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## Hamiltonicity of 3-arc graphs

Guangjun Xu · Sanming Zhou

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**Abstract** An arc of a graph is an oriented edge and a 3-arc is a 4-tuple  $(v, u, x, y)$  of vertices such that both  $(v, u, x)$  and  $(u, x, y)$  are paths of length two. The 3-arc graph of a graph  $G$  is defined to have vertices the arcs of  $G$  such that two arcs  $uv, xy$  are adjacent if and only if  $(v, u, x, y)$  is a 3-arc of  $G$ . We prove that any connected 3-arc graph is hamiltonian, and all iterative 3-arc graphs of any connected graph of minimum degree at least three are hamiltonian. As a corollary we obtain that any vertex-transitive graph which is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three must be hamiltonian. This confirms the conjecture, for this family of vertex-transitive graphs, that all vertex-transitive graphs with finitely many exceptions are hamiltonian. We also prove that if a graph with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs.

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## 1 Introduction

A path or cycle which contains every vertex of a graph is called a *Hamilton path* or *Hamilton cycle* of the graph. A graph is *hamiltonian* if it contains a Hamilton cycle, and is *Hamilton-connected* if any two vertices are connected by a Hamilton path. The hamiltonian problem, that of determining when a graph is hamiltonian, is a classical problem in graph theory with a long history. The reader is referred to [3], [4, Chapter 18], [8, Chapter 10] and [10] for results on Hamiltonicity of graphs.

In this paper we present a large family of hamiltonian graphs. Such graphs are defined by means of a graph operator, called the 3-arc graph construction, which bears some similarities with the line graph operator. This construction was first introduced in [17,24] in studying a family of arc-transitive graphs whose automorphism group contains a subgroup acting imprimitively on the vertex set. (A graph is *arc-transitive* if its automorphism group is transitive on the set of oriented edges.) It was used in classifying or characterizing certain families of arc-transitive graphs [9,12,17,18,23,25].

All graphs in this paper are finite and undirected without loops. We use the term *multigraph* when parallel edges are allowed. An *arc* of a graph  $G = (V(G), E(G))$  is an ordered pair of adjacent vertices, or equivalently an oriented edge. For adjacent vertices  $u, v$  of  $G$ , we use  $uv$  to denote the arc from  $u$  to  $v$ ,  $vu$  ( $\neq uv$ ) the arc from  $v$  to  $u$ , and  $\{u, v\}$  the edge between  $u$  and  $v$ . A *3-arc* of  $G$  is a 4-tuple of vertices  $(v, u, x, y)$ , possibly with  $v = y$ , such that both  $(v, u, x)$  and  $(u, x, y)$  are paths of  $G$ .

**Notation:** We follow [4] for graph-theoretic terminology and notation. The degree of a vertex  $v$  in a graph  $G$  is denoted by  $d(v)$ , and the minimum degree of  $G$  is denoted by  $\delta(G)$ . The set of arcs of  $G$  with tail  $v$  is denoted by  $A(v)$ , and the set of arcs of  $G$  is denoted by  $A(G)$ .

The general 3-arc construction [17, 24] involves a self-paired subset of the set of 3-arcs of a graph. The following definition is obtained by choosing this subset to be the set of all 3-arcs of the graph.

**Definition 1** *Let  $G$  be a graph. The 3-arc graph of  $G$ , denoted by  $X(G)$ , is defined to have vertex set  $A(G)$  such that two vertices corresponding to two arcs  $uv$  and  $xy$  are adjacent if and only if  $(v, u, x, y)$  is a 3-arc of  $G$ .*

It is clear that  $X(G)$  is an undirected graph with  $2|E(G)|$  vertices and  $\sum_{\{u,v\} \in E(G)} (d(u) - 1)(d(v) - 1)$  edges. We can obtain  $X(G)$  from the line graph  $L(G)$  of  $G$  by the following operations [14]: split each vertex  $\{u, v\}$  of  $L(G)$  into two vertices, namely  $uv$  and  $vu$ ; for any two vertices  $\{u, v\}, \{x, y\}$  of  $L(G)$  that are distance two apart in  $L(G)$ , say,  $u$  and  $x$  are adjacent in  $G$ , join  $uv$  and  $xy$  by an edge. On the other hand, the quotient graph of  $X(G)$  with respect to the partition  $\mathcal{P} = \{\{uv, vu\} : \{u, v\} \in E(G)\}$  of  $A(G)$  is isomorphic to the graph obtained from the square of  $L(G)$  by deleting the edges of  $L(G)$ . The reader is referred to [14, 13, 2] respectively for results on the diameter and connectivity, the independence, domination and chromatic numbers, and the edge-connectivity and restricted edge-connectivity of 3-arc graphs.

The following is the first main result in this paper.

**Theorem 1** *Let  $G$  be a graph without isolated vertices. The 3-arc graph of  $G$  is hamiltonian if and only if*

- (a)  $\delta(G) \geq 2$ ;
- (b) *no two degree-two vertices of  $G$  are adjacent; and*
- (c) *the subgraph obtained from  $G$  by deleting all degree-two vertices is connected.*

We remark that Theorem 1 can not be obtained from known results on the hamiltonicity of line graphs, though  $X(G)$  and  $L(G)$  are closely related as mentioned above. As a matter of fact, even if  $L(G)$  is hamiltonian,  $X(G)$  is not necessarily hamiltonian, as witnessed by stars  $K_{1,t}$  with  $t \geq 3$ .

We define the *iterative 3-arc graphs* of  $G$  by

$$X^1(G) = X(G), \quad X^{i+1}(G) = X(X^i(G)), \quad i \geq 1.$$

Theorem 1 together with [14, Theorem 2] implies the following result.

**Theorem 2** (a) *A 3-arc graph is hamiltonian if and only if it is connected.*

(b) *If  $G$  is a connected graph with  $\delta(G) \geq 3$ , then  $X^i(G)$  is hamiltonian for every integer  $i \geq 1$ .*

We will prove Theorems 1 and 2 in Section 3. In Section 4 we will prove the following result.

**Theorem 3** *Let  $G$  be a 2-edge connected graph with  $\delta(G) \geq 3$ . If  $G$  contains a path of odd length between any two distinct vertices, then its 3-arc graph is Hamilton-connected.*

A basic strategy in the proof of Theorems 1 and 3 is to find an Eulerian tour or an open Eulerian trail in a properly defined multigraph that produces the required Hamilton cycle or path. This is similar to the observation [5] that an Eulerian tour of a graph produces a Hamilton cycle of its line graph.

Theorem 3 implies the following result.

**Theorem 4** *If a graph  $G$  with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs  $X^i(G)$ ,  $i \geq 1$ .*

Given vertex-disjoint graphs  $G$  and  $H$ , the join  $G \vee H$  of them is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$ . Theorem 3 implies the following result.

**Corollary 1** *Let  $G$  and  $H$  be graphs such that  $\max\{\delta(G), \delta(H)\} \geq 2$ . Then  $X(G \vee H)$  is Hamilton-connected.*

In the case when  $G$  has a large order but small maximum degree,  $X(G)$  has a large order but relatively small maximum degree. In this case the Hamiltonicity of  $X(G)$  may not be derived from known sufficient conditions for Hamilton

cycles such as the degree conditions in the classical Dirac's or Ore's Theorem (see [3, 4, 8, 10]).

In spirit, Theorems 1 and 2 are parallel to the well-known conjecture of Thomassen [20] which asserts that every 4-connected line graph is hamiltonian. This conjecture is still open; see [6, 10, 11, 16, 22]. In contrast, Theorem 1 solves the hamiltonian problem for 3-arc graphs completely.

A well-known conjecture due to Lovász, formulated by Thomassen [21], asserts that all connected vertex-transitive graphs, with finitely many exceptions, are hamiltonian. Since the 3-arc graph of an arc-transitive graph is vertex-transitive, Theorem 2 implies the following result, which confirms this conjecture for a large family of vertex-transitive graphs. (The family of arc-transitive graphs is large from a group-theoretic point of view [19].)

**Corollary 2** *If a vertex-transitive graph is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three, then it is hamiltonian.*

The Lovász conjecture has been confirmed for several families of vertex-transitive graphs [15], including connected vertex-transitive graphs of order  $kp$ , where  $k \leq 4$ , (except for the Petersen graph and the Coxeter graph) of order  $p^j$ , where  $j \leq 4$ , and of order  $2p^2$ , where  $p$  is prime, and some families of Cayley graphs. Tools from group theory were used in the proof of almost all these results. Corollary 2 has a different flavour and its proof does not rely on group theory.

There has also been considerable interest on Hamilton-connectedness of vertex-transitive graphs. Theorem 4 implies that if a vertex-transitive graph (with at least four vertices) is Hamilton-connected, then so are its iterative 3-arc graphs. For example, it is known that every connected non-bipartite Cayley graph of degree at least three on a finite abelian group [7] or a Hamiltonian group [1] is Hamilton-connected. (A finite non-abelian group in which every subgroup is normal is called a Hamiltonian group.) From this and Theorem 4 we know immediately that all iterative 3-arc graphs of such a Cayley graph are also Hamilton-connected.

## 2 Preliminaries

Let  $G^*$  be a multigraph. A *walk* in  $G^*$  of length  $l$  is a sequence  $v_0, e_1, v_1, \dots, v_{l-1}, e_l, v_l$ , whose terms are alternately vertices and edges of  $G^*$  (not necessarily distinct), such that  $v_{i-1}$  and  $v_i$  are the end-vertices of  $e_i$ ,  $1 \leq i \leq l$ . A walk is *closed* if its initial and terminal vertices are identical, is a *trail* if all its edges are distinct, and is a *path* if all its vertices are distinct. Often we present a trail by listing its sequence of vertices only, with the understanding that the edges used are distinct. A trail that traverses every edge of  $G^*$  is called an *Eulerian trail* of  $G^*$ , and a closed Eulerian trail is called an *Eulerian tour*. A multigraph is *Eulerian* if it admits an Eulerian tour. It is well known that a multigraph is Eulerian if and only if all its vertices have even degrees.

A *2-trail* of  $G^*$  is a trail of length two (and so is a path or cycle of length two). We call a 2-trail  $(u, x, v)$  with mid-vertex  $x$  a *visit to  $x$*  (if  $u = v$ , then  $(u, x, u)$  is thought as entering and leaving  $x$  on parallel edges). When there is no need to make distinction between  $(u, x, v)$  and  $(v, x, u)$ , or the orientation of the visit is unknown, we write  $[u, x, v]$ . Two visits  $(u, x, v)$  and  $(u', x, v')$  are called *twin visits* if  $\{u, v\} = \{u', v'\}$  and the four edges involved are distinct. In particular, when  $u = v$ , two twin visits  $(u, x, u)$  and  $(u, x, u)$  use four parallel edges between  $u$  and  $x$ .

Denote by  $E^*(x)$  the set of edges of  $G^*$  incident with  $x \in V(G^*)$ , and  $d^*(x) = |E^*(x)|$  the degree of  $x$  in  $G^*$ . In the case when  $d^*(x)$  is even, a decomposition of  $E^*(x)$  into a set of visits to  $x$  is called a *visit-decomposition* of  $E^*(x)$  (at  $x$ ). In this definition the orientations of the visits in the decomposition are not important in our subsequent discussion. So we may view each visit  $(u, x, v)$  in such a visit-decomposition as a non-oriented path (if  $u \neq v$ ) or cycle (if  $u = v$ ) of length two. As an example, if  $E^*(x) = \{\{x, y\}, \{x, y\}, \{x, z\}, \{x, z\}\}$ , where  $\{x, y\}$  and  $\{x, y\}$  are viewed as distinct edges between  $x$  and  $y$ , then both  $\{[y, x, y], [z, x, z]\}$  and  $\{[y, x, z], [y, x, z]\}$  are visit-decompositions of  $E^*(x)$ .

**Definition 2** Given a visit-decomposition  $J(x)$  of  $E^*(x)$ , define  $H(x)$  to be the bipartite graph with vertex bipartition  $\{J(x), A(x)\}$  such that  $p \in J(x)$

and  $xy \in A(x)$  are adjacent if and only if  $y$  is not in  $p$ , where  $A(x)$  is the set of arcs of the underlying simple graph of  $G^*$  with tail  $x$ .

We emphasize that  $H(x)$  relies on  $J(x)$ . One can verify the following result by using Hall's marriage theorem.

**Lemma 1** *Suppose  $x$  is a vertex of  $G^*$  such that  $d^*(x) \geq 6$  is even and either  $x$  is joined to every neighbour of  $x$  by exactly two parallel edges, or  $x$  is joined to one of its neighbours by exactly three parallel edges, another neighbour by a single edge, and each of the remaining neighbours by exactly two parallel edges. Let  $J(x)$  be a visit-decomposition of  $E^*(x)$ . Then the bipartite graph  $H(x)$  with respect to  $J(x)$  has no perfect matchings if and only if  $d^*(x) = 6$  and  $J(x)$  contains two twin visits.*

**Proof** We have  $|J(x)| = |A(x)| = d^*(x)/2$  and  $\delta(H(x)) \geq (d^*(x)/2) - 2 \geq 1$ . One can show that, if  $d^*(x) \geq 8$ , then the neighbourhood  $N_{H(x)}(S)$  in  $H(x)$  of each  $S \subseteq J(x)$  has size at least  $|S|$ . Thus, by Hall's marriage theorem,  $H(x)$  has a perfect matching when  $d^*(x) \geq 8$ .

Suppose  $H(x)$  has no perfect matchings, so that  $d^*(x) = 6$  and  $|J(x)| = |A(x)| = 3$ . Then there exists  $S \subseteq J(x)$  such that  $|N_{H(x)}(S)| < |S|$ . This implies  $|S| = 2$  and so  $|N_{H(x)}(S)| \leq 1$ . Denote  $S = \{(u, x, v), (y, x, z)\}$ , where  $u, v, y, z \in N(x)$  (the neighbourhood of  $x$  in  $G^*$ ). Then  $N_{H(x)}(S) = (A(x) - \{xu, xv\}) \cup (A(x) - \{xy, xz\}) = A(x) - (\{xu, xv\} \cap \{xy, xz\})$ . Since  $|N(x)| = 3$  and  $|N_{H(x)}(S)| \leq 1$ , it follows that  $\{u, v\} = \{y, z\}$ , and therefore  $(u, x, v)$  and  $(y, x, z)$  are twin visits.

Conversely, if  $d^*(x) = 6$  and  $J(x)$  contains twin visits, then  $H(x)$  consists of two paths of length two and hence has no perfect matchings.  $\square$

**Definition 3** Let  $C : v_0, e_1, v_1, e_2, v_2, \dots, v_{l-2}, e_{l-1}, v_{l-1}, e_l, v_l$  be an Eulerian trail of  $G^*$ , possibly with  $v_l = v_0$ . The visit  $(v_{i-1}, v_i, v_{i+1})$  to  $v_i$  is said to be induced by  $C$ ,  $1 \leq i \leq l-1$ . In addition, if  $C$  is an Eulerian tour, then  $(v_{l-1}, v_0, v_1)$  is also a visit to  $v_0$  induced by  $C$ .

Denote by  $C(x)$  the set of visits to  $x \in V(G^*)$  induced by  $C$ .



where  $P^-$  is the trail obtained from  $P$  by reversing its direction, and  $e_2^{-1}$  and  $e_3^{-1}$  are the same edges as  $e_2$  and  $e_3$  but with reversed orientations, respectively. (See Figure 1 (a).)

We call  $C \rightarrow C((x_1, x, x_2), (x_3, x, x_4))$  the bow-tie operation on  $C$  with respect to  $(x_1, x, x_2)$  and  $(x_3, x, x_4)$ .

**Definition 6** Let

$$C_1 : x_1, e_1, x, e_2, \overbrace{x_2, \dots, x_1}^P; \quad C_2 : x_3, e_3, x, e_4, \overbrace{x_4, \dots, x_3}^Q.$$

be edge-disjoint closed trails of  $G^*$  with  $x$  as a common vertex. Define

$$C^1 : x_1, e_1, x, e_3^{-1}, \overbrace{x_3, \dots, x_4}^{Q^{-1}}, e_4^{-1}, x, e_2, \overbrace{x_2, \dots, x_1}^P.$$

We call  $(C_1, C_2) \rightarrow C^1$  the concatenation operation with respect to  $(C_1, C_2, (x_1, x, x_2), (x_3, x, x_4))$ . (See Figure 1 (b).)

*Remark 1* Some of  $x_1, x_2, x_3, x_4$  or even all of them in Definitions 5 and 6 are allowed to be the same vertex. Each of  $P, Q$  (and  $R$  in Definition 5) may visit some of  $x, x_1, x_2, x_3, x_4$  several times, and they may have common vertices.

In each operation above, the visits  $(x_1, x, x_2)$ ,  $(x_3, x, x_4)$  are replaced by  $(x_1, x, x_3)$ ,  $(x_4, x, x_2)$ , respectively. All other visits induced by  $C$  (in Definition 5) or  $C_1 \cup C_2$  (in Definition 6) are retained or with orientation reversed.

In Definition 6,  $C^1$  is a closed trail which covers every edge covered by  $C_1$  and  $C_2$ . In particular, if  $C_1$  and  $C_2$  collectively cover all edges of  $G^*$ , then  $C^1$  is an Eulerian tour of  $G^*$ .

### 3 Proof of Theorems 1 and 2

**Proof of Theorem 1** Denote by  $S_i$  the set of vertices of  $G$  with degree  $i$ , for  $i \geq 1$ .

Suppose that  $G$  has no isolated vertices and  $X(G)$  is hamiltonian. We show that (a), (b) and (c) hold. Note first that if  $G$  has a degree-one vertex, then

the unique arc emanating from it gives rise to an isolated vertex of  $X(G)$ . Similarly, if  $x, y \in S_2$  are adjacent, say,  $N(x) = \{u, y\}, N(y) = \{x, v\}$ , then the edge of  $X(G)$  between  $xu$  and  $yv$  is an isolated edge no matter whether  $u \neq v$  or not. Since  $X(G)$  is assumed to be hamiltonian, it follows that  $G$  is connected with  $\delta(G) \geq 2$  and  $S_2$  is an independent set of  $G$ .

It remains to prove that  $G - S_2$  is connected. Suppose otherwise. Then we can choose a minimal subset  $S$  of  $S_2$  such that  $G - S$  is disconnected. Note that  $S \neq \emptyset$  as  $G$  is connected. Let  $H$  be a component of  $G - S$ . The minimality of  $S$  implies that each vertex of  $S$  has exactly one neighbour in  $V(H)$ , and each vertex of  $S_2$  with both neighbours in  $H$  (if such a vertex exists) is contained in  $V(H)$ . Denote by  $A_1$  the set of arcs of  $G$  with tails in  $S$  and heads outside of  $V(H)$ . Denote by  $A_2$  the set of arcs of  $G$  with tails in  $V(H)$  (and heads in  $V(H)$  or  $S$ ). One can verify that the subgraph of  $X(G)$  induced by  $A_1 \cup A_2$  is a connected component of  $X(G)$ . Since there are arcs of  $G$  not in  $A_1 \cup A_2$ , it follows that  $X(G)$  is disconnected, contradicting our assumption. Hence  $G - S_2$  is connected.

Suppose that  $G$  satisfies (a), (b) and (c). We aim to prove that  $X(G)$  is hamiltonian. Note that  $G$  is connected by (c). Let  $G^*$  be the multigraph obtained from  $G$  by doubling each edge. Then the degree  $d^*(v) = 2d(v)$  of each  $v \in V(G)$  in  $G^*$  is even. Hence  $G^*$  is Eulerian. We will prove the existence of an Eulerian tour of  $G^*$  such that the corresponding bipartite graph (see Definition 3) at each vertex has a perfect matching. We will then exploit such an Eulerian tour to construct a Hamilton cycle of  $X(G)$ .

We claim first that there exists an Eulerian tour  $C$  of  $G^*$  such that

$$\text{if } v \in S_2 \text{ with } N(v) = \{u, w\}, \text{ then } C(v) \prec \{(u, v, u), (w, v, w)\}. \quad (1)$$

To construct such an Eulerian tour, we can start from any vertex and travel as far as possible without repeating any edge such that, whenever the tour reaches a vertex of  $S_2$ , it returns to the previous vertex immediately. Since  $G - S_2$  is connected, an Eulerian tour  $C$  of  $G^*$  satisfying (1) can be constructed this

way. Note that  $G^* - S_2$  is Eulerian because it is connected and all its vertices have even degrees.

For an Eulerian tour  $C$  of  $G^*$  satisfying (1), let  $Z(C)$  denote the set of vertices  $x$  such that  $H_C(x)$  has no perfect matchings. Since for every  $x \in S_2$ ,  $H_C(x) \cong 2K_2$  is a perfect matching, by Lemma 1 we have  $Z(C) \subseteq S_3$ .

Now we choose an Eulerian tour  $C$  of  $G^*$  satisfying (1) such that  $|Z(C)|$  is minimum. We claim that  $Z(C) = \emptyset$ . Suppose otherwise. Then by Lemma 1,  $C(x)$  contains twin visits for each  $x \in Z(C)$ . Denote  $N(x) = \{x_1, x_2, x_3\}$  for a fixed  $x \in Z(C)$ , and assume without loss of generality that  $C(x) = \{(x_1, x, x_2), (x_1, x, x_2), (x_3, x, x_3)\}$ . Denote  $C' = C((x_1, x, x_2), (x_3, x, x_3))$ . Then  $C'$  is an Eulerian tour of  $G^*$  and  $C'(x) = \{(x_1, x, x_2), (x_1, x, x_3), (x_2, x, x_3)\}$ . One can see that  $H_{C'}(x)$  is a perfect matching of three edges, and  $H_{C'}(y)$  is isomorphic to  $H_C(y)$  for each  $y \neq x$ . Thus  $Z(C')$  is a proper subset of  $Z(C)$ , and moreover (1) is respected by  $C'$  at every  $v \in S_2$ . Since this contradicts the choice of  $C$ , we conclude that  $Z(C) = \emptyset$ ; that is,  $H_C(v)$  has a perfect matching for each  $v \in V(G)$ .

Let  $C$  be a fixed Eulerian tour of  $G^*$  satisfying (1) such that  $Z(C) = \emptyset$ . Let us fix a perfect matching of  $H_C(v)$  for each  $v \in V(G)$ . Every traverse of  $C$  to  $v$  corresponds to a visit to  $v$ , say,  $(u, v, w)$ , and in the chosen perfect matching of  $H_C(v)$ ,  $(u, v, w)$  is matched to an arc of  $A(v)$  other than  $vu$  and  $vw$ . Denote this arc by  $\phi(u, v, w)$ . Then for any two consecutive visits  $(u, v, w), (v, w, x)$  induced by  $C$  (that is,  $(u, v, w, x)$  is a segment of  $C$ ),  $\phi(u, v, w)$  and  $\phi(v, w, x)$  are adjacent in  $X(G)$ . Since  $C$  is an Eulerian tour of  $G^*$  and a perfect matching of each  $H_C(v)$  is used, every arc of  $G$  is of the form  $\phi(u, v, w)$  for some segment  $(u, v, w)$  of  $C$ . Therefore, if, say,  $C = (u, v, w, x, y, \dots, a, b, c, u)$ , then the sequence

$$\phi(u, v, w), \phi(v, w, x), \phi(w, x, y), \dots, \phi(a, b, c), \phi(b, c, u), \phi(c, u, v), \phi(u, v, w)$$

of arcs of  $G$  gives rise to a Hamilton cycle of  $X(G)$ . □

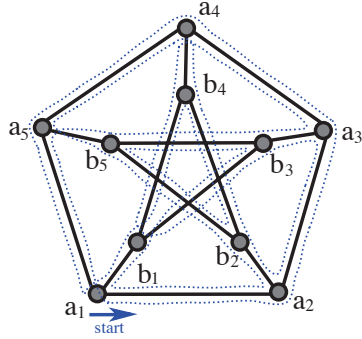
We illustrate the proof above by the following example.

*Example 1* Since the Petersen graph  $PG$  (see Figure 2) satisfies the conditions in Theorem 1, its 3-arc graph  $X(PG)$  is hamiltonian. Let

$$C : a_1, a_2, a_3, a_4, a_5, a_1, b_1, b_4, b_2, b_5, b_3, b_1, a_1, a_2, b_2, \\ b_5, a_5, a_4, b_4, b_2, a_2, a_3, b_3, b_1, b_4, a_4, a_3, b_3, b_5, a_5, a_1.$$

Then  $C$  is an Eulerian tour of the multigraph  $PG^*$  obtained from  $PG$  by doubling each edge. One can verify that at each  $a_i$  or  $b_i$ ,  $H_C(a_i)$  or  $H_C(b_i)$  has a perfect matching. In  $H_C(a_2)$  the ‘vertex’  $(a_1, a_2, a_3)$  is matched to the ‘vertex’  $a_2b_2$ , and in  $H_C(a_3)$ ,  $(a_2, a_3, a_4)$  is matched to  $a_3b_3$ , and so on. Continuing, one can verify that  $C$  gives rise to the following Hamilton cycle of  $X(PG)$ :

$$a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_1a_2, b_1b_3, b_4a_4, b_2a_2, b_5a_5, b_3a_3, b_1b_4, a_1a_5, a_2a_3, b_2b_4, b_5b_3, \\ a_5a_1, a_4a_3, b_4b_1, b_2b_5, a_2a_1, a_3a_4, b_3b_5, b_1a_1, b_4b_2, a_4a_5, a_3a_2, b_3b_1, b_5b_2, a_5a_4, a_1b_1, a_2b_2.$$



**Fig. 2** An Eulerian tour of  $PG^*$  which produces a Hamilton cycle of the 3-arc graph of the Petersen graph  $PG$ .

**Proof of Theorem 2** (a) Let  $G$  be a graph. Define  $\hat{G}$  to be the graph obtained from  $G$  by replacing each degree-two vertex  $v$  by a pair of nonadjacent vertices each joining to exactly one neighbour of  $v$  in  $G$ . In [14, Theorem 2] it is proved that, if  $\delta(G) \geq 2$ , then  $X(G)$  is connected if and only if  $\hat{G}$  is connected. One can verify that  $\delta(G) \geq 2$  and  $\hat{G}$  is connected if and only if (a), (b) and (c)

in Theorem 1 hold. Thus, by Theorem 1, if  $X(G)$  is connected, then it is hamiltonian. The converse of this statement is obvious.

(b) If  $G$  is connected with  $\delta(G) \geq 3$ , then  $\hat{G} = G$  and so  $X(G)$  is connected by [14, Theorem 2]. Hence, by (a),  $X(G)$  is hamiltonian. Since  $\delta(G) \geq 3$ , we have  $\delta(X(G)) \geq 3$ . Thus, by applying (a) to  $X(G)$ , we see that  $X^2(G)$  is hamiltonian. Continuing, by induction we can prove that  $X^i(G)$  is hamiltonian for every  $i \geq 1$ .  $\square$

#### 4 Proof of Theorems 3 and 4

Let us first introduce an operation that will be used in the proof of Theorem 3. Let  $G^*$  be an Eulerian multigraph and  $C$  an Eulerian tour of  $G^*$ . Let  $(z_1, x, z_2)$  be a visit of  $C$  to  $x$ . Write

$$C : z_1, e_1, x, e_2, \overbrace{z_2, \dots, z_1}^T,$$

where  $e_1$  is the oriented edge from  $z_1$  to  $x$ ,  $e_2$  the oriented edge from  $x$  to  $z_2$ , and  $T$  the segment of  $C$  from  $z_2$  to  $z_1$  covering all edges of  $G^*$  except  $e_1$  and  $e_2$ . Add two new vertices  $t, t'$  to  $G^*$  and join them to  $x$  by edges  $e_t, e_{t'}$ , respectively, with orientation towards  $x$ . Denote the resultant multigraph by  $G_C^*(z_1, x, z_2)$ . Set

$$W = W_C(z_1, x, z_2) : t, e_t, x, e_2, \overbrace{z_2, \dots, z_1}^T, e_1, x, e_{t'}^{-1}, t'.$$

Since  $C$  is an Eulerian tour of  $G^*$ ,  $W$  is an open Eulerian trail of  $G_C^*(z_1, x, z_2)$ . Denote by  $W(x)$  the set of visits to  $x$  induced by  $W$ . As the first and last visits induced by  $W$ ,  $(t, e_t, x, e_2, z_2)$  and  $(z_1, e_1, x, e_{t'}^{-1}, t')$  are members of  $W(x)$ . Note that  $xt, xt' \notin A(x)$ .

**Definition 7** Define  $K_C(z_1, x, z_2)$  to be the bipartite graph with bipartition  $\{W(x), A(x)\}$  such that an arc in  $A(x)$  is adjacent to a visit  $p \in W(x)$  if and only if its head does not appear in  $p$ . Denote by  $L_C(z_1, x, z_2)$  the graph obtained from  $K_C(z_1, x, z_2)$  by deleting the vertices  $(t, e_t, x, e_2, z_2)$ ,  $(z_1, e_1, x, e_{t'}^{-1}, t')$ ,  $xz_1$  and  $xz_2$ .

To prove Theorem 3, we need to prove that, for any two distinct arcs  $xy, uv$  of  $G$ , there exists a Hamilton path of  $X(G)$  between  $xy$  and  $uv$ . We will prove the existence of such a path by constructing a specific Eulerian trail in a certain auxiliary multigraph  $G^*$ . We treat the cases  $x = u$  and  $x \neq u$  separately in the next two lemmas.

**Lemma 2** *Under the condition of Theorem 3, for any distinct arcs  $xy, xv \in A(G)$  with the same tail, there exists a Hamilton path of  $X(G)$  between  $xy$  and  $xv$ .*

**Proof** By our assumption there exists a path in  $G$  of odd length connecting  $y$  and  $v$ . Let

$$P : y = x_0, x_1, x_2, \dots, x_{l-1}, x_l = v$$

be a path in  $G$  between  $y$  and  $v$  with minimum possible odd length  $l \geq 1$ . Denote  $E_0(P) = \{\{x_j, x_{j+1}\} \mid j = 0, 2, \dots, l-1\}$  and  $E_1(P) = \{\{x_j, x_{j+1}\} \mid j = 1, 3, \dots, l-2\}$ .

**Case 1.**  $x \notin V(P)$ . In this case let  $G^*$  be obtained from  $G$  by doubling each edge of  $E(G) - (E(P) \cup \{\{x, y\}, \{x, v\}\})$  and tripling each edge of  $E_0(P)$ .

**Case 2.**  $x \in V(P)$ . In this case we have  $l \geq 3$  and  $x = x_j$  for some  $1 \leq j \leq l-1$ . If  $2 \leq j \leq l-2$ , then since  $l$  is odd, one of the two paths  $y, x_1, \dots, x_{j-1}, x, v$  and  $y, x, x_{j+1}, \dots, x_{l-1}, v$  would be a path of odd length connecting  $y$  and  $v$  that is shorter than  $P$ , contradicting the choice of  $P$ . Therefore, either  $x = x_1$  or  $x = x_{l-1}$ . Assume without loss of generality that  $x = x_1$ . Define  $G^*$  to be the multigraph obtained from  $G$  by doubling each edge of  $E(G) - [(E(P) - \{\{x, y\}\}) \cup \{\{x, v\}\}]$  and tripling each edge of  $E_0(P) - \{\{x, y\}\}$ .

In each case above,  $d^*(x) = 2d(x) - 2$  and  $d^*(z) = 2d(z)$  for every  $z \neq x$ , and hence  $G^*$  is Eulerian.

Set  $a = y$  in Case 1 and  $a = x_2$  in Case 2. By extending the 2-path  $a, x, v$  to an Eulerian tour, we see that there are Eulerian tours of  $G^*$  which pass through  $(a, x, v)$ . Choose  $C$  to be an Eulerian tour of  $G^*$  with  $(a, x, v) \in C(x)$

such that  $|Z(C)|$  is minimum, where  $Z(C)$  is the set of vertices  $w \neq x$  of  $G^*$  such that  $H_C(w)$  has no perfect matching.

**Claim 1.**  $Z(C) = \emptyset$ ; that is,  $H_C(w)$  has a perfect matching for every  $w \neq x$ .

*Proof of Claim 1.* We prove this by way of contradiction. Suppose  $H_C(w)$  has no perfect matching for some  $w \neq x$ . By Lemma 1,  $d^*(w) = 6$  and  $C(w)$  contains twin visits. Since  $w \neq x$ , we have  $d(w) = 3$  by the construction of  $G^*$ . Denote  $N(w) = \{w_1, w_2, w_3\}$ . In the case when each of  $w_1, w_2$  and  $w_3$  is joined to  $w$  by two parallel edges, we apply the bow-tie operation at  $w$  with respect to one of the twin visits and the third visit of  $C(w)$ . Similar to the proof of Theorem 1, for the resultant Eulerian tour  $C'$  of  $G^*$ ,  $H_{C'}(w)$  has a perfect matching, and the visit-decomposition at any other vertex is unchanged. Thus  $(a, x, v) \in C'(x)$  and  $Z(C')$  is a proper subset of  $Z(C)$ , contradicting the choice of  $C$ .

It remains to consider the case where exactly one vertex of  $N(w)$  is joined to  $w$  by one, two or three (parallel) edges, respectively. Without loss of generality we may assume that there is one edge between  $w_3$  and  $w$ , two parallel edges between  $w_1$  and  $w$ , and three parallel edges between  $w_2$  and  $w$ . Then  $C(w) = \{(w_1, w, w_2), [w_1, w, w_2], [w_3, w, w_2]\}$ . Reversing the orientation of  $C$  when necessary, we may assume  $(w_1, w, w_2) \in C(w)$ . Denote by  $e_1, e_3$  the oriented parallel edges from  $w_1$  to  $w$ , by  $e_2, e_4, e_6$  the oriented parallel edges from  $w$  to  $w_2$ , and by  $e_5$  the oriented edge from  $w$  to  $w_3$ .

Case (a):  $C(w) = \{(w_1, w, w_2), (w_1, w, w_2), [w_3, w, w_2]\}$ . We may assume

$$C : w_1, e_1, w, e_2, w_2, f, \dots, g, w_1, e_3, w, e_4, w_2, h, \dots, k, w_1.$$

Let

$$C' : w_1, e_1, w, e_3^{-1}, w_1, g^{-1}, \dots, f^{-1}, w_2, e_2^{-1}, w, e_4, w_2, h, \dots, k, w_1.$$

Then  $C'$  is an Eulerian tour of  $G^*$  and  $C'(w) = \{(w_1, w, w_1), (w_2, w, w_2), [w_3, w, w_2]\}$ . Moreover,  $H_{C'}(w)$  has a perfect matching which matches  $(w_1, w, w_1)$ ,  $(w_2, w, w_2)$ ,  $[w_3, w, w_2]$  to  $ww_2$ ,  $ww_3$ ,  $ww_1$  respectively.

Case (b):  $C(w) = \{(w_1, w, w_2), (w_2, w, w_1), [w_3, w, w_2]\}$ . We may assume

$$C : w_1, e_1, w, e_2, w_2, f, \dots, g, w_2, e_4^{-1}, w, e_3^{-1}, w_1, h, \dots, k, w_1.$$

Denote

$$C_1 : w_1, e_1, w, e_3^{-1}, w_1, h, \dots, k, w_1; \quad C_2 : w_2, e_4^{-1}, w, e_2, w_2, f, \dots, g, w_2.$$

Note that each of  $C_1$  and  $C_2$  is a closed trail, and  $[w_3, w, w_2]$  is a segment of exactly one of  $C_1$  and  $C_2$ .

In the case when  $(w_3, w, w_2) \in C(w)$  and it is in  $C_2$ , we first rewrite  $C_2$  to highlight the position of  $(w_3, w, w_2)$  in  $C_2$ :

$$C'_2 : w_3, e_5^{-1}, w, e_6, w_2, \dots, w_3.$$

Applying the concatenation operation to  $(C_1, C'_2, (w_1, w, w_1), (w_3, w, w_2))$  yields:

$$C' : w_1, e_1, w, e_5, w_3, \dots, w_2, e_6^{-1}, w, e_3^{-1}, w_1, h, \dots, k, w_1.$$

We have  $C'(w) = \{(w_1, w, w_3), (w_2, w, w_1), [w_2, w, w_2]\}$ . Hence  $H_{C'}(w)$  has a perfect matching which matches  $(w_1, w, w_3)$ ,  $(w_2, w, w_1)$ ,  $[w_2, w, w_2]$  to  $ww_2, ww_3, ww_1$  respectively.

In the case when  $(w_3, w, w_2) \in C(w)$  and it is in  $C_1$ , we first rewrite  $C_1$  to highlight the position of  $[w_3, w, w_2]$  in  $C_1$ :

$$C'_1 : w_3, e_5^{-1}, w, e_6, w_2, \dots, w_3.$$

Applying the concatenation operation to  $(C_2, C'_1, (w_2, w, w_2), (w_3, w, w_2))$  yields:

$$C' : w_2, e_4^{-1}, w, e_5, w_3, \dots, w_2, e_6^{-1}, w, e_2, w_2, f, \dots, g, w_2.$$

Since  $C'(w) = \{(w_2, w, w_3), (w_2, w, w_2), [w_1, w, w_1]\}$ ,  $H_{C'}(w)$  has a perfect matching which matches  $(w_2, w, w_3)$ ,  $(w_2, w, w_2)$ ,  $[w_1, w, w_1]$  to  $ww_1, ww_3, ww_2$  respectively.

The remaining case when  $(w_2, w, w_3) \in C(w)$  can be dealt with similarly.

In all possibilities above we obtain a new Eulerian tour  $C'$  of  $G^*$  such that  $H_{C'}(w)$  has a perfect matching whilst the visit-decomposition at any other

vertex is unchanged. Thus  $(a, x, v) \in C'(x)$  and  $Z(C')$  is a proper subset of  $Z(C)$ , contradicting the choice of  $C$ . This completes the proof of Claim 1.

**Claim 2.** There exists an Eulerian tour  $C^*$  of  $G^*$  together with a visit  $(u_1, x, u_2) \in C^*(x)$  such that (i)  $H_{C^*}(z)$  has a perfect matching for every  $z \neq x$ , and (ii) the bipartite graph  $K_{C^*}(u_1, x, u_2)$  (as defined in Definition 7) has a perfect matching under which the first and last visits induced by  $W_{C^*}(u_1, x, u_2)$  are matched to  $xy$  and  $xv$  respectively.

Note that, for  $z \neq x$ ,  $H_{C^*}(z) = H_W(z)$ , where  $W = W_{C^*}(u_1, x, u_2)$ .

*Proof of Claim 2.* We will prove the existence of  $C^*$  and  $(u_1, x, u_2) \in C^*(x)$  based on  $C$  as in Claim 1.

Case (a):  $G^*$  was constructed in Case 1. Then  $(a, x, v) = (y, x, v) \in C(x)$  and all edges of  $G$  incident with  $x$  except  $\{x, y\}$  and  $\{x, v\}$  were doubled.

In the case when  $d(x) = 3$ , let  $z_1$  be the neighbour of  $x$  in  $G$  other than  $y$  and  $v$ . One can see that  $K_C(z_1, x, z_1)$  has a perfect matching which matches  $(t, x, z_1)$ ,  $(y, x, v)$ ,  $(z_1, x, t')$  to  $xy$ ,  $xz_1$ ,  $xv$ , respectively.

In the case when  $d(x) = 4$ , let  $z_1$  and  $z_2$  be the neighbours of  $x$  in  $G$  other than  $y$  and  $v$ . Since  $(y, x, v) \in C(x)$ , without loss of generality we may assume  $C(x) \prec \{(z_1, x, z_1), (z_2, x, z_2), (y, x, v)\}$  or  $\{(z_1, x, z_2), [z_1, x, z_2], (y, x, v)\}$ . If  $C(x) \prec \{(z_1, x, z_1), (z_2, x, z_2), (y, x, v)\}$ , then  $K_C(y, x, v)$  has a perfect matching which matches  $(t, x, v)$ ,  $(z_1, x, z_1)$ ,  $(z_2, x, z_2)$ ,  $(y, x, t')$  to  $xy$ ,  $xz_2$ ,  $xz_1$ ,  $xv$ , respectively. In the case when  $C(x) \prec \{(z_1, x, z_2), [z_1, x, z_2], (y, x, v)\}$ , by applying the bow-tie operation at  $x$  with respect to  $((z_1, x, z_2), (y, x, v))$  we obtain a new Eulerian tour  $C' = C((z_1, x, z_2), (y, x, v))$  for which  $C'(x) = \{[z_1, x, z_2], (z_j, x, y), (z_{j'}, x, v)\}$ , where  $\{j, j'\} = \{1, 2\}$ . Without loss of generality we may assume  $C'(x) = \{(z_1, x, z_2), (z_j, x, y), (z_{j'}, x, v)\}$ . One can see that  $K_{C'}(z_1, x, z_2)$  contains a perfect matching which matches  $(t, x, z_2)$ ,  $(z_j, x, y)$ ,  $(z_{j'}, x, v)$ ,  $(z_1, x, t')$  to  $xy$ ,  $xz_{j'}$ ,  $xz_j$ ,  $xv$ , respectively.

Assume  $d(x) \geq 5$ . If  $L_C(y, x, v)$  has a perfect matching, then adding the edges  $\{(t, x, v), xy\}$ ,  $\{(y, x, t'), xv\}$  to it yields a perfect matching of  $K_C(y, x, v)$  which matches the first and last visits of  $W_C(y, x, v)$  to  $xy, xv$ , respectively.

Suppose that  $L_C(y, x, v)$  has no perfect matchings. Similar to Lemma 1, by using Hall's marriage theorem we can prove that  $d(x) = 5$  and  $C(x)$  contains twin visits, say,  $[z_1, x, z_2]$ ; that is,  $C(x) \prec \{[z_1, x, z_2], [z_1, x, z_2], [z_3, x, z_3], (y, x, v)\}$ . Without loss of generality we may assume  $(z_1, x, z_2) \in C(x)$ . It is not hard to see that  $K_C(z_1, x, z_2)$  has a perfect matching which matches  $(t, x, z_2)$ ,  $(z_1, x, z_2)$ ,  $[z_3, x, z_3]$ ,  $(y, x, v)$ ,  $(z_1, x, t')$  to  $xy, xz_3, xz_2, xz_1, xv$ , respectively.

Case (b):  $G^*$  was constructed in Case 2. Then  $(x_2, x, v) \in C(x)$  and all edges of  $G$  incident with  $x$  except  $\{x, x_2\}$  and  $\{x, v\}$  were doubled.

In the case when  $d(x) = 3$ , we have  $C(x) \prec \{(x_2, x, v), (y, x, y)\}$  and  $K_C(x_2, x, v)$  has a perfect matching which matches  $(t, x, v)$ ,  $(y, x, y)$ ,  $(x_2, x, t')$  to  $xy, xx_2, xv$ , respectively.

In the case when  $d(x) = 4$ , we have  $C(x) \prec \{(x_2, x, v), [z_1, x, y], [z_1, x, y]\}$  or  $C(x) \prec \{(x_2, x, v)(z_1, x, z_1), (y, x, y)\}$ , where  $z_1$  is the neighbour of  $x$  other than  $y, v, x_2$ . If  $C(x) \prec \{(x_2, x, v), [z_1, x, y], [z_1, x, y]\}$ , let  $(z_1, x, y) \in C(x)$ , say. Then  $K_C(y, x, z_1)$  has a perfect matching, namely  $(t, x, z_1)$ ,  $(x_2, x, v)$ ,  $[z_1, x, y]$ ,  $(y, x, t')$  are matched to  $xy, xz_1, xx_2, xv$ , respectively. If  $C(x) \prec \{(x_2, x, v)(z_1, x, z_1), (y, x, y)\}$ , then  $K_C(z_1, x, z_1)$  has a perfect matching which matches  $(t, x, z_1)$ ,  $(x_2, x, v)$ ,  $(y, x, y)$ ,  $(z_1, x, t')$  to  $xy, xz_1, xx_2, xv$ , respectively.

Assume  $d(x) \geq 5$  hereafter. In the case when  $L_C(x_2, x, v)$  has a perfect matching, say,  $M$ , let  $xy$  be matched to  $(w_1, x, w_2)$  by  $M$ , where  $w_1, w_2 \in N(x) - \{x_2, v, y\}$ . Deleting  $\{(w_1, x, w_2), xy\}$  from  $M$  and then adding  $\{(w_1, x, w_2), xx_2\}$ ,  $\{(t, x, v), xy\}$  and  $\{(x_2, x, t'), xv\}$  yields a perfect matching of  $K_C(x_2, x, v)$  satisfying (ii) in Claim 2.

Suppose  $L_C(x_2, x, v)$  has no perfect matchings. Similar to Lemma 1, we can prove that  $d(x) = 5$  and  $C(x)$  contains twin visits. Denote by  $z_1, z_2 \neq y, v, x_2$  the other two neighbours of  $x$ . Let  $(w_1, x, w_2)$  be one of the twin visits in  $C(x)$ , where  $w_1, w_2 \in \{y, z_1, z_2\}$  are distinct, and let  $w_3$  denote the unique vertex in  $\{y, z_1, z_2\} - \{w_1, w_2\}$ . Then  $C(x) \prec \{(x_2, x, v), (w_1, x, w_2), [w_1, x, w_2], (w_3, x, w_3)\}$ . Since  $w_1$  and  $w_2$  are distinct, one of them, say,  $w_2$ , is not equal to  $y$ .

Thus  $K_C(w_1, x, w_2)$  has a perfect matching which matches  $(t, x, w_2)$ ,  $(x_2, x, v)$ ,  $[w_1, x, w_2]$ ,  $(w_3, x, w_3)$ ,  $(w_1, x, t')$  to  $xy$ ,  $xw_2$ ,  $xw_3$ ,  $xx_2$ ,  $xv$ , respectively.

Since  $H_C(z)$  has a perfect matching for every  $z \neq x$ , one can see that in all possibilities above, condition (i) in Claim 2 is satisfied by the underlying Eulerian tour (which is  $C$  or  $C'$ ). This proves Claim 2.

Choose an Eulerian tour  $C^* : w_l, x, w_1, w_2, w_3, \dots, w_l$  of  $G^*$  together with a visit  $(w_l, x, w_1) \in C^*(x)$  satisfying the conditions of Claim 2. Then  $W = W_{C^*}(w_l, x, w_1) : t, x, w_1, w_2, w_3, \dots, w_{l-1}, w_l, x, t'$ . Denote by  $\phi(t, x, w_1)$  ( $\phi(w_l, x, t')$ , respectively) the arc of  $G$  with tail  $x$  that is matched to  $(t, x, w_1)$  ( $(w_l, x, t')$ , respectively) by a perfect matching of  $K_{C^*}(w_l, x, w_1)$  satisfying (ii) in Claim 2. Let  $\phi(x, w_1, w_2)$  denote the arc matched to  $(x, w_1, w_2)$  in a perfect matching of  $H_{C^*}(w_1)$  ( $= H_W(w_1)$ ), and let  $\phi(w_1, w_2, w_3), \dots, \phi(w_{l-1}, w_l, x)$  be interpreted similarly. Conditions (i) and (ii) in Claim 2 ensure that

$$xy = \phi(t, x, w_1), \phi(x, w_1, w_2), \phi(w_1, w_2, w_3), \dots, \phi(w_{l-1}, w_l, x), \phi(w_l, x, t') = xv$$

is a Hamilton path of  $X(G)$  connecting  $xy$  and  $xv$ .  $\square$

**Lemma 3** *Under the condition of Theorem 3, for distinct  $xy, uv \in A(G)$  with  $x \neq u$ , there exists a Hamilton path of  $X(G)$  between  $xy$  and  $uv$ .*

**Proof** We have five possibilities to consider:  $x = v$  and  $y = u$ ;  $x, y, u, v$  are pairwise distinct;  $x = v$  and  $y \neq u$ ;  $y = v$  and  $x \neq u$ ;  $y = u$  and  $x \neq v$ . The following treatment covers all of them.

By our assumption there exists a path of odd length connecting  $x$  and  $u$  in  $G$ . Let

$$P : x = x_0, x_1, x_2, \dots, x_{l-1}, x_l = u \quad (2)$$

be such a path with shortest (odd) length  $l \geq 1$ . (It may happen that  $y = x_1$  and/or  $v = x_{l-1}$ .) Define  $G^*$  to be the multigraph obtained from  $G$  by doubling each edge of  $G$  outside of  $P$  and tripling each edge  $\{x_j, x_{j+1}\}$  for  $j = 1, 3, \dots, l-2$ . Then  $d^*(x) = 2d(x) - 1$ ,  $d^*(u) = 2d(u) - 1$  and  $d^*(z) = 2d(z)$  for  $z \neq x, u$ .

Let  $G_{x,u}^*(t, t')$  be the multigraph obtained from  $G^*$  by adding two new vertices  $t, t'$  and joining them to  $x, u$  respectively by a single edge. Then all vertices of  $G_{x,u}^*(t, t')$  except  $t$  and  $t'$  have even degrees in  $G_{x,u}^*(t, t')$ . Hence  $G_{x,u}^*(t, t')$  has Eulerian trails connecting  $t$  and  $t'$ .

Since  $\delta(G) \geq 3$ , we can choose  $x'$  to be a neighbour of  $x$  other than  $y$  and  $x_1$ , and  $u'$  a neighbour of  $u$  other than  $v$  and  $x_{l-1}$ . In addition, if  $d(x) = d(u) = 3$ ,  $y = x_1$  and  $v = x_{l-1}$ , say,  $N(x) = \{y, x', z\}$  and  $N(u) = \{v, u', w\}$ , then we can choose  $x'$  and  $u'$  in such a way that the edges  $\{x, z\}$  and  $\{u, w\}$  do not form an edge cut of  $G$ . In fact, if  $\{\{x, z\}, \{u, w\}\}$  is an edge cut of  $G$  in this case, then since  $G$  is assumed to be 2-edge connected,  $G - \{\{x, z\}, \{u, w\}\}$  has two connected components, say,  $G_0$  and  $G_1$  with  $z, w \in V(G_0)$  and  $P$  in  $G_1$ . Since  $x'$  is in  $G_1$  and removal of  $\{x, x'\}$  does not disconnect  $G$ , one can see that  $\{\{x, x'\}, \{u, w\}\}$  is not an edge-cut of  $G$ . Thus interchanging the roles of  $x'$  and  $z$  produces the desired  $x'$  and  $u'$ . (In general, at most one of  $x'$  and  $u'$  lies on  $P$  since  $P$  is a path between  $x$  and  $u$  with minimum odd length.)

With  $x'$  and  $u'$  as above, let

$$W' : t, x, x', \overbrace{x, x_1, x_2, \dots, x_{l-1}}^P, u, u', u, t',$$

where  $P$  is the path given in (2). Then  $W'$  is a trail of  $G_{x,u}^*(t, t')$ . Let  $W$  be an Eulerian trail of  $G_{x,u}^*(t, t')$  obtained by extending  $W'$  to cover all edges of  $G_{x,u}^*(t, t')$  while maintaining  $(t, x, x')$  and  $(u', u, t')$  as its first and last visits respectively. Such a trail  $W$  exists because removing the four edges in  $(t, x, x')$  and  $(u', u, t')$  from  $G_{x,u}^*(t, t')$  results in a connected multigraph with  $x'$  and  $u'$  as the only odd-degree vertices. In addition, if  $d(x) = 3$  and  $y = x_1$ , say,  $N(x) = \{y, x', z\}$ , since  $\{\{x, z\}, \{u, w\}\}$  is not an edge cut of  $G$  by our choices of  $x'$  and  $u'$ , we can choose  $W$  in such a way that  $(x', x, x_1)$  is a visit induced by  $W$ ; similarly, we can choose  $W$  such that  $(u', u, x_{l-1})$  is a visit induced by  $W$ , if  $d(u) = 3$  and  $v = x_{l-1}$ , say,  $N(u) = \{v, u', w\}$ . (Such a  $W$  can be constructed as follows: extend  $W'$  to an Eulerian trail of the multigraph obtained by deleting the parallel edges between  $x$  and  $z$  and/or that between

$u$  and  $w$ , and then insert the visits  $(z, x, z)$  and/or  $(w, u, w)$  to this trail.) In this way we obtain an Eulerian trail  $W$  of  $G_{x,u}^*(t, t')$  such that

- (A)  $(t, x, x')$  and  $(u', u, t')$  are its first and last visits, respectively; and
- (B) if  $d(x) = 3$  and  $y = x_1$ , say,  $N(x) = \{y, x', z\}$ , then  $(x', x, x_1) \in W(x)$ ; and, if  $d(u) = 3$  and  $v = x_{l-1}$ , say,  $N(u) = \{v, u', w\}$ , then  $(u', u, x_{l-1}) \in W(x)$ .

Similar to Claim 1, one can show that there exists an Eulerian trail of  $G_{x,u}^*(t, t')$ , denoted by  $W$  hereafter, satisfying (A), (B) and

- (C)  $H_W(z)$  has a perfect matching for every  $z \in V(G) - \{x, u\}$ .

Note that  $|W(z)| = |A(z)| = d(z)$  for every  $z \in V(G)$ .

**Claim 3.** There exists an Eulerian trail  $W^*$  of  $G_{x,u}^*(t, t')$  such that (i)  $(t, x, x')$  and  $(u', u, t')$  are its first and last visits, respectively; (ii)  $H_{W^*}(x)$  has a perfect matching under which  $(t, x, x')$  is matched to  $xy$ ; (iii)  $H_{W^*}(u)$  has a perfect matching under which  $(u', u, t')$  is matched to  $uv$ ; and (iv)  $H_{W^*}(z)$  has a perfect matching for every  $z \in V(G) - \{x, u\}$ .

*Proof of Claim 3.* Let  $p = (t, x, x')$  denote the first visit of  $W$ , and let  $L_W(x) = H_W(x) - \{p, xy\}$  be the subgraph of  $H_W(x)$  obtained by deleting vertices  $p$  and  $xy$ . For  $S \subseteq W(x) - \{p\}$ , denote by  $N_{L_W(x)}(S)$  the neighbourhood of  $S$  in  $L_W(x)$ .

Case (a):  $y \neq x_1$ . If  $d(x) \geq 5$ , then  $|N_{L_W(x)}(S)| \geq |S|$  for any  $S$ , and so  $L_W(x)$  contains a perfect matching by Hall's marriage theorem.

Suppose  $d(x) = 4$ . Then  $|N_{L_W(x)}(S)| \geq |S|$  for every  $S$  with  $|S| = 1$  or  $3$ . Suppose  $|S| = 2$  and  $S = \{(a, x, b), (a', x, b')\}$ . Then  $N_{L_W(x)}(S) = [(A(x) - \{xy\}) - \{xa, xb\}] \cup [(A(x) - \{xy\}) - \{xa', xb'\}] = [(A(x) - \{xy\})] - (\{xa', xb'\} \cap \{xa, xb\})$ . Thus, if  $|\{xa', xb'\} \cap \{xa, xb\}| \leq 1$ , then  $|N_{L_W(x)}(S)| \geq |S|$ . If  $|\{xa', xb'\} \cap \{xa, xb\}| = 2$ , then  $\{a, b\} = \{a', b'\}$  and  $\{x', x_1\} \cap \{a, b\} = \emptyset$ , which implies  $y \in \{a, b\}$  and  $|N_{L_W(x)}(S)| = |(A(x) - \{xa, xb\})| = 2$ . Hence  $L_W(x)$  contains a perfect matching by Hall's theorem.

Suppose  $d(x) = 3$ . Then  $W(x) = \{p, (x', x, y), (y, x, x_1)\}$  or  $W(x) = \{p, (x', x, x_1), (y, x, y)\}$ . In the former case  $L_W(x)$  clearly has a perfect matching.

In the latter case, apply the bow-tie operation to  $W$  with respect to  $(x', x, x_1)$  and  $(y, x, y)$  to obtain a new Eulerian trail  $W_0$  such that  $L_{W_0}(x)$  has a perfect matching.

Case (b):  $y = x_1$ . Similar to Case (a), if  $d(x) \geq 5$ , then  $L_W(x)$  has a perfect matching. If  $d(x) = 4$ , let  $N(x) = \{x', x_1, z_1, z_2\}$ . Then  $|N_{L_W(x)}(S)| \geq |S|$  unless  $S = \{(z_1, x, z_2), [z_1, x, z_2]\}$ . In this exceptional case,  $W(x) = \{p, (x', x, x_1), (z_1, x, z_2), [z_1, x, z_2]\}$ , and we apply the bow-tie operation to  $W$  with respect to  $(x', x, x_1)$  and  $(z_1, x, z_2)$  to obtain a new Eulerian trail  $W_0$ . One can show that  $L_{W_0}(x)$  has a perfect matching.

If  $d(x) = 3$ , let  $N(x) = \{x', x_1, z\}$ . By (B),  $(x', x, x_1)$  is a visit to  $x$  induced by  $W$ . Hence  $W(x) = \{p, (x', x, x_1), (z, x, z)\}$  and  $L_W(x)$  has a perfect matching.

So far we have proved that there exists an Eulerian trail  $W_1$  of  $G_{x,u}^*(t, t')$  (which is either  $W$  or  $W_0$ ) satisfying (A) such that  $L_{W_1}(x)$  has a perfect matching. This matching together with the edge between  $(t, x, x')$  and  $xy$  is a perfect matching of  $H_{W_1}(x)$ . Moreover, since  $W$  satisfies (C), from the proof above one can see that  $W_1$  satisfies (C) as well. If  $H_{W_1}(u)$  has a perfect matching which matches  $(u', u, t')$  to  $uv$ , then set  $W^* = W_1$  and we are done. Otherwise, beginning with  $W_1$  and using similar arguments as above, we can construct an Eulerian trail  $W^*$  of  $G_{x,u}^*(t, t')$  satisfying all requirements in Claim 3. This completes the proof of Claim 3.

Similar to the proof of Lemma 2, we can show that the Eulerian trail  $W^*$  in Claim 3 produces a Hamilton path in  $X(G)$  connecting  $xy$  and  $uv$ .  $\square$

**Proof of Theorem 3** This follows from Lemmas 2 and 3 immediately.  $\square$

In the proof of Theorem 4 we will use the following lemma which may be known in the literature. We give its proof since we are unable to allocate a reference.

**Lemma 4** *In any Hamilton-connected graph with at least four vertices, there exists a path of odd length connecting any two distinct vertices.*

**Proof** Let  $G$  be such a graph. Then for any distinct  $u, v \in V(G)$  there exists a Hamilton path  $P : u = x_0, x_1, x_2, \dots, x_{n-1}, x_n = v$ , where  $n = |V(G)| - 1$ . It suffices to consider the case when  $n$  is even. Denote  $A = \{x_0, x_2, \dots, x_n\}$  and  $B = \{x_1, x_3, \dots, x_{n-1}\}$ . Since  $\{A, B\}$  is a partition of  $V(G)$  and any bipartite graph other than  $K_2$  is not Hamilton-connected, there exist adjacent vertices  $x_i, x_j$  both in  $A$  or  $B$ , where  $j \geq i+2$ . Thus  $x_0, x_1, \dots, x_{i-1}, x_i, x_j, x_{j+1}, \dots, x_n$  is a path of odd length between  $u$  and  $v$ .  $\square$

**Proof of Theorem 4** It can be verified that any Hamilton-connected graph with at least four vertices is 2-edge connected and has minimum degree at least three. Hence Theorem 3 and Lemma 4 together imply that the 3-arc graph of such a graph is Hamilton-connected (with more than four vertices). Applying this iteratively, we obtain Theorem 4.  $\square$

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