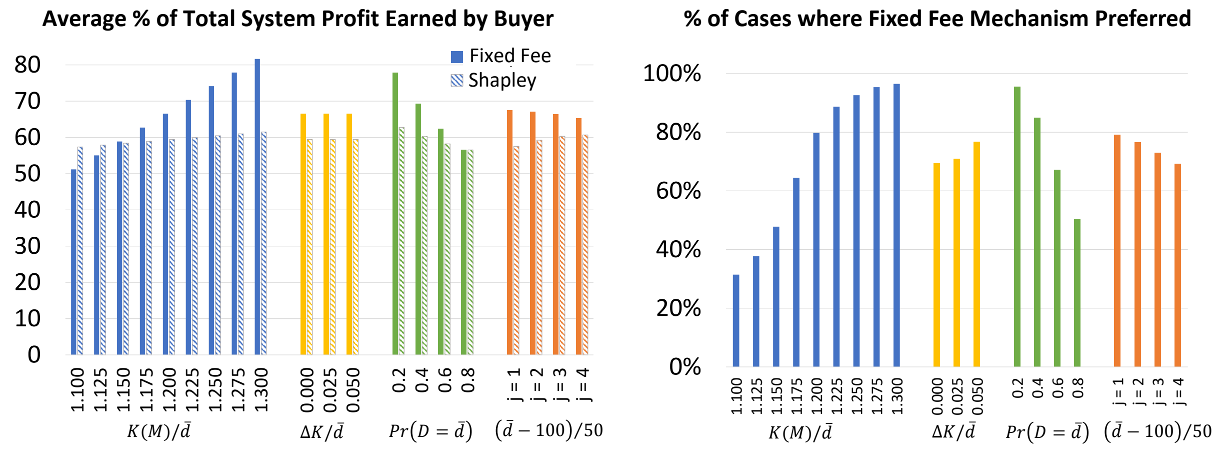
Corollary 5. *Suppose* $\sum_{i\in M} K_i \leq \underline{d}$. *Then* $\phi_i < \pi_i^*$ *for* $i \in M$ *and* $\phi_B > \pi_B^*$.

In this case, there is no competition between the suppliers. Further, each supplier's profit under the Shapley value mechanism is half of that under the fixed fee mechanism (see Corollaries 3 and 4) due to the fact that the Shapley value is a weighted average of the supplier's marginal contributions across all coalitions that include him. Since the marginal contributions are zero when the coalitions do not include the buyer, each supplier's allocation under the Shapley value mechanism is smaller than under the fixed fee mechanism. Thus, the suppliers prefer the fixed fee mechanism (which is ineffective at inducing competition), while the buyer prefers the Shapley value mechanism.

4.5. Numerical Results

To demonstrate the results and investigate the impact of key problem parameters on the buyer's profit allocation under the fixed fee mechanism and Shapley value allocation, we next present a numerical study. We consider a setting with four suppliers (i.e., $m = 4$). We vary the unit price as $r \in \{6, 8, 10, 12, 14\}$ and we set $p_o = r$, which implies that the buyer's backup source is not profitable. We use a binary demand distribution, with $\bar{d} = 500 + 50 \times j$ and $\underline{d} = 500 - 50 \times j$, for $j \in \{1, 2, 3, 4\}$, where j measures the amount of uncertainty in demand. We consider $Pr(D = \bar{d}) \in \{0.2, 0.4, 0.6, 0.8\}$. The suppliers' unit costs are $c_i = 3 + (i-3)a$, for $i = 1, \ldots, 4$ and $a \in \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$, where a measures the amount of variation in the costs. The suppliers' capacities are $K_i = 0.25 \times \bar{d} \times b_i$, for $b_i \in \{1.3, 1.2, 1.1\}$, for $i = 1, \ldots, 4$, which ensures that the assumption regarding the magnitude of capacity relative to demand is satisfied. Overall, we have $5 \times 4 \times 4 \times 6 \times 3^4 = 38,880$ cases.

As will be seen, the magnitude of the available capacity, relative to demand, is a key factor in the relative performance of the two mechanisms from the perspective of the buyer. Therefore, we let $K(M) = \sum_{i=1}^{4} K_i$ denote the aggregate capacity across the suppliers and $\Delta(K) = \max_{i=1,\ldots,4}\{K_i\} - \min_{i=1,\ldots,4}\{K_i\}$ be a measure of the amount of variation in capacity across the suppliers. When presenting the results, we scale both of these measures by the maximum demand, $\bar{d}$.



(a) Sensitivity Analysis on $\%(B)$ (b) Fixed Fee Mechanism vs. Shapley Value Allocation

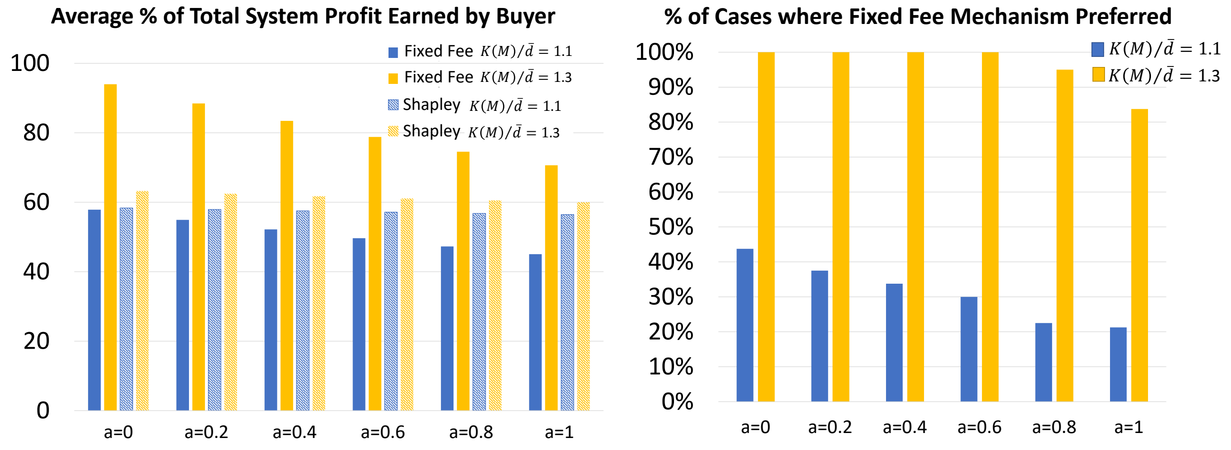
Figure 1 **Sensitivity Analysis on Performance of Fixed Fee Mechanism Relative to Shapley Value Allocation.**

Figures 1 and 2 show the impact of the key model parameters on the percentage of the total system profit earned by the buyer, which is computed as $\%(B) = \frac{\pi_B^*}{\pi_B^* + \sum \pi_i^*}$ for the fixed fee mechanism, and analogously for the Shapley value allocation. The figures also present the percentage of cases in which the fixed fee mechanism is preferred by the buyer over the Shapley value allocation. Note that the fixed fee mechanism and the Shapley value allocation result in the same total network profit and the same allocation of demand across the suppliers. However, the mechanisms differ in how they allocate the total system profit across the suppliers and the buyer. Since the buyer determines the mechanism by which the suppliers bid and/or are compensated, we are interested in which of these two mechanisms would be selected by the buyer.

Figure 1(a) shows $\%(B)$, while Figure 1(b) shows the percentage of cases in which the fixed fee mechanism is preferred by the buyer, as functions of the scaled aggregate capacity, $K(M)/\bar{d}$, the scaled variation in capacity, $\Delta K/\bar{d}$, the probability of high demand, $Pr(D = \bar{d})$, and the measure of uncertainty in demand, $j = (\bar{d} - 500)/50$. Figure 1(a) indicates that the performance of the Shapley value allocation mechanism is less sensitive to the model parameters than the performance of the fixed fee mechanism. The parameter with the most significant impact on $\%(B)$ under the Shapley value allocation is the probability of high demand. As high demand becomes more likely, the performance of the Shapley value allocation mechanism decreases. High demand implies a tighter capacity, which tends to benefit the suppliers relative to the buyer and, thus, the buyer's profit allocation decreases when high demand becomes more likely. The performance of the fixed fee mechanism also decreases with the probability of high demand, and that decrease is more significant than for the Shapley value allocation. As a result, Figure 1(b) shows that the percentage of cases in which the fixed fee mechanism is preferred decreases as the probability of high demand increases.

Similarly, Figure 1(a) indicates that the performance of fixed fee mechanism from the perspective of the buyer increases significantly as capacity becomes less tight relative to demand (i.e., as $K(M)/\bar{d}$ increases), while Figure 1(b) shows that the fixed fee mechanism tends to be preferred by the buyer, compared to the Shapley value allocation, when capacity is large relative to demand and when there is more variation in capacity across the suppliers.

Figure 1(a) also shows that the performance of the fixed fee mechanism decreases slightly as uncertainty in demand (i.e., the parameter j) increases, while the performance of the Shapley value allocation increases slightly in the uncertainty in demand. To understand this, note that higher uncertainty implies that the maximum possible demand, i.e., $\bar{d}$, is larger. Hence, the figure implies that, from the perspective of the buyer, the fixed fee mechanism performs worse when the maximum possible demand is large, while the Shapley value allocation performs better. Thus, Figure 1(b) shows that the percentage of cases in which the fixed fee mechanism is preferred decreases as the uncertainty in demand increases. These observations are consistent with the impact of $K(M)/\bar{d}$ and $Pr(D=\bar{d})$ on the relative performance of the two mechanisms. As discussed in Section 4.3.1, under the Shapley value allocation, the buyer's cost of purchasing from supplier i, i.e., the profit allocation $\phi_i(d)$, is based on the realized demand and allocation. In contrast, in the fixed fee mechanism, the fee paid to supplier i, i.e., f_i, is determined before demand is realized. As a result, one would expect that in settings with more demand uncertainty, the Shapley value allocation would perform better relative to the fixed fee mechanism because the Shapley value allocation allows for profit to be allocated after demand is realized. This intuition matches the results shown in Figure 1(b).



(a) Sensitivity Analysis on $\%(B)$ (b) Fixed Fee Mechanism vs. Shapley Value Allocation

Figure 2 **Impact of Supplier Capacity and Cost Variation on Performance of Fixed Fee Mechanism.**

Similarly, Figure 2(a) shows $\%(B)$, while Figure 2(b) shows the percentage of cases in which the fixed fee mechanism is preferred by the buyer, but now focusing on the impact of a, which measures the amount of variation that exists in the unit costs of the suppliers. When capacity is high relative to demand, $\%(B)$ is large for both mechanisms. However, Figure 2(a) shows that $K(M)/\bar{d}$ has a significant impact on $\%(B)$ for the fixed fee mechanism, but only minimal impact for the Shapley value allocation. There is also an interaction shown in Figure 2(a), i.e., $\%(B)$ is consistently lower for the Shapley value allocation than for the fixed fee mechanism when there is excess capacity ($K(M)/\bar{d} = 1.3$), while $\%(B)$ is higher for the Shapley value allocation than for the fixed fee mechanism when there is tighter capacity ($K(M)/\bar{d} = 1.1$). Finally, the figure shows that variation in cost has a more significant impact on the fixed fee mechanism than on Shapley value allocation, i.e., $\%(B)$ decreases with the variation in costs for the fixed fee mechanism, but is relatively unaffected by the variation in costs for the Shapley value allocation.

We can use Figures 1(a) and 2(a) to assess how the key model parameters affect the allocation of the total system profit between the buyer and suppliers, where large (small) values of $\%(B)$ imply that a small (large) amount of the system profit is allocated to the suppliers. When the overall capacity is high relative to the demand, the buyer is able to earn a larger portion of the total network profit. When the overall capacity is tighter, the suppliers will be able to extract more of the network profit. In other words, when there is significant excess capacity, the buyer is able to extract more of the profit and the suppliers earn little profit. The suppliers can earn a larger portion of the profits when the probability of high demand is large and when there is more variation in the costs of the suppliers. Intuitively, when the costs vary significantly, the competition between the suppliers will be less intense and, thus, the buyer will not be able to extract as much profit.

These results provide a consistent message, i.e., when aggregate capacity of the supplier network is high relative to demand, or demand is more likely to be small, the fixed fee mechanism will perform well for the buyer, and is likely to outperform the Shapley value allocation. This is because the larger aggregate capacity creates more competition between the suppliers, which benefits the buyer under the fixed fee mechanism. When capacity is tight or demand is likely to be large, there will be less competition between the suppliers (because all suppliers will receive a significant allocation), and the buyer will not be able to extract much profit. In this case, the buyer should consider the Shapley value allocation. Given the additional complexity required to implement that Shapley value allocation, as well as the need for cooperation between the buyer and suppliers to achieve this allocation, identifying conditions under which that allocation can provide additional value to the buyer is a key contribution provided by the analysis presented in this paper.

5. Model Extension: Outside Option for Suppliers

We can extend the results presented in this paper to settings in which some of the model assumptions are relaxed. For example, in the appendix, we present two extensions: (1) the suppliers' production costs are nonlinear in quantity and (2) the buyer has her own production capacity. In both cases, we demonstrate that the submodularity property of the optimal network profit continues to hold and, thus, the main results for our base model will also continue to hold.

In the remainder of this section, we consider a setting in which the suppliers have the option of entering into a relationship with another customer, which we refer to as the *outside option*, if they are unable to come to an agreement with the buyer. We will describe how these outside options can be considered when determining the set of suppliers, M, with which the buyer seeks to contract, as defined in Section 2. Once M is determined, the remainder of the analysis in Section 3 will hold.

Suppose that there are $\bar{m}$ *potential suppliers*, where the set of suppliers is denoted by $\bar{M} = \{1, \ldots, \bar{m}\}$, and that the suppliers are sorted such that $c_1 \leq c_2 \leq \ldots \leq c_{\bar{m}}$. As an alternative to contracting with the buyer, supplier i can sell his entire capacity, K_i, to another firm to earn a fixed reservation profit, denoted by R_i. For example, supplier i may be able to sell his capacity to the outside option at the price of p_b per unit, implying that $R_i = (p_b - c_i) \times K_i$.

Each supplier must determine whether it is better to contract with the buyer, given the fixed fee mechanism proposed by the buyer, or use the outside option. In addition, to implement the fixed fee mechanism, the buyer must be able to identify the subset of suppliers, denoted by $M \subseteq \bar{M}$, that would be willing to participate in the supplier network under that mechanism. Thus, we next specify a method for identifying the subset of suppliers, M, that is taken as given in Section 3.

Under the fixed fee mechanism with a subset of suppliers M, supplier i earns equilibrium profit equal to the upfront fee, $f_i^*(M) = \Pi(M) - \Pi(M \setminus \{i\})$, which is the supplier's marginal contribution to the network[11]. Thus, supplier i will choose to contract with the buyer if and only if $f_i^*(M) \geq R_i$. Further, due to the submodularity of the network profit, $M_1 \subset M_2$ implies that $f_i^*(M_1) \geq f_i^*(M_2)$ for all $i \in M_1$. Therefore, the buyer can use the following forward selection algorithm to determine the subset of suppliers, $M \subseteq \bar{M}$, to include in the supplier network for the analysis in Section 3.

- Let $M_0 = \emptyset$. Assume the suppliers in $\bar{M}$ are ordered such that $c_1 \leq c_2 \leq \ldots \leq c_{\bar{m}}$.
- For $i = 1, \ldots, \bar{m}$:
  —Compute $f_i^*(M_{i-1} \cup \{i\})$ for all $i \in M_{i-1} \cup \{i\}$.
  —Set $M_i = \{i : f_i^*(M_{i-1} \cup \{i\}) \geq R_i\}$.
- Set $M = M_{\bar{m}}$.

If $i \in M$, supplier i will find contracting with the buyer under the fixed fee mechanism to be more profitable than his outside option and, thus, supplier i should be included in the supplier network.

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[11] Notice that we have added the argument M to the upfront fee to indicate that supplier i's marginal contribution to the network is dependent on the specific subset of suppliers, M.

6. Concluding Remarks

In this paper, we consider the contract design problem faced by a buyer who experiences uncertain demand and sources a key input of production from multiple competing heterogeneous suppliers. Since the suppliers have limited capacity, the buyer must form a network of suppliers in order to meet all demand for the final product. Our analysis addresses two key questions: (1) How should the buyer design the contracting mechanism to ensure a stable sourcing network? and (2) Given that more than one possible mechanism can achieve a stable network, how do the profit allocations between the buyer and suppliers differ under those alternatives, and which mechanism is preferred by the buyer?

For (1), we identify two mechanisms, i.e., the competitive fixed fee mechanism and the cooperative Shapley value allocation, that can be used by the buyer to achieve a stable sourcing network, i.e., can achieve a profit allocation that motivates all members to join and stay in the network. However, these mechanisms differ in the type of stability achieved, with the fixed fee mechanism achieving myopic stability, while the Shapley value allocation achieves farsighted stability.

For (2), we demonstrate some key differences between these two mechanisms, which can impact their ease of implementation. Specifically, the fixed fee mechanism is easier to implement since it does not assume cooperation between the suppliers and only requires the specification of fixed fees, which do not depend on the realization of demand. In contrast, the Shapley value allocation assumes the suppliers will cooperate in the sourcing network and results in payments to the suppliers that are dependent on the realized demand. We also compare the performance of these two mechanisms from the perspective of the buyer's profit. When the aggregate capacity of the suppliers is high relative to demand, or when demand is more likely to be small, the fixed fee mechanism will be preferred by the buyer. This result is due to the fact that, when there is abundant capacity, the suppliers must compete to earn a portion of the demand and the fixed fee mechanism takes advantage of this competition, while the Shapley value allocation does not. However, when capacity is tight or demand is likely to be large, all suppliers will be allocated a portion of the demand. Therefore, encouraging competition between the suppliers is less critical, and the buyer may prefer the Shapley value allocation. Further, in settings with high demand uncertainty, the Shapley value allocation can outperform the fixed fee mechanism due to the fact that the Shapley value allocates profit to the suppliers after demand is realized, while the fixed fee mechanism sets the suppliers' profits (i.e., the fees) before the demand realization.

Overall, our results provide valuable insights to managers considering how to design a contracting mechanism that will support a stable sourcing network. In particular, buyers will need to consider the trade-offs between the two mechanisms in terms of ease of implementation, as well as the magnitude of the profit earned under each mechanism. Our results provide some guidance regarding those trade-offs. For example, as noted in Section 1, the players in the iron ore industry have demonstrated an interest in investigating new contractual forms that will ensure more stable and long-term relationships between producers and purchasers. Further, the industry is currently experiencing an oversupply (Lau 2021). Thus, our results imply that the fixed fee mechanism, which can help the buyer achieve a stable sourcing network, is likely to be preferred by buyers in that industry since it is easier to implement than the Shapley value allocation. Further, it allows the buyer to take advantage of competition between the suppliers for the limited demand, providing a larger allocation of the network profit. This example demonstrates a key contribution of the research presented in this paper, i.e., assisting buyers in identifying conditions under which they can focus on the simple fixed fee mechanism, rather than investing effort in the implementation of the more complex Shapley value allocation.

Further, a recent study used behavioral lab experiments to consider myopic and farsighted stability in network formation games (Teteryatnikova and Tremewan 2020). The study finds evidence to support both types of stability behavior, but the results also indicate that the most stable networks are those predicted by myopic stability (such as the core). However, cautious farsighted stability concepts (such as the largest consistent set) were better able to identify all networks that have the potential to be stable. The authors also find that networks which are stable in the farsighted sense require more knowledge and experience than those that are myopically stable since the players must understand the sequence of other players' reactions. Given these insights, we would anticipate that larger and more experienced buyers, i.e., those who possess a more sophisticated understanding of their supplier network, might prefer the mechanism based on the Shapley value allocation, while smaller and less experienced buyers may prefer the fixed fee mechanism.

Finally, this paper provides opportunities for future research. In particular, while our model captures many of the key aspects of the iron ore industry, the model necessarily makes some assumptions that are not entirely consistent with that practical setting. For example, we assume a static selling price for the final product (e.g., steel), while in reality the industry experiences significant price volatility. Further, our analysis has assumed that the backup source is uncapacitated and has a unit cost that is less than the unit selling price for the final product. Without the latter assumption, the backup source would never be used in our model setting. However, practically, when faced with a large order from an important customer, the buyer may be willing to purchase from the backup source at a loss in order to avoid the penalty costs associated with an unfilled or partially-filled order. Such a possibility is not captured in our current model setting since we do not include a penalty cost for unfilled orders. However, such a setting would be an important topic for future research. Finally, buyers and suppliers in the iron ore industry do not currently make use of a contracting mechanism, such as the fixed fee mechanism proposed in this paper, under which the profits earned by the suppliers are fixed regardless of the quantity of demand allocated to each supplier. However, that type of mechanism could be considered by the industry players if it is shown to be effective in creating more stable sourcing networks. Thus, demonstrating the practical effectiveness of the fixed fee mechanism is another important research direction.

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Appendix

Definitions

The following definitions, drawn from Edmonds (2003) and He et al. (2012), are used in the analysis presented in this paper. Both are standard concepts in combinatorial optimization.

Definition 1 (Rank Function). For a given finite set S, set function $z: 2^S \to \mathbb{R}$ is a *rank function* if the following conditions are satisfied:

- z is normalized, i.e., $z(\emptyset) = 0$;
- z is nondecreasing, i.e., $z(A) \leq z(B)$ for $A \subseteq B \subseteq S$;
- z is submodular, i.e., $z(A \cup B) + z(A \cap B) \leq z(A) + z(B)$ for $A, B \subseteq S$.

Definition 2 (Polymatroid). For a given finite set S and a rank function $z: 2^S \to \mathbb{R}$, the polyhedron, defined as

$$\mathbb{P}(S, z) = \left\{ \mathbf{x} \in \mathbb{R}_+^{|S|} : \sum_{i \in A} x_i \leq z(A), \forall A \subseteq S \right\}, \tag{15}$$

is called a *polymatroid*.

Proofs and Technical Results

Proof of Lemma 1 We first show the necessity and then the sufficiency of the constraints.

Necessity: From constraint (2), we obtain $\sum_{i \in A} x_i \leq \sum_{i \in A} K_i$ for any $A \subseteq S$. Furthermore, the non-negativity of $\mathbf{x}$ implies that $\sum_{i \in A} x_i \leq \sum_{i \in S} x_i$ for any $A \subseteq S$, which together with constraint (3), leads to $\sum_{i \in A} x_i \leq d$. Therefore, we have established the necessity of the conditions in (4).

Sufficiency: First, let A be a singleton. Without loss of generality, we suppose $A = \{i\}$ for $i \in S$. Then constraint (4) becomes $x_i \leq \min(d, K_i)$, from which we obtain $x_i \leq K_i$, which leads to constraint (2). Second, let $A = S$. Constraint (4) then becomes $\sum_{i \in S} x_i \leq \min\left(d, \sum_{i \in S} K_i\right)$, from which we obtain $\sum_{i \in S} x_i \leq d$. This leads to constraint (3). Thus, we have established the sufficiency of the conditions in (4). $\square$

Proof of Lemma 2 First, we know that $z(\emptyset) = 0$, implying that $z(A)$ is normalized. Second, it is straightforward that $z(A)$ is nondecreasing. Third, we show that $z(A)$ is submodular. For any $A \subset S$, $l, m \notin A$ and $l, m \in S$, define the following notions:

$$\Delta_1 := z(A \cup \{l\}) - z(A) = \min\left(d, \sum_{j \in A} K_j + K_l\right) - \min\left(d, \sum_{j \in A} K_j\right)$$

$$\Delta_2 := z(A \cup \{l, m\}) - z(A \cup \{m\}) = \min\left(d, \sum_{j \in A} K_j + K_l + K_m\right) - \min\left(d, \sum_{j \in A} K_j + K_m\right).$$

It is without loss of generality that we assume $K_m \leq K_l$, since otherwise we could simply swap the indices of l and m. To show the submodularity of $z(A)$, it suffices to show that $\Delta_1 \geq \Delta_2$ always holds. We consider five cases below, depending on the value of d.

Case (1): $d \leq \sum_{j\in A} K_j$. In this case, we obtain $\Delta_1 = d - d = 0$ and $\Delta_2 = d - d = 0$. Thus, the required result $\Delta_1 \geq \Delta_2$ holds.

Case (2): $\sum_{j\in A} K_j < d \leq \sum_{j\in A} K_j + K_m$. In this case we obtain $\Delta_1 = d - \sum_{j\in A} K_j > 0$ and $\Delta_2 = d - d = 0$. Thus, we have $\Delta_1 > \Delta_2$ and, hence, the required result $\Delta_1 \geq \Delta_2$ holds.

Case (3): $\sum_{j\in A} K_j + K_m < d \leq \sum_{j\in A} K_j + K_l$. In this case we obtain $\Delta_1 = d - \sum_{j\in A} K_j$ and $\Delta_2 = d - \sum_{j\in A} K_j - K_m$. Thus, we have $\Delta_1 > \Delta_2$ and, hence, the required result $\Delta_1 \geq \Delta_2$ holds.

Case (4): $\sum_{j\in A} K_j + K_l < d \leq \sum_{j\in A} K_j + K_l + K_m$. In this case we obtain $\Delta_1 = \sum_{j\in A} K_j + K_l - \sum_{j\in A} K_j = K_l$ and $\Delta_2 = d - \sum_{j\in A} K_j - K_m \leq K_l$ where the inequality follows from the condition we suppose, i.e., $d \leq \sum_{j\in A} K_j + K_l + K_m$. Thus, we have $\Delta_1 \geq \Delta_2$ and, hence, the required result.

Case (5): $d > \sum_{j\in A} K_j + K_l + K_m$. In this case we obtain $\Delta_1 = \sum_{j\in A} K_j + K_l - \sum_{j\in A} K_j = K_l$ and $\Delta_2 = \sum_{j\in A} K_j + K_l + K_m - \sum_{j\in A} K_j - K_m = K_l$. Thus, we have $\Delta_1 = \Delta_2$ and, hence, the required result $\Delta_1 \geq \Delta_2$ holds.

From the five cases above, we conclude that for any $A \subset S$, $l, m \notin A$ and $l, m \in S$, the result $\Delta_1 \leq \Delta_2$ holds. Therefore, $z(A)$ is submodular in A. This means that $z(A)$ is a rank function. Having shown that $z(A)$ is a rank function, by Definition 2, $\mathbb{P}(S, z)$ is a polymatroid. □

Proof of Lemma 3 Please refer to Theorem (The Greedy Algorithm) of Edmonds (2003) and Lemma 1 of He et al. (2012). □

Proof of Theorem 1 Note first that the objective function in (1) is linearly separable. Furthermore, using the results in Lemma 1 and Lemma 2, we know the feasible set is a polymatroid. Therefore, the optimal demand allocation problem is to maximize a linearly separable objective function over a polymatroid. For the following problem

$$\begin{aligned} g(\alpha) := \max \quad & (r - p_o)d + \sum_{i\in S} \alpha_i x_i, \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{P}(z, S), \end{aligned}$$

where $\alpha_i = p_o - c_i \geq 0$. Theorem 1 of He et al. (2012) shows that

$$g(\alpha \wedge \beta) + g(\alpha \vee \beta) \leq g(\alpha) + g(\beta).$$

For a given $A \subseteq S$, let $\alpha^A \in \mathbb{R}_+^{|S|}$ such that for each $i \in S$,

$$\alpha_i^A = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g(\alpha^A) = \pi(A, d)$, and for any $A, B \subseteq S$, we have $\alpha^{A\cup B} = \alpha^A \vee \alpha^B$ and $\alpha^{A\cap B} = \alpha^A \wedge \alpha^B$. We obtain

$$\pi(A \cup B, d) + \pi(A \cap B, d) = g(\alpha^{A\cup B}) + g(\alpha^{A\cap B})$$

$$
\begin{aligned}
&= g(\alpha^A \vee \alpha^B) + g(\alpha^A \wedge \alpha^B) \\
&\leq g(\alpha^A) + g(\alpha^B) \\
&= \pi(A,d) + \pi(B,d).
\end{aligned}
$$

Therefore, $\pi(S,d)$ is a submodular set function of S for $S \subseteq M$. Since, for a given d, $\pi(S,d)$ is a submodular function, we have that $-\pi(S,d)$ is a supermodular function, and that the expected value, i.e., $-\mathbb{E}[\pi(S,D)]$ is also supermodular. See Corollary 2.6.2 of Topkis (1998). Therefore, $\Pi(S) := \mathbb{E}[\pi(S,D)]$ is a submodular function. $\square$

Proof of Corollary 1 We prove this by induction. It is trivial when $|S| \leq 1$, so we begin with $|S| = 2$. If $|S| = 2$, suppose $S = \{j,k\}$. From (9), we obtain $\Pi(M) + \Pi(\{j,k\}) \leq \Pi(M \setminus \{j\}) + \Pi(M \setminus \{k\})$, which can be rearranged to obtain the result required.

Suppose $\Pi(M) - \Pi(S) \geq \sum_{i \in M \setminus S} (\Pi(M) - \Pi(M \setminus \{i\}))$ holds for any S with $|S| > 2$. Then for $j \in M \setminus S$ we have,

$$
\begin{aligned}
\sum_{i \in S \cup \{j\}} (\Pi(M) - \Pi(M \setminus \{i\})) &= \sum_{i \in S} (\Pi(M) - \Pi(M \setminus \{i\})) + \Pi(M) - \Pi(M \setminus \{j\}) \\
&\leq \Pi(M) - \Pi(M \setminus S) + \Pi(M) - \Pi(M \setminus \{j\}) \\
&\leq 2\Pi(M) - (\Pi(M) + \Pi(M \setminus (S \cup \{j\}))) \\
&= \Pi(M) - \Pi(M \setminus (S \cup \{j\})),
\end{aligned}
$$

where the first inequality follows from what we suppose and the second inequality follows from the submodularity property. Hence, by induction, we complete the proof. $\square$

Proof of Lemma 4 Theorem 1 has shown that the optimal supply chain profit $\Pi(S)$ is submodular, that is, for any $C, D \subseteq S$, we obtain

$$
\Pi(C \cap D) + \Pi(C \cup D) \leq \Pi(C) + \Pi(D). \tag{16}
$$

Define $\mu(S) = \sum_{i \in S} f_i$. Observe that $C = (C \setminus D) \cup (C \cap D)$ and, thus, $\mu(C) = \mu(C \setminus D) + \mu(C \cap D)$. Similarly observe that $D = (D \setminus C) \cup (C \cap D)$ and, thus, $\mu(D) = \mu(D \setminus C) + \mu(C \cap D)$. Therefore, we obtain

$$
\begin{aligned}
\mu(C) + \mu(D) &= \mu(C \setminus D) + \mu(C \cap D) + \mu(D \setminus C) + \mu(C \cap D) \\
&= \mu(C \cap D) + \mu((C \setminus D) \cup (D \setminus C) \cup (C \cap D)) \\
&= \mu(C \cap D) + \mu(C \cup D).
\end{aligned}
$$

That is,

$$
\mu(C \cap D) + \mu(C \cup D) = \mu(C) + \mu(D). \tag{17}
$$

We then obtain

$$\begin{aligned}\Pi_B(C\cap D)+\Pi_B(C\cup D) &= \Pi(C\cap D)+\Pi(C\cup D)-\mu(C\cap D)-\mu(C\cup D)\\ &\le \Pi(C)+\Pi(D)-\mu(C)-\mu(D)\\ &= \Pi_B(C)+\Pi_B(D),\end{aligned}$$

where the inequality follows from (16) and (17). Therefore, Π_B is also submodular. $\square$

Proof of Theorem 2 Given $\hat{f}_i$ and f_j for $j\neq i$, the buyer's optimal profit is $\hat{\Pi}^*_B(M)-\hat{f}_i=\Pi^*_B(M\setminus\{i\})$. If supplier i is not selected by the buyer, the buyer's optimal profit is $\Pi^*_B(M\setminus\{i\})$. According to our tie-breaking assumption, the buyer will choose supplier i. Therefore, we have established that the optimal fee is $\hat{f}_i=\hat{\Pi}^*_B(M)-\Pi^*_B(M\setminus\{i\})$. $\square$

Proof of Theorem 3 First we show that, given the offers $\{f^*_i : i\in M\}$, the buyer will select the system optimal choice of suppliers, which yields a total network profit of $\Pi(M)$. Since each supplier i's offer specifies a fee from the buyer of $\Pi(M)-\Pi(M\setminus\{i\})$, the buyer's profit is given by $\Pi(M)-\sum_{i\in M}(\Pi(M)-\Pi(M\setminus\{i\}))$. Following the same logic, if the buyer selects suppliers in $S\subset M$ only, the buyer's profit becomes

$$\begin{aligned}\Pi(S)-\sum_{i\in S}(\Pi(M)-\Pi(M\setminus\{i\})) &= \Pi(S)+\sum_{i\in M\setminus S}(\Pi(M)-\Pi(M\setminus\{i\}))-\sum_{i\in M}(\Pi(M)-\Pi(M\setminus\{i\}))\\ &\le \Pi(M)-\sum_{i\in M}(\Pi(M)-\Pi(M\setminus\{i\})),\end{aligned}$$

where the inequality follows from Corollary 1 that $\Pi(M)-\Pi(S)\ge\sum_{i\in M\setminus S}(\Pi(M)-\Pi(M\setminus\{i\}))$. This shows that the buyer's expected profit, when selecting suppliers only from S, is no greater than that when selecting the suppliers in M. Therefore, according to our assumption, the buyer will select the optimal subset of suppliers.

Second, we show that no supplier has any incentive to deviate from the proposed offer. It is obvious that no supplier $k\in M$ will deviate by setting a lower fee than $\Pi(M)-\Pi(M\setminus\{k\})$. Indeed, if the supplier does, the buyer will still select the system optimal subset of suppliers, but this leads to a lower profit for him. On the other hand, if the supplier charges a higher fee, the buyer will not select him, but instead, the other suppliers in $N\setminus\{k\}$, thus providing the buyer with the same profit of $\Pi(M)-\sum_{i\in M}(\Pi(M)-\Pi(M\setminus\{i\}))$. To see this, when choosing suppliers in $M\setminus\{k\}$ only, the buyer's profit is

$$\begin{aligned}&\Pi(M\setminus\{k\})-\sum_{i\in M\setminus\{k\}}(\Pi(M)-\Pi(M\setminus\{i\}))\\ =&\Pi(M\setminus\{k\})+\Pi(M)-\Pi(M\setminus\{k\})-\sum_{i\in M}(\Pi(M)-\Pi(M\setminus\{i\}))\\ =&\Pi(M)-\sum_{i\in M}(\Pi(M)-\Pi(M\setminus\{i\})).\end{aligned}$$

Thus, the buyer makes the same profit as before by simply excluding supplier k. We have, therefore, established the result that neither supplier has any incentive to unilaterally change his strategy. Finally, the equilibrium profit allocations follow naturally from the equilibrium strategies. $\square$

Proof of Corollary 2 The proof follows directly from Theorem 3. $\square$

Proof of Corollary 3 When $\sum_{i\in M} K_i \le \underline{d}$, from (5) we obtain $z(A) = \sum_{i\in A} K_i$ for all $\emptyset \subset A \subseteq M$ and all $d \in [\underline{d}, \bar{d}]$. Lemma 3 then shows that $x_i^*(M,d) = K_i$ for any $i \in M$. Furthermore, we obtain $\Pi(M) = (r - p_0)\mathbb{E}[D - \sum_{i\in M} K_i] + \sum_{i\in M}(r - c_i)K_i$, and $\Pi(M \setminus \{j\}) = (r - p_0)\mathbb{E}[D - \sum_{i\in M\setminus\{j\}} K_i] + \sum_{i\in M, i\neq j}(r - c_i)K_i$ for any $j \in M$. Therefore, using the results in Theorem 3, we obtain the profit split as follows: $\pi_j^* = \Pi(M) - \Pi(M \setminus \{j\}) = (p_o - c_j)K_j$ for any $j \in M$ and $\pi_B^* = (r - p_0)\mathbb{E}[D]$. $\square$

Proof of Theorem 4 We first prove part (a). For $S \subseteq T \subseteq M$, by definition $v(S \cup \{B\}) = \Pi(S)$, $v(T \cup \{B\}) = \Pi(T)$, and $v(S) = v(T) = 0$. From the definition of the optimal supply chain profit, as given in (8), we have that $\Pi(T) \ge \Pi(S)$. Hence, $v(S \cup \{B\}) - v(S) \le v(T \cup \{B\}) - v(S)$.

For part (b), we first note that $v(S) = 0$ for any $S \subseteq M$. We consider three cases: (1) if $B \in S \subseteq T$, the required result follows directly from the submodularity of $\Pi(S)$ for $S \subseteq M$; (2) if $B \notin T$, we obtain $v(S \cup \{i\}) = v(S) = v(T \cup \{i\}) = v(T) = 0$ and, thus, the result holds; and (3) if $B \notin S$, but $B \in T$, we obtain $v(T \cup \{i\}) - v(T) \ge 0$ and $v(S \cup \{i\}) - v(S) = 0$. Thus, we have $v(T \cup \{i\}) - v(T) \ge v(S \cup \{i\}) - v(S)$. $\square$

Proof of Theorem 5 We first prove the non-emptiness of the core using the construction approach. Consider the following allocation: $a_B = v(N)$ and $a_i = 0$ for all $i \in M$. First it is straightforward that $a(N) = v(N)$. Second, if $S \subseteq N$ and $B \notin S$, we know that $a(S) = 0$ by construction and $v(S) = 0$ by definition. Thus, $v(S) \le a(S)$ holds. If $S \subseteq N$ and $B \in S$, then $a(S) = v(N)$ by construction, and $v(S) \le v(N)$ from the optimality of the problem, i.e., since $v(S)$ is the optimal supply chain profit (with B included in S), $v(S)$ cannot be greater than $v(N)$ because S is a subset of N. Thus, we obtain $v(S) \le a(S)$. Combining the above, we show that the conditions for the core are all met by the constructed allocation. This shows that the sourcing game has a non-empty core.

Next we show the equivalence between the conditions in (12) and the conditions in (11). First, we prove the "if" part. Consider the following two cases.

Case (1): If $B \in S$, then we obtain

$$\begin{aligned}\sum_{j\in S} a_j &= \sum_{j\in N} a_j - \sum_{j\in N\setminus S} a_j \\ &\ge v(N) - \sum_{j\in N\setminus S} (v(N) - v(N \setminus \{j\})) \\ &\ge v(N) - (v(N) - v(N \setminus S)) \\ &= v(S),\end{aligned}$$

where the first inequality follows from the conditions in (12) and the second inequality follows from the submodularity property of $v(\cdot)$ as shown in Corollary 1. Note for the latter inequality we have used the definition that $v(N) = \Pi(M)$ and $v(N \setminus \{j\}) = \Pi(M \setminus \{j\})$ for $j \in M$. Therefore, we have established the result that $a(S) \geq v(S)$.

Case (2): If $B \notin S$, then we obtain $\sum_{j\in S} a_j \geq 0$ from the conditions in (12). We also know that $v(S) = 0$. Therefore, we obtain $a(S) \geq v(S)$ as required.

Combining the above two cases, we have established the sufficiency of the conditions in (12).

Second we prove the "only if" part. For the second condition in (11), we first let $S = N \setminus \{i\}$ and obtain the following result:

$$\sum_{j\in N\setminus\{i\}} a_j \geq v(N \setminus \{i\}),$$

which together with the first condition in (11) (i.e., $\sum_{j\in N} a_j = v(N)$) yields $a_i \leq v(N) - v(N \setminus \{i\})$. Then we let $S = \{i\}$ where $i \in N$, and obtain $a_i \geq v(\{i\}) = 0$. Therefore, we have established the result that $0 \leq a_i \leq v(N) - v(N \setminus \{i\})$ and, thus, the necessity of the conditions in (12). □

Proof of Theorem 6 This can be shown by using the definition of the core. First, we know that

$$\begin{aligned}\pi_B^* + \sum_{i\in M} \pi_i^* &= \Pi(M) - \sum_{i\in M} \Pi(M) - \Pi(M \setminus \{i\}) + \sum_{i\in M} \Pi(M) - \Pi(M \setminus \{i\}) \\ &= \Pi(M) = v(N).\end{aligned}$$

Second, for any $S \subseteq N$, we consider the following two cases:

Case (1): if $B \notin S$, then $v(S) = 0$. We know that $\sum_{i\in S} \pi_i^* = \sum_{i\in S} \Pi(M) - \Pi(M \setminus \{i\}) \geq 0$, thus $\sum_{i\in S} \pi_i^* \geq v(S)$;

Case (2): if $B \in S$, then $v(S) = \Pi(S \setminus \{B\})$. We have

$$\begin{aligned}\sum_{i\in S} \pi_i^* &= \pi_B^* + \sum_{i\in S\setminus\{B\}} \pi_i^* \\ &= \Pi(M) - \sum_{i\in M} \Pi(M) - \Pi(M \setminus \{i\}) + \sum_{i\in S\setminus\{B\}} \Pi(M) - \Pi(M \setminus \{i\}) \\ &= \Pi(M) - \sum_{i\in M\cup\{B\}\setminus S} \Pi(M) - \Pi(M \setminus \{i\}) \\ &\geq \Pi(S \setminus \{B\}) = v(S),\end{aligned}$$

where the inequality follows from the submodularity property of $\Pi(\cdot)$.

Combining the above two, we have shown that $\sum_{i\in S} \pi_i^* \geq v(S)$ for any $S \subseteq N$. This establishes that the equilibrium lies in the core of the corresponding cooperative game. □

Proof of Theorem 7 To prove this theorem, we must first demonstrate that, under the Shapley value allocations, (a) $\phi_B^T > \phi_B^{T\setminus\{i\}}$, for all $i \in N \setminus T$, and (b) $\phi_i^N > 0$ for all $i \in N$.

We first prove (a). Let $T_B := T \setminus \{B\}$ denote the set of suppliers in T. Further denote by k the size of T_B, i.e., $|T_B| = k$, and thus $|T| = k+1$. By the definition of the Shapley value, we obtain:

$$\begin{aligned}
\phi_B^T - \phi_B^{T\setminus\{i\}} =& \sum_{S\subseteq T_B} \frac{|S|!(k-|S|)!}{(k+1)!}\left(v(S\cup\{B\})-v(S)\right) - \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-1-|S|)!}{k!}\left(v(S\cup\{B\})-v(S)\right)\\
=& \sum_{S\subseteq T_B} \frac{|S|!(k-|S|)!}{(k+1)!}\Pi(S) - \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-1-|S|)!}{k!}\Pi(S)\\
=& \sum_{S\subseteq T_B, i\in S} \frac{|S|!(k-|S|)!}{(k+1)!}\Pi(S) + \sum_{S\subseteq T_B, i\notin S} \frac{|S|!(k-|S|)!}{(k+1)!}\Pi(S) - \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-1-|S|)!}{k!}\Pi(S)\\
=& \frac{|\{i\}|!(k-|\{i\}|)!}{(k+1)!}\Pi(\{i\}) + \sum_{\{i\}\subset S\subseteq T_B} \frac{|S|!(k-|S|)!}{(k+1)!}\Pi(S) + \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-|S|)!}{(k+1)!}\Pi(S)\\
&- \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-1-|S|)!}{k!}\Pi(S)\\
=& \frac{1}{k(k+1)}\Pi(\{i\}) + \sum_{S\subseteq T_B\setminus\{i\}} \frac{(|S|+1)!(k-1-|S|)!}{(k+1)!}\Pi(S\cup\{i\}) + \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-|S|)!}{(k+1)!}\Pi(S)\\
&- \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-1-|S|)!}{k!}\Pi(S)\\
\geq& \frac{1}{k(k+1)}\Pi(\{i\}) + \sum_{S\subseteq T_B\setminus\{i\}} \frac{(|S|+1)!(k-1-|S|)!}{(k+1)!}\Pi(S) + \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-|S|)!}{(k+1)!}\Pi(S)\\
&- \sum_{S\subseteq T_B\setminus\{i\}} \frac{|S|!(k-1-|S|)!}{k!}\Pi(S)\\
=& \frac{1}{k(k+1)}\Pi(\{i\}) + \sum_{S\subseteq T_B\setminus\{i\}} \left(\frac{(|S|+1)!(k-1-|S|)!}{(k+1)!} + \frac{|S|!(k-|S|)!}{(k+1)!} - \frac{|S|!(k-1-|S|)!}{k!}\right)\Pi(S)\\
=& \frac{1}{k(k+1)}\Pi(\{i\})\\
=& \frac{1}{k(k+1)}\left(\mathbb{E}[(r-p_o)D + (p_o-c_i)\min(K_i, D)]\right) > 0,
\end{aligned}$$

where the first inequality follows from $\Pi(S\cup\{i\}) \geq \Pi(S)$. Therefore, we have established the result that $\phi_B^T > \phi_B^{T\setminus\{i\}}$, for all $i \in N\setminus T$.

We next prove (b). Under the grand coalition, the profit allocated to supplier $i \in M$ is given by

$$\begin{aligned}
\phi_i^N =& \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}\left(v(S\cup\{i\})-v(S)\right)\\
\geq& \frac{|\{B\}|!(|N|-|\{B\}|-1)!}{|N|!}\left(v(\{B,i\})-v(\{B\})\right)\\
=& \frac{1}{m(m+1)}(\Pi(\{i\})-0)\\
=& \frac{1}{m(m+1)}\mathbb{E}[(r-p_o)D + (p_o-c_i)\min(D,K_i)] > 0,
\end{aligned}$$

where the first inequality follows from the fact that the Shapley value is a weighted sum of the marginal values of supplier i when he is a member of different coalitions. Similarly, we can show that $\phi_B^N > 0$.

We next formally describe the concept of LCS, which requires us to introduce additional notation. A partition on the set of players is referred to as a coalition structure, which we denote by $\mathscr{Z}$. Formally, $\mathscr{Z} := \{Z_1, Z_2, \cdots, Z_k\}$ where Z_i is a coalition of players for $i = 1, \cdots, k$ such that $\cup_{i=1}^k Z_i = N$ and $Z_i \cap Z_j = \emptyset$ for any i, j such that $i \neq j$. Denote by $\prec_i$ player i's strong preference relation. That is, for any two coalition structures $\mathscr{Z}_1$ and $\mathscr{Z}_2$, we write $\mathscr{Z}_1 \prec_i \mathscr{Z}_2$ if and only if $u_i^{\mathscr{Z}_1} < u_i^{\mathscr{Z}_2}$, where $u_i^{\mathscr{Z}}$ is the profit allocated to player i under the coalition structure $\mathscr{Z}$ where $\mathscr{Z} = \mathscr{Z}_1$ or $\mathscr{Z}_2$. Extending the notation of strong preference to an arbitrary coalition S, we write $\mathscr{Z}_1 \prec_S \mathscr{Z}_2$ if and only if $u_i^{\mathscr{Z}_1} < u_i^{\mathscr{Z}_2}$ for all $i \in S$. For a given coalition S, let $G_S(\mathscr{Z})$ be the set of all coalition structures that can be achieved by a one-step deviation of S from the coalition structure $\mathscr{Z}$. Denote by $\rightarrow_S$ a one-step deviation of S, and we write $\mathscr{Z}_1 \rightarrow_S \mathscr{Z}_2$ if the deviation of S results in the change of the coalition structure from $\mathscr{Z}_1$ to $\mathscr{Z}_2$. Thus, we obtain $\mathscr{Z}_2 \in G_S(\mathscr{Z}_1)$.

Equipped with the above notation, we introduce the following definitions which are standard in the literature.

Definition 3. $\mathscr{Z}_1$ is *directly* dominated by $\mathscr{Z}_2$, denoted by $\mathscr{Z}_1 < \mathscr{Z}_2$, if there exists a coalition S such that $\mathscr{Z}_1 \rightarrow_S \mathscr{Z}_2$ and $\mathscr{Z}_1 \prec_S \mathscr{Z}_2$. Also, $\mathscr{Z}_1$ is *indirectly* dominated by $\mathscr{Z}_2$, denoted by $\mathscr{Z}_1 \ll \mathscr{Z}_k$, where $\mathscr{Z}_k = \mathscr{Z}_2$, if there exist coalitions $S_1, S_2, \cdots, S_{k-1}$ and $\mathscr{Z}_1, \mathscr{Z}_2, \cdots, \mathscr{Z}_k$, such that $\mathscr{Z}_i \rightarrow_{S_i} \mathscr{Z}_{i+1}$ and $\mathscr{Z}_i \prec_{S_i} \mathscr{Z}_k$ for $i = 1, 2, \cdots, k-1$.

Definition 4. A set Y is consistent if the following condition holds: $\mathscr{Z} \in Y$ if and only if, for all S and all $\hat{\mathscr{Z}} \in G_S(\mathscr{Z})$, there exists $\mathscr{S} \in Y$, where $\hat{\mathscr{Z}} = \mathscr{S}$ or $\hat{\mathscr{Z}} \ll \mathscr{S}$, such that $\mathscr{Z} \not\prec_S \mathscr{S}$.

Following the definition of LCS, we will show that no coalitions will deviate from the grand coalition assuming that all the players are farsighted. We start by defining our notation system for coalition structures: the coalitions are separated by a semicolon, and if a coalition consists of two or more players we collect them into a round bracket. For example, we write $\{(1, 2, \cdots, m, B)\}$ as the grand coalition, where i represents the ith supplier and B represents the buyer. Similarly, we let $\{(1, 2, \cdots, m); B\}$ denote the coalition structure, where all the suppliers form a coalition with the buyer acting alone. For convenience, let $\mathscr{Z} = \{(1, 2, \cdots, m, B)\}$ denote the grand coalition.

In our model, there are three types of coalitions that may initially deviate from the grand coalition, depending on whether the coalition members are the buyer and/or the suppliers.

Case (1): The buyer deviates on her own.

Case (2): A coalition of suppliers deviates together. Here the term coalition is defined in a board sense and may include only one supplier.

Case (3): A coalition of suppliers and the buyer deviates.

Note that in the second and third cases, there are multiple scenarios in which coalition structures may be formed. In what follows, we will carefully construct, for each case, a sequence of deviations triggered by the initial deviations, and show that the players who make the initial deviations are not better off at the end of the constructed sequence of deviations. Once this is done, we know from Definition 4 that we have established the result required.

Case (1): After the deviation of the buyer, the coalition structure becomes $\{(1,2,\cdots,m);B\}$. Define the following coalitions: $S_0=\{B\}$, $S_1=\{1\}$, $S_2=\{2\}$, ..., $S_m=\{m\}$, and $S_{m+1}=\{1,2,\cdots,m,B\}$. Further, define the following coalition structures: $\mathscr{X}_0=\{(1,2,\cdots,m);B\}$, $\mathscr{X}_1=\{1;(2,\cdots,m);B\}$, $\mathscr{X}_2=\{1;2;(3,\cdots,m);B\}$, ..., and $\mathscr{X}_m=\{1;2;3;\cdots;m;B\}$. Thus, if the deviations are organized in the orders of S_0, S_1, ..., S_{m+1} (one coalition at a time), then the sequence of resulting coalitions is given by $\mathscr{X}_0$, $\mathscr{X}_1$, ..., $\mathscr{X}_m$, and $\mathscr{X}$. That is, we have

$$\mathscr{X}\to_{S_0}\mathscr{X}_0\to_{S_1}\mathscr{X}_1\to_{S_2}\mathscr{X}_2\cdots\to_{S_m}\mathscr{X}_m\to_{S_{m+1}}\mathscr{X}.$$

We will show that the above sequence of deviations will deter the initial deviation by the buyer. We know that when a supplier is part of a coalition that does not contain the buyer, he makes a profit of zero. This together with (a) and (b) shows that $\mathscr{X}_0\prec_{S_1}\mathscr{X}$, $\mathscr{X}_1\prec_{S_2}\mathscr{X}$, ..., and $\mathscr{X}_m\prec_{S_{m+1}}\mathscr{X}$. From Definition 3, $\mathscr{X}_0\ll\mathscr{X}$. Since $\mathscr{X}\not\prec_{S_0}\mathscr{X}$, we have shown that the initial deviation by the buyer is deterred.

Case (2): Suppose the initially deviating suppliers are in the subset $J:=\{j_1,\cdots,j_k\}$, where $1\le k\le m$ and $J\subseteq M$. That is, there are k suppliers that deviate from the grand coalition. The set of non-deviating suppliers is denoted by $\bar{J}:=M\setminus J$ and we rename these suppliers such that $\bar{J}=\{\bar{j}_1,\cdots,\bar{j}_{m-k}\}$. Define the following coalitions: $S_0=\{j_1,\cdots,j_k\}$, $S_1=\{B\}$, $S_2=\{j_1\}$, $S_3=\{j_2\}$, ..., $S_{k+1}=\{j_k\}$, $S_{k+2}=\{\bar{j}_1\}$, $S_{k+3}=\{\bar{j}_2\}$, ..., $S_{m+1}=\{\bar{j}_{m-k}\}$, and $S_{m+2}=\{1,2,\cdots,m,B\}$. Further, define the following coalition structures: $\mathscr{X}_0=\{(j_1,j_2,\cdots,j_k);(\bar{j}_1,\cdots,\bar{j}_{m-k},B)\}$, $\mathscr{X}_1=\{(j_1,j_2,\cdots,j_k);(\bar{j}_1,\cdots,\bar{j}_{m-k});B\}$, $\mathscr{X}_2=\{j_1;(j_2,\cdots,j_k);(\bar{j}_1,\cdots,\bar{j}_{m-k});B\}$, ..., $\mathscr{X}_{k+1}=\{j_1;j_2;\cdots;j_k;(\bar{j}_1,\bar{j}_2,\cdots,\bar{j}_{m-k});B\}$, $\mathscr{X}_{k+2}=\{j_1;j_2;\cdots;j_k;\bar{j}_1;(\bar{j}_2,\cdots,\bar{j}_{m-k});B\}$, ..., $\mathscr{X}_m=\{j_1;j_2;\cdots;j_k;\bar{j}_1;\bar{j}_2;\cdots;(\bar{j}_{m-k-1},\bar{j}_{m-k});B\}$, and $\mathscr{X}_{m+1}=\{j_1;j_2;\cdots;j_k;\bar{j}_1;\bar{j}_2;\cdots;\bar{j}_{m-k};B\}$. Thus, if the deviations are organized in the orders of S_0, S_1, ..., S_{m+2} (one coalition at a time), then the sequence of resulting coalitions is given by $\mathscr{X}_0$, $\mathscr{X}_1$, ..., $\mathscr{X}_{m+1}$, and $\mathscr{X}$. That is, we have

$$\mathscr{X}\to_{S_0}\mathscr{X}_0\to_{S_1}\mathscr{X}_1\to_{S_2}\mathscr{X}_2\cdots\to_{S_{m+1}}\mathscr{X}_{m+1}\to_{S_{m+2}}\mathscr{X}.$$

Similar to Case (1), from (a) and (b) we obtain that $\mathscr{X}_0\prec_{S_1}\mathscr{X}$, $\mathscr{X}_1\prec_{S_2}\mathscr{X}$, ..., and $\mathscr{X}_{m+1}\prec_{S_{m+2}}\mathscr{X}$. From Definition 3, $\mathscr{X}_1\ll\mathscr{X}$. This together with $\mathscr{X}\not\prec_{S_0}\mathscr{X}$ shows that the initial deviation by any subset of suppliers will be deterred.

Case (3): The proof is similar to Case (2), and only differs in the construction of the sequence of deviations. Again we can show that the initial deviations by a coalition of any suppliers and the buyer will be deterred. $\square$

Proof of Corollary 4 Since the aggregate capacity is insufficient, even for the lowest possible demand, we obtain from Lemma 3 that the optimal demand allocation for supplier $i \in M$ is $x_i^*(T,d) = K_i$ for all $T \subseteq M$ and $d \in [\underline{d}, \bar{d}]$. Thus, the optimal profit when suppliers in T are present is given by $\Pi(T) = \mathbb{E}[(r-p_o)D] + \sum_{j\in T}(p_o - c_j)K_j$. The characteristic function of the cooperative game (N,v) has $v(T\cup\{B\}) = \mathbb{E}[(r-p_o)D] + \sum_{j\in T}(p_o-c_j)K_j$ and $v(T) = 0$, for all $\emptyset \subset T \subseteq M$. We also obtain that $v(\{B\}) = \mathbb{E}[(r-p_o)D]$.

Thus, for each supplier $i \in M$, we obtain: $v(S\cup\{i\}) - v(S)) = 0$ for all $S \subseteq N \setminus \{i\}$ and $B \notin S$; and $v(S\cup\{i\}) - v(S)) = (p_o - c_i)K_i$ for all $S \subseteq N\setminus\{i\}$ and $B \in S$. For the buyer, we obtain $v(S\cup\{B\}) - v(S)) = v(S\cup\{B\}) = \mathbb{E}[(r-p_o)D] + \sum_{j\in S}(p_o - c_j)K_j$ for all $S \subseteq M$. From the definition of the Shapley value, each supplier $i \in M$ is allocated a profit of

$$\begin{aligned}
\phi_i &= \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}\left(v(S\cup\{i\}) - v(S)\right)\\
&= \sum_{S\subseteq N\setminus\{i\}, B\in S} \frac{|S|!(|N|-|S|-1)!}{|N|!}(p_o - c_i)K_i\\
&= \sum_{S\subseteq N\setminus\{i\}, B\in S} \frac{|S|!(m+1-|S|-1)!}{(m+1)!}(p_o - c_i)K_i\\
&= \sum_{j=0}^{m-1} \frac{(m-1)!}{j!(m-1-j)!}\frac{(m-j)!j!}{(m+1)!}(p_o-c_i)K_i\\
&= \frac{1}{2}(p_o - c_i)K_i.
\end{aligned}$$

Similarly, we can show that the buyer makes a profit of

$$\begin{aligned}
\phi_B &= \sum_{S\subseteq N\setminus\{B\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}\left(v(S\cup\{B\}) - v(S)\right)\\
&= \sum_{\emptyset\subset S\subseteq M} \frac{|S|!(m-|S|)!}{(m+1)!}\left(v(S\cup\{B\}) - v(S)\right) + \frac{0!m!}{(m+1)!}v(\{B\})\\
&= \sum_{\emptyset\subset S\subseteq M} \frac{|S|!(m-|S|)!}{(m+1)!}\left(\mathbb{E}[(r-p_o)D] + \sum_{i\in S}(p_o-c_i)K_i\right) + \frac{1}{m+1}\mathbb{E}[(r-p_o)D]\\
&= \sum_{\emptyset\subset S\subseteq M} \frac{|S|!(m-|S|)!}{(m+1)!}\left(\sum_{i\in S}(p_o-c_i)K_i\right) + \mathbb{E}[(r-p_o)D]\\
&= \sum_{i=1}^{m}\left(\sum_{j=0}^{m-1} \frac{(m-1)!}{j!(m-1-j)!}\frac{(m-j)!j!}{(m+1)!}(p_o-c_i)K_i\right) + \mathbb{E}[(r-p_o)D]\\
&= \sum_{i=1}^{m}\left(\frac{1}{2}(p_o-c_i)K_i\right) + \mathbb{E}[(r-p_o)D].
\end{aligned}$$

□

Proof of Corollary 5 The proof follows directly from Corollary 4 and Corollary 3. □

Model Extensions

Nonlinear Costs

In some practical applications, the suppliers' costs may be nonlinear in quantity. To capture this possibility, we denote each supplier's cost function by $C_i(x)$, for $x \geq 0$. Since the key building block for our analysis in Sections 3 and 4 is the submodularity property of the optimal network profit, $\Pi(S)$, as shown in Theorem 1, in this section, we demonstrate that this property continues to hold with nonlinear costs. As in the baseline model, we first consider the optimal allocation problem for any subset of suppliers $S \subseteq M$ and any realized demand d:

$$\max_{\mathbf{x} \in \mathbb{R}_+^{|S|}} \left\{ rd - \sum_{i \in S} C_i(x_i) - \left(d - \sum_{i \in S} x_i \right) p_o \right\}, \tag{18}$$

where the maximization is subject to the set of constraints defined in (2) and (3). If $m_i(x) := -C_i(x) + p_o\, x$ is concave in x for all $i \in M$, then the above optimization problem is a convex program with a separable and concave objective function. Further, the constraints in (2) and (3) are linear. Thus, the KKT conditions are necessary and sufficient to determine the optimal allocation policy (Mas-Colell et al. 1995). Let λ_i be the Lagrangian multiplier for the ith constraint in (2), where $i \in S$, and let λ_0 be the Lagrangian multiplier for the constraint $\sum_{i \in S} x_i \leq d$. Then the optimal allocation is characterized by the following lemma:

Lemma 5. *There exists a unique optimal allocation $\mathbf{x}^*$, which together with the optimal multipliers λ_i^* and λ_0^*, must satisfy the following conditions:*

$$\begin{aligned} p_o - \lambda_i^* - \lambda_0^* - C_i'(x_i^*) \leq 0, \quad x_i^* \geq 0, \quad &\text{and } x_i^*(p_o - \lambda_i^* - \lambda_0^* - C_i'(x_i^*)) = 0, \quad \forall i \in S, \\ K_i - x_i^* \geq 0, \quad \lambda_i^* \geq 0, \quad &\text{and } \lambda_i^*(K_i - x_i^*) = 0, \quad \forall i \in S, \\ d - \sum_{i \in S} x_i^* \geq 0, \quad \lambda_0^* \geq 0, \quad &\text{and } \lambda_0^* \left(d - \sum_{i \in S} x_i^* \right) = 0. \end{aligned}$$

Proof of Lemma 5 Given $m_i(x)$ is concave in x for all $i \in M$, the objective function is a separable, concave function. This, together with the fact that all the constraints are linear, implies that the optimization problem is a convex program. Thus, there exists a unique global optimal solution for which the KKT conditions are necessary and sufficient (Mas-Colell et al. 1995).

Let λ_i be the Lagrangian multiplier for the ith constraint in (2), where $i \in S$, and let λ_0 be the Lagrangian multiplier for constraint $\sum_{i \in S} x_i \leq d$. The Lagrangian function $\mathcal{L}$ is written as follows:

$$\mathcal{L} = (r - p_o)d + \sum_{i \in S} (p_o x_i - C_i(x_i) + \lambda_i(K_i - x_i)) + \lambda_0 \left(d - \sum_{i \in S} x_i \right).$$

Taking the first derivatives of $\mathcal{L}$ with respect to each variable, we obtain:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_i} &= p_o - \lambda_i - \lambda_0 - C_i'(x_i), \quad \forall i \in S,\\
\frac{\partial \mathcal{L}}{\partial \lambda_i} &= K_i - x_i, \quad \forall i \in S,\\
\frac{\partial \mathcal{L}}{\partial \lambda_0} &= d - \sum_{i\in S} x_i.
\end{aligned}$$

Substituting the above equations into the KKT conditions below

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x_i} \le 0, \quad x_i \ge 0, \quad \text{and } x_i \frac{\partial \mathcal{L}}{\partial x_i} = 0, \quad \forall i \in S,\\
&\frac{\partial \mathcal{L}}{\partial \lambda_i} \ge 0, \quad \lambda_i \ge 0, \quad \text{and } \lambda_i \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0, \quad \forall i \in S,\\
&\frac{\partial \mathcal{L}}{\partial \lambda_0} \ge 0, \quad \lambda_0 \ge 0, \quad \text{and } \lambda_0 \frac{\partial \mathcal{L}}{\partial \lambda_0} = 0,
\end{aligned}$$

yields the results in the lemma. □

If $m_i(x)$ is concave in x, the objective function (18) is a separable concave function. Since the constraint set is a polymatroid, we can use Theorem 3 of He et al. (2012) to show that the optimal profit function is submodular, i.e., we can replicate the result presented in Theorem 1.

THEOREM 8. *Suppose $m_i(x)$ is concave in x for all $i \in M$. Then the optimal network profit, $\Pi(S)$, is submodular in S for $S \subseteq M$.*

The theorem indicates that, as long as the individual profit function for each supplier is concave, the submodularity result will hold.

Positive Capacity at the Buyer

We next consider a setting in which the buyer has her own capacity and demonstrate that the submodularity property of the optimal network profit continues to hold.

Let $K_B > 0$ denote the buyer's capacity and $c_B > 0$ denote the buyer's unit production cost. In the analysis, we consider the buyer's capacity as if it is owned by a nominal supplier who does not compete with the other suppliers. Consider a given subset $S \subseteq M$ of suppliers and define $L := S \cup \{B\}$. Given the realized demand, d, the allocation problem can be formulated as follows:

$$\max_{\mathbf{x}\in\mathbb{R}_+^{|L|}} \left\{ rd - \sum_{i\in L} c_i x_i - \left(d - \sum_{i\in L} x_i \right) p_o \right\}, \tag{19}$$

subject to:

$$x_i \le K_i, \quad \forall i \in L, \tag{20}$$

$$\sum_{i\in L} x_i \le d. \tag{21}$$

This problem differs from (1) in that the buyer may also produce up to her capacity, K_B. Let $\tilde{\Pi}(L)$ be the optimal network profit in this setting, where we use the tilde to differentiate from the optimal network profit for the case in which the buyer has no capacity. Following the approach in Theorem 1, we can show that $\tilde{\Pi}(L)$ is submodular in L for $L \subseteq N$, which implies that

$$\tilde{\Pi}(L \cup \{i\}) - \tilde{\Pi}(L) \geq \tilde{\Pi}(G \cup \{i\}) - \tilde{\Pi}(G), \quad \forall L \subseteq G \subset N \text{ and } i \in M.$$

Since the subsequent analysis builds on this submodularity property, the main results for our baseline model carry over to this extension.

Although the main results are structurally unchanged, each supplier's marginal contribution is reduced when the buyer has positive capacity. To illustrate this, we note for $i \in M$ and $i \notin S \subset M$,

$$\begin{aligned}\tilde{\Pi}(S \cup \{B, i\}) - \tilde{\Pi}(S \cup \{B\}) &\leq \tilde{\Pi}(S \cup \{i\}) - \tilde{\Pi}(S) \\ &= \Pi(S \cup \{i\}) - \Pi(S),\end{aligned}$$

where the inequality follows from the submodularity of $\tilde{\Pi}(\cdot)$, and the equality follows from the definitions of $\tilde{\Pi}(\cdot)$ and $\Pi(\cdot)$. This demonstrates that if the buyer has her own capacity, she can use this capacity to meet some of the demand, leaving less demand for the suppliers to fill, thus intensifying the competition between the suppliers. As a result, each supplier earns less profit under the fixed fee mechanism.

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