Author/s:
Nesic, D;Loria, A;Panteley, E;Teel, AR

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Persistent Link:

On stability of sets for sampled-data nonlinear inclusions via their approximate discrete-time models

Dragan Nešić   Antonio Loría   Elena Panteley   Andrew R. Teel

***Abstract*— We generalize previous results on stability of sampled-data systems based on the approximate discrete-time models: we consider stabilization of arbitrary closed sets (not necessarily compact), plants described as sampled-data differential inclusions and arbitrary dynamic controllers in the form of difference inclusions. Our result does not require the knowledge of a Lyapunov function for the approximate model, which is a standing assumption in previous papers. We present checkable conditions that one can use to conclude semi-global asymptotic (SPA) stability, or global exponential stability (GES), of the sampled-data system via appropriate properties of its approximate discrete-time model. Thus, we provide a framework for stabilization of arbitrary closed sets for sampled-data nonlinear differential inclusions via their approximate discrete-time models.**

I. INTRODUCTION

Although most controllers are nowadays implemented digitally using sample and hold devices, sampled data nonlinear control has received much less attention than continuous time nonlinear control. The controller design problem for sampled-data systems can be carried out in essentially three different ways: (i) emulation; (ii) discrete-time design; (iii) sampled-data design. For nonlinear systems, some results on emulation can be found in [9], while we are not aware of any results on sampled data design for nonlinear systems (details on the sampled data method for linear systems can be found in [3], see also [2], [6]).

For nonlinear plants, the discrete-time design is frustrated by the fact that it is typically not possible to analytically find the *exact* discrete-time model of the plant and in such situation an *approximate* discrete-time model is the only alternative to use for controller design. However, it was shown in [11], [12] that there are situations where a controller stabilizes an approximate discrete-time plant model for all small sampling periods but at the same *destabilizes* the exact discrete-time plant model for all small sampling periods – stability of sampled-data systems under mild conditions was addressed in [10]. This has motivated the development of a *prescriptive* framework for control design; in particular, it has been established which sufficient conditions the controller and the approximate model need to satisfy for the design to be successful –*cf.* [11], [12]. Particular controller design techniques within this framework have been developed for nonholonomic systems [7] and port controlled Hamiltonian systems [8] model predictive control [5], to mention a few.

In this paper we contribute novel results that are inscribed in the mentioned framework for stabilization via approximate discrete-time models. We extend previous results to the case of stability of arbitrary closed sets for difference inclusions. In particular, Theorem 1 is a generalization of [11, Theorem 1] in several directions: we consider semi-global practical (SPA) stability of arbitrary (not necessarily compact) sets, plants modeled as differential inclusions and arbitrary dynamic controllers modeled as difference inclusions. Motivation for considering such general stability properties, classes of plants and controllers is given in [12], [15]. Theorem 2 provides stronger conditions under which one can conclude global exponential stability (GES) for the exact discrete-time model and we are not aware of similar results even in the simpler setting of [11]. We emphasize that these results are different from the main results in [12] that assume existence of an appropriate Lyapunov function for the approximate model. Proofs in this paper are purely trajectory based and they do not need such Lyapunov functions. We also note that our results may be used for the (rare) cases when the exact discrete-time model of the plant is known. Our experience shows that these tools may be useful for controller design via approximate discrete-time models: see for instance [4] where we present summation-type characterizations of GAS and GES for difference inclusions; these results, together with the main theorem in this paper serve to conclude stability for sampled-data systems.

The paper is organized as follows. Section II contains mathematical preliminaries and the description of the mathematical set-up that we use. Our main results, which relate stability properties of sampled-data inclusions and stability properties of their approximate discrete time models, are presented in Section III. All proofs are presented in Section IV and Conclusions are given in the last section.

II. PRELIMINARIES

Sets of real and natural numbers are respectively denoted as $\mathbb{R}$ and $\mathbb{N}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, $\gamma(0) = 0$ and strictly increasing. γ is said to be of class $\mathcal{K}_\infty$, denoted as $\gamma \in \mathcal{K}_\infty$, if $\gamma \in \mathcal{K}$ and it is unbounded. Class $\mathcal{K}_\infty$ functions are globally invertible. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$, denoted as $\beta \in \mathcal{KL}$, if for each fixed $t \geq 0$ we have that $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \geq 0$ we have that

Dragan Nešić is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia. E-mail: d.nesic@ee.mu.oz.au. The work of D. Nešić was partially supported by the Australian Research Council under the Australian Professorial Fellow and Discovery Grant schemes and partly by EGIDE, France.

Antonio Loría and Elena Panteley are with CNRS–LSS, Supélec, 3, Rue Joliot Curie, 91192 Gif s/Yvette, France. E-mail: loria@lss.supelec.fr, lamnabhi@lss.supelec.fr. A. R. Teel is with the Dept. of Electrical Engg., of the University of California, Santa Barbara, CA, USA. E-mail: teel@ece.ucsb.edu.

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$\lim_{t\to\infty}\beta(s,t)=0$. For arbitrary positive L,T we define:

$$\ell_{L,T} := \left\lfloor \frac{L}{T} \right\rfloor ,$$

where for arbitrary $x\in\mathbb{R}$ we have that $\lfloor x\rfloor := \max\{z\in\mathbb{N} : z\le x\}$. Given a closed (not necessarily compact) set $\mathcal{A}\subset\mathbb{R}^n$, we denote the distance of a point $x\in\mathbb{R}^n$ to the set as:

$$|x|_{\mathcal{A}} := \inf_{z\in\mathcal{A}} |z-x| .$$

We often use the well known fact that $|\cdot|_{\mathcal{A}}$ is globally Lipschitz with the Lipschitz constant equal to one, that is for all $x,y\in\mathbb{R}^n$ we have:

$$||x|_{\mathcal{A}} - |y|_{\mathcal{A}}| \le |x-y| .$$

We consider nonlinear control systems of the form

$$\dot{x}_p \in F(x_p,u) \tag{1}$$

where $x_p\in\mathbb{R}^{n_p}$ and the set-valued map $F(\cdot,u)$ is assumed to have enough regularity to guarantee existence (but not necessarily uniqueness) of solutions:

Assumption 1 For each $u\in\mathbb{R}^m$, the set-valued map $F(\cdot,u)$ satisfies the following basic conditions: 1) it is upper semi-continuous, i.e., for each $x_p\in\mathbb{R}^{n_p}$ and each $\varepsilon>0$ there exists $\delta>0$ such that, for all $\xi\in\mathbb{R}^{n_p}$ satisfying $|\xi-x_p|\le\delta$ we have $F(\xi,u)\subseteq F(x_p,u)+\varepsilon\overline{\mathcal{B}}_{n_p}$, where $\overline{\mathcal{B}}_{n_p}$ denotes the closed unit ball in $\mathbb{R}^{n_p}$, 2) for each $x_p\in\mathbb{R}^{n_p}$ the set $F(x_p,u)$ is nonempty, compact and convex.

This assumption guarantees that for each fixed u there exists at least one solution to (1) (see [1]). We will use $\mathcal{S}(x_p,u)$ to denote the set of solutions to (1) starting at x_p with constant input u. For a given $t>0$ and $(x_p,u)\in\mathbb{R}^{n_p}\times\mathbb{R}^m$ we use the following notation $F^e_t(x_p,u) := \{\xi\in\mathbb{R}^n : \xi=\phi(t,x_p,u),\ \phi\in\mathcal{S}(x_p,u)\}$.

The exact discrete-time model of the sampled-data system is given by:

$$x_p^+ \in F^e_T(x_p,u) , \tag{2}$$

where $F^e_T(x_p,u)$ is the set of values the solutions to (1) can take at time T when starting at x_p and with the constant input u applied. The parameter $T>0$ represents the sampling period. We will consider the case where the sampling period T can be adjusted to arbitrarily small values. Hence, (2) represents a family of systems. We note that since F in (1) is in general nonlinear, it is not possible to analytically determine F^e_T in (2). Instead, we assume that the family of approximate discrete-time models

$$x_p^+ \in F^a_T(x_p,u) , \tag{3}$$

which approximates the exact discrete-time model (2), is used in the control design. In particular, we assume that a family of, possibly discontinuous, discrete-time controllers

$$x_c^+ \in G_T(x_c,x_p); \qquad u \in H_T(x_c,x_p) , \tag{4}$$

where $x_c\in\mathbb{R}^{n_c}$, has been designed to (approximately) asymptotically stabilize a nonempty closed set $\mathcal{A}\subset\mathbb{R}^n$, where $n:=n_c+n_p$, for the family (3). Our object of study is the stability of the system (2), (4) with respect to a nonempty closed set $\mathcal{A}\subset\mathbb{R}^n$, based on the stability of the system (3), (4). To shorten notation, we introduce $x=(x_p^T\ x_c^T)^T$, $\mathcal{H}_{\mathcal{A}}(\delta,\Delta) := \{x\in\mathbb{R}^n : \delta\le|x|_{\mathcal{A}}\le\Delta\}$ and

$$\mathcal{F}^a_T(x) := \begin{pmatrix} F^a_T(x_p,H_T(x_c,x_p)) \\ G_T(x_c,x_p) \end{pmatrix} ,$$

$$\mathcal{F}^e_T(x) := \begin{pmatrix} F^e_T(x_p,H_T(x_c,x_p)) \\ G_T(x_c,x_p) \end{pmatrix} .$$

Then, we write

$$x^+ \in \mathcal{F}^\star_T(x) \tag{5}$$

and denote as $\mathcal{S}^\star_T(x_\circ)$ the set of all solutions $\phi^\star_T(k,x_\circ)$ initialized at $x_\circ$. The symbol $\star=e$ is used for the exact closed loop (2), (4) and $\star=a$ for the approximate closed loop (3), (4).

Remark 1 In general, it is possible to consider more complex classes of approximate discrete-time models of the form $x^+\in F^a_{T,h}(x,u)$, where T is the sampling period and h is a modeling parameter that can be used to reduce the mismatch between the approximate and exact models (usually, it is an integration period of the numerical integration scheme). The case when $T\ne h$ is useful in situations when the structure of the underlying approximate model is not exploited in controller design, such as in model predictive control. When $T=h$, then we write $x^+\in F^a_{T,T}(x,u) =: F^a_T(x,u)$ and such situations typically lead to approximate models with simpler structure that are amenable to constructive nonlinear control techniques. For more details on this general case, see [12].

In the sequel, we need the following definition:

Definition 1 [Forward completeness] *Consider the family of systems (5), where $\star\in\{a,e\}$. The family of systems (5) is said to be forward complete if there exist strictly positive numbers T^*,c and $\sigma_1,\sigma_2\in\mathcal{K}_\infty$ such that for all $T\in(0,T^*)$ and $x_\circ\in\mathbb{R}^n$ we have that all solutions $\phi^\star_T\in\mathcal{S}^\star_T(x_\circ)$ of the family (5) satisfy:*

$$|\phi_T(k,x_\circ)| \le \sigma_1(|x_\circ|)+\sigma_2(kT)+c \qquad \forall k\ge 0 . \tag{6}$$

Remark 2 The definition of forward completeness given above was first used in [13] to treat stability of time-varying discrete-time parameterized cascaded systems. Lyapunov like sufficient conditions that guarantee forward completeness in the sense of Definition 1 can be found in [13]. This property impedes (1), with constant us, to have finite-escape times.

III. STABILIZATION VIA APPROXIMATE DISCRETE-TIME MODELS

Consider the following: *if there exists a (not necessarily compact) set $\mathcal{A}$ such that the system (3), (4) is asymptotically/exponentially stable with respect to $\mathcal{A}$ for all small T, then under which conditions is the family of exact discrete-time models (2), (4) also (approximately) asymptotically/exponentially stable with respect to the set $\mathcal{A}$ for sufficiently small values T?*

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The above question was answered in [12] for the same set-up as in this paper but with the assumption that an appropriate family of strict Lyapunov functions can be constructed for the family (3), (4). Constructing such families of Lyapunov functions is in general hard and the question arises whether one can answer the above question without knowledge of appropriate Lyapunov functions for (3), (4). We present several such non-Lyapunov based results in this section.

A. SPA stability via approximate discrete-time models

In order to state the main result of this subsection we first need to define an appropriate stability property and a consistency property that quantifies the mismatch between the approximate and exact closed loop systems.

Definition 2 [SPA stability] *Consider the family of systems (5), where $\star \in \{a, e\}$. Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ be given. The family of systems (5) is said to be (β,$\mathcal{A}$)-semi-globally practically asymptotically (SPA) stable if the system is forward complete and there exists $\beta \in \mathcal{KL}$ such that for any pair of strictly positive numbers (Δ, ν) there exists $T^* > 0$ such that for all $T \in (0, T^*)$, all $x_\circ \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$ and all solutions $\phi^\star_T(\cdot, x_\circ)$ of the family (5) we have:*

$$|\phi^\star_T(k, x_\circ)|_{\mathcal{A}} \le \beta(|x_\circ|_{\mathcal{A}}, kT) + \nu, \qquad \forall k \in \mathbb{N}\,. \tag{7}$$

Moreover, if the system is forward complete and there exists $T^* > 0$ such that for all $T \in (0, T^*)$ we have that (7) holds for all $x_\circ \in \mathbb{R}^n$ and with $\nu = 0$, then we say that the system (5) is (β, $\mathcal{A}$)-globally asymptotically stable.

Definition 3 [Multi-step upper consistency] *The family $\mathcal{F}^a_T$ is said to be $\mathcal{A}$-multi-step upper semi-consistent with $\mathcal{F}^e_T$ if, for each triple of strictly positive real numbers (L, η, Δ) there exist a function $\alpha : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0} \cup \{\infty\}$ and $T^* > 0$ such that, for all $T \in (0, T^*)$ we have*

$$\{x, y \in \mathcal{H}_{\mathcal{A}}(0, \Delta)\,,\ |x - y| \le \delta\} \qquad \Longrightarrow$$
$$\mathcal{F}^e_T(x) \subseteq \mathcal{F}^a_T(y) + \alpha(\delta, T)\overline{\mathcal{B}}_n$$

and

$$k \in [0, \ell_{L,T}] \qquad \Longrightarrow$$
$$\alpha^k(0, T) := \overbrace{\alpha\left(\cdots \alpha\left(\alpha\right.\right.}^{k}\left.\left.(0, T)\,, T\right) \cdots, T\right) \le \eta\,. \tag{8}$$

Remark 3 We present sufficient conditions for multi-step upper semi-consistency in Subsection III-C. We emphasize that this property can be checked without knowing the exact discrete-time model. The notion of consistency is adapted from numerical analysis literature [14], [16] and was already used in [11], [12].

With these definitions, we can state the main result of this subsection:

Theorem 1 *Let $\beta \in \mathcal{KL}$ and let a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ be given. If the following holds:*

1) *$\mathcal{F}^a_T$ us multi-step upper semi-consistent with $\mathcal{F}^e_T$;*
2) *The approximate closed loop system (3),(4) is $(\beta, \mathcal{A})$-SPA stable (or $(\beta, \mathcal{A})$-GAS)*

then, the family of exact closed loop systems (4), (2) is $(\beta, \mathcal{A})$-SPA stable.

Remark 4 Theorem 1 presents stability conditions that can be verified without knowledge of the exact discrete-time model. Indeed, we already noted that the consistency property can be checked without knowing the exact discrete-time model of the system (item 1). Hence, we only need to verify an appropriate stability of the approximate model (item 2) to conclude a corresponding stability property of the exact discrete-time system. Note that stability for the exact closed loop can be guaranteed only for sufficiently small sampling periods T.

Remark 5 Several examples in [11] and [12] illustrate that if the item 1 in Theorem 1 does not hold while the item 2 holds, it may happen that the exact discrete-time model can not be stabilized by sufficiently reducing T. Also, it is trivial to see that we do need the item 2 to state the result. Hence, while our conditions are only sufficient, they are tight since if one of them does not hold there are examples for which the conclusion does not hold.

Remark 6 The condition of multi-step upper consistency may be removed if one has a Lyapunov function to conclude GAS. This has been done in [12] where Lyapunov conditions that can be used to verify SPA stability (or GAS) of arbitrary sets for parameterized inclusions of the form (5) are presented. These results constitute a toolbox for controller design for sampled-data nonlinear systems via their approximate discrete-time models.

Remark 7 Theorem 1 generalizes Theorem [11, Theorem 1] in several different directions: it covers differential inclusions, it is given for stability with respect to arbitrary sets and the controllers are allowed to be dynamic. We note that Theorem 1 differs from results presented in [12] because we do not use a family of Lyapunov functions for the approximate model to state the result. In particular, results in [12] generalize [11, Theorem 2] whereas Theorem 1 generalizes [11, Theorem 1].

B. GES via approximate discrete-time models

In some cases, it is possible to establish stronger global exponential stability for the family of approximate models and it is natural to look for appropriate consistency conditions that will guarantee that the family of exact closed loops will also be globally exponentially stable. We summarize such a result (Theorem 2) in this section. We use the following definitions:

Definition 4 [GES stability] *Consider the family of systems (5), where $\star \in \{a, e\}$. Let a nonempty closed set $\mathcal{A} \subset \mathbb{R}^n$ be given. The family of systems (5) is said to be (K,λ,$\mathcal{A}$)-globally exponentially stable (GES) if the system is forward complete and there exist positive numbers K, λ and T^* such that for all $T \in (0, T^*)$, all $x_\circ \in \mathbb{R}^n$ and all solutions $\phi^\star_T(\cdot, x_\circ)$ of the family (5) we have:*

$$|\phi^\star_T(k, x_\circ)|_{\mathcal{A}} \le K \exp(-\lambda kT)\,|x_\circ|_{\mathcal{A}}\,, \qquad \forall k \in \mathbb{N}\,. \tag{9}$$

Definition 5 [Linear gain multi-step upper consistency] *The family $\mathcal{F}^a_T$ is said to be linear gain $\mathcal{A}$-multi-step upper* *semi-consistent with $\mathcal{F}^e_T$ if for each pair of positive numbers (L, η) there exists $T^* > 0$ and a function $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for all $T \in (0, T^*)$ and all $\Delta > 0$ we have*

$$\{x, \tilde{y} \in \mathcal{H}_{\mathcal{A}}(0, \Delta)\ ,\ |x - \tilde{y}| \leq \delta\} \qquad \Longrightarrow$$

$$\mathcal{F}^e_T(x) \subseteq \mathcal{F}^a_T(\tilde{y}) + \alpha(\delta, T, \Delta)\overline{\mathcal{B}}_n \tag{10}$$

and

$$kT \leq L \qquad \Longrightarrow$$

$$\alpha^k(0,T,\Delta) := \overbrace{\alpha\,(\cdots\alpha\,(\alpha}^{k}\,(0,T,\Delta)\,,T,\Delta)\cdots,T,\Delta) \leq \eta\Delta\,. \tag{11}$$

Theorem 2 *Let positive K, λ and a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ be given. If the following holds:*

1) *$\mathcal{F}^a_T$ is linear-gain multi-step upper semi-consistent with $\mathcal{F}^e_T$;*
2) *The approximate closed loop system (3),(4) is $(K, \lambda, \mathcal{A})$-GES*

then, there exist positive K_1, λ_1 such that family of exact closed loop systems (4), (2) is $(K_1, \lambda_1, \mathcal{A})$-GES.

Remark 8 Theorem 2 can be used to conclude stronger stability property (GES) of the exact discrete-time model if the approximate discrete-time model is GES and a stronger linear gain multi-step upper semi-consistency holds.

Remark 9 We are not aware whether Theorem 2 has been proved even in the case of sampled-data differential equations, static state feedback controllers and stability of the origin.

C. Sufficient conditions for consistency

In this subsection we present several different conditions to guarantee the consistency properties that we used in Theorems 1 and 2. We emphasize that all these conditions can be checked without knowing the exact discrete-time model of the system. First, we present sufficient conditions for the consistency property needed in Theorem 1.

Proposition 1 *If, for each $\Delta > 0$, there exist $K > 0$, $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that for all $T \in (0, T^*)$ and all $x, y \in \mathcal{H}_{\mathcal{A}}(0, \Delta)$ we have*

$$\mathcal{F}^e_T(x) \subseteq \mathcal{F}^a_T(y) + [(1 + KT)\,|x - y| + T\rho(T)]\overline{\mathcal{B}}_n \tag{12}$$

then $\mathcal{F}^a_T$ is multi-step upper semi-consistent with $\mathcal{F}^e_T$.

Proof of Proposition 1: Let (L, η, Δ) be given. From the assumption of the lemma, let Δ generate $K > 0$, $\rho \in \mathcal{K}_\infty$ and $T^*_1 > 0$. Define

$$\alpha(\delta, T) := (1 + KT)\delta + T\rho(T);$$

$$T^* := \min\left\{T^*_1, \rho^{-1}\left(\frac{\eta K}{\exp(KL) - 1}\right)\right\}\ .$$

With these definitions, the condition (3) is satisfied. Also note that for all k such that $kT \leq L$ we have:

$$\begin{aligned}
\alpha^k(0, T) &= T\rho(T)\sum_{j=0}^{k-1}(1 + KT)^j \\
&= \frac{\rho(T)}{K}\left[(1 + KT)^k - 1\right] \\
&\leq \frac{\rho(T)}{K}\left[\exp(KTk) - 1\right] \\
&\leq \frac{\rho(T)}{K}\left[\exp(KL) - 1\right]
\end{aligned}$$

and so (8) is satisfied.

Remark 10 It should be noted that one can state sufficient conditions for multi-step upper semi-consistency in terms of another (one step) consistency condition that characterizes the mismatch between the open loop exact $F^e_T(x, u)$ and approximate $F^a_T(x, u)$ plant models, a Lipschitz property on F^a_T and uniform boundedness of the control law (4). Such conditions can be found in [11] and [12].

Next, we present sufficient conditions for the consistency property used in Theorem 2.

Proposition 2 *If there exist positive numbers K and T^* and $\rho \in \mathcal{K}$ such that, for all $T \in (0, T^*)$ and all $x, \tilde{y} \in \mathbb{R}^n$ we have*

$$\begin{aligned}
\mathcal{F}^e_T(x)\ \subseteq\ \mathcal{F}^a_T(\tilde{y}) + [&(1 + KT)|x - \tilde{y}| \\
&+ T\rho(T)\max\{|x|_{\mathcal{A}}, |\tilde{y}|_{\mathcal{A}}\}]\overline{\mathcal{B}}_n\,.
\end{aligned}$$

Then, $\mathcal{F}^a_T$ is linear-gain multi-step upper semi-consistent with $\mathcal{F}^e_T$.

Proof of Proposition 2: Let (L, η) be given. Let $K > 0$, $\rho \in \mathcal{K}_\infty$ and $T^*_1 > 0$ come from the conditions in the lemma. Define

$$\alpha(\delta, T, \Delta) := (1 + KT)\delta + T\rho(T)\Delta;$$

$$T^* := \min\left\{T^*_1, \rho^{-1}\left(\frac{\eta K}{\exp(KL) - 1}\right)\right\}\ .$$

With these definitions, the condition (3) is satisfied. Also note that for all k such that $kT \leq L$ we have:

$$\begin{aligned}
\alpha^k(0, T, \Delta) &= T\rho(T)\Delta\sum_{j=0}^{k-1}(1 + KT)^j \\
&= \frac{\rho(T)\Delta}{K}\left[(1 + KT)^k - 1\right] \\
&\leq \frac{\rho(T)\Delta}{K}\left[\exp(KTk) - 1\right] \\
&\leq \frac{\rho(T)\Delta}{K}\left[\exp(KL) - 1\right] \leq \eta\Delta
\end{aligned}$$

and so (11) is satisfied.

IV. Proofs of main results

Lemma 1 *If $\mathcal{F}^a_T$ is multi-step consistent with $\mathcal{F}^e_T$, then for each strictly positive triple (L, η, Δ) there exists $T^* > 0$*

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such that, if for all $T \in (0, T^*)$ all solutions of the approximate closed loop (3),(4) satisfy

$$\phi_T^a(k,\xi) \in \mathcal{H}_{\mathcal{A}}(0,\Delta) \quad \forall k \in [0, \ell_{L,T}],$$

then for any solution of the exact closed loop (4), (2), there exists a solution of the approximate closed loop (4), (3) such that

$$|\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \le \eta \qquad \forall k \in [0, \ell_{L,T}] \ .$$

Proof of Lemma 1: Let (L, η, Δ) be given. Define $\Delta_1 := \Delta + \eta$. Since $\mathcal{F}_T^a$ is multi-step upper semi-consistent with $\mathcal{F}_T^e$, there exist a function $\alpha(\cdot,\cdot)$ and a strictly positive real number T^* such that (3) and (8) are satisfied for the triple (L, η, Δ_1). We now prove the result by induction. First we have $|\phi_T^e(0,\xi) - \phi_T^a(0,\xi)| = |\xi - \xi| = 0 \le \alpha^1(0,T) \le \eta$. Next, suppose that for every $\phi_T^e(k,\xi)$ there exists $\phi_T^a(k,\xi)$ such that $|\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \le \alpha^k(0,T) \le \eta$ and $k+1 \in [0, \ell_{L,T}]$. Since $\phi_T^a(k,\xi) \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$, it follows from the definition of Δ_1 that all solutions of exact and approximate closed loops satisfy $\phi_T^a(k,\xi), \phi_T^e(k,\xi) \in \mathcal{H}_{\mathcal{A}}(0,\Delta_1)$. It then follows from (3) that for any solution $\phi_T^e(k+1,\xi)$ there exists $\phi_T^a(k+1,\xi)$ such that $|\phi_T^e(k+1,\xi) - \phi_T^a(k+1,\xi)| \le \alpha(\alpha^k(0,T),T) = \alpha^{k+1}(0,T)$. Since $k+1 \in [0, \ell_{L,T}]$ it follows from (8) that $\alpha^{k+1}(0,T) \le \eta$.

Proof of Theorem 1: Let (Δ, ν) be given. Let β come from the item 2 of the theorem. Let $\eta > 0$ and $\epsilon \in (0,1)$ be such that[2]:

$$\begin{aligned} \beta(2\eta + \epsilon\nu, 0) + 2\eta + \epsilon\nu &\le \nu & (13)\\ 2\eta + \epsilon\nu &\le \Delta\ . & (14)\end{aligned}$$

Let $L > 1$ be such that

$$\beta(\Delta, t) \le \eta \qquad \forall t \ge L - 1\ .$$

Let

$$\Delta_1 := \beta(\Delta, 0)\ .$$

Let (L, η, Δ_1) generate $T_1^* > 0$ via the item 1 of the the theorem and let $(\Delta, \epsilon\nu)$ generate $T_2^* > 0$ via the item 2. Let

$$T^* := \min\{T_1^*, T_2^*, 1\} \tag{15}$$

and $T \in (0, T^*)$. From the item 2 and the choice of T, we have that:

$$\xi \in \mathcal{H}_{\mathcal{A}}(0,\Delta) \implies \phi_T^a(k,\xi) \in \mathcal{H}_{\mathcal{A}}(0,\Delta_1) \qquad \forall k \in \mathbb{N}\ .$$

Using the item 1 of the theorem and Lemma 1 we have that for all $\xi \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$, $T \in (0,T^*)$ and any solution $\phi_T^e(k,\xi)$ there exists a solution $\phi_T^a(k,\xi)$ such that for all k with $k \in [0, \ell_{L,T}]$ we have:

$$|\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \le \eta\ . \tag{16}$$

Thus, for any such $\phi_T^e(k,\xi)$ there exists $\phi_T^a(k,\xi)$ such that

$$\begin{aligned} |\phi_T^e(k,\xi)|_{\mathcal{A}} &\le |\phi_T^a(k,\xi)|_{\mathcal{A}} + |\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \\ &\le \beta\left(|\xi|_{\mathcal{A}}, kT\right) + \epsilon\nu + \eta \quad \forall k \in [0, \ell_{L,T}]\ .\end{aligned}$$

[2]Since $\beta(s,0) \in \mathcal{K}$, it is always possible to find such numbers.

Since (13) implies that $\epsilon\nu + \eta < \nu$, we have that the desired bound (7) holds for all k such that $k \in [0, \ell_{L,T}]$. Now we need to prove that the desired bound holds for all $k \ge 0$. Then, since $T < T^* \le 1$, we can write:

$$\ell_{L,T} T > L - T > L - 1\ .$$

Define $k_i := i \cdot \ell_{L_1,T}$. Thus, using the definition of L, η and ϵ, we get from (14) that for all $\xi \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$ we have:

$$|\phi_T^e(k_1,\xi)|_{\mathcal{A}} \le \beta(\Delta, L-1) + \epsilon\nu + \eta \le 2\eta + \epsilon\nu \le \Delta\ . \tag{17}$$

Now consider those k such that $k \in [k_1, k_2]$. We have, using time-invariance, (16), (17) and the fact that $\phi_T^e(k_1,\xi) \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$, we get that for each ϕ_T^e there exists ϕ_T^a such that

$$\begin{aligned} |\phi_T^e(k,\xi)|_{\mathcal{A}} &= |\phi_T^e(k-k_1, \phi_T^e(k_1,\xi))|_{\mathcal{A}} \\ &\le |\phi_T^a(k-k_1, \phi_T^e(k_1,\xi))|_{\mathcal{A}} + \\ &\quad |\phi_T^e(k-k_1, \phi_T^e(k_1,\xi)) - \phi_T^a(k-k_1, \phi_T^e(k_1,\xi))| \\ &\le \beta(2\eta + \epsilon\nu, (k-k_1)T) + \epsilon\nu + \eta \end{aligned}$$

from which it follows (using (13)) that for all $k \in [k_1, k_2]$,

$$|\phi_T^e(k,\xi)|_{\mathcal{A}} \le \beta(2\eta + \epsilon\nu, 0) + \epsilon\nu + \eta < \nu \tag{18}$$

and, using the definition of k_i and (14) we have that

$$|\phi_T^e(k_2,\xi)|_{\mathcal{A}} \le \beta(2\eta + \epsilon\nu, L-1) + \epsilon\nu + \eta \le 2\eta + \epsilon\nu \le \Delta\ .$$

The result then follows by induction.

In terms of trajectory error over "continuous-time" intervals with length of order one, multi-step consistency gives the following:

Lemma 2 *Suppose that $\mathcal{F}_T^a$ is linear-gain multi-step consistent with $\mathcal{F}_T^e$ and there exist positive T_1^*, B such that that for each $L > 0$ and for all $T \in (0, T_1^*)$ all solutions of the approximate closed loop satisfy*

$$|\phi_T^a(k,\xi)|_{\mathcal{A}} \le B \cdot |\xi|_{\mathcal{A}} \quad \forall k \in [0, \ell_{L,T}]. \tag{19}$$

Then, for each strictly positive pair (L, η) there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and for any solution of the exact closed loop (4), (2), there exists a solution of the approximate closed loop (4), (3) such that

$$|\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \le \eta \cdot |\xi|_{\mathcal{A}} \qquad \forall k \in [0, \ell_{L,T}]\ .$$

Proof of Lemma 2: Let (L, η) be given. Let L generate $B > 0$ and $T_1^* > 0$ using the conditions of the lemma. Let $B_1 := B + 1$ and $\eta_1 := \min\{\frac{\eta}{B_1}, 1\}$. Let (L, η_1) generate $T_2^* > 0$ and $\alpha(\cdot,\cdot,\cdot)$ via the linear multi-step upper semi-consistency. Let $T^* := \min\{T_1^*, T_2^*\}$ and $T \in (0, T^*)$ and define $\Delta := B_1 |\xi|_{\mathcal{A}}$. The proof is completed by induction. First we have $|\phi_T^e(0,\xi) - \phi_T^a(0,\xi)| = |\xi - \xi| = 0 \le \alpha^1(0,T,\Delta) \le \eta_1 \Delta = \eta_1 \cdot B_1 \cdot |\xi|_{\mathcal{A}} \le \eta |\xi|_{\mathcal{A}}$ (which follows from the definition of Δ and η_1). Next, suppose that for every $\phi_T^e(k,\xi)$ there exists $\phi_T^a(k,\xi)$ such that $|\phi_T^e(k,\xi) - \phi_T^a(k,\xi)| \le \alpha^k(0,T,\Delta) \le \eta_1 \cdot \Delta$ and $k+1 \in [0, \ell_{L,T}]$. Since $|\phi_T^a(k,\xi)| \le B\,|\xi|_{\mathcal{A}}$ by assumption, it follows from the definition of B_1 and $\eta_1 \le 1$ that all solutions of exact and approximate closed loops satisfy $\max\{|\phi_T^a(k,\xi)|_{\mathcal{A}}, |\phi_T^e(k,\xi)|_{\mathcal{A}}\} \le B_1\,|\xi|_{\mathcal{A}}$. It then follows from (3) that for any solution $\phi_T^e(k+1,\xi)$ there exists

$\phi_T^a(k+1,\xi)$ such that $|\phi_T^e(k+1,\xi) - \phi_T^a(k+1,\xi)| \le \alpha(\alpha^k(0,T,\Delta),T,\Delta) = \alpha^{k+1}(0,T,\Delta)$. Since $k+1 \in [0,\ell_{L,T}]$ it follows from (8) that $\alpha^{k+1}(0,T,\Delta) \le \eta_1\Delta = \eta_1 \cdot B_1|\xi|_{\mathcal{A}} \le \eta|\xi|_{\mathcal{A}}$.

Proof of Theorem 2: Let $c \in (0,1)$ and $\delta \in (0,c)$ be arbitrary. Let K, λ, T_1^* come from the item 2. Let $L := \frac{1}{\lambda}\ln\left(\frac{K}{c-\delta}\right)$. Define $L_1 := L+1$ and let L_1 and δ generate T_2^* via the item 1. Let $T^* := \min\{T_1^*, T_2^*, 1\}$ and let $T \in (0,T^*)$ be arbitrary. Note from the definitions and the fact that $T < T^* \le 1$, we have:

$$L = L_1 - 1 \le \ell_{L_1,T} \cdot T \le L_1 \; . \tag{20}$$

Define $k_i := i \cdot \ell_{L_1,T}$. From the item 1 (with Lemma 1), we can write that for every $x_\circ$ and every $\phi_T^e \in \mathcal{S}_T^e(x_\circ)$ there exists $\phi_T^a \in \mathcal{S}_T^a(x_\circ)$ such that:

$$\begin{aligned} |\phi_T^e(k_{i+1},x_\circ)|_{\mathcal{A}} &= |\phi_T^e(k_{i+1}-k_i, \phi_T^e(k_i,x_\circ))|_{\mathcal{A}} \\ &\le |\phi_T^a(k_{i+1}-k_i, \phi_T^e(k_i,x_\circ))|_{\mathcal{A}} + \\ &\quad |\phi_T^e(k_{i+1}-k_i, \phi_T^e(k_i,x_\circ)) - \phi_T^a(k_{i+1}-k_i, \phi_T^e(k_i,x_\circ))| \\ &\le |\phi_T^a(k_{i+1}-k_i, \phi_T^e(k_i,x_\circ))|_{\mathcal{A}} + \delta\,|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \; . \end{aligned} \tag{21}$$

Using the item 2, (21), (20) and definitions of L_1 and L we can write:

$$\begin{aligned} |\phi_T^e(k_{i+1},x_\circ)|_{\mathcal{A}} &= K\mathrm{e}^{-\lambda(k_{i+1}-k_i)T}\,|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} + \\ &\quad \delta\,|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \\ &= \left[K\mathrm{e}^{-\lambda\ell_{L_1,T}T} + \delta\right]|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \\ &= \left[K\mathrm{e}^{-\lambda L} + \delta\right]|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \\ &\le \left[(c-\delta)+\delta\right]|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \\ &= c\,|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \; . \end{aligned} \tag{22}$$

From (22), we conclude that for all $x_\circ$ and all $\phi_T^e \in \mathcal{S}_T^e(x_\circ)$

$$|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \le c^i \cdot |x_\circ|_{\mathcal{A}} = \exp(-\tilde{\lambda}_1 i)\,|x_\circ|_{\mathcal{A}} \; , \tag{23}$$

for $\tilde{\lambda}_1 := \ln(\frac{1}{c}) > 0$. Using the definitions of k_i and $\ell_{L_1,T}$ and (20) we can write:

$$-\tilde{\lambda}_1 i = -\tilde{\lambda}_1 \frac{k_i}{\ell_{L_1,T}} \le -\frac{\tilde{\lambda}_1}{L_1} k_i T = -\lambda_1 k_i T \; ,$$

where $\lambda_1 := \frac{\tilde{\lambda}_1}{L_1}$, and using (23), we obtain:

$$|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \le \exp(-\lambda_1 k_i T)\,|x_\circ|_{\mathcal{A}} \qquad \forall i = 0,1,\ldots \tag{24}$$

Again, using items 1 with Lemma 1 and 2, we have for all $x_\circ$ and $\phi_T^e \in \mathcal{S}_T^e(x_\circ)$ that there exists $\phi_T^a \in \mathcal{S}_T^a(x_\circ)$ such that for all $k \in [k_i, k_{i+1}]$ we have:

$$\begin{aligned} |\phi_T^e(k,x_\circ)|_{\mathcal{A}} &= |\phi_T^e(k-k_i, \phi_T^e(k_i,x_\circ))|_{\mathcal{A}} \\ &\le (K+\delta)\,|\phi_T^e(k_i,x_\circ))|_{\mathcal{A}} \; . \end{aligned} \tag{25}$$

Finally, using (24) and (25), we obtain:

$$\begin{aligned} |\phi_T^e(k,x_\circ)|_{\mathcal{A}} &\le (K+\delta)\,|\phi_T^e(k_i,x_\circ)|_{\mathcal{A}} \\ &\le (K+\delta)\mathrm{e}^{-\lambda_1 k_i T}\,|x_\circ|_{\mathcal{A}} \\ &\le (K+\delta)\mathrm{e}^{\lambda_1 \ell_{L_1,T} T}\mathrm{e}^{-\lambda_1 k T}\,|x_\circ|_{\mathcal{A}} \\ &\le (K+\delta)\mathrm{e}^{\lambda_1 L_1}\mathrm{e}^{-\lambda_1 k T}\,|x_\circ|_{\mathcal{A}} \\ &=: K_1\mathrm{e}^{-\lambda_1 k T}\,|x_\circ|_{\mathcal{A}} \end{aligned}$$

which completes the proof.

V. Conclusion

We presented sufficient conditions for stability of nonlinear sampled-data differential inclusions; semiglobal practical asymptotic stability and global exponential stability of arbitrary closed sets for the exact discrete-time model of the sampled-data inclusion. These conditions may be verified without knowledge of the exact discrete-time model. While our theorem for GAS generalizes previous results we are not aware of a result for exponential stability in the simpler case of sampled-data differential equations, static controllers and stability of the origin. The trajectory-based proofs make our results suitable for further analysis, for instance of complex systems such as cascades.

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