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Input-Output Stability of Wireless Networked Control Systems[†]

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***Abstract*— This paper provides a general framework for analyzing the stability of general nonlinear networked control systems (NCS) with disturbances in the setting of $\mathcal{L}_p$ stability. With general, weak sufficient conditions on the NCS, we are able to conclude much tighter performance bounds than [1] and [2] for a very wide class of NCS. We also define the notion of persistently exciting (PE) scheduling protocols, discuss how PE leads to $\mathcal{L}_p$ stability for high enough transmission rates and show that PE is a natural property to demand, especially in the design of wireless scheduling protocols. Our results are of wide applicability as the analyses require neither the existence nor explicit construction of Lyapunov functions and the performance and stability bounds are completely parameterized by properties of the "network-free" system and a single NCS parameter chosen in the design of the scheduling protocol.**

I. INTRODUCTION

Although networked control systems (NCS) are a relatively new arrival to the world of automatic control, an increasing number of practical implementations and their respective traffic scheduling protocols now exist, particularly in the automotive and aeronautical industries that are seeing high adoption-rates of drive-by-wire and fly-by-wire designs. Standards-based component connectivity offers lower implementation costs, greater interoperability and a wide range of choices in developing control systems. The price paid for these advantages is the added complexity in the initial design and analysis of NCS.

The fundamental difference between NCS and traditional control systems is that numerous point-to-point connections between system nodes are replaced with a single shared serial bus. In the most general scenario, disparate actuators and sensors vie for exclusive access to the network at transmission instants presenting the immediate problem of how such access is arbitrated. The arbitration of network access amongst links, or *scheduling* is of fundamental importance to NCS design and analysis. A survey of scheduling and scheduling protocols is provided in [3] and stability and performance results of wireline NCS have been examined in [3], [4], [5], [6], [1] and [7]. Although round-robin (RR) scheduling is used almost exclusively in practice, the aforementioned works present various alternative protocols that demonstrate a performance gain over RR in simulations and, in special cases, demonstrate the superiority of the alternative protocols analytically.

The NCS design approach adopted in [6], [2], [4], [7], [1] and this paper consists of the following steps: (i) design of a stabilizing controller ignoring the network; (ii) and analysis of robustness of stability with respect to effects that scheduling within a network introduce. The principal advantage of this approach is its simplicity – the designer of the NCS can exploit familiar tools for controller design and select an appropriate scheduling protocol and transmission rate such that the desired properties of the network-free system are preserved.

Our analysis framework analyzes the input-output $\mathcal{L}_p$ stability (IOS) of NCS, the essence of which is that outputs (or state) of an NCS verify a robustness property with respect to exogenous disturbances. This notion of stability is the cornerstone of modern robust and optimal control (see [8], for instance) and a notion that we believe will be useful in the design of NCS. In fact, if the network-free system is $\mathcal{L}_p$ stable, we show that the NCS remains so with any scheduling protocol that always visits its links within every $T < \infty$ consecutive transmissions, whenever transmissions occur faster than once every τ^* secs, where[1]

$$\tau^* = \frac{\ln(z)}{|A|T}, \quad z \in [1,2] \tag{1}$$

and satisfies $T\gamma|A|z^{1+1/T} + (|A| - \gamma T)z - 2|A| = 0$. The parameter γ captures robustness properties of the network-free system and $|A|$ captures the speed at which the NCS departs from it nominal network-free behavior in the absence of exogenous disturbances. In particular, when $\gamma = 0$, $\tau^* = \ln(2)/(|A|T)$. Under additional technical conditions, $\mathcal{L}_p$ stability specializes to uniform global exponential stability (UGES) in the absence of exogenous disturbances.

The condition that a scheduling protocol visits all its links is particularly important in our analysis and, for reasons that will become apparent, we say that a scheduling protocol is uniformly persistently exciting in time T, or just PE_T, if *there is a fixed number $T < \infty$ such that all links of the NCS have transmitted within every T consecutive transmissions.* The technological limitations of communications channels,

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[1]When there is no ambiguity, $|\cdot|$ denotes both the 2-norm on $\mathbb{R}^n$ and the induced matrix 2-norm.

especially wireless channels, leave us with few guarantees about what information is available to the scheduler for making its scheduling decisions or it may be the case that the communication channel is using a collision protocol, in which case we have very few guarantees at all. Save for RR, the protocols described and analyzed in the literature (see [1], [3] and [9], for instance) require more information (NCS state information) than is typically available, particularly in a wireless setting. The results presented in [2] partially address the issue, concluding UGES for linear systems on wireless networks when using the protocols described therein. The results in this paper address the issue more precisely and in greater generality through PE and $\mathcal{L}_p$ stability.

PE in the sense we have described is verified by many network technologies[2] and many protocols described in the aforementioned NCS literature. Building on the informal definition of PE, the primary contributions of this paper are: **1.)** that we (formally) introduce the fundamental notion of PE scheduling protocols and demonstrate the importance of PE in stability analysis; **2.)** we generalize the NCS model presented in [1] to handle a wider class of scheduling protocols with a focus on PE and protocols implementable on wireless channels – in particular, RR, every protocol presented in [2] and the hybrid-TOD protocol that we introduce are all PE and can be analyzed with our framework; **3.)** under general conditions we show that every PE scheduling protocol leads to the $\mathcal{L}_p$ stability of general nonlinear NCS with disturbances whenever the network-free system is $\mathcal{L}_p$ stable; **4.)** and we provide significantly better analytic performance bounds than [1] for RR scheduling and significantly better analytic performance bounds for every protocol considered in [2].

The paper is divided into three additional sections: Section II presents our model for NCS and outlines the limitations inherent in the design of scheduling protocols. The notion of PE is also described in detail and characterized. Section III presents our main result and several remarks on its use in examples and Section IV presents a case-study with applications of our results demonstrating large improvements over existing results in the literature.

Preliminaries: Consider the system $\Sigma_z : \dot{z} = f(t, z, w)$ initialized at (t_0, z_0) with input w and a prescribed output $y = g(t, z)$. We say that Σ_z is $\mathcal{L}_p$ stable from w to y with gain γ if[3]

$$\exists\, K \geq 0 : \|y[t_0, t]\|_p \leq K|z_0| + \gamma\|w[t_0, t]\|_p.$$

The state z of Σ_z is said to be $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from output y with gain γ if

$$\exists\, K \geq 0 : \|z[t_0, t]\|_q \leq K|z_0| + \gamma\|y[t_0, t]\|_p + \gamma\|w[t_0, t]\|_p.$$

[2]Ethernet and 802.11 are examples of CSMA/CD protocols where it is known (see [10], for instance) that for a finite number of users (links), the expected waiting time for a link is finite. Although our results are cast in a deterministic setting, we mention collision protocols to motivate how natural the notion of PE is in the context of real networks and to suggest that similar results maybe pursued in a stochastic setting.

[3]Specifically, $\|y[t_0, t]\|_p := \left(\int_{t_0}^{t} |y(s)|^p ds\right)^{1/p}$ for $p \in [1, \infty)$ and $\|y[t_0, t]\|_\infty = \text{ess.}\sup\{|y(s)| : s \in [t_0, t]\}$.

An exposition of these ideas as they pertain to NCS can be found in[1, Section II].

II. WIRED AND WIRELESS NETWORKED CONTROL SYSTEMS

Computer networks and communications systems present rich and sophisticated models of varying degrees of complexity, within stochastic and deterministic settings, and of various underlying physical communication media. The network model presented in this paper aims to capture the essential aspects of control over networks in a deterministic setting. In that vein, we assume:

Assumption 1: All NCS links are connected via a shared, serial bus on which such links either have exclusive access to the network medium or no access at all. ■

Assumption 2: Transmitted frames are received error-free and network throughput is high enough to ignore delays due to transmission time.[4] ■

A. A Model for NCS

We consider general nonlinear NCS with disturbances conceptually depicted in Figure 1, where x_P and x_C are, respectively, states of the plant and controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of the most recently transmitted plant and controller output values via the network; e is the network-induced error defined as $e(t) := \begin{pmatrix} \hat{y}(t)-y(t) \\ \hat{y}(t)-y(t) \end{pmatrix}$. The key difference between this model and that presented in [1] is the addition of the *scheduler* and its corresponding decision-vector $\hat{e}$ that are represented by "dashed" elements in Figure 1. Special cases of $\hat{e}$-based scheduling were first considered in [2]. The model we introduce formalizes the $\hat{e}$ based scheduling that was considered in [2] and it generalizes the NCS models considered in [1]. Motivation for considering NCS in Figure 1 is given later in Assumption 3 and the surrounding discussion.

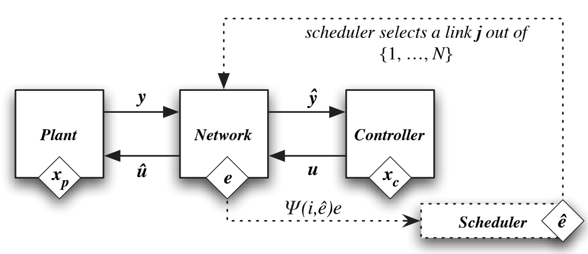

Fig. 1. Conceptual diagram of scheduled NCS.

We model the NCS in Figure 1 as a so-called jump-continuous (hybrid) system. Node data (controller and sensor values) are transmitted at transmission instants at times $\{t_0, t_1, \ldots, t_i\}, i \in \mathbb{N}$ satisfying $0 < \epsilon < t_{j+1} - t_j \leq \tau$

[4]This is especially true for wireline channels but some wireless channels may exhibit notable delays. As the focus of this paper is scheduling, we defer the treatment of delays and appeal to the robustness properties verified by the class of systems considered to assert that the results in this paper remain true for sufficiently small delays.

for all $j \geq 0$ where $\tau > 0$ and $\epsilon > 0$.[5] The constant τ is the *maximum allowable transmission interval* (MATI). The system is initialized at time t_s with $t_s \in [0, t_0], t_0 - t_s < \tau$.

Our NCS model is prescribed by the following dynamical and jump equations. In particular, for all $t \in [t_{i-1}, t_i]$:

$$\dot{x}_P = f_P(t, x_P, \hat{u}, w) \quad (2)$$
$$\dot{x}_C = f_C(t, x_C, \hat{y}, w) \quad (3)$$
$$u = g_C(t, x_C) \quad y = g_P(t, x_P) \quad (4)$$
$$\dot{\hat{y}} = 0 \quad \dot{\hat{u}} = 0 \quad \dot{\hat{e}} = 0, \quad (5)$$

and at each transmission instant[6] t_i,

$$e(t_i^+) = (I - \Psi(i, \hat{e}(t_i)))e(t_i) \quad (6)$$
$$\hat{e}(t_i^+) = \Lambda(i, \Psi(i, \hat{e}(t_i))e(t_i), \hat{e}(t_i)) \quad (7)$$

and we refer to Ψ as the scheduling function and Λ as the decision-update function. The effect of the protocol on the error is such that if the mth to nth nodes are scheduled[7] at transmission instant t_i the corresponding components of error, $e_n, \ldots, e_m$, experience a "jump". For transmission of nodes mth to nth nodes, we will always assume that $e_n(t_i^+), \ldots, e_m(t_i^+) = 0$ and, hence, $\Psi(i, \hat{e}(t_i))$ is a diagonal matrix consisting of entries $[a_{ij}]$, where $a_{ii} = 1$ for $n \leq i \leq m$ and 0 elsewhere. We group the nodes that are scheduled together into logical links, associating a partition of size N_i, denoted by $\mathbf{e}_i = (e_{i1}, e_{i2}, \ldots, e_{iN_i})$, of the error vector e such that we can write $e = (\mathbf{e}_1, \ldots, \mathbf{e}_N)$. We say that the NCS has N links and $\sum_{i=1}^{N} N_i$ nodes. Note that this is purely a notational convenience and simplifies the description of scheduling protocols and the NCS itself. We combine the controller and plant states into a vector $x = (x_P, x_C)$ and similarly to [1, pp. 1653], we can rewrite (2)-(7):

$$\dot{x} = f(t, x, e, w) \quad t \in [t_{i-1}, t_i] \quad (8)$$
$$\dot{e} = g(t, x, e, w) \quad t \in [t_{i-1}, t_i] \quad (9)$$
$$\dot{\hat{e}} = 0 \quad t \in [t_{i-1}, t_i] \quad (10)$$
$$e(t_i^+) = (I - \Psi(i, \hat{e}(t_i)))e(t_i) \quad (11)$$
$$\hat{e}(t_i^+) = \Lambda(i, \Psi(i, \hat{e}(t_i))e(t_i), \hat{e}(t_i)) \quad (12)$$

where $x \in \mathbb{R}^{n_x}, e \in \mathbb{R}^{n_e}, w \in \mathbb{R}^{n_w}, \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$. Note that implicit to our model is the following:

Assumption 3: The vector $e(t_i)$ is unavailable for scheduling. Moreover, only the transmitted value $\Phi(i, \hat{e}(t_i))e(t_i)$ is available for scheduling. ■

The above assumption is quite realistic in a range of wireline and, more typically, wireless NCS and was the main motivation for introducing $\hat{e}$ based scheduling in [2]. Moreover, Assumption 3 does not apply to any of the protocols considered in [1] that assume that $e(t_i)$ is available for scheduling, where (11), (12) are replaced by:

$$e(t_i^+) = (I - \Psi(i, e(t_i)))e(t_i) \ . \quad (13)$$

Example 2.1: The (maximum-error-first) try-once-discard (MEF-TOD or just TOD) protocol operates as follows [6]. At a transmission instant t_j, the scheduler transmits link i with largest maximum $|e_i(t_j)|$ so that the (instantaneous) error is minimized after the jump. TOD can be written in the form (13), where $\Psi(i, e) = \Psi(e) = diag\{d_j(e) I_{n_j}\}, j = 1, 2, \ldots, N$, I_{n_j} are identity matrices of appropriate dimensions and

$$d_j(\hat{e}) = \begin{cases} 1 & \text{if } j = \min\left(\arg\max_{1 \leq k \leq N} |\mathbf{e}_k|\right) \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Clearly, TOD does not satisfy our Assumption 3 since it must have access to the full error vector e. It is easy to see this is equivalent having access to, for example, every *remote* sensors' measurements *prior to transmission* which is quite unreasonable for most NCS.[8] ■

B. Persistently Exciting Scheduling Protocols

Motivated by the results in [1], we consider the following auxiliary discrete-time system induced by the protocol:

$$e^+ = (I - \Psi(i, \hat{e}))e \quad (15)$$
$$\hat{e}^+ = \Lambda(i, \Psi(i, \hat{e})e, \hat{e}) \quad (16)$$

Henceforth, we will "disconnect" the protocol from the NCS and refer to (15)-(16) as the protocol. It was shown in [1] that stability properties of the auxiliary system $e^+ = (I - \Psi(i, e))e$ induced by (13) is instrumental for proving $\mathcal{L}_p$ stability properties of a class of NCS for which Assumption 3 does not hold. In our case, it turns out that partial stability of the system (15)-(16) (in the variable e only) is crucial for stability of NCS. To motivate our PE definition, note that setting $V(i, e, \hat{e}) = |e|$ in (15)-(16), we have $\Delta V = -|\Psi(i, \hat{e})e|$ and if there exists a $T < \infty$ such that, for every $k \in \mathbb{N}$, $\hat{e}(k) = \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$, $e(k) = e \in \mathbb{R}^{n_e}$:

$$\sum_{i=k}^{k+T-1} |\Psi(i, \phi_{\hat{e}}(i))e| \geq |e| \ , \quad (17)$$

where $\phi_{\hat{e}}(i) := \phi_{\hat{e}}(i, e, \hat{e})$, we can conclude partial stability (in e) of the system (15)-(16) using the persistency of excitation (PE) arguments (see [12], for instance). Note that PE can be rephrased by saying that each node needs to be visited within T transmissions. Bearing in mind results in [1], it is tempting to conjecture that (17) is enough for proving stability of (8)-(12). However, this is not true in general.

Indeed, it can be shown that if we integrate the equations (9)-(10) on the interval $[t_{i-1}^+, t_i]$ and then use (11), (12) at t_i, the NCS induces the following discrete-time system:

$$e^+ = (I - \Psi(i, \hat{e}))(e + d) \quad (18)$$
$$\hat{e}^+ = \Lambda(i, \Psi(i, \hat{e})(e + d), \hat{e}) \ , \quad (19)$$

[5] This ensures that Zeno solutions cannot occur.

[6] By $\hat{e}(t^+)$ and $e(t^+)$ we mean "evaluated just after the jump": $e(t^+) = \lim_{s \to t, s > t} e(s)$ (resp. for $\hat{e}$).

[7] It may be the case that a single logical node (a "link") consists of several sensors or several actuators or both with the scheduling of that node having the effect of setting multiple components of e to zero. It may also be the case that the network allows the scheduling of more than one node at each transmission and our model allows for this extra degree of freedom.

[8] TOD is implementable through binary countdown in CAN-like medium access protocols – see [11] and [3] for details.

where d captures the inter-sample behavior of $e(\cdot)$. Indeed, for specific initializations $(i, e(i), \hat{e}(i))$ and specific (bounded) values of $d(j), j = i, i+1, \ldots$ the solution of the system (18)-(19) *exactly* coincides with the solution of the NCS (8)-(12) at time instants $t_j^+, j = i, i+1, \ldots$. Moreover, we note that while system (15)-(16) may satisfy the PE condition (17) and, hence, is partially stable in e, there may exist a bounded disturbance d that destroys the PE property (17) for the system (18)-(19), potentially destroying the partial stability in e. Hence, our formal definition of PE needs to posses appropriate robustness and will be stated using (18)-(19):

Definition 2.2: The protocol (15)-(16) is said to be (robustly) persistently exciting in T (PE_T) if there exists $T \in (0, \infty)$ such that (17) holds for all $k \in \mathbb{N}$, $\hat{e}(k) = \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$, $e(k) = e \in \mathbb{R}^{n_e}$ and all $d \in \ell_\infty$, where $\phi_{\hat{e}}(i) := \phi_{\hat{e}}(i, e, \hat{e}, d_{[k,i]})$ is the $\hat{e}$ solution of the system (18)-(19). ■

Two PE_T protocols that satisfy Assumption 3 are presented next.

Example 2.3 (Hybrid RR-TOD Scheduling Protocol): The hybrid RR-TOD scheduling protocol enforces PE in a time-periodic manner. Let an integer $M \geq 1$ be given. The protocol takes the form:

$$e^+ = (I - \Omega(i, \hat{e}))(e + d) \tag{20}$$
$$\hat{e}^+ = (I - \Omega(i, \hat{e}))\hat{e} + \Omega(i, \hat{e})(e + d)\ , \tag{21}$$

$$\Omega(i, \hat{e}) := \begin{cases} \text{diag}\{p_1(i) I_{n_1}, \ldots, p_N(i) I_{n_N}\}, \ \text{mod}(i, M) = 0 \\ \text{diag}\{d_1(\hat{e}) I_{n_1}, \ldots, d_N(\hat{e}) I_{n_N}\}, \ \text{otherwise}, \end{cases}$$

where, $p_n(i) = 1$ when $\text{mod}(i/M, N) = n-1$ and $p_n(i) = 0$ otherwise; d_j are defined in (14). Hybrid RR-TOD protocol (20)-(21) is PE_T with $T := MN$. In particular, when $M = 1$, we obtain the classical RR protocol which is the simplest PE_T protocol with $T = N$.

Example 2.4 (Constant-Penalty TOD): The protocols in [2] use the mechanism of "silent-time" to enforce PE: each link j has a counter r_j that is incremented at every transmission instant that it is *not* scheduled and reset to zero when it *is* scheduled. Irrespective of the underlying scheduling protocol, when a link's counter reaches a predetermined threshold, say M, it will be transmitted. This ensures that every link is scheduled within $M + N - 1$ transmission instants, hence, $T = M + N - 1$. The underlying scheduler in this example is TOD and corresponds to the constant-penalty TOD scheme in [2] with a penalty (vector) of Θ:

$$e^+ = (I - \Phi(r, \zeta))(e + d) \tag{22}$$
$$\zeta^+ = (I - \Phi(r, \zeta))(\zeta + \Theta) + \Phi(r, \zeta)(e + d) \tag{23}$$
$$r^+ = (I - \Phi(r, \zeta))(r + \mathbf{1}), \tag{24}$$

where $\mathbf{1} = [1\ \ldots\ 1]^T$, the scheduling function Φ is given by $\Phi(r, \zeta) = \text{diag}\{\varphi_j(r, \zeta) I_{n_j}\}, j = 1, \ldots, N$ and $\varphi_n(r, \zeta) = 1$ if $[(n = \min\{m : r_m \geq M\}) \vee \left(n = \min\left(\arg\max_{1 \leq j \leq N} |\boldsymbol{\zeta}_j|\right)\right)]$ and $(\forall\ m \in \{1, \ldots, N\})(r_m < M))$; $\varphi_n(r, \zeta) = 0$ otherwise.

III. $\mathcal{L}_p$ STABILITY OF NCS

The framework of $\mathcal{L}_p$ stability and detectability lies at the core of the following theorem that is the main result of the paper and confirms that PE protocols lead to $\mathcal{L}_p$ stability of NCS when the network-free system is $\mathcal{L}_p$ stable from disturbance to state. In particular, this is true for RR, hybrid RR-TOD, CP-TOD and every protocol described in [6]. Note that the conditions of the theorem can be checked, essentially, by inspection of (8)-(12).

Theorem 3.1: Consider NCS (8)-(12) and suppose that:

1) the NCS scheduling protocol (8)-(12) is PE_T;
2) there exists a symmetric positive semi-definite $n_e \times n_e$ matrix with nonnegative entries A and a continuous $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_e}$ so that the error dynamics (9) satisfy
$$\overline{g}(t, x, e, w) \preceq A\overline{e} + \tilde{y}(x(t), w(t)) \tag{25}$$
for[9] $t \in (t_i, t_{i+1})$, for all $i \in \mathbb{N}$;
3) system (8) is $\mathcal{L}_p$ stable from (e, w) to x with gain γ for some $p \in [1, \infty]$ and x in (8) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $\tilde{y}$;
4) and MATI satisfies $\tau \in (\epsilon, \tau^*)$, $\epsilon \in (0, \tau^*)$, where
$$\tau^* = \frac{\ln(z)}{|A|T} \quad z \in [1, 2], \tag{26}$$
where z solves
$$T\gamma|A|z^{1+1/T} + (|A| - \gamma T)z - 2|A| = 0 \tag{27}$$
and $|A|$ comes from (25).

Then, the NCS is $\mathcal{L}_p$-stable from w to (x, e) with linear gain.

Proof: It can be shown (see [13]) that conditions 1 and 2 ensure that the NCS error subsystem (9)-(12) is $\mathcal{L}_p$ stable from $\tilde{y}$ to e for $p \in [1, \infty]$ with gain

$$\tilde{\gamma}(\tau) = \frac{T\exp(|A|T\tau)(\exp(|A|\tau) - 1)}{|A|(2 - \exp(|A|T\tau))}. \tag{28}$$

Solving for τ^* in $\tilde{\gamma}(\tau^*)\gamma = 1$ is equivalent to applying the transformation $z = \exp(|A|T\tau^*)$ to (28) and solving for z in (27).[10] As $\tilde{\gamma}(0) = 0$, and by monotonicity of $\tilde{\gamma}$, we have that $\tau^* > 0$ for every $\gamma < \infty$ and $\tilde{\gamma}(\tau)\gamma < 1$ for any $\tau \in [0, \tau^*)$. Together with detectability assumptions, the proof is complete in light of the small-gain theorem for jump-discontinuous systems discussed in [1, Theorem 1] and [1, Theorem 6], ■

Remark 1: Suppose that $g(t, x, e, w) = Bx + Ce + w$ and let $A = \overline{[a_{ij}]}$, where $a_{ij} = \max\{|c_{ij}|, |c_{ji}|\}$ and $\tilde{y}(x, w) = \overline{Bx + w}$. We immediately have that A and $\tilde{y}(x, w)$

[9] Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The *vector* partial order $\preceq$ is given by $x \preceq y \iff (x_1 \leq y_1) \wedge \cdots \wedge (x_n \leq y_n)$ and $\overline{e}$ and $\overline{g}$ are given by $\overline{e} := (|e_1|, \ldots, |e_{n_e}|)^T$ and $t \overset{\overline{g}}{\mapsto} \overline{g(t)}$, respectively. That is, $\overline{e}$ is the vector that results from taking the absolute value of each scalar component of e and $\overline{g}$ does operates analogously on the image of g.

[10] It can be show (see [13]) that there exists a unique positive solution to (27).

satisfy condition 2 of Theorem 3.1.[11] Moreover, it is clear that $\|\tilde{y}(x,w)\|_p = \|Bx + w\|_p$, that $|A| \leq 6|C|$ for all C and that $|A| \leq 2|C|$ when C has no negative entries.

Remark 2: Under weak conditions, it can be shown (see [1, Section II]) that if the conditions of Theorem 3.1 hold, then the NCS is UGES when $w \equiv 0$. In particular, this is true whenever there exists an $L > 0$ such that $f(t,x,0,0) \leq L|x|$, where f comes from the right-hand side of (8).

Remark 3: It is relatively straightforward to show (see [13]) that the $\mathcal{L}_\infty$ IOS asserted by Theorem 3.1 can be strengthened to input-to-state stability (ISS). In particular, it is possible to conclude ISS with an $\exp$-$\mathcal{KL}$ function β and linear gains:

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma\|w\|_\infty.$$

Remark 4: 3. a remark on proving UGAS as opposed to UGES.

Remark 5: 4. A possibility to let T=T(e). Note that in my Automatica paper with Andy we also have an example of a protocol that is PET but T=T(e) which leads to UGAS as opposed to UGES. This can be another remark in the main section.

6. Our results can be tightened to give guaranteed performance bounds - if network free system had a certain gain gamma, then given any gamma*¿gamma there exists MATI so that,.... This is very useful for engineers that want to design optimal/robust controllers.

7. Re-emphasize that the main attraction in applying these results that they are ”easy to use” and they are derived under realistic assumptions (ignoring some important issues like delays).

8. Mention that dropouts can probably be tackled - PE on average may be enough. This may be closer to stochastic treatment that we regard as an important future problem.

5. A remark on improvements of MATI - the analytic formula for $\gamma = 0$ may be useful. Mention that the bulk of comparisons is done in the next section.

IV. CASE STUDY: CH-47 TANDEM-ROTOR HELICOPTER

We consider the networked control of a CH-47 tandem-rotor helicopter in horizontal motion about a nominal airspeed of 40 knots as discussed in [14]:

$$\dot{x}_P = Ax_P + Bu; \quad y = Cx_P, \tag{29}$$

where $C_P = \left[\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{smallmatrix}\right]$,

$$A_P = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B_P = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix}.$$

The outputs y_1, y_2 are vertical velocity (knots/hr) and pitch altitude (radians) respectively and the inputs u_1, u_2 are collective rotor thrust and differential collective rotor thrust respectively. Let the stabilizing output feedback be $u = Ky$ where K is given by $K = \left[\begin{smallmatrix} -12.7177 & -45.0824 \\ 63.5123 & 25.9144 \end{smallmatrix}\right]$. The design of K is due to [15].

We assume that only outputs are transmitted via the network thus $e = \hat{y} - y$ with two links ($N = 2$) in the NCS and, hence, the simplified NCS equations take the form

$$\dot{x} = A_{11}x + A_{12}e \quad \dot{e} = A_{21}x + A_{22}e \tag{30}$$

where $A_{11} = A_P + B_PKC_P$, $A_{12} = B_PK$, $A_{21} = -C_PA_{11}$, and $A_{22} = -C_PA_{12}$.

Consider the Lyapunov matrix equation $A_{11}^TP + PA_{11} = -I$ and let σ_1 and σ_2 to be the largest and smallest singular value of P, respectively. We set $\tilde{y} = A_{21}x$, and let $k_h = \left|\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right| = 19482$. For the purposes of applying Remark 1, we select $A = \left[\begin{smallmatrix} 567.98 & 239.09 \\ 239.09 & 0 \end{smallmatrix}\right]$ with $|A| = 655.23$ and $A_{22} \preceq A$. We can readily compute the gain from $\tilde{y}$ to e and have $\gamma = 10397$.[12]

We focus on comparing MATI bounds required to achieve global exponential stability (see Remark 2) and compare the results of this paper, [1] and [2]: From the discussion in [1, Section VII], we have that the MATI bound stated therein, which we will refer to as $\tau^*_{[1]}$ is given by

$$\tau^*_{[1]} = \frac{1}{k_h\sqrt{N}} \ln\left(\frac{k_h + \gamma}{k_h\sqrt{(N-1)/N} + \gamma}\right), \tag{31}$$

the MATI bound,[13] $\tau^*_{[2]}$, obtained via [2][Theorem 1] is given by

$$\tau^*_{[2]} = \min\left\{\frac{\ln(2)}{k_hT}, \frac{S}{8}, \frac{S}{16\sigma_2\sqrt{\sigma_2/\sigma_1}k_h}\right\} \tag{32}$$

where $S = [k_h\sqrt{\sigma_2/\sigma_1}\sum_{i=1}^{N}(i + T - N)]^{-1}$, and that of this paper, τ^*_{new} is given by Theorem 3.1 – readily obtained by solving (27) numerically. [14] From the perspective of MATI bounds, neither the results of this paper nor those in [2] distinguish between different PE_T scheduling protocols. The comparison results are summarized below:

1) Performance figures are shown in Figure 3 that apply to any PE_T protocol for the original two-link system (30). The bound $\tau^*_{[1]}$ only applies to RR ($T = N$).
2) The improvements are significant and to put them into perspective, when using RR, τ^*_{new} that achieves UGES is equivalent to a network throughput of 24 Mbps (assuming 128 byte frames) while $\tau^*_{[1]}$ requires an effective network throughput of over 1003 Gbps.
3) In fact, we demonstrate in [13] that our MATI bounds are analytically better than those provided in [1] for RR (that are, in turn, analytically better than the bounds provided in [2]) for every scheduling protocol

[11] It is straightforward to demonstrate the existence of a A and $\tilde{y}$ that satisfy condition 2 of Theorem 3.1 whenever g satisfies a linear growth bound of the form $|g(t,x,e,w)| \leq L(|x| + |e| + |w|)$. Setting $A = [a_{ij}]$, where $a_{ij} = L$, and $\tilde{y}(x,w) = g(t,x,0,w)$ is one possibility with $|A| = n_eL$. This will almost certainly not be the "best" A or $\tilde{y}$ for a given NCS but it shows that, even for nonlinear systems, existence of A and $\tilde{y}$ is generally not an issue.

[12] In light of Remark 1, we would actually set $\tilde{y} = \overline{A_{21}x}$ in our analysis framework though this has no effect on the calculated γ.

[13] In principle, different choices of Lyapunov function and "output" $\tilde{y}$ in the framework presented in [1] may lead to improved MATI bounds. We have followed [1, Section VII] in choices of Lyapunov function amd $\tilde{y}$ – these were choices that lead to the best previously obtainable MATI bounds for RR.

[14] Closed-form bounds for MATI are derived in [13].

considered therein by a factor of $N^{1/2}$ as $N \to \infty$ for arbitrary systems.

4) We formally fix σ_1, σ_2, γ, k_h and $|A|$ and plot $\tau^*_{[1]}$ and $\tau^*_{[2]}$ with $T = N \in [1, 1000]$ in Figure 2 to examine the behavior of the bounds as the number of links grow. We also fix $N = 2$ and allow $T \geq 2$ to vary for $\tau^*_{[2]}$ and τ^*_{new}.[15] The differences are marked, even on the $\log_{10}(T) \times \log_{10}(\tau^*)$ scale used in Figure 2.

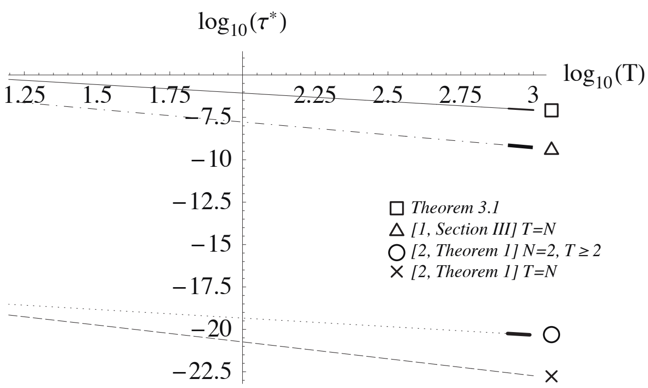

Fig. 2. MATI bounds comparison for PE_T protocols, $T \in [1, 1000]$.

	$T = 2$	$T = 6$	$T = 50$
τ^*_{new}	4.23×10^{-5}	1.42×10^{-5}	1.71×10^{-6}
$\tau^*_{[1]}$	1.02×10^{-9}	N/A	N/A
$\tau^*_{[2]}$	3.08×10^{-18}	8.41×10^{-19}	9.35×10^{-20}
$\tau^*_{\text{new}}/\tau^*_{[2]}$	1.37×10^{13}	1.68×10^{13}	1.83×10^{13}
$\tau^*_{\text{new}}/\tau^*_{[1]}$	4.147×10^{4}	N/A	N/A

Fig. 3. MATI bounds achieving UGES for the system in (30) with PE_T protocols.

V. Conclusion

This paper highlighted the importance of the PE property in scheduling protocols – a property naturally verified by many scheduling protocols presented in the literature and a property that is crucial in the design of wireless NCS. We presented results where the MATI bounds obtained are completely parameterized by T and properties of the network-free system for arbitrary PE_T protocols, as are the bounds that characterize the robustness of the NCS to exogenous disturbances. We believe that the results will be particularly useful in practical applications in light of the significant improvements over existing results, the generality of the model and the fact that every condition required to apply our results can be checked directly and easily from properties of the network-free system and the protocol alone.

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[15]For the purposes of applying Theorem 3.1, it is immaterial as to whether N is fixed and $T \geq N$ is varied or whether $T = N$ is varied.