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A note on preservation of dissipation inequalities under sampling: the dynamic feedback case

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Abstract

We present a general and unified framework for the design of nonlinear digital controllers using the emulation method for nonlinear systems with disturbances. It is shown that if a (dynamic) continuous-time controller, which is designed so that the continuous-time closed-loop system satisfies a certain dissipation inequality, is appropriately discretized and implemented using the sampler and zero-order-hold, then the discrete-time model of the closed-loop sampled-data system satisfies a dissipation inequality in a semiglobal practical sense (sampling period is the parameter that we can adjust). We consider two different forms of dissipation inequalities for the discrete-time model: the "weak" form and the "strong" form. These inequalities turn out to be equivalent if the disturbances are Lipschitz but they are not equivalent in general.

Keywords: Dissipation; nonlinear; emulation design; sampled-data; dynamic feedback.

1 Introduction

Emulation is a well-established method to design digital controllers for continuous-time plants (see, for instance [1, 3, 4]). The first step in the emulation method is to design a continuous-time controller for a continuous-time plant using some known continuous-time design method; sampling is completely ignored at this stage. Then, in the second step, the continuous-time controller is discretized and implemented using the sampler and hold devices. Digital controllers designed using emulation have been proved to perform well for a number of control problems under sufficiently fast sampling. Indeed, the following problems have been addressed in the literature: stability for linear [2] and nonlinear [12, 14, 21] plants, $\mathcal{L}_p$ stability of linear systems [2], input-to-state stability (ISS) of nonlinear systems [15, 19], adaptive stabilization of nonlinear systems [5] and general dissipativity of nonlinear systems using static state feedback [7]. In particular, it was shown in [7] that if in the first step of the emulation design the continuous-time system satisfies a certain dissipation inequality, then a similar dissipation inequality holds for the discrete-time model of the closed-loop sampled-data system in a semiglobal practical sense (sampling is the parameter that we can adjust). In [7] two cases were considered: the case when the controls are regarded as open-loop inputs to the system and the case of static state feedback. Finally, similar ideas were exploited in [13], where dissipativity property is investigated using discrete observation of the storage function.

In this paper we generalize the results of [7] by considering preservation of dissipation inequalities in the emulation design for the case of *dynamic state feedback*. Hence, the results of this paper are important whenever an observer is used to estimate some of the states. In the case of static state

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feedback considered in [7], discretization of the controller is trivial since the controller has no dynamics. On the other hand, in the case of dynamic state feedback considered here the discretization should be carried out carefully[1]. We introduce properties that the discretized controller should satisfy in order to have preservation of the dissipation inequality under sampling. These properties, which are called one-step strong and weak consistency, are specified in Definitions 2.3 and 2.4 and sufficient conditions for these properties to hold are given in the Appendix.

Our results unify the emulation design for a large number of different properties. Indeed, the special cases of the notion of dissipativity that we consider are dissipation inequalities used to investigate stability, L_p stability, passivity, input-to-state stability, integral input-to-state stability, forward completeness, detectability, etc. (see for instance [6, 16, 20]). An application on input-to-state stability is presented in this paper. Moreover, our results are also applicable to a large class of nonlinear plants and controllers.

In our main results we explore two types of dissipation inequalities for the discrete-time model of the closed-loop sampled-data system: the weak and strong form. In Definition 2.5 and 2.6, we introduce properties associated to the weak and strong dissipation inequalities. Relationship among the properties is given in Theorem 2.1. For the weak dissipation result to hold, the discretized controller needs to satisfy the one-step weak consistency condition (Definition 2.3) and the disturbances need to be Lipschitz (Theorem 3.1). It is shown in Proposition 3.2 that Lipschitz disturbances can be obtained by filtering out bounded measurable disturbances through a strictly proper input-to-state stable (ISS) filter. The strong dissipation inequality holds if the discretized controller satisfies the one-step strong consistency condition (Definition 2.4) and in this case disturbances are allowed to be only measurable (see Theorem 3.3). In general, strong and weak dissipation inequalities do not imply each other and this is illustrated by Example 3.1. An important feature of our results is that the required sampling period can be computed using our method, although it may be conservative (smaller than necessary) which is a consequence of the conservative Lipschitz bounds that we are using in the proofs. This is a common problem in numerical analysis literature [18] and the emulation design in sampled-data systems [5].

The paper is organized as follows. In Section 2 we present preliminaries. Main results are stated and discussed in Section 3. Proofs of the main results and an application of the main results are presented in Section 4 and Section 5 respectively. Conclusions are given in the last section. Sufficient conditions for one-step weak and strong consistency properties are stated and proved in the Appendix. Due to limited space, proof of results which can be carried out in a similar manner as proof of other results are omitted.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{N}$ if it is continuous and non-decreasing. It is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing; it is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. For a given function $d(\cdot)$, we use the following notation $d[t_1, t_2] := \{d(t) : t \in [t_1, t_2]\}$. If $t_1 = kT, t_2 = (k+1)T$, we use the shorter notation $d[k]$, and take the norm of $d[k]$ to be the supremum of $d(\cdot)$ over $[kT, (k+1)T]$, that is $\|d[k]\|_\infty = \operatorname{ess\,sup}_{\tau \in [kT,(k+1)T]} |d(\tau)|$.

Consider the continuous-time nonlinear plant model:

$$\dot{x} = f(x, u, d_c, d_s) \tag{1}$$
$$y = h(x, u, d_c, d_s) \ , \tag{2}$$

with dynamic state feedback control:

$$\begin{aligned} \dot{z} &= g(x, z, d_c, d_s) \\ u &= u(x, z, d_c, d_s) \ , \end{aligned} \tag{3}$$

[1] It is true, however, that if one wants to implement the dynamic controller using the Euler approximation, then the results on static state feedback still apply (see Remark 2.2). Therefore, the main contribution of this paper is to present results that apply to situations when higher order approximations are used to approximate controller dynamics.

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state of the plant, state of the controller, control input and output of the plant. $d_c \in \mathbb{R}^{n_c}$ and $d_s \in \mathbb{R}^{n_s}$ are disturbance inputs to the system. It is assumed that f, g and u are locally Lipschitz and h is continuous. We also assume that $f(0,0,0,0) = 0$, $h(0,0,0,0) = 0$, $g(0,0,0,0) = 0$ and $u(0,0,0,0) = 0$. The controller (3) covers the case of dynamic output feedback:

$$\begin{aligned} \dot{z} &= \tilde{g}(y, z, d_c, d_s) =: g(x, z, d_c, d_s) \\ u &= \tilde{u}(y, z, d_c, d_s) =: u(x, z, d_c, d_s) \ , \end{aligned} \tag{4}$$

where we assume that the feedback system (1), (2), (3) is Lipschitz well posed, that is the equations:

$$\begin{aligned} y &= h(x, u(y, z, d_c, d_s), d_c, d_s) \\ u &= \tilde{u}(h(x, u, d_c, d_s), z, d_c, d_s) \end{aligned}$$

have unique solutions $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ so that (1), (2) and (4) can be written in the form $\dot{\eta} = \mathcal{F}(\eta, d_c, d_s)$, $\psi = \mathcal{H}(\eta, d_c, d_s)$ where $\eta := (x^T \quad z^T)^T$, $\psi := (y^T \quad u^T)^T$ and $\mathcal{F}$ and $\mathcal{H}$ are locally Lipschitz.

The following definitions are used in the sequel.

Definition 2.1 *The system (1), (2), (3) is said to be (V, w)-dissipative if there exist a continuously differentiable function $V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}$, called the storage function, and a continuous function $w : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \to \mathbb{R}$, called the dissipation rate, such that for all $x \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, d_c \in \mathbb{R}^{n_c}, d_s \in \mathbb{R}^{n_s}$ the following holds:*

$$\frac{\partial V}{\partial x} f(x, u(x, z, d_c, d_s), d_c, d_s) + \frac{\partial V}{\partial z} g(x, z, d_c, d_s) \leq w(x, z, d_c, d_s) \ . \tag{5}$$

■

Definition 2.2 *The system $\dot{x} = f(x, d)$ is input-to-state stable if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in \mathbb{R}^n$ and all $d \in \mathcal{L}_\infty$, the solutions of the system satisfy:*

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|d\|_\infty), \quad \forall t \geq 0 \ . \tag{6}$$

■

Emulation procedure: Suppose that, as a first step in the emulation design, we designed a controller (3) for the plant (1), (2) in continuous-time domain, so that the closed-loop continuous-time system is (V, w)-dissipative.

As a second step in the emulation design, we discretize the controller and implement it using a sampler and zero order hold. The discretization of the controller is carried out as follows. First, we assume that $x(t) = x(kT) =: x(k)$, $d_s(t) = d_s(kT) =: d_s(k)$ for all $t \in [kT, (k+1)T)$ in the differential equation (3), where $T > 0$ is the sampling period. Consider the following initial value problem:

$$\dot{z}(t) = g(x(k), z(t), d_c(t), d_s(k)) \ , \qquad z_0 = z(k) \tag{7}$$

where $x(k), z(k), d_c[k], d_s(k)$ are given. Denote the solution of the initial value problem (7) as $z(t)$, and then we obtain the exact discretization of the controller (3) (see also [2]):

$$\begin{aligned} z(k+1) &= z(k) + \int_{kT}^{(k+1)T} g(x(k), z(\tau), d_c(\tau), d_s(k)) d\tau =: G_T^e(x(k), z(k), d_c[k], d_s(k)) \\ u(k) &= u(x(k), z(k), d_c(k), d_s(k)) \ . \end{aligned} \tag{8}$$

Note that in general the discretization (8) can not be implemented directly since G_T^e in (8) is usually impossible to compute exactly (since we need to solve the nonlinear initial value problem (7) explicitly over one sampling interval), so we need to use instead an approximate discrete-time model of the controller:

$$\begin{aligned} z(k+1) &= G_T^a(x(k), z(k), d_c(k), d_s(k)) \\ u(k) &= u(x(k), z(k), d_c(k), d_s(k)) \ , \end{aligned} \tag{9}$$

which is obtained from (7) using one of the numerical integration methods (e.g. Runge-Kutta). For instance, if we use the forward Euler method, we obtain $G_T^a(x, z, d_c, d_s) := x + Tg(x, z, d_c, d_s)$. Note that for some classes of systems, such as linear systems, considered in [2] or the case of nonlinear systems with static state feedback considered in [7], one can obtain G_T^e and there is no need to use an approximate discrete-time model of the controller. It is obvious that in general we will have to use a sufficiently small sampling period T, since the approximate discrete-time model (9) is usually a good approximation of the exact discrete-time model (8) typically only for small T.

The sampled-data closed-loop system consists of the controller (9), which is between a sampler and zero order hold and the continuous-time plant (1), (2). In the sequel, we use the discrete-time model of this sampled-data system, which consists of (9) and the exact discrete-time model of the plant, which is obtained as follows. We assume that $u(t) = u(kT) =: u(k)$, $d_s(t) = d_s(kT) =: d_s(k)$ for all $t \in [kT, (k+1)T]$ and consider the initial value problem

$$\dot{x}(t) = f(x(t), u(k), d_c(t), d_s(k)) \ , \qquad x_0 = x(k) \tag{10}$$

where $x(k)$, $u(k)$, $d_c[k]$ and $d_s(k)$ are given. The output y is measured only at sampling instants kT, $k \geq 0$. Denote the solution of the initial value problem (10) as $x(t)$. Then the exact discrete-time model of the plant can be written as:

$$\begin{aligned} x(k+1) &= x(k) + \int_{kT}^{(k+1)T} f(x(\tau), u(k), d_c(\tau), d_s(k))d\tau =: F_T(x(k), u(k), d_c[k], d_s(k)) \\ y(k) &= h(x(k), u(k), d_c(k), d_s(k)) \ . \end{aligned} \tag{11}$$

The discrete-time model of the sampled-data closed-loop system consists of (9) and (11).

In order to prove our main results, we need to guarantee that the mismatch between the exact discrete-time model of the controller (8) and its approximation (9) is small in some sense. We define two consistency properties that are used to limit the mismatch. Different forms of the consistency property are used in numerical analysis literature (see Definition 2 [8], Definition 1 [10] and Definition 3.4.2 [18]).

Definition 2.3 (One-step weak consistency) *The family G_T^a is said to be one-step weakly consistent with G_T^e if given any quintuple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$, there exist a function $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that, for all $T \in (0, T^*)$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $|d_s| \leq \Delta_{d_s}$ and functions $d_c(\cdot)$ that are Lipschitz and satisfy $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, we have*

$$|G_T^e - G_T^a| \leq T\rho(T) \ . \tag{12}$$

■

Definition 2.4 (One-step strong consistency) *The family G_T^a is said to be one-step strongly consistent with G_T^e if given any quadruple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{d_s})$, there exists a function $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that, for all $T \in (0, T^*)$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$, we have*

$$|G_T^e - G_T^a| \leq T\rho(T) \ . \tag{13}$$

■

Remark 2.1 *Consistency properties specify how the controller should be discretized for the emulation procedure to yield desired results. In the Appendix, we present and prove general checkable conditions under which one-step weak and strong consistency properties hold. It is important to emphasize that if the exact discrete-time model of the controller can be obtained, then we do not have to use an approximate discrete-time model of the controller and consistency definitions become superfluous, i.e., they hold automatically. Two important such cases were considered in the literature: emulation for linear systems was considered in [2] and emulation for static state feedback controllers was considered in [7]. Finally, note that the weak and strong consistency definitions become equivalent when G_T^e and G_T^a are independent of d_c.* ■

Remark 2.2 *Note that if we want to implement the controller so that* $G_T^a = z + Tg(x, z, d_c, d_s)$*, then we can regard the closed-loop system (1), (2) and (3) as an augmented plant of the form*

$$\begin{aligned} \dot{x} &= f(x, u, d_c, d_s) \\ \dot{z} &= v \end{aligned}$$

controlled by the static state feedback controller of the form:

$$\begin{aligned} u &= u(x, z, d_c, d_s) \\ v &= g(x, z, d_c, d_s) \end{aligned}$$

which is implemented between the sampler(s) and zero order hold(s). In this case, one can use [7] on emulation results for static state feedback controllers. However, if we want to use G_T^a *other than Euler, this method is not applicable and we need to consider results proved in this paper that use the notion of consistency.* ■

In order to precisely state the main results, we introduce the following properties:

Definition 2.5 *Let* V *be continuously differentiable and* w *be continuous. The system (9), (11) is said to have Property P1 (respectively, have Property P2) if given any 6-tuple of strictly positive real numbers* $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$*, there exists* $T^* > 0$ *such that for all* $T \in (0, T^*)$ *and all* $|x| \leq \Delta_x$*,* $|z| \leq \Delta_z$*,* $|d_s| \leq \Delta_{d_s}$ *and for all disturbances* $d_c(\cdot)$ *that satisfy* $\|d_c[0]\|_\infty \leq \Delta_{d_c}$*,* $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$ *the following holds:*

$$\frac{V(F_T(x, u(x, z, d_c, d_s), d_c[0], d_s), G_T^a(x, z, d_c, d_s)) - V(x, z)}{T} \leq \frac{1}{T}\int_0^T w(x, z, d_c(\tau), d_s)d\tau + \nu \ , \tag{14}$$

(respectively the following holds for the system (9), (11):

$$\frac{V(F_T(x, u(x, z, d_c, d_s), d_c[0], d_s), G_T^a(x, z, d_c, d_s)) - V(x, z)}{T} \leq w(x, z, d_c, d_s) + \nu \ ). \tag{15}$$

■

The following preliminary result that is proved in Section 4 shows that Properties P1 and P2 in Definition 2.5 are equivalent.

Theorem 2.1 *The system (9), (11) has Property P1 if and only if the system has Property P2.* ■

Definition 2.6 *Let* V *be continuously differentiable and* w *be continuous. The system (9), (11) is said to have Property P3 if given any quintuple of strictly positive real numbers* $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{d_s}, \nu)$*, there exists* $T^* > 0$ *such that for all* $T \in (0, T^*)$ *and all* $|x| \leq \Delta_x$*,* $|z| \leq \Delta_z$*,* $\|d_c[0]\|_\infty \leq \Delta_{d_c}$*,* $|d_s| \leq \Delta_{d_s}$ *the inequality (14) holds.* ■

The main difference between the Properties P1 and P3 (or P2 and P3, since Properties P1 and P2 are equivalent) is that Property P1 (Property P2) requires the disturbances d_c to be Lipschitz for the inequality (14) to hold, whereas the inequality (14) in Property P3 must hold for non Lipschitz disturbances as well. The dissipation inequalities in Properties P1 and P2 (since they are equivalent) are said to have the "weak" form (since they hold for a smaller class of disturbances) and the dissipation inequality in Property P3 is said to have the "strong" form (since it holds for a larger class of disturbances).

3 Main results

In this section we state the main results (Theorem 3.1 and 3.3) which assume that the continuous-time system is (V, w)-dissipative. Theorem 3.1 states that if one-step weak consistency holds and disturbances $d_c(\cdot)$ are Lipschitz, then the (equivalent) Properties P1 and P2 hold for discrete-time model of the sampled-data system. Since in most cases we do not know whether the disturbances are Lipschitz or not, in Proposition 3.2 we prove that if we filter out a bounded measurable signal using a strictly proper input-to-state stable filter, we obtain a filtered signal which is bounded and Lipschitz. If disturbances are only measurable (but not Lipschitz) then the inequality (15) may not hold in a semiglobal practical sense while the inequality (14) still holds (see Example 3.1). In Theorem 3.3 we show that for a smaller class of controllers, if $d_c(\cdot)$ are measurable (but not Lipschitz) and one-step strong consistency holds then the discrete-time model has Property P3.

Theorem 3.1 *(Weak form of dissipation) If the system (1), (2), (3) is (V, w)-dissipative, then given any approximate discrete-time model of the controller G^a_T (9) that is one-step weakly consistent with the exact discrete-time model of the controller G^e_T (8) the system (9), (11) has Property P1 (equivalently, Property P2).* ■

Note that Properties P1 and P2 require $d_c(\cdot)$ to be Lipschitz. The following example shows that indeed the Lipschitz condition on $d_c(\cdot)$ is necessary, since the inequality (15) may not hold if $d_c(\cdot)$ is not Lipschitz.

Example 3.1 *[7] Consider the continuous-time system $\dot{x} = u(x) + d_c = -x + d_c$, where $x, d_c \in \mathbb{R}$. Using the storage function $V = \frac{1}{2}x^2$, the derivative of V is $\dot{V} = -x^2 + xd_c \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2$, and the system is ISS. It was shown in [7] that if a family of bounded disturbances $d_c(t) = \cos\left(\frac{t+2T}{T}\right)$ is considered, then the inequality $\frac{\Delta V}{T} \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2 + \nu$ does not hold in a semiglobal practical sense, which implies that Property P1, as well as P2, does not hold! This is due to the fact that the family of disturbances is not Lipschitz, uniformly in T, since $\left\|\dot{d}_c\right\|_\infty = 1/T$. This illustrates that, in general, the Lipschitz condition on $d_c(\cdot)$ in Theorem 3.1 is necessary for the result to hold.* ▲

The following result shows that if we can filter out any bounded measurable disturbances using a strictly proper input-to-state stable filter, then the filtered disturbances are bounded and Lipschitz. This further motivates Theorems 2.1 and 3.1 that require disturbances to be Lipschitz.

Proposition 3.2 *Consider any nonlinear filter:*

$$\dot{\xi} = f(\xi, d_c) \tag{16}$$

$$\upsilon = h(\xi)\ , \tag{17}$$

which is input-to-state stable with respect to input d_c and where f and h are locally Lipschitz. Then, given any $d_c(\cdot) \in \mathcal{L}_\infty$ and any $\xi_0 \in \mathbb{R}^{n_\xi}$ we have that the output $\upsilon(\cdot)$ is bounded, that is $\upsilon(\cdot) \in \mathcal{L}_\infty$. Moreover, $\dot{\upsilon}(\cdot) \in \mathcal{L}_\infty$, which implies that υ is Lipschitz. ■

The use of filters in sampled-data systems is standard (see for instance [2]). In particular, filters that are strictly proper, stable, linear and time invariant:

$$\dot{\xi} = A\xi + Bd_c \tag{18}$$

$$\upsilon = C\xi\ , \tag{19}$$

were considered in [2] in the context of $\mathcal{L}_p$ stability of linear sampled-data systems. In this case, we have that the filter satisfies all conditions of Proposition 3.2 and consequently for any ξ_0 and $d_c \in \mathcal{L}_\infty$ we have that $\upsilon, \dot{\upsilon} \in \mathcal{L}_\infty$.

Example 3.1 showed that if disturbances $d_c(\cdot)$ are not Lipschitz, then Property P1 and P2 may not hold. It is of interest to investigate that Property P3 still holds, for the case when $d_c(\cdot)$ are not Lipschitz.

To prove a general result for this case it is necessary to restrict our attention to the controllers of the following form (see Example 3.2 below):

$$\begin{aligned} \dot{z} &= g(x, z, d_s) \\ u &= u(x, z, d_s) \ . \end{aligned} \tag{20}$$

We assume that g and u are locally Lipschitz, $g(0, 0, 0) = 0$ and $u(0, 0, 0) = 0$. In a similar manner as for controller (3), we define the exact discrete-time model of the controller (20) as:

$$\begin{aligned} z(k+1) = z(k) + \int_{kT}^{(k+1)T} g(x(k), z(\tau), d_s(k)) d\tau =: G_T^e(x(k), z(k), d_s(k)) \\ u(k) = u(x(k), z(k), d_s(k)) \ , \end{aligned} \tag{21}$$

and its approximate discrete-time model:

$$\begin{aligned} z(k+1) = G_T^a(x(k), z(k), d_s(k)) \\ u(k) = u(x(k), z(k), d_s(k)) \ . \end{aligned} \tag{22}$$

Note that the controller (20) does not have $d_c(\cdot)$ as its input and the following example shows that this is necessary in general if we want to prove that the discrete-time model of the sampled-data system has Property P3.

Example 3.2 *[7] Consider the system $\dot{x} = u$, where $u = -d_c$. The storage function that we consider is $V(x) = x$, so that the derivative: $\frac{\partial V}{\partial x}(-d_c) = -d_c$, and hence the dissipation rate is $w(x, d_c, d_s) = -d_c$. It can be shown (see [7]) that Property P3 does not hold in this case (consider disturbance $d_c(0) = 0$ and $d_c(t) = 1, \ \forall t > 0$ to show contradiction).* ▲

Compared to Theorem 3.1, the following result on strong form of dissipation considers a larger class of measurable disturbances d_c.

Theorem 3.3 *(Strong form of dissipation) If the system (1), (2), (20) is (V, w)-dissipative, then given any approximate discrete-time model of the controller G_T^a (22) that is one-step strongly consistent with the exact discrete-time model of the controller G_T^e (21) the system (11), (22) has Property 3.* ■

Example 3.1 continued *Note that since the state feedback of the system in Example 3.1 is static and it does not depend on d_c, all conditions of Theorem 3.3 are satisfied and the discrete-time system has Property P3.* ▲

4 Proofs of main results

Proof of Theorem 2.1:
(P1) $\Longrightarrow$ **(P2)** Suppose that Property P1 holds. Let $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu_w)$ be given and let $T_s^* > 0$ (from Property P1) be such that for all $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, $|d_s| \leq \Delta_{d_s}$ and all $T \in (0, T_s^*)$ the following holds:

$$\begin{aligned} \frac{\Delta V}{T} &\leq \frac{1}{T} \int_0^T w(x, z, d_c(\tau), d_s) d\tau + \frac{\nu_w}{2} \\ &\leq w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{1}{T} \int_0^T |w(x, z, d_c(\tau), d_s) - w(x, z, d_c, d_s)| \, d\tau \ , \end{aligned} \tag{23}$$

where the second inequality was obtained by adding and subtracting $w(x, z, d_c, d_s)$. Since $d_c(\cdot)$ is Lipschitz with Lipschitz constant $\Delta_{\dot{d}_c}$, we can write $|d_c(\tau) - d_c| \leq \Delta_{\dot{d}_c} \tau$. Moreover, since w is continuous, it is uniformly continuous on compact sets, and given any $\varepsilon > 0$ there exists $T_s > 0$ such that for any $\tau \in [0, T_s]$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, $|d_s| \leq \Delta_{d_s}$ we have that $|w(x, z, d_c(\tau), d_s) - w(x, z, d_c, d_s)| \leq \varepsilon$. Let $\varepsilon = \frac{\nu_w}{2}$ and let this fix T_s. Let $T_w^* = \min\{T_s, T_s^*\}$. Then using (23) we have that for all $T \in (0, T_w^*)$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$, $|d_s| \leq \Delta_{d_s}$:

$$\frac{\Delta V}{T} \leq w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{1}{T}\int_0^T \frac{\nu_w}{2} d\tau = w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{\nu_w}{2} \ , \tag{24}$$

which shows that Property P2 holds.
(**P2**) $\Longrightarrow$ (**P1**) follows a similar way as the proof for (**P1**) $\Longrightarrow$ (**P2**), to show that if Property P2 holds, then Property P1 holds. ■

Proof of Theorem 3.1: To shorten the notation we define $u := u(x, z, d_c, d_s)$, $f := f(x, u, d_c, d_s)$, $g := g(x, z, d_c, d_s)$, $F_T := F_T(x, u, d_c[0], d_s)$, $G_T^e := G_T^e(x, z, d_c[0], d_s)$ and $G_T^a := G_T^a(x, z, d_c, d_s)$.

Definition of T^*: Suppose that the continuous-time system (1), (2), (3) is (V, w)-dissipative, that is for all $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, $d_c \in \mathbb{R}^{n_c}$, $d_s \in \mathbb{R}^{n_s}$, the inequality (5) holds. Let G_T^a be one-step weakly consistent with G_T^e, and let a 6-tuple of strictly positive real numbers (Δ_x, Δ_z, Δ_{d_c}, $\Delta_{\dot{d}_c}$, Δ_{d_s}, ν) be given. Let these data generate $\rho \in \mathcal{K}_\infty$ from the definition of one-step weak consistency. Define $R_x := \Delta_x + 1$ and $R_z := \Delta_z + 1$. Let $L > 0$ be the Lipschitz constant of f and g, and let $b > 0$ be a number that satisfies $\max\left\{\left|\frac{\partial V}{\partial x}\right|, \left|\frac{\partial V}{\partial z}\right|, |f|, |g|\right\} \leq b$ for all $|x| \leq R_x$, $|z| \leq R_z$, $|d_c| \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$. Define $\Delta := \Delta_x + \Delta_z + \Delta_{d_c} + \Delta_{d_s}$.

We assume without loss of generality that $\nu \leq 1$ and $b \geq 1$ and define

$$T_1^* := \min\left\{\frac{1}{2b}, \rho^{-1}\left(\frac{\nu}{2b}\right)\right\} \ . \tag{25}$$

Note that $T_1^* \leq \frac{1}{2b} \leq \frac{1}{2} < 1$. Let $T_2^* > 0$ be such that the following holds:

$$bL\left[\Delta\frac{\exp(LT) - 1 - LT}{LT} + \frac{1}{2}\Delta_{\dot{d}_c}T\right] \leq \frac{\nu}{8} \ , \ \forall T \in (0, T_2^*) \ . \tag{26}$$

It is easy to see that such a T_2^* always exists. Let $x_1 := x + \theta_1 T f$ and $z_1 := z + \theta_2 T g$ where $\theta_1, \theta_2 \in (0, 1)$. Let $T_3^* > 0$ be such that:

$$b\left|\left.\frac{\partial V}{\partial x}\right|_{(x_1, z+Tg)} - \left.\frac{\partial V}{\partial x}\right|_{(x,z)}\right| \leq \frac{\nu}{8} \ , \tag{27}$$

for all $T \in (0, T_3^*)$, $|x| \leq R_x$, $|z| \leq R_z$, $|d_s| \leq \Delta_{d_s}$, and $d_c(\cdot)$ such that $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$. The required T_3^* always exists, which can be proved as follows. From the continuity of $\frac{\partial V}{\partial x}$, which implies that $\frac{\partial V}{\partial x}$ is uniformly continuous on the compact sets, and since $|x_1 - x| \leq T|f| \leq Tb$ and $|(z + Tg) - z| = T|g| \leq Tb$, it follows that given any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $\left|\left.\frac{\partial V}{\partial x}\right|_{(x_1, z+Tg)} - \left.\frac{\partial V}{\partial x}\right|_{(x,z)}\right| \leq \epsilon$, $\forall T \in (0, T_\epsilon)$, $|x| \leq R_x$, $|z| \leq R_z$, $|d_c| \leq \Delta_{d_c}$ and $|d_s| \leq \Delta_{d_s}$. Hence, we can choose $\epsilon^* := \nu/(8b)$ and let this fix the desired $T_3^* := T_{\epsilon^*}$ for which (27) holds.

In exactly the same way we choose $T_4^* > 0$ such that

$$b\left|\left.\frac{\partial V}{\partial z}\right|_{(x, z_1)} - \left.\frac{\partial V}{\partial z}\right|_{(x,z)}\right| \leq \frac{\nu}{8} \ , \tag{28}$$

for all $T \in (0, T_4^*)$, $|x| \leq R_x$, $|z| \leq R_z$, $|d_s| \leq \Delta_{d_s}$, and $d_c(\cdot)$ such that $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$. Finally, we define

$$T^* := \min\{T_1^*, T_2^*, T_3^*, T_4^*\} \ . \tag{29}$$

Proof that Property P1 (P2) holds: We will show first, that Property P2 holds. Consider arbitrary $T \in (0, T^*)$, $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $|d_s| \leq \Delta_{d_s}$, and $d_c(\cdot)$ such that $\|d_c[0]\|_\infty \leq \Delta_{d_c}$, and $\left\|\dot{d}_c[0]\right\|_\infty \leq \Delta_{\dot{d}_c}$.

Since $T < T^* \leq \frac{1}{2b}$, the solutions $x(t)$ and $z(t)$ of the initial value problems (10) and (7) exist and $|x(t)| \leq \Delta_x + \frac{1}{2}$, $|z(t)| \leq \Delta_z + \frac{1}{2}$, $\forall t \in [0, T]$, which implies

$$\begin{aligned} |F_T| &\leq \Delta_x + \frac{1}{2} < R_x \ , \\ |G^e_T| &\leq \Delta_z + \frac{1}{2} < R_z \ . \end{aligned} \tag{30}$$

From the second inequality in (30), one-step weak consistency and the choice of T_1^* we have:

$$\begin{aligned} |G^a_T| &\leq |G^e_T| + |G^a_T - G^e_T| \\ &< \Delta_z + \frac{1}{2} + \rho(T_1^*) \\ &\leq \Delta_z + \frac{1}{2} + \frac{1}{2} \\ &= R_z \ . \end{aligned} \tag{31}$$

From the local Lipschitz properties of f and g, we can write

$$|x(\tau) - x| \leq \Delta[\exp(L\tau) - 1] \ , \ \forall \tau \in [0, T] \tag{32}$$

$$|z(\tau) - z| \leq \Delta[\exp(L\tau) - 1] \ , \ \forall \tau \in [0, T] \tag{33}$$

and since $d_c(\cdot)$ is Lipschitz, with Lipschitz constant $\Delta_{\dot{d}_c}$, we can write that for all τ

$$|d_c(\tau) - d_c| = |d_c(\tau) - d_c(0)| \leq \Delta_{\dot{d}_c} \tau \ . \tag{34}$$

We consider

$$\begin{aligned} \frac{\Delta V}{T} &= \frac{V(F_T, G^a_T) - V(x,z)}{T} \\ &= \underbrace{\left.\frac{\partial V}{\partial x}\right|_{(x,z)} f + \left.\frac{\partial V}{\partial z}\right|_{(x,z)} g}_{\mathbf{1}} + \underbrace{\frac{1}{T}\left\{V(F_T, G^a_T) - V(x + Tf, z + Tg)\right\}}_{\mathbf{2}} \\ &\quad + \underbrace{\frac{1}{T}\left\{V(x + Tf, z + Tg) - V(x,z) - \left.\frac{\partial V}{\partial x}\right|_{(x,z)} Tf - \left.\frac{\partial V}{\partial z}\right|_{(x,z)} Tg\right\}}_{\mathbf{3}} \ , \end{aligned} \tag{35}$$

where the second equality holds since we just added and subtracted $\frac{1}{T}V(x+Tf, z+Tg)$, $\left.\frac{\partial V}{\partial x}\right|_{(x,z)} f$ and $\left.\frac{\partial V}{\partial z}\right|_{(x,z)} g$. Now we bound each term in (35).

Term 1: It follows from (V, w)-dissipativity of the continuous-time system (1), (2), (3) that:

$$\left.\frac{\partial V}{\partial x}\right|_{(x,z)} f + \left.\frac{\partial V}{\partial z}\right|_{(x,z)} g \leq w(x, z, d_c, d_s) \ . \tag{36}$$

Term 2: Applying the Mean Value Theorem to the Term **2**, we have by adding and subtracting $V(x + Tf, G^a_T)$:

$$\begin{aligned} &\frac{1}{T}\left\{V(F_T, G^a_T) - V(x + Tf, z + Tg)\right\} \\ &\qquad \leq \underbrace{\frac{1}{T}\left|\left.\frac{\partial V}{\partial x}\right|_{(x_2, G^a_T)}\right| |F_T - (x + Tf)|}_{\mathbf{2a}} + \underbrace{\frac{1}{T}\left|\left.\frac{\partial V}{\partial z}\right|_{(x+Tf, z_2)}\right| |G^a_T - (z + Tg)|}_{\mathbf{2b}} \ , \end{aligned} \tag{37}$$

where $x_2 = \theta_3 F_T + (1-\theta_3)(x+Tf)$ and $z_2 = \theta_4 G_T^a + (1-\theta_4)(z+Tg)$ and $\theta_3, \theta_4 \in (0,1)$.

Since $\max\{|F_T|, |x+Tf|\} \le R_x$ (see (30)), then $|x_2| \le R_x$. Moreover, since $\max\{|G_T^a|, |z+Tg|\} \le R_z$ (see (30) and (31)), this implies $|z_2| \le R_z$. Hence, we have that $\left| \frac{\partial V}{\partial x}\Big|_{(x_2, G_T^a)} \right| \le b$ and $\left| \frac{\partial V}{\partial z}\Big|_{(x+Tf, z_2)} \right| \le b$.

Term **2a**: Since $\left| \frac{\partial V}{\partial x}\Big|_{(x_2, G_T^a)} \right| \le b$ and f is locally Lipschitz, we can write

$$\begin{aligned}
\frac{1}{T}\left| \frac{\partial V}{\partial x}\Big|_{(x_2,G_T^a)} \right| |F_T - (x+Tf)| &\le \frac{b}{T}\,|F_T - (x+Tf)| \\
&= \frac{b}{T}\left| \int_0^T f(x(\tau), u, d_c(\tau), d_s)d\tau - \int_0^T f(x,u,d_c,d_s)d\tau \right| \\
&\le \frac{b}{T}\left(L\int_0^T |x(\tau)-x|\,d\tau + L\int_0^T |d_c(\tau)-d_c|\,d\tau \right) \\
&\le \frac{bL}{T}\left\{ \Delta \int_0^T [\exp(L\tau)-1]d\tau + \Delta_{\dot{d}_c}\int_0^T \tau d\tau \right\} \\
&= bL\left\{ \Delta \frac{\exp(LT)-1-LT}{LT} + \frac{1}{2}\Delta_{\dot{d}_c} T \right\} \\
&\le \frac{\nu}{8}\,,
\end{aligned} \tag{38}$$

where we first added and subtracted $\frac{b}{T}\int_0^T f(x,u,d_c(\tau),d_s)d\tau$, then used Lipschitz property of f, then used bounds (32) and (34) and finally exploited the definition of T_2^*.

Term **2b**: We use the fact that $\left| \frac{\partial V}{\partial z}\Big|_{(x+Tf, z_2)} \right| \le b$ and add and subtract G_T^e to the last factor of Term **2b** to obtain:

$$\begin{aligned}
\frac{1}{T}\left| \frac{\partial V}{\partial z}\Big|_{(x+Tf,z_2)} \right| |G_T^a - (z+Tg)| &\le \frac{b}{T}\,|G_T^a - z - Tg| \\
&\le \frac{b}{T}\,|G_T^a - G_T^e| + \frac{b}{T}\,|G_T^e - z - Tg| \\
&\le b\rho(T) + \frac{b}{T}\left| \int_0^T g(x, z(\tau), d_c(\tau), d_s)d\tau - Tg(x,z,d_c,d_s) \right| \\
&\le b\rho(T) + \frac{b}{T}\int_0^T L\,|z(\tau)-z|\,d\tau + \frac{b}{T}\int_0^T L\,|d_c(\tau)-d_c|\,d\tau \\
&\le b\rho(T) + bL\left[\Delta \frac{\exp(LT)-1-LT}{LT} + \frac{1}{2}\Delta_{\dot{d}_c} T \right] \\
&\le \frac{\nu}{2} + \frac{\nu}{8}\,,
\end{aligned} \tag{39}$$

where we first used one-step weak consistency, then Lipschitz property of g, then inequalities (33) and (34) and finally definitions of T_1^* and T_2^*.

Term 3: From the differentiability of V, we apply the Mean Value Theorem to Term **3** (where x_1 and

z_1 are defined just before (27)) to obtain:

$$\begin{aligned}
\frac{1}{T}\Big\{V(x+Tf, z+Tg) - V(x,z) - \frac{\partial V}{\partial x}\Big|_{(x,z)} Tf - \frac{\partial V}{\partial z}\Big|_{(x,z)} Tg\Big\} \\
&\leq \frac{\partial V}{\partial x}\Big|_{(x_1,z+Tg)} f + \frac{\partial V}{\partial z}\Big|_{(x,z_1)} g - \frac{\partial V}{\partial x}\Big|_{(x,z)} f - \frac{\partial V}{\partial z}\Big|_{(x,z)} g \\
&\leq |f|\cdot\left|\frac{\partial V}{\partial x}\Big|_{(x_1,z+Tg)} - \frac{\partial V}{\partial x}\Big|_{(x,z)}\right| + |g|\cdot\left|\frac{\partial V}{\partial z}\Big|_{(x,z_1)} - \frac{\partial V}{\partial z}\Big|_{(x,z)}\right| \\
&\leq b\left|\frac{\partial V}{\partial x}\Big|_{(x_1,z+Tg)} - \frac{\partial V}{\partial x}\Big|_{(x,z)}\right| + b\left|\frac{\partial V}{\partial z}\Big|_{(x,z_1)} - \frac{\partial V}{\partial z}\Big|_{(x,z)}\right| \\
&\leq \frac{\nu}{8} + \frac{\nu}{8} \ .
\end{aligned} \tag{40}$$

In deriving (40) we first used the definition of b and then definitions of T_3^* and T_4^*. Combining (35), (36), (38), (39) and (40) complete the proof that Property P2 holds. The proof for Property P1 follows directly, as the consequence of Theorem 2.1. ■

Proof of Proposition 3.2: It is trivial: since $d_c \in \mathcal{L}_\infty$ and (16) is ISS, then $\xi \in \mathcal{L}_\infty$. Since f and h are continuous, then $\dot{\xi} \in \mathcal{L}_\infty$ and $v \in \mathcal{L}_\infty$. Finally, since h is locally Lipschitz, then

$$\begin{aligned}
|\dot{v}| &= \left|\lim_{\delta\to 0}\frac{h(\xi(t+\delta)) - h(\xi(t))}{\delta}\right| \\
&\leq L\lim_{\delta\to 0}\left|\frac{\xi(t+\delta)-\xi(t)}{\delta}\right| \\
&\leq L\left|\dot{\xi}\right| \ ,
\end{aligned}$$

which implies $\dot{v} \in \mathcal{L}_\infty$. ■

Sketch of proof of Theorem 3.3: The proof follows the same steps as that of Theorem 3.1. First, define all parameters and choose T^* in a similar way as in the proof of Theorem 3.1, and then show that Property P3 holds, using the chosen T^*. The only difference from the proof of Theorem 3.1 is that instead of using one-step weak consistency, we use the one-step strong consistency property. ■

5 An application: input-to-state stability

It was shown in [19] that if an ISS controller is emulated then ISS is preserved in a semiglobal practical sense for the sampled-data system. Detailed proofs were given in [19] only for the static state feedback case and dynamic case was only commented on. Below we use results of this paper to provide a sketch of proof for the case of emulation of dynamic ISS controllers. Suppose that the nonlinear plant

$$\dot{x} = f(x, u, d_c) \tag{41}$$

can be rendered ISS using dynamic feedback

$$\begin{aligned}
\dot{z} &= g(x,z) \\
u &= u(x,z) \ ,
\end{aligned} \tag{42}$$

where f, g, and u are locally Lipschitz. Suppose that the dynamic feedback controller is emulated and then implemented digitally using a sampler and zero order hold, where we use an approximation of the dynamic controller, so that:

$$\begin{aligned}
z(k+1) &= G_T^a(x(k), z(k)) \\
u(k) &= u(x(k), z(k)) \ ,
\end{aligned} \tag{43}$$

Assume that the approximate discrete-time model of the dynamic controller G_T^a is one-step strongly consistent with the exact discrete-time model G_T^e (see Definition 2.4). In order to shorten notation we write (x, z) to denote the vector $(x^T\ z^T)^T$.

We assume also that:

Assumption 5.1 *There exists $\gamma_g \in \mathcal{K}_\infty$ such that given any $\Delta > 0$ there exists $T^* > 0$ such that for all $|(x, z)| \leq \Delta$ and $T \in (0, T^*)$ we have:*

$$|G_T^a| \leq \gamma_g(|(x, z)|) \ . \tag{44}$$

■

Remark 5.1 *Since f and u are locally Lipschitz and zero at zero, the following is true: there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that given any strictly positive numbers Δ_1, Δ_2, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and $t_\circ \geq 0$ the solutions of the initial value problem:*

$$\dot{x}(t) = f(x(t), u(x(t_\circ), z(t_\circ)), d_c(t)), \quad x_\circ = x(t_\circ)$$

satisfy:

$$|x(t)| \leq \gamma_1(|(x(t_\circ), z(t_\circ))|) + \gamma_2(\|d_c\|_\infty), \ \forall t \in [t_\circ, t_\circ + T] \ ,$$

whenever $|(x(t_\circ), z(t_\circ))| \leq \Delta_1$ and $\|d_c\|_\infty \leq \Delta_2$. This, together with Assumption 5.1, guarantees that given any compact set of initial conditions $(x_\circ, z_\circ)$ and inputs d_c, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ the state of the sampled-data system $(x(t), z(t))$ is bounded uniformly in $t \in (0, T)$. This conditions is referred to as uniform boundedness over T (UBT) in [11]. ■

We can state and prove the following result using Theorem 3.3:

Corollary 5.1 *If the continuous time system (41), (42) with f, g and u locally Lipschitz is ISS, then given any approximate discrete-time model of the dynamic controller G_T^a which satisfies Assumption 5.1 and is one-step strongly consistent with the exact discrete-time model of the dynamic controller G_T^e, there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ such that given any quadruple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \nu)$, there exists $T^* > 0$ such that $\forall T \in (0, T^*)$, $|x(t_0)| \leq \Delta_x$, $|z(t_0)| \leq \Delta_z$, $\|d_c\|_\infty \leq \Delta_{d_c}$, the solutions of the sampled-data system (41), (43) satisfy:*

$$|(x(t), z(t))| \leq \beta(|(x(t_0), z(t_0))|, t - t_0) + \gamma(\|d_c\|_\infty) + \nu, \qquad \forall t \geq t_0 \geq 0 \ . \tag{45}$$

■

Sketch of proof of Corollary 5.1: Since the continuous time system (41), (42) is ISS, it implies (see Theorem 1 in [17]) that the system (41), (42) is (V, w)-dissipative, where V is smooth and there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty, \gamma_1 \in \mathcal{K}$ such that

$$\begin{aligned} \alpha_1(|(x, z)|) \leq V(x, z) &\leq \alpha_2(|(x, z)|) \\ w(x, z, d_c) &= -\alpha_3(|(x, z)|) + \gamma_1(|d_c|) \\ \left|\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial z}\right)\right| &\leq \alpha_4(|(x, z)|) \ . \end{aligned} \tag{46}$$

Then it follows from Theorem 3.3, that given any G_T^a which is one-step strongly consistent with G_T^e, and given any $(\Delta_1, \Delta_2, \Delta_3, \nu_1)$ there exists $T_1^* > 0$ such that for all $T \in (0, T_1^*)$ and $|x| \leq \Delta_1, |z| \leq \Delta_2, \|d_c[0]\|_\infty \leq \Delta_3$, the discrete-time model of (41), (43) satisfies:

$$\begin{aligned} \frac{\Delta V}{T} &\leq \frac{1}{T}\int_0^T [-\alpha_3(|(x, z)|) + \gamma_1(|d_c(\tau)|)]\, d\tau + \nu_1 \\ &\leq -\alpha_3(|(x, z)|) + \gamma_1(\|d_c[0]\|_\infty) + \nu_1 \ . \end{aligned} \tag{47}$$

This implies (see Lemma 4 of [9]) that there exists $\beta_2 \in \mathcal{KL}, \gamma_2 \in \mathcal{K}$ such that if all the assumptions on G_T^a hold and given any $(\Delta_4, \Delta_5, \Delta_6, \nu_2)$ there exists $T_2^* > 0$ such that for all $T \in (0, T_2^*)$ and $|x(0)| \leq \Delta_4$, $|z(0)| \leq \Delta_5$, $\|d_c\|_\infty \leq \Delta_6$, the discrete-time model of (41), (43) satisfies:

$$|(x(k), z(k))| \leq \beta_2(|(x(0), z(0))|, kT) + \gamma_2(\|d_c\|_\infty) + \nu_2, \quad \forall k \geq 0 \ . \tag{48}$$

Finally, from Assumption 5.1 it follows that solutions of the sampled-data system are UBT (see Remark 5.1 and Definition 2 in [11]) and then using results in Section 3 in [11], there exists $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ such that given any G_T^a which is one-step strongly consistent with G_T^a and any $(\Delta_x, \Delta_z, \Delta_{d_c}, \nu)$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and $|x(t_0)| \leq \Delta_x$, $|z(t_0)| \leq \Delta_z$, $\|d_c\|_\infty \leq \Delta_{d_c}$, the solutions of (41), (43) satisfy:

$$|(x(t), z(t))| \quad \leq \quad \beta(|(x(t_0), z(t_0))|, t - t_0) + \gamma(\|d_c\|_\infty) + \nu, \quad \forall t \geq t_0 \geq 0 \ , \tag{49}$$

which completes the proof. ■

It is important to note that we could not use Theorem 3.1 instead of Theorem 3.3 to prove semiglobal practical ISS of the sampled-data system in Corollary 5.1. Indeed, Theorem 3.1 requires us to impose an additional condition on disturbances to be Lipschitz and hence the bound (49) would hold for a smaller set of disturbances (bounded and Lipschitz) than measurable bounded disturbances for which the ISS property is defined.

6 Conclusions

We have presented general results on preservation of dissipation inequalities under sampling in the emulation controller design for the case of dynamic feedback control. This result generalizes all available results on emulation design in the sampled-data literature that we are aware of (see [2, 7, 12, 14, 15, 19, 21]) and provides a unified framework for digital controller design using the emulation method for general nonlinear systems.

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A Appendix

A sufficient condition for one-step weak consistency is the following:

Lemma A.1 *Consider G_T^e and G_T^a of the controller (3). If*

1. *G_T^a is one-step weakly consistent with G_T^{Euler} where $G_T^{Euler} := z + Tg(x, z, d_c, d_s)$,*
2. *given any quintuple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$, there exist $\rho_1 \in \mathcal{K}_\infty$, $\rho_2 \in \mathcal{K}_\infty$, $M > 0$, $T^* > 0$, such that, for all $T \in (0, T^*)$ and for all $|x| \leq \Delta_x$, $|z| \leq \Delta_z$, $|d_c| \leq \Delta_{d_c}$, $|d_s| \leq \Delta_{d_s}$,*

$$(a) \quad \max_{\{|x|\leq\Delta_x, |z|\leq\Delta_z, |d_c|\leq\Delta_{d_c}, |d_s|\leq\Delta_{d_s}\}} |g(x, z, d_c, d_s)| \leq M \ ,$$

$$(b) \quad |g(x, z_1, d_{c1}, d_s) - g(x, z_2, d_{c2}, d_s)| \leq \rho_1(|z_1 - z_2|) + \rho_2(|d_{c1} - d_{c2}|) \ ,$$

then G_T^a is one-step weakly consistent with G_T^e. ■

Proof of Lemma A.1: Let $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$ be given. Using $(\Delta_x, R_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$, where $R_z = \Delta_z + 1$, let the second condition of the lemma generate $T^* > 0$, $M > 0$, $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{K}_\infty$. Let $T_1^* := \min\{T^*, 1/M\}$. It follows from condition 2a of the lemma that, for each $|x| \le \Delta_x$, $|z| \le \Delta_z$, $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $|d_s| \le \Delta_{d_s}$ and all $t \in [0, T]$, where $T \in (0, T_1^*)$, the solution $z(t)$ of

$$\dot{z}(t) = g(x, z(t), d_c(t), d_s) \ , \qquad\qquad z(0) = z \tag{50}$$

satisfies $|z(t)| \le R_z$ and $|z(t) - z| \le Mt$. Condition 3 guarantees that $|d_c(t)| \le \Delta_{d_c}$ and $|d_c(t) - d_c(0)| \le \Delta_{\dot{d}_c} t$, for all $t \in [0, T_1^*]$. It then follows from condition 2b of the lemma that, for all $|z| \le \Delta_z$, $|x| \le \Delta_x$, $\|d_c[0]\|_\infty \le \Delta_{d_c}$, $\left\|\dot{d}_c[0]\right\|_\infty \le \Delta_{\dot{d}_c}$, $|d_s| \le \Delta_{d_s}$ and all $T \in (0, T_1^*)$,

$$\begin{aligned} \left| \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)] d\tau \right| &\le \int_0^T \rho_1(|z(\tau) - z|) d\tau + \int_0^T \rho_2(|d_c(\tau) - d_c|) d\tau \\ &\le T\rho_1(MT) + T\rho_2(\Delta_{\dot{d}_c} T) \ . \end{aligned} \tag{51}$$

Hence, we can write

$$\left| \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)] d\tau \right| \le T\tilde{\rho}(T) \ , \tag{52}$$

where $\tilde{\rho}(s) := \rho_1(Ms) + \rho_2(\Delta_{\dot{d}_c} s)$ is a $\mathcal{K}_\infty$ function since ρ_1 and ρ_2 are $\mathcal{K}_\infty$. Since

$$G_T^e(x, z, d_c[0], d_s) = z + Tg(x, z, d_c, d_s) + \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)] d\tau \ , \tag{53}$$

the result follows from (52) and the first condition of the lemma, which implies the existence of $\tilde{\rho}_1 \in \mathcal{K}_\infty$, such that

$$\left| G_T^a - G_T^{Euler} \right| \le T\tilde{\rho}_1(T) \ .$$

Finally, by letting $\rho = \tilde{\rho} + \tilde{\rho}_1$ we prove that G_T^a is one-step weakly consistent with G_T^e. ■

A sufficient condition for one-step strong consistency is the following:

Lemma A.2 *Consider G_T^e and G_T^a of the controller (20). If*

1. *G_T^a is one-step strongly consistent with G_T^{Euler} where $G_T^{Euler} := z + Tg(x, z, d_s)$,*
2. *given any triple of strictly positive real numbers $(\Delta_x, \Delta_z, \Delta_{d_s})$, there exist $\rho_1 \in \mathcal{K}_\infty$, $M > 0$, $T^* > 0$, such that, for all $T \in (0, T^*)$ and for all $|x| \le \Delta_x$, $|z| \le \Delta_z$, $|d_s| \le \Delta_{d_s}$,*

$$\begin{aligned} &(a) \quad \max_{\{|x| \le \Delta_x, |z| \le \Delta_z, |d_s| \le \Delta_{d_s}\}} |g(x, z, d_s)| \le M \ , \\ &(b) \quad |g(x, z_1, d_s) - g(x, z_2, d_s)| \le \rho_1(|z_1 - z_2|) \ . \end{aligned}$$

then G_T^a is one-step strongly consistent with G_T^e. ■

The proof of Lemma A.2 follows closely the steps of the proof of Lemma A.1.