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Diagonal stability for a class of graphs with connected circles

Wei Wang and Dragan Nešić

***Abstract*— Diagonal stability for a class of matrices having strongly connected graphs is considered, in which each pair of distinct simple circles have at most one common edge or a common vertex. We apply the obtained results to analyze stability of a class of nonlinear dynamical networked systems, for which each subsystem is output strictly passive and the storage function is available. We show that diagonal stability of the dissipativity matrix that includes the information of interconnection structure of subsystems implies that the sum of weighted storage functions is a storage Lyapunov function for this class of networks.**

I. Introduction

Stability analysis of nonlinear networked continuous-time systems is essential for understanding of their behavior and is invaluable for design of practical, high-performance and robust control strategies for system networks. Literature on this topic is diverse and most of it is based on either the small gain theorem [12], [19], the passivity theorem [4]–[6] or their combinations [9].

These strategies are often developed for networks of certain topology that is characterized with a graph of special structure. For example, graphs with structures like string [18], tree [19], acyclic [7], cyclic [6], [12] and cactus [5] have been investigated in the literature. In particular, assuming that each subsystem is input-to-state stable (ISS) and for which ISS-Lyapunov function is available, ISS properties are considered in [12] for a class of nonlinear dynamical networked systems when each group of their subsystems has cyclic interconnection structure and satisfies the small gain condition, i.e., the "loop gain" is less than 1. In contrast, with the assumption that subsystems are output strictly passive, [5] and [6] consider stability for networked systems with cyclic and cactus interconnection structure. The secant condition, like [5], [6], and a condition similar to small gain, see [2] for example, arise in this context naturally.

To compare the small gain condition with the secant condition, consider a matrix of the cyclic structure:

$$E = \begin{bmatrix} -1 & 0 & \cdots & e_{1,n} \\ e_{2,1} & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & e_{n,n-1} & -1 \end{bmatrix}, \tag{1}$$

where $e_{i,j} \neq 0$ for all $1 \leq i,j \leq n$. Let $\gamma := e_{2,1} \cdots e_{n,n-1} e_{1,n}$ that is the circle gain for the cyclic matrix E. Then, we know from [6, Theorem 1] that the matrix E in (1) is diagonally stable if and only if

$$|\gamma|\Phi(\mathrm{sgn}(\gamma), n) < 1 \tag{2}$$

where

$$\Phi(\mathrm{sgn}(\gamma), n) := \begin{cases} \cos^n(\pi/n) & if \quad \gamma < 0 \\ 1 & if \quad \gamma > 0\ . \end{cases}$$

When $\gamma < 0$, condition (2) is called the secant condition. For linear stability, it says that a linear stable system with distinct real eigenvalues and no zeros can tolerate negative feedback with a gain larger than that provided by the small gain theorem. The concept of secant condition takes the advantage simultaneously of phase and gain information on the open-loop system [17], which is employed to present diagonal stability for matrices with cyclic and cactus structure in [5], [6]. The cactus structure implies that distinct circles in the graph have at most one common vertex. It is challenging to extend results based on the secant condition to systems with general interconnection structure.

The requirement (2) when $\gamma > 0$ is the small gain condition. Although the bound of gains is comparatively restrictive than the secant condition, the small gain condition guarantees quasidominant properties for the matrix $-E$, or further M matrices properties if $-E$ is characterized by nonpositive off-diagonal entries, that is $e_{i,j} > 0$ for all $1 \leq i,j \leq n$. Note that properties of M matrices are quite useful in stability analysis for nonlinear interconnected systems, see examples in [2], [3] using the composite-system method and [1] based on decentralized techniques for systems with multiple model representations.

In this paper, we apply the small gain condition to consider diagonal stability for the interconnection structure that is more general than cyclic and cactus; cyclic and cactus topologies are special cases of the structure that we consider. Indeed, we present sufficient conditions to guarantee that high dimension matrices allowing for nonpositive off-diagonal entries are M matrices. Comparing with previous results on M matrices, see [14], [15], we decentralize a matrix E with strong connected graphs into reduced-order matrices E_i of the form (1). The graph for each of these matrices is a circle that is included in the graph for the original matrix E. We show that diagonal stability of E can be studied with these E_i. We also apply the obtained results to analyze stability of nonlinear dynamical networks.

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The outline of this paper is as follow. In Section II, some preliminary results on diagonal stability of matrices and some useful definitions are given. In Section III, we consider diagonal stability for a class of matrices and in Section IV we apply main results of Section III to analyze a class of nonlinear interconnected dynamical systems. Section V contains the conclusion.

II. Preliminaries

For matrices A and $B \in \mathbb{R}^{n\times n}$, let $A \succeq B$ ($A \succ B$) denote the case that all entries of $A - B$ are nonnegative (positive), in particular, $A \succeq 0$ means that A has nonnegative entries. A matrix $E \in \mathbb{R}^{n\times n}$ is a Z-matrix if $E = sI - A$ for some nonnegative matrix $A \succeq 0$. A Z-matrix $E := sI - A \in \mathbb{R}^{n\times n}$ with matrix $A \succeq 0$ and the real number s is an M-matrix if the condition that $s \geq \rho(A)$ holds for the spectral radius $\rho(A)$ of A. For a set S, $|S|$ denotes the cardinality of this set. A matrix $A \in \mathbb{R}^{n\times n} = \{a_{ij}\}$ is a quasidominant matrix if there exists a positive diagonal matrix $D = \mathrm{diag}\{d_1, d_2, \cdots, d_n\}$ such that $a_{ii}d_i \geq \sum_{j\neq i} |a_{ij}|d_j$ for all i. [1]. A matrix $E \in \mathbb{R}^{n\times n}$ is called diagonally stable if there exists a diagonal matrix $D > 0$ such that

$$DE + E^T D < 0 \ . \tag{3}$$

Such a matrix is also called Volterra-Lyapunov stable or Lyapunov diagonally stable, see more details on diagonal stability of matrices in [6].

Note that the definition in (3) implies that a necessary condition to guarantee diagonal stability for a matrix $E \in \mathbb{R}^{n\times n}$ with entries e_{ij} for $i, j \in \{1, \cdots, n\}$, is that all its diagonal entries e_{ii} are negative. Since scaling the rows of E by positive constants or permutations do not change diagonal stability properties, we consider a matrix E with diagonal entries $e_{ii} = -1$ in the sequel.

To relate the matrix E to a graph of certain topology, we define its graph $G(E)$ as follows in the following sense.

Definition 1: The graph for the matrix $E := \{e_{ij}\} \in \mathbb{R}^{n\times n}$, denoted as $G(E)$, is defined as a weighted directed graph, for which there are n vertices denoting the dimension of E. An edge with weights e_{ij} that is a link leads from vertex j to vertex i is used to represent each of nonzero off diagonal entries e_{ij} of E. Self loops corresponding to the non-zero diagonal entries e_{ii} are excluded.

For example: the graphs for matrices

$$E_1 = \begin{bmatrix} -1 & 0 & 0 & e_{14} & 0 \\ e_{21} & -1 & e_{23} & 0 & 0 \\ 0 & e_{32} & -1 & 0 & e_{35} \\ 0 & e_{42} & 0 & -1 & e_{45} \\ 0 & 0 & 0 & e_{54} & -1 \end{bmatrix} \tag{4}$$

[1]Note that this quasidominant matrix definition coming from [13] is slightly stronger than the usual definition [16] that requires $|a_{ii}|d_i \geq \sum_{j\neq i} |a_{ij}|d_j$ for all i. The quasidominant matrix defined in the present paper requires that all diagonal entries of A are positive since matrices with nonpositive diagonal entries are of no interest in the sequel.

and

$$E_2 = \begin{bmatrix} -1 & 0 & 0 & e_{14} & 0 \\ e_{21} & -1 & e_{23} & 0 & 0 \\ 0 & e_{32} & -1 & 0 & 0 \\ 0 & e_{42} & 0 & -1 & e_{45} \\ 0 & 0 & 0 & e_{54} & -1 \end{bmatrix} \tag{5}$$

are given in Fig. 1. Note that E_1 is the dissipativity matrix for mitogen-activated protein kinase model [4] that is used later to illustrate how to apply our main results.

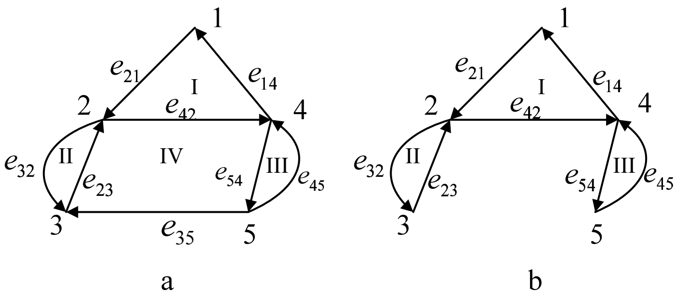

Fig. 1. a. Graph $G(E_1)$, b. Cactus example $G(E_2)$.

A simple circle refers to the circle with no repeated vertices other than the starting and ending vertex. A directed graph is called strongly connected if there is a path from each vertex in the graph to every other vertex. We assume that the matrix E is irreducible, or its graph $G(E)$ is strongly connected. We need the following technical results on quasidominant and M matrices properties.

Claim 1 ([13]): For a matrix $A \in \mathbb{R}^{n\times n} := \{a_{ij}\}$, let a matrix $\tilde{A} \in \mathbb{R}^{n\times n} := \{\tilde{a}_{ij}\}$ be such that

$$\begin{aligned} \tilde{a}_{ii} &:= a_{ii} \\ \tilde{a}_{ij} &:= |a_{ij}| \quad \text{for } j \neq i \ . \end{aligned} \tag{6}$$

Then, the matrix A is quasidominant if and only if $\tilde{A}$ is an M matrix.

Claim 2 ([13]): If A is quasidominant then $-A$ is diagonally stable. □

Claim 3 ([10]): Let $n \times n$ matrices $A, B \in Z$ be given. Assume that A is an M-matrix and $A \preceq B$, then B is an M-matrix.

III. Main results

In this section, we present diagonal stability analysis results for a class of interconnected structure that is more general than cyclic or cactus. Noting that permutation and scaling the rows of a matrix by positive constants do not change diagonal stability properties, we consider matrices satisfying the following conditions.

Assumption 1: The matrix $E \in \mathbb{R}^{n\times n} := \{e_{ij}\}$ has diagonal entries $e_{ii} = -1$ for $i = 1, \cdots, n$ and strongly connected graph $G(E)$, in which any pair of distinct simple circles have at most one common edge or a common vertex. □

Assumption 2: The matrix $E \in \mathbb{R}^{n\times n} := \{e_{ij}\}$ has nonnegative off diagonal entries. □

The strong connection requirement in Assumption 1 implies that the matrix E is irreducible. There is no loss of generality since one can permutate a reducible matrix into a diagonal block matrix with irreducible blocks as its entries. Then, analysis of diagonal stability for a reducible matrix turns into studying diagonal stability for irreducible blocks [5]. With the strong connection assumption, we have that the graph of the matrix E includes at least one circle and each link is included at least in one circle.

For the graph $G(E)$, we denote nodes $i = 1, \cdots, n$, links $l = 1, \cdots, N$ and circles $c = 1, \cdots, p$. Let ij denote a directed edge from vertex j to vertex i. Let L_c and I_c be the set of links ij and vertices respectively that are traversed by the c-th circle. Let $J_i = \{c \in \{1, \cdots, p\} : i \in I_c\}$ denote the set of circles that vertex i belongs to. We next consider the matrix E_1 in (4) as an example to illustrate the notation L_c, I_c and J_i.

Example 1: For the matrix E_1 in (4) and its graph given in Fig. 1-a, noting that L_c is the set of links traversed by the c-th circle, one gets $L_I = \{21, 42, 14\}$ and $L_{II} = \{23, 32\}$.

From the definition of I_c, we have that $I_I = \{1, 2, 4\}$ and $I_{II} = \{2, 3\}$, $I_{III} = \{2, 3, 4, 5\}$, $I_{IV} = \{4, 5\}$. Since J_i is the sets of circles that vertex i belongs to, we have that $J_1 = \{I\}$, $J_2 = \{I, II, IV\}$, $J_3 = \{II, IV\}$, $J_4 = \{I, III, IV\}$, $J_5 = \{III, IV\}$. Note that the sum of cardinality of sets I_c and J_i for $c = 1, \cdots, 4$ and $i = 1, \cdots, 5$ satisfies

$$\sum_{c=1}^{4} |I_c| - \sum_{i=1}^{5}(|J_i| - 1) = 11 - 6 = 5 \ ,$$

which agrees with the dimension of E_1 or the number of vertices in $G(E_1)$. This fact is true in general and it is useful to prove the main results. □

We next define the circle gain γ_c for the c-th circle of the $G(E)$ to present the main results.

Definition 2: For a matrix $E \in \mathbb{R}^{n\times n}$ and the c-th circle in its graph $G(E)$, let $E(I_c) \in \mathbb{R}^{|I_c|\times|I_c|}$ be the matrix obtained by selecting the rows and columns of E whose indices belong to I_c and letting $e_{ij} := 0$ for the link $ij \notin L_c$. Letting a proper permutation bring the cyclic matrix E_{I_c} to the form (1), the gain for the c-th circle is the product of its nonzero off diagonal entries:

$$\gamma_c := \prod_{l\in L_c} e_l \tag{7}$$

Note that in the above definition, we use the fact that for each link $ij \in L_c$ and the matrix $E(I_c)$, there exists a nonzero entry e_{ij} and all other off diagonal entries are zero.

We first present some useful results on diagonal stability for matrices of cactus structure. The strong connected graph $G(E)$ has a cactus structure, if each pair of distinct simple circles in $G(E)$ have at most one common vertex. Fig. 1-b is an example of cactus graph. In contrast, cyclic is the special case of cactus structure for which the graph $G(E)$ contains a single circle. The following results come directly from [5, Theorem 1].

Proposition 1: A matrix $E \in \mathbb{R}^{n\times n} := \{e_{ij}\}$ satisfying Assumption 2, having diagonal entries $e_{ii} = -1$ for $i = 1, \cdots, n$ and strongly connected cactus graph $G(E)$ is diagonally stable if and only if there exist constants $\theta_i^c > 0$ such that

$$\prod_{i\in I_c} \theta_i^c > \gamma_c \quad c = 1, \cdots, p \tag{8}$$

$$\sum_{c\in J_i} \theta_i^c = 1 \quad i = 1, \cdots, n \tag{9}$$

with γ_c given by (7), where p is the number of circles in $G(E)$. □

Note that the results for cactus structure in [5, Theorem 1] apply the concept of the secant condition and Assumption 2 is not required in that case, which provides a more relaxed bound for γ_c than Proposition 1. However, for the class of matrices having the graph that is also strongly connected but there are more than one vertex shared by distinct simple circles, it not easy to consider diagonal stability of matrices using secant condition. Assumption 2, on the other hand, implies that diagonal stability of E is equivalent to the fact that $-E$ is an M-matrix. This is helpful to show diagonal stability for matrices satisfying Assumptions 1 and 2. Indeed, we need the fact:

Claim 4 ([11]): A matrix $E \in \mathbb{R}^{n\times n}$ satisfying Assumption 2 is diagonally stable if and only if $-E$ is an M matrix. □

The above claim is used to show the following diagonal stability results for the matrices with graphs more general than cactus or cyclic. The proof is omitted due to page limit.

Theorem 1: A matrix $E \in \mathbb{R}^{n\times n}$ satisfying Assumptions 1 and 2 is diagonally stable if there exist constants $\theta_i^c > 0$ such that

$$\prod_{i\in I_c} \theta_i^c > \gamma_c \quad c = 1, \cdots, p \tag{10}$$

$$\sum_{c\in J_i} \theta_i^c = 1 \quad i = 1, \cdots, n \tag{11}$$

with γ_c in (7), where p is the number of circles in $G(E)$.

For matrices such that Assumption 1 holds but Assumption 2 might not hold, a corollary can be derived from Theorem 1 to show that under conditions (12) and (13) the sign pattern of nondiagonal entries of E does not influence the main results we obtained. This is due to the fact that we apply the small gain condition which is determined by the gain not the phase.

Corollary 1: A matrix $E \in \mathbb{R}^{n\times n} := \{e_{ij}\}$ satisfying Assumption 1 is diagonally stable if there exist constants $\theta_i^c > 0$ such that

$$\prod_{i\in I_c} \theta_i^c > |\gamma_c| \quad c = 1, \cdots, p \tag{12}$$

$$\sum_{c\in J_i} \theta_i^c = 1 \quad i = 1, \cdots, n \tag{13}$$

with γ_c in (7), where p is the number of circles in $G(E)$.

Proof: Let E be given and γ_c come from (7). Suppose that (12) and (13) hold. For the matrix $-E$, let a matrix

$\tilde{A} \in \mathbb{R}^{n\times n}$ be defined as (6). Since the matrix $-\tilde{A}$ satisfies Assumptions 1 and 2, (10) and (11) hold for $-\tilde{A}$ as $\gamma_c = |\gamma_c|$, we know that $-\tilde{A}$ is diagonally stable from Theorem 1. Note that $\tilde{A}$ is an M matrix from Claim 4, and then $-E$ is quasidominant from Claim 1. Then, Claim 2 concludes that E is diagonally stable. □

IV. Stability analysis for networked systems

The results obtained in last section can be applied to analysis of networked systems, for which each subsystem is output strictly passive (OSP). To apply the diagonal stability results for matrices, constructing a dissipativity matrix that includes information on passivity gains of each subsystem and their interconnection structure is needed. The following techniques to construct dissipativity matrices for some classes of nonlinear networked systems, for which each subsystem has single output or multiple outputs, come from [6] and [4] respectively. They are given here for completeness.

For networks with SISO subsystems H_i, suppose that H_i is OSP with the input and output pair (u_i, y_i) and the gain $\gamma_i > 0$ for $i = \{1, \cdots, n\}$:

$$||y_i||^2 \le \gamma_i < u_i, y_i > +\delta_i \ , \tag{14}$$

where $\delta_i \ge 0$ represents a bias due to initial conditions. Suppose that for each subsystem H_i there exists a storage function V_i, which with (14) implies that V_i satisfies

$$\dot{V}_i \le -\frac{1}{\gamma_i} y_i^2 + u_i y_i \ , \tag{15}$$

and subsystems H_i are coupled according to the feedback law

$$u = (K \otimes I_m) y \ , \tag{16}$$

where $\otimes$ refers to Kronecker product, $u := [u_1^T, \cdots, u_n^T]^T$, $y := [y_1^T, \cdots, y_n^T]^T$, $K \in \mathbb{R}^{n\times n}$ and I_m is the $m \times m$ identity matrix.

The dissipativity matrix is defined as

$$E := -I + \operatorname{diag}(\gamma_1, \cdots, \gamma_n)\, K \tag{17}$$

that includes passivity gains γ_i in (15) and the interconnection structure denoted by the matrix K. Then, diagonal stability of E guarantees the existence of a positive diagonal matrix $D := \operatorname{diag}\{d_i\}$ such that $V := \sum_i^n k_i V_i$ is the Lyapunov function for the original system and $k_i := d_i \gamma_i$ since

$$\dot{V} = \sum_i^n k_i \dot{V}_i \le \frac{1}{2} y^T ((DE + E^T D) \otimes I_m) y < 0 \ \ \forall\ y \ne 0 \ .$$

For networks with SISO subsystems, each node in $G(E)$ for the dissipativity matrix $E \in \mathbb{R}^{n\times n}$ represents a subsystem and the links denote the feedback connections among subsystems. On the other hand, for an interconnected system with subsystems having possible multiple outputs, it is useful to construct the dissipativity matrix so that the node of its graph is the feedback connection between subsystems. In particular, for a system that has n subsystems, let $l := ij$ for $i, j \in \{1, \cdots, n\}$ denote a link that reflects one of the outputs of subsystem H_j (a separate link is used for each output) acting as part of the input to subsystem H_i. If there are N such links connecting subsystems, then the dissipativity matrix E is of order $N \times N$. Note that it is essential but not easy to verify (15) for constructing such a E for complex dynamical systems. We next consider a class of nonlinear systems from [4] and show that (15) holds with nonlinearity satisfying sector conditions. To provide details on constructing matrix $E \in \mathbb{R}^{N\times N}$, some notation is needed.

For a link $l := ij$ directed from j to i, j is referred as the source and i as the sink of the link. For the $i = 1, \cdots, n$ and $l = 1, \cdots, N$, let $\mathcal{L}_i^+ \subseteq \{1, \cdots, N\}$ be the subset of link for which i is the sink and $\mathcal{L}_i^- \subseteq \{1, \cdots, N\}$ is the subset of links for which i is the source. Then, $i = \text{source}(l)$ if $l \in \mathcal{L}_i^-$ and $i = \text{sink}(l)$ if $l \in \mathcal{L}_i^+$. With this representation, we can consider dynamical systems of the form:

$$H_i: \quad \dot{x}_i = -f_i(x_i) + g_i(x_i) \sum_{l \in \mathcal{L}_i^+} h_l(x_{\text{source(l)}}) \tag{18}$$

for $i = 1, \cdots, n$, where f_i, g_i and h_l are locally Lipschitz continuous functions. For subsystem H_i, its input is $u_i = \sum_{l\in\mathcal{L}_i^+} h_l(x_{\text{source(l)}})$, and there are multiple outputs $y_i = h_l(x_i)$ for $l \in \mathcal{L}_i^-$ if $|\mathcal{L}_i^-| > 1$. Without loss of generality, let the origin be the equilibrium. For each $i = 1, \cdots, n$ and $l \in \mathcal{L}_i^-$, suppose that the following sector conditions hold:

$$\begin{array}{ll} x_i \frac{f_i(x_i)}{g_i(x_i)} > 0 & \\ x_i \frac{h_l(x_i)}{g_i(x_i)} > 0 & \end{array} \quad \forall\ x_i \ne 0 \tag{19}$$

and there exists $\gamma_l > 0$ such that

$$\frac{h_l(x_i)}{\frac{f_i(x_i)}{g_i(x_i)}} \le \gamma_l \quad \forall\ x_i \ne 0 \ . \tag{20}$$

We next show that under conditions (19) and (20), for each H_i in (18) and each of its output $y_l(x_i) := h_l(x_i)$ for $l \in \mathcal{L}_i^-$, H_i is OSP and there exists a storage function $V_l(x_{\text{source}(l)})$ satisfying (15). Let

$$V_l(x_{\text{source}(l)}) = \int_0^{x_{\text{source}(l)}} \frac{h_l(s)}{g_{\text{source}(l)}(s)} ds \ , \tag{21}$$

which is positive definite due to sector condition in (19). Noting that $i = \text{source}(l)$ for $l \in \mathcal{L}_i^-$ and (18), we have that

$$\dot{V}_l(x_i) = h_l(x_i) \left(-\frac{f_i(x_i)}{g_i(x_i)} + \sum_{k\in\mathcal{L}_i^+} h_k(x_{\text{source}(k)}) \right) \ , \tag{22}$$

which with the definition of u_i and

$$-h_l(x_i) \frac{f_i(x_i)}{g_i(x_i)} \le -\frac{1}{\gamma_l} h_l^2(x_i) \ ,$$

that comes from (20) shows that (22) satisfies (15).

For the networked system in (18) with passivity gains γ_l for $l = 1, \cdots, N$, let the dissipativity matrix $E_l \in \mathbb{R}^{N\times N}$ with entries e_{lk} be defined as

$$e_{lk} = \begin{cases} -1/\gamma_l & l = k \\ 1 & \text{source}(l) = \text{sink}(k) \\ 0 & \text{otherwise .} \end{cases} \qquad (23)$$

Then, we next show that diagonal stability of E_l ensures asymptotic stability for the interconnected system with noting the results in Corollary 1.

Theorem 2: The networked dynamical system with n subsystems H_i in (18) and N links is asymptotically stable, if H_i satisfies conditions (19) and (20) for $i = 1, \cdots, n$, its dissipativity matrix $E_l := \{e_{lk}\} \in \mathbb{R}^{N\times N}$ with e_{kl} defined in (23) satisfies Assumption 1 and is diagonally stable. □

Proof: Since E_l satisfies Assumption 1, one knows that the subsystems are strongly connected, that is, the set $\mathcal{L}_i^-$ is nonempty for all $i = 1, \cdots, n$. Thus, for each link $l = 1, \cdots, N$, there exists a subsystem H_i with $i = \text{source}(l)$ for which its output is $y_l(x_i)$ and a storage functions $V_l(x_i)$ in (21) satisfying (15). Let a Lyapunov candidate function V be defined as

$$V = \sum_{l=1}^{N} d_l V_l$$

for some constants $d_l > 0$. It is positive definite since each V_l is positive definite and each subsystem is the input for at least another subsystem (each node is the source of at least one link). From the definition of the dissipativity matrix $E_l := \{e_{lk}\}$ in (23), one gets that $\dot{V}_l \leq -\frac{1}{\gamma_l} y_l^2 + y_l \sum_{k\in\mathcal{L}_i^+} h_k(x_{\text{source}(k)})$ is equivalent to

$$\dot{V}_l \leq y_l \sum_{l=1}^{N} e_{lk} y_k \ ,$$

where $y_k := h_k(x_{\text{source}(k)})$. This gives that

$$\dot{V} = \sum_{l=1}^{N} d_l \dot{V}_l \leq \sum_{l=1}^{N} d_l y_l \sum_{l=1}^{N} e_{lk} y_k = \frac{1}{2} y^T (E^T D + DE) y \ ,$$

where $y := [y_1, \cdots, y_N]$. Noting that condition (19) implies zero state detectability for the state x_i to y_i, diagonal stability of E yields that $\dot{V}$ is negative definite and asymptotic stability of the interconnected system (18). □

Global asymptotic stability (GAS) can be concluded when storage function V_l satisfies further the condition:

$$\lim_{|x_i|\to\infty} \int_0^{x_i} \frac{h_l(s)}{g_i(s)} ds = \infty$$

which implies that the composite Lyapunov function is radially unbounded and gives GAS.

Remark 1: The results of Theorem 2 can be extended to consider dynamical systems with nonnegative state variables, that is motivated by biochemical reaction models, see details of extension method in [4]. For such special class of systems formed as H_i in (18), we assume that $x_i \in \mathbb{R}_{\geq 0}$, $g(x_i) > 0$ and $h_l(x_i) \geq 0$ for all $x_i \geq 0$ and $i = 1, \cdots, n$. Moreover, for each $i = 1, \cdots, n$ and $l \in \mathcal{L}_i^-$ and the equilibrium $x_i^* \in \mathbb{R}_{\geq 0}$, the sector conditions in (19) and (20) turn into:

$$(x_i - x_i^*)\left(\frac{f_i(x_i)}{g_i(x_i)} - \frac{f_i(x_i^*)}{g_i(x_i^*)}\right) > 0 \ \ \forall\, x_i \in \mathbb{R}_{\geq 0} - \{x_i^*\} \qquad (24)$$

and

$$\frac{h_l(x_i) - h_l(x_i^*)}{\frac{f_i(x_i)}{g_i(x_i)} - \frac{f_i(x_i^*)}{g_i(x_i^*)}} \leq \gamma_l \quad \forall\, x_i \in \mathbb{R}_{\geq 0} - \{x_i^*\} \ . \qquad (25)$$

We next consider a biochemical reaction example, for which the interconnection structure of subsystems is not cyclic or cactus, to illustrate how to apply Theorem 2 to consider asymptotic stability for dynamical networks.

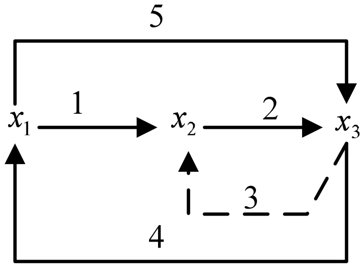

Fig. 2. Feedback configuration for MAPK networks listed in [4, Example 2-c].

Example 2: Consider the mitogen-activated protein kinase (MAPK) model for which the feedback configuration is given in Fig. 2:

$$\begin{aligned} \dot{x}_1 &= -\tfrac{b_1 x_1}{c_1 + x_1} + \tfrac{d_1(1-x_1)}{e_1 + (1-x_1)} x_3 \\ \dot{x}_2 &= -\tfrac{b_2 x_2}{c_2 + x_2} + \tfrac{d_2(1-x_2)}{e_2 + (1-x_2)} \left(x_1 + \tfrac{\mu}{1+kx_3}\right) \\ \dot{x}_3 &= -\tfrac{b_3 x_3}{c_3 + x_3} + \tfrac{d_3(1-x_3)}{e_3 + (1-x_3)} (x_1 + x_2) \end{aligned} \qquad (26)$$

where the variables $x_i \in [0, 1]$ denote the active forms of the proteins, and the term $1 - x_i$ indicates the inactive form; a_i, b_i, c_i, d_i and $e_i \in (0, 1]$ are some given positive parameters; μ and k are positive real numbers. The first term in each equation indicates the inactivation of the respective protein and the second term is the rate at which the inactive form of the protein is being converted to the active form.

Since the functions $\theta_i(x_i) : [0, 1] \to [0, \infty)$ with $\theta_i(x_i) := \frac{f_i(x_i)}{g_i(z_i)}$ are strictly increasing, we have that conditions (24) and (25) hold. Let the set of links $\{21, 32, 23, 13, 31\}$ be indexed with $l = 1, \cdots, 5$ as showed in Fig.2. Then, we can define $\gamma_l > 0$ for $l = 1, \cdots, 5$ as (20). From the definition of the dissipativity matrix in (23), one gets

$$E_l = \begin{bmatrix} -\frac{1}{\gamma_1} & 0 & 0 & 1 & 0 \\ 1 & -\frac{1}{\gamma_2} & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{\gamma_3} & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{\gamma_4} & 1 \\ 0 & 0 & 0 & 1 & -\frac{1}{\gamma_5} \end{bmatrix} . \qquad (27)$$

This matrix E_l can be formed into (4) with multiplying l-th columns with $\gamma_l > 0$ for $l = 1, \cdots, 5$ and get that $e_l = \gamma_{\text{source(l)}}$: $e_{21} = \gamma_1$, $e_{32} = e_{42} = \gamma_2$, $e_{23} = \gamma_3$, $e_{14} =$

$e_{54} = \gamma_4$ and $e_{35} = e_{45} = \gamma_5$. Noting the circles of the graph $G(E_l)$ in Fig. 1-a, we have circle gains satisfy $\tilde{\gamma}_I = \gamma_1\gamma_2\gamma_4$, $\tilde{\gamma}_{II} = \gamma_2\gamma_3$, $\tilde{\gamma}_{III} = \gamma_4\gamma_5$ and $\tilde{\gamma}_{IV} = \gamma_2\gamma_3\gamma_4\gamma_5$. The circle gain $\tilde{\gamma}_{IV} = \tilde{\gamma}_{II}\tilde{\gamma}_{III}$ is redundant.

In order to get diagonal stability of E_l as required in Theorem 2, one has to find constants $\theta_c^l > 0$ satisfying conditions (10) and (11). Since for $G(E_l)$ with E_l in (27), circle IV is redundant and connection structure for circles $I - III$ are cactus, one can follow the procedure provided in [5, Proposition 3] to search proper parameters θ_c^i. As nodes 1, 3 and 5 of $G(E_l)$ are only included in one circle for Fig .1-b, we have $\theta_I^1 = \theta_{II}^3 = \theta_{III}^5 = 1$ from condition (11). Following the procedure in [5, Proposition 3], let $\theta_{II}^2 := \tilde{\gamma}_{II} < 1$ and $\theta_{III}^4 := \tilde{\gamma}_{III} < 1$. For the circle I, noting conditions (10) and (11) in Theorem 1, we have that

$$\tilde{\gamma}_1 < \theta_I^1\theta_I^2\theta_I^4 = (1 - \tilde{\gamma}_{II})(1 - \tilde{\gamma}_{III})$$

and then

$$\tilde{\gamma}_1 + \tilde{\gamma}_{II} + \tilde{\gamma}_{III} < 1 \qquad (28)$$

is the condition that E is diagonally stable and consequently guarantees asymptotic stability of system (26) from Theorem 2.

The same example is considered in [4], where a necessary condition $\tilde{\gamma}_1 + \tilde{\gamma}_{III} < 1$ is given for diagonal stability of E_l. Through numerical calculation, it is also showed in [4] that the condition $\tilde{\gamma}_1 + \tilde{\gamma}_{III} < 1$ can not guarantee diagonal stability of E_l and the gain $\tilde{\gamma}_{III}$ impacts diagonal stability of E_l. On the other hand, condition (28) is obtained through analysis and it shows that the magnitude of the gain $\tilde{\gamma}_{III}$ does influence diagonal stability of the matrix E_l, that verifies the same conclusion obtained through numerical calculation in [4].

V. Conclusion

We extended the diagonal stability results for matrices in cyclic and cactus structure in [5], [6] to a more general class of interconnected structure, for which each pair of neighbor simple circles in its graph have at most one common edge or a common vertex. We applied the small gain condition that allows for matrices having M-matrix property and presented sufficient conditions that guarantee diagonal stability of graphs that are not cyclic or cactus. We also illustrated how to apply the diagonal stability results obtained in the present paper for analysis of networked systems. Searching necessary conditions for such class of graphs is the next step for the future work.

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