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Trajectory based small gain theorems for parameterized systems

Romain Postoyan and Dragan Nešić

Abstract— **In this paper, trajectory based small gain theorems are developed for parameterized families of continuous-time systems. We show that the interconnection of two input-to-state stable (ISS) parameterized systems is semiglobally and practically ISS, when ISS gain functions are appropriately parameterized, and we provide explicit sufficient conditions on the parameter. A small gain theorem is also presented for the case where two parameterized systems that satisfy some input-to-state and input-to-output stability properties are interconnected with a bounded-input-bounded-state system. Obtained results are applied to several stabilization and tracking problems.**

I. INTRODUCTION

In a number of situations, the domain of attraction and/or the size of the stable set of a system can be adjusted by tuning certain parameters, such as controller gain, time-scale for averaging techniques or sampling period for sampled-data systems for instance. When dealing with parameterized families of systems, classical stability analysis tools need to be modified to handle the parameter dependence and its impact on system performance.

In this paper, we present trajectory based small gain theorems for parameterized families of continuous-time systems. The small gain theorem is a key tool for analyzing input-output stability of control systems. A particularly useful version is given in [3] that is based on the concept of input-to-state stability (ISS) introduced by Sontag [11]. This work paved the way to many results among them [2], [12], [14], [15], [16]. Several studies propose small gain theorems for parameterized families of systems. In [13], a local and practical trajectory based small gain theorem is developed and applied to extremum seeking control problems. The stability of parameterized interconnected discrete-time systems is analysed in [6] using a small gain theorem that is based on the construction of an ISS Lyapunov function for the overall system.

In this study, we prove that the interconnection of two ISS continuous-time parameterized systems is semiglobally practically ISS (SP-ISS) when ISS gain functions are appropriately parameterized. The small gain condition is parameter-dependent and can always be ensured by tuning the parameter contrary to [6]. Compared to [13], [15], we provide explicit relationships between the parameter value and the obtained stability properties in terms of the ball of initial conditions and steady-state error. We illustrate the relevance of our results by applying them to a couple of stabilization problems.

A small gain theorem is also developed for the case where two parameterized systems that satisfy some input-to-state and input-to-output stability (IOS) properties are interconnected with a bounded-input-bounded-state system. This type of non-standard interconnection arises for instance when studying tracking problems for nonlinear sampled-data systems as we show it, and for the observer design for networked control systems, see [10]. It has to be noted that the small gain theorems developed in [1], [4] for multiple systems interconnections do not cover this case since here systems are parameterized and do not exhibit the same stability properties.

The paper is organized as follows. Having defined the notations and recalled some stability definitions in Section II, a small gain theorem for the interconnection of two ISS parameterized systems is developed in Section III and applied to the semiglobal practical stabilization of a class of nonlinear systems and the stabilization of a class of sampled-data systems using emulated controllers. We present the small gain theorem for the non-standard interconnection in Section IV and show how it can be applied to a sampled-data tracking problem.

II. PRELIMINARIES

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}_{>0} = (0, \infty)$, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$. Let $a \in \mathbb{R}_{>0} \cup \{\infty\}$, a function $\gamma : [0, a) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing and of class $\mathcal{K}_\infty$ if, in addition, $a = \infty$ and it is also unbounded. By extension, for $a, b \in \mathbb{R}_{>0} \cup \{\infty\}$, $\gamma : [0, a) \times [0, b) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KK}$ if, for any $(s_1, s_2) \in [0, a) \times [0, b)$, $\gamma(s_1, \cdot)$ and $\gamma(\cdot, s_2)$ are of class $\mathcal{K}$. A continuous function $\gamma : [0, a) \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_{\geq 0}$, $\gamma(\cdot, t)$ is of class $\mathcal{K}$, and, for each $s \in [0, a)$, $\gamma(s, \cdot)$ is decreasing to zero. For $a, b \in \mathbb{R}_{>0} \cup \{\infty\}$, we say that $\gamma : [0, a) \times [0, b) \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KKL}$ if $\gamma(\cdot, \cdot, t)$ is of class-$\mathcal{KK}$ for each $t \in \mathbb{R}_{\geq 0}$ and $\gamma(s_1, \cdot, \cdot)$ is of class $\mathcal{KL}$ for any $s_1 \in [0, a)$. Considering a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $n \in \mathbb{Z}_{>0}$, the notation $f(t^+)$ is used to denote $\lim\limits_{\substack{s \to t \\ s > t}} f(s)$, $t \in \mathbb{R}_{\geq 0}$ (if it exists). The Euclidean norm of a vector is denoted by $|\cdot|$. Let $f : \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{Z}_{>0}$, be a (Lebesgue) measurable function and define, for $t_1 \leq t_2 \in \mathbb{R}$, $\|f\|_{[t_1, t_2]} = \operatorname{ess.sup}_{\tau \in [t_1, t_2]} |f(\tau)|$ and $\|f\|_\infty = \operatorname{ess.sup}_{\tau \in [t_0, \infty)} |f(\tau)|$, $t_0 \in \mathbb{R}_{\geq 0}$. The set $\mathcal{L}^n_\infty$ denotes the

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set of functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\|f\|_\infty < r$, for some $r \in \mathbb{R}_{\geq 0}$. For $(x, y) \in \mathbb{R}^{n+m}$, the notation (x, y) stands for $[x^{\mathrm{T}},\, y^{\mathrm{T}}]^{\mathrm{T}}$. For a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(A)$ denotes the minimum eigenvalue of A. For $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, we write that $f(s) = \mathcal{O}(g(s))$ as $s \to a$ if there exists $M \in \mathbb{R}_{\geq 0}$ and a neighbourhood V of a such that $f(s) \leq Mg(s)$ for $s \in \mathbb{R}_{\geq 0} \cap V$.

Consider the parameterized family of systems:

$$\begin{aligned} \dot{x} &= f(x, u, \theta) && (1)\\ y &= h(x) && (2) \end{aligned}$$

where $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$, $u \in \mathbb{R}^{n_u}$ are, respectively, the state, the output and the input, $\theta \in (0, \vartheta)$ is a constant parameter where $\vartheta \in (0, \infty]$.

Definition 1 (**(U)IOS / (U)ISS**). *System (1)-(2) is* uniformly input-to-output stable (UIOS) *from u to y with (β, γ) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that for any $x(t_0) \in \mathbb{R}^{n_x}$, $u \in \mathcal{L}_\infty^{n_u}$, solutions to (1)-(2) satisfy:*

$$|y(t)| \quad \leq \quad \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|u\|_{[t_0,t]})\}, \qquad (3)$$

for any $\theta \in (0, \vartheta)$ and $t \geq t_0 \geq 0$. System (1)-(2) is input-to-output stable (IOS) *if for any $\theta \in (0, \vartheta)$ there exist $\beta_\theta \in \mathcal{KL}$ and $\gamma_\theta \in \mathcal{K}$ such that (3) holds. If $y = x$, then system (1) is* (uniformly) input-to-state stable ((U)ISS) *with input u. We refer to γ as the* IOS / ISS gain.

Definition 2 (**SP-IOS / SP-ISS**). *System (1)-(2) is* semiglobally practically IOS (SP-IOS) *from u to y w.r.t. θ if for any $\Delta, \varepsilon \in \mathbb{R}_{>0}$ there exist $\theta^* \in (0, \vartheta)$, $\beta_{\theta^*} \in \mathcal{KL}$ and $\gamma_{\theta^*} \in \mathcal{K}$ such that, for any $x(t_0) \in \mathbb{R}^{n_x}$, $u \in \mathcal{L}_\infty^{n_u}$ with $\max\{|x(t_0)|, \|u\|_\infty\} < \Delta$, solutions to (1)-(2) satisfy for all $t \geq t_0 \geq 0$:*

$$|y(t)| \leq \max\{\beta_{\theta^*}(|x(t_0)|, t - t_0), \gamma_{\theta^*}(\|u\|_{[t_0,t]}), \varepsilon\}. \qquad (4)$$

If $y = x$, system (1) is semiglobally practically ISS (SP-ISS) *with input u, if, in addition, $u = 0$, we say that system (1) is* semiglobally practically asymptotically stable (SP-AS). *When (4) holds for $\Delta = \infty$ or $\varepsilon = 0$, we respectively say that system (1)-(2) is* practically *or* semiglobally IOS (*or* ISS).

Definition 3 (**(U/S)-BIBS**). *System (1) is said to be* uniformly bounded-input-bounded-state (UBIBS) *with input u if there exist $\alpha, \eta \in \mathcal{K}$, such that, for any $x(t_0) \in \mathbb{R}^{n_x}$, $u \in \mathcal{L}_\infty^{n_u}$, solutions to (1) satisfy:*

$$|x(t)| \quad \leq \quad \max\{\alpha(|x(t_0)|), \eta(\|u\|_{[t_0,t]})\}, \qquad (5)$$

for any $\theta \in (0, \vartheta)$ and $t \geq t_0 \geq 0$. System (1) is said to be semiglobally bounded-input-bounded-state (S-BIBS) *with input u w.r.t. θ if for any $\Delta \in \mathbb{R}_{>0}$ there exist $\theta^* \in (0, \vartheta)$, $\alpha_{\theta^*}, \eta_{\theta^*} \in \mathcal{K}$, such that, for any $x(t_0) \in \mathbb{R}^{n_x}$, $u \in \mathcal{L}_\infty^{n_u}$ with $\max\{|x(t_0)|, \|u\|_\infty\} < \Delta$, (5) holds. We refer to η as the* BIBS gain.

III. STANDARD INTERCONNECTIONS

A. Main result

Consider the following parameterized family of systems:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u, \theta) && (6)\\ \dot{x}_2 &= f_2(x_1, x_2, u, \theta), && (7) \end{aligned}$$

where $x_1 \in \mathbb{R}^{n_{x_1}}$, $x_2 \in \mathbb{R}^{n_{x_2}}$ are the state vectors, $u \in \mathcal{L}_\infty^{n_u}$ is an exogenous input, $\theta \in (0, \vartheta)$ is a known constant parameter with $\vartheta \in (0, \infty]$, f_1 and f_2 are continuous functions for any $\theta \in (0, \vartheta)$.

The proof of the following theorem is provided in the Appendix.

Theorem 1. *Consider system (6)-(7) and suppose that we have the following.*

1) *System (6) is ISS with inputs (x_2, u) with $(\beta_1, \gamma_1^2, \gamma_1^u)$ that are functions of class-$\mathcal{K}$ in θ.*
2) *System (7) is UISS with inputs (x_1, u) with $(\beta_2, \gamma_2^1, \gamma_2^u)$.*

Then system (6)-(7) is SP-ISS with input u w.r.t. θ. Indeed, for any $\Delta, \varepsilon \in \mathbb{R}_{>0}$ and $\bar{\theta} \in (0, \vartheta)$, define $M \in \mathbb{R}_{>0}$ (sufficiently big) and $m \in \mathbb{R}_{>0}$ (sufficiently small) such that:

$$\begin{cases} \delta^m(\bar{\theta}, m) \leq \varepsilon \\ \max\left\{m, \tilde{\sigma}^m(\bar{\theta}, m), \nu^x(\bar{\theta}, \Delta), \nu^u(\bar{\theta}, \Delta)\right\} \leq M, \end{cases} \qquad (8)$$

where for $(\theta, s) \in (0, \bar{\theta}) \times \mathbb{R}_{\geq 0}$ and $\eta > 1$,

$$\begin{aligned} \delta^m(\theta, s) &= 2\max\left\{\eta\tilde{\sigma}^m(\theta, s), \eta s\right\}\\ \tilde{\sigma}^m(\theta, s) &= \max\big\{\beta_1(\theta, s, 0), \gamma_1^2(\theta, \beta_2(s, 0)), \beta_2(s, 0),\\ &\qquad \gamma_2^1(\beta_1(\theta, s, 0))\big\}\\ \nu^x(\theta, s) &= \max\big\{\beta_1(\theta, s, 0), \gamma_1^2(\theta, \beta_2(s, 0)), \beta_2(s, 0),\\ &\qquad \gamma_2^1(\beta_1(\theta, s, 0)), \tilde{\beta}(\theta, s, 0), s\big\}\\ \nu^u(\theta, s) &= \max\big\{\gamma_1^u(\theta, s), \gamma_1^2(\theta, \gamma_2^u(s)), \gamma_2^u(s),\\ &\qquad \gamma_2^1(\gamma_1^u(\theta, s)), \tilde{\gamma}^u(\theta, s)\big\}, \end{aligned}$$

with $\tilde{\beta}$ and $\tilde{\gamma}^u$ respectively given in (52) and (57). Taking $\theta^* \in (0, \bar{\theta}]$ such that:

$$\max\left\{\gamma_1^2(\theta^*, \gamma_2^1(s)), \gamma_2^1(\gamma_1^2(\theta^*, s))\right\} < s \quad \forall s \in [m, M], \qquad (9)$$

then there exist $\beta_{\theta^*} \in \mathcal{KL}$, $\sigma_{\theta^*} \in \mathcal{K}$ such that for all $(x_1(t_0), x_2(t_0)) \in \mathbb{R}^{n_{x_1}+n_{x_2}}$ and $u \in \mathcal{L}_\infty^{n_u}$ where $\max\left\{|x_1(t_0)|, |x_2(t_0)|, \|u\|_\infty\right\}) < \Delta$, solutions to (6)-(7) satisfy for all $t \geq t_0 \geq 0$,

$$\begin{aligned} |(x_1(t), x_2(t))| \leq \max\big\{&\beta_{\theta^*}\big(|(x_1(t_0), x_2(t_0))|, t - t_0\big),\\ &\sigma_{\theta^*}(\|u\|_{[t_0,t]}), \varepsilon\big\}. \end{aligned} \qquad (10)$$

Theorem 1 relies on the fact that the ISS gains of system (6) are of class-$\mathcal{K}$ in θ. In that way, we can always find a sufficiently small θ^* that guarantees the small gain condition (9) (see the Appendix). It has to be noticed that Theorem 1 gives us a method for estimating parameter θ^* for any radius of the ball of initial conditions Δ and any desired steady state error ε: first, fix a constant $\bar{\theta}$ and the desired Δ and ε. Second, choose constants m and M such that (8) holds. Finally, take θ^* sufficiently small such that (9) is satisfied.

Remark 1. *Note that there is an error in the statement of Lemma 2 in [13] as the needed condition (8) was not used in the statement of the lemma. However, the main results in [13] still hold as the condition (8) can always be satisfied in that paper.*

Remark 2. *When both systems (6) and (7) are ISS with a $\mathcal{KL}$-function and ISS gains that are functions of class $\mathcal{K}$ in θ (like in condition 1) of Theorem 1), similar results to Theorem 1 can be immediately derived.*

The following proposition follows from Theorem 1.

Proposition 1. *Consider system (6)-(7) and suppose that all conditions of Theorem 1 are satisfied. Define* $\gamma(\theta,s) = \max\left\{\gamma_1^2(\theta,\gamma_2^1(s)),\gamma_2^1(\gamma_1^2(\theta,s))\right\}$ *for* $(\theta,s)\in(0,\vartheta)\times\mathbb{R}_{\geq 0}$. *If for any* $\theta\in(0,\vartheta)$*:*

- $\gamma(\theta,s)=\mathcal{O}(s)$ *as* $s\to\infty$*, then system (6)-(7) is practically ISS with input* u *w.r.t.* θ*.*
- $\gamma(\theta,s)=\mathcal{O}(s)$ *as* $s\to 0$*, then system (6)-(7) is semiglobally ISS with input* u *w.r.t.* θ*.*

B. Applications

1) Semiglobal practical stabilization of a class of nonlinear systems: Consider the systems of the form:

$$\begin{aligned}\dot z &= f(x,z) \qquad (11)\\ \dot x &= u+g(z), \qquad (12)\end{aligned}$$

where $(x,z)\in\mathbb{R}^{n_x+n_z}$ are the state variables, f and g are continuous functions, the feedback law is:

$$u = -Kx, \qquad (13)$$

where the controller gain K is a real symmetric positive definite matrix of appropriate dimensions. Suppose that function g is such that $g(z)\leq\gamma(|z|)$ for any $z\in\mathbb{R}^{n_z}$ with $\gamma\in\mathcal{K}$. Consider the Lyapunov function $V=\frac{1}{2}|x|^2$, we can show that along solutions to (12):

$$\left(|x|\geq\tfrac{2}{\lambda_{\min}(K)}\gamma(|z|)\right)\Rightarrow\left(\dot V\leq-\lambda_{\min}(K)V\right). \qquad (14)$$

Using the comparison principle (Lemma 3.4 in [5]), it can then be deduced from (14) that system (12) is ISS with input z with $\beta_1:(\theta,s,t)\mapsto\exp(-\frac{t}{\theta})s\in\mathcal{KKL}$ and gain $\gamma_1:(\theta,s)\mapsto\theta\gamma(s)\in\mathcal{KK}$ where $\theta=\frac{2}{\lambda_{\min}(K)}\in\mathbb{R}_{>0}$. The following proposition is a direct consequence of Theorem 1 by identifying $x_1=x$, $x_2=z$, $\beta_1=\beta_1$ and $\gamma_1^2=\gamma_1$.

Proposition 2. *Consider system (11)-(12) as parameterized in* θ*. Suppose that the following conditions hold:*

a) *There exists* $\gamma\in\mathcal{K}$ *such that* $g(z)\leq\gamma(|z|)$ *for any* $z\in\mathbb{R}^{n_z}$*.*
b) *System (11) is UISS with input* x*.*

Then system (11)-(12) is SP-AS w.r.t. θ*.*

2) Sampled-data stabilization using emulated controllers: Although Theorem 1 is stated for continuous-time systems, it can be applied to the following class of hybrid systems since the states are continuous between sampling instants and their norms do not increase at sampling instants (see the Appendix). Consider the plant:

$$\begin{aligned}\dot x_P &= f_P(x_P,u,w) \qquad (15)\\ y &= h_P(x_P), \qquad (16)\end{aligned}$$

where $x_P\in\mathbb{R}^{n_{x_P}}$, $y\in\mathbb{R}^{n_y}$, $u\in\mathbb{R}^{n_u}$ are, respectively, the state, the output and the control input, $w\in\mathcal{L}_\infty^{n_w}$ is an exogenous input, f_P is a continuous function and h_P is a continuously differentiable function. Suppose that we know a stabilizing dynamic controller for system (15)-(16):

$$\begin{aligned}\dot x_C &= f_C(x_C,y,w) \qquad (17)\\ u &= h_C(x_C), \qquad (18)\end{aligned}$$

where $x_C\in\mathbb{R}^{n_{x_C}}$ is the controller state, f_C is a continuous function and h_C is a continuously differentiable function. The objective is to study the stability of the closed-loop system (15)-(18) when control input and system output are sampled and hold at a constant period $\tau\in\mathbb{R}_{>0}$. According to [8], the problem can be modeled as follows:

$$\begin{aligned}\dot x &= f(x,e,w) && \forall t\in[t_{i-1},t_i] \qquad (19)\\ \dot e &= g(x,e,w) && \forall t\in[t_{i-1},t_i] \qquad (20)\\ x(t_i^+) &= x(t_i) && \qquad (21)\\ e(t_i^+) &= 0, && \qquad (22)\end{aligned}$$

where $t_i=t_0+i\tau$, $i\in\mathbb{Z}_{>0}$, $t_0\in\mathbb{R}_{\geq0}$ is the initial time, $x=(x_P,x_C)\in\mathbb{R}^{n_x}$, $e=(\hat u-u,\hat y-y)\in\mathbb{R}^{n_e}$ where $\hat u$ and $\hat y$ are, respectively, the last control input and system output transmitted, i.e. $\hat y(t)=y(t_{i-1})$ and $\hat u(t)=u(t_{i-1})$ for all $t\in[t_{i-1},t_i]$.

Proposition 3. *Consider system (19)-(22) as parameterized in* τ*. Suppose that the following conditions hold:*

a) *There exist* $L\in\mathbb{R}_{\geq0}$ *and* $\gamma^x,\gamma^w\in\mathcal{K}$ *such that for any* $(x,e,w)\in\mathbb{R}^{n_x+n_e+n_w}$, $|g(x,e,w)|\leq L|e|+\gamma^x(|x|)+\gamma^w(|w|)$*;*
b) *System (19) is UISS with inputs* (e,w)*;*

Then system (19)-(22) is SP-ISS with input w *w.r.t.* τ*.*

Proof. In view of condition a) in Proposition 3 and Proposition 6 in [7], there exists $\beta_1\in\mathcal{KKL}$ such that, for all $e(t_0)\in\mathbb{R}^{n_e}$, $t\geq t_0\geq0$:

$$\begin{aligned}|e(t)| \leq{}& \beta_1(\theta,|e(t_0)|,t-t_0)\\ &+\zeta(\tau)\left[\gamma^x(\|x\|_{[t_0,t]})+\gamma^w(\|w\|_{[t_0,t]})\right], \qquad (23)\end{aligned}$$

with $\zeta:\tau\mapsto\frac{1}{L}(\exp(L\tau)-1)\in\mathcal{K}$ ($\zeta:\tau\mapsto\tau$ if $L=0$). Using the fact that $a+b\leq\max\{2a,2b\}$ for any $a,b\in\mathbb{R}_{\geq0}$ and in view of (23), we have that system (20), (22) satisfies condition 1) of Theorem 1 by identifying $x_1=e$, $x_2=x$, $u=w$, $\beta_1=2\beta_1$, $\gamma_1^2(\tau,s)=4\zeta(\tau)\gamma^x(s)$ and $\gamma_1^u(\tau,s)=4\zeta(\tau)\gamma^w(s)$ for any $(\tau,s)\in\mathbb{R}^2_{\geq0}$. Consequently, since condition b) in Proposition 3 ensures that condition 2) in Theorem 1 holds, Theorem 1 is finally applied to obtain the desired result. ☐

IV. NON-STANDARD INTERCONNECTIONS

A. Main result

Consider the following parameterized family of systems:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3, u, \theta) && (24)\\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u, \theta) && (25)\\ \dot{x}_3 &= f_3(x_1, x_2, x_3, u, \theta), && (26) \end{aligned}$$

where $x_1 \in \mathbb{R}^{n_{x_1}}$, $x_2 \in \mathbb{R}^{n_{x_2}}$, $x_3 \in \mathbb{R}^{n_{x_3}}$, are the state vectors, $u \in \mathcal{L}_\infty^{n_u}$ is an exogenous input, $\theta \in (0, \vartheta)$ is a known constant parameter where $\vartheta \in (0, \infty]$, f_1, f_2 and f_3 are continuous functions for any $\theta \in (0, \vartheta)$.

The proof of the following theorem is provided in [9] (see the proof of Theorem D.3.2).

Theorem 2. *Suppose that we have the following.*

1) *System (24) is ISS with inputs (x_2, x_3, u) with $(\beta_1, \gamma_1^2, \gamma_1^3, \gamma_1^u)$ that are functions of class $\mathcal{K}$ in θ.*
2) *System (25)-(26) is UIOS from (x_1, u) to x_2 with gains (γ_2^1, γ_2^u).*
3) *System (26) is UBIBS with inputs (x_1, x_2, u) with gains $(\eta_3^1, \eta_3^2, \eta_3^u)$.*

Then system (24)-(26) is S-BIBS with input u w.r.t. θ and SP-IOS from u to (x_1, x_2) w.r.t. θ.

Similar to Theorem 1, sufficient conditions on parameter θ can be derived from the small gain analysis (see Theorem D.3.2 in [9]). The proposition below follows from Theorem 2.

Proposition 4. *Consider system (24)-(26) and suppose that all conditions of Theorem 2 are satisfied. Define $\tilde{\gamma}(\theta, s)$ $= \max\Big\{\gamma_1^2(\theta, \gamma_2^1(s)), \gamma_1^3(\theta, \eta_3^2(\gamma_2^1(s))), \gamma_1^3(\theta, \eta_3^1(s)),$ $\gamma_2^1(\gamma_1^2(\theta, s)), \gamma_2^1(\gamma_1^3(\theta, \eta_3^2(s))), \gamma_2^1(\gamma_1^3(\theta, \eta_3^1(s)))\Big\}$ for $(\theta, s) \in (0, \vartheta) \times \mathbb{R}_{\geq 0}$. If for any $\theta \in (0, \vartheta)$, $\tilde{\gamma}(\theta, s) = \mathcal{O}(s)$ as $s \to \infty$, then system (24)-(26) is UBIBS with input u and is practically IOS from u to (x_1, x_2) w.r.t. θ.*

B. Application to the tracking control of a class of nonlinear sampled-data systems

Consider the plant:

$$\begin{aligned} \dot{x}_P &= f_P(x_P, u, w) && (27)\\ y_P &= h_P(x_P), && (28) \end{aligned}$$

where $x_P \in \mathbb{R}^{n_{x_P}}$, $y_P \in \mathbb{R}^{n_y}$, $u \in \mathbb{R}^{n_u}$ are, respectively, the state, the output and the control input of the plant, $w \in \mathcal{L}_\infty^{n_w}$ is an exogenous input, f_P is continuous and h_P is continuously differentiable. The reference system is defined as:

$$\begin{aligned} \dot{x}_R &= f_R(x_R, w) && (29)\\ y_R &= h_R(x_R), && (30) \end{aligned}$$

where $x_R \in \mathbb{R}^{n_{x_P}}$, $y_R \in \mathbb{R}^{n_y}$ are, respectively, the state and the output, f_R is continuous and h_R is continuously differentiable. We assume that we know a controller:

$$\begin{aligned} \dot{x}_C &= f_C(x_C, y_P, y_R) && (31)\\ u &= h_C(x_C), && (32) \end{aligned}$$

where $x_C \in \mathbb{R}^{n_{x_C}}$ is the state, f_C is continuous and h_C a continuously differentiable, that ensures the tracking goal $\xi(t) = y_R(t) - y_P(t) \to 0$ as $t \to \infty$. Our objective is to study the stability properties when u, y_P, y_R are sampled and hold at a period $\tau \in \mathbb{R}_{>0}$. Similarly to [8], the problem can be written as:

$$\begin{aligned} \dot{\xi} &= f_\xi(\xi, x, e, w) && \forall t \in [t_{i-1}, t_i] \quad (33)\\ \dot{x} &= f_x(\xi, x, e, w) && \forall t \in [t_{i-1}, t_i] \quad (34)\\ \dot{e} &= g(\xi, x, e, w) && \forall t \in [t_{i-1}, t_i] \quad (35)\\ \xi(t_i^+) &= \xi(t_i) && (36)\\ x(t_i^+) &= x(t_i) && (37)\\ e(t_i^+) &= 0, && (38) \end{aligned}$$

where $t_i = t_0 + i\tau$, $i \in \mathbb{Z}_{>0}$, denotes the sampling instants, $t_0 \in \mathbb{R}_{\geq 0}$ is the initial time, $x = (x_P, x_R, x_C) \in \mathbb{R}^{n_x}$, $e = (\hat{u} - u, \hat{y}_P - y_P, \hat{y}_R - y_R) = (e_u, e_{y_P}, e_{y_R}) \in \mathbb{R}^{n_e}$ where $\hat{u}$, $\hat{y}_P$, $\hat{y}_R$ are, respectively, the last control input, plant output and reference output transmitted, i.e. $\hat{u}(t) = u(t_{i-1})$, $\hat{y}_P = y_P(t_{i-1})$, $\hat{y}_R = y_R(t_{i-1})$ for all $t \in [t_{i-1}, t_i]$ and:

$$\begin{aligned} f_\xi(\xi, x, e, w) &= \tfrac{\partial h_R}{\partial x_R}(x_R) f_R(x_R, w)\\ &\quad - \tfrac{\partial h_P}{\partial x_P}(x_P) f_P(x_P, e_u + h_C(x_C), w)\\ f_x(\xi, x, e, w) &= \begin{pmatrix} f_P(x_P, e_u + h_C(x_C), w) \\ f_R(x_R, w) \\ f_C(x_C, e_{y_P} + h_P(x_P), e_{y_R} + h_P(x_P) + \xi) \end{pmatrix}\\ g(\xi, x, e, w) &= \\ &\begin{pmatrix} -\tfrac{\partial h_C}{\partial x_C}(x_C) f_C(x_C, e_{y_P} + h_P(x_P), e_{y_R} + h_P(x_P) + \xi) \\ -\tfrac{\partial h_P}{\partial x_P}(x_P) f_P(x_P, e_u + h_C(x_C), w) \\ -\tfrac{\partial h_R}{\partial x_R}(x_R) f_R(x_R, w) \end{pmatrix}. \end{aligned}$$

Proposition 5. *Consider system (33)-(38) as parameterized in τ. Suppose that the following conditions hold:*

a) *There exist $L \in \mathbb{R}_{\geq 0}$ and $\gamma^\xi, \gamma^x, \gamma^w \in \mathcal{K}$ such that for any $(\xi, x, e, w) \in \mathbb{R}^{n_y + n_x + n_e + n_w}$, $|g(\xi, x, e, w)| \leq L|e| + \gamma^\xi(|\xi|) + \gamma^x(|x|) + \gamma^w(|w|)$.*
b) *System (33)-(34) is UIOS from (e, w) to ξ.*
c) *System (34) is UBIBS with inputs (ξ, e, w).*

Then system (33)-(38) is S-BIBS with input w w.r.t. θ and is SP-IOS from w to (ξ, e) w.r.t. θ.

Sketch of proof. It can be shown, using similar arguments than in the proof of Proposition 3, that system (35),(38) is ISS with inputs (ξ, x, w) and therefore satisfies condition 1) of Theorem 2 by identifying $x_1 = e$, $x_2 = \xi$ and $x_3 = x$. On the other hand, we have that conditions 2) and 3) of Theorem 2 are respectively satisfied in view of conditions b) and c) of Proposition 5. The desired results are then obtained by invoking Theorem 2, noting that, although system (33)-(38) is not purely continuous-time, Theorem 2 can still be applied since the norms of variables e, ξ, x do not increase at sampling times in view of (36)-(38) (see footnote in the Appendix). □

V. CONCLUSION

In this paper, small gain theorems for parameterized families of systems are developed. It is shown that the ISS stability of two interconnected parameterized systems is maintained semiglobally and practically under some conditions and we provide explicit sufficient conditions on the parameter. This result is applied to the semiglobal practical stabilization of a class of continuous-time systems and the stabilization of a class of sampled-data systems using emulated controllers. A small gain theorem is also presented for the case where two parameterized systems that satisfy some ISS and IOS properties are interconnected with a third system that only exhibits a boundedness property. We show how this theorem can be used to analyse tracking problems for sampled-data systems.

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APPENDIX - PROOF OF THEOREM 1

First, the following proposition is introduced that is a local version of Theorem 1 in [12]. Its proof is given in Appendix D in [9] (see Proposition D.1.3).

Proposition 6. *Suppose there exist* $\beta \in \mathcal{KL}$, γ^y, γ^u, σ^t, σ^x, σ^y, $\sigma^u \in \mathcal{K}$, $c_x, c_y, m, M, \Delta \in \mathbb{R}_{>0}$ *such that, for all* $x(t_0) \in \mathbb{R}^{n_x}$ *and* $u \in \mathcal{L}_\infty^{n_u}$ *with* $\max\{|x(t_0)|, \|u\|_\infty\} < \Delta$, *along solutions to (1)-(2), for any* $t \geq t_0 \geq 0$ *this holds:*

$$\begin{aligned} |y(t)| &\leq \max\{\beta(|x(t_0)|, t-t_0), \gamma^y(\|y\|_{[t_0,t]}), \\ &\qquad \gamma^u(\|u\|_{[t_0,t]}), c_y\} \qquad (39)\\ |y(t)| &\leq M \qquad (40)\\ |x(t)| &\leq \max\{\sigma^t(t-t_0), \sigma^x(|x(t_0)|), \sigma^y(\|y\|_{[t_0,t]}), \\ &\qquad \sigma^u(\|u\|_{[t_0,t]}), c_x\}, \qquad (41) \end{aligned}$$

and

$$\gamma^y(s) \quad < \quad s \qquad \forall s \in [m, M], \qquad (42)$$

with

$$c = \max\{m, c_y\} \quad < \quad M, \qquad (43)$$

define $\tilde{\gamma}^u(s) = \max\{\beta(\sigma^y \circ \gamma^u(s), 0), \beta(\sigma^u(s), 0), \gamma^u(s)\}$ *for* $s \in \mathbb{R}_{\geq 0}$, *and* $\bar{\Delta} \in (0, \Delta]$ *such that* $\max\{\beta(\bar{\Delta}, 0), \tilde{\gamma}^u(\bar{\Delta})\} < M$ *then, for any* $\eta \in (1, \infty)$, *there exists* $\tilde{\beta} \in \mathcal{KL}$ *such that for all* $x(t_0) \in \mathbb{R}^{n_x}$ *and* $u \in \mathcal{L}_\infty^{n_u}$ *with* $\max\{|x(t_0)|, \|u\|_\infty\} < \bar{\Delta}$, *for all* $t \geq t_0 \geq 0$*:*

$$|y(t)| \leq \max\{\tilde{\beta}(|x(t_0)|, t-t_0), \tilde{\gamma}^u(\|u\|_{[t_0,t]}), \eta c\}. \qquad (44)$$

Proof of Theorem 1. After having shown that there exists θ^* that ensures (9), forward completeness of system (6)-(7) is proved when initial conditions and inputs are appropriately bounded. Afterwards, the stability property (10) is obtained by invoking Proposition 6.

Step 1. Existence of θ^*.

Let $\Delta, \varepsilon \in \mathbb{R}_{>0}$, $\bar{\theta} \in (0, \vartheta)$ and define $m, M \in \mathbb{R}_{>0}$ such that (8) holds. We introduce the following function (like in Proposition 1): $\gamma : (\theta, s) \mapsto \max\{\gamma_1^2(\theta, \gamma_2^1(s)), \gamma_2^1(\gamma_1^2(\theta, s))\}$ that is of class $\mathcal{KK}$. We choose $\theta^* \leq \bar{\theta}$ sufficiently small such that, for instance, $\gamma(\theta^*, M) < m$, thus (9) is satisfied.

Step 2. Forward completeness and boundedness of the state variables.

Let $t_0 \in \mathbb{R}_{\geq 0}$, $(x_1(t_0), x_2(t_0)) \in \mathbb{R}^{n_{x_1}+n_{x_2}}$ and $u \in \mathcal{L}_\infty^{n_u}$ be such that:

$$\max\{|x_1(t_0)|, |x_2(t_0)|, \|u\|_\infty\} \quad < \quad \Delta, \qquad (45)$$

as a consequence, according to the definition of M,

$$\begin{aligned} &\nu^x(\theta^*, \max\{|x_1(t_0)|, |x_2(t_0)|\}) < M \\ &\nu^u(\theta^*, \|u\|_\infty) < M. \end{aligned} \qquad (46)$$

Let $[t_0, t_{\max})$ be the maximum existence interval of system (6)-(7) with $t_{\max} \in (t_0, \infty]$. In view of conditions 1) and 2) of Theorem 1, we have for any $t \in [t_0, t_{\max})$:

$$\begin{aligned} \|x_1\|_{[t_0,t]} \leq \max\{&\beta_1(\theta^*, |x_1(t_0)|, 0), \gamma_1^2(\theta^*, \beta_2(|x_2(t_0)|, 0)), \\ &\gamma_1^2(\theta^*, \gamma_2^1(\|x_1\|_{[t_0,t]})), \gamma_1^2(\theta^*, \gamma_2^u(\|u\|_{[t_0,t]})), \\ &\gamma_1^u(\theta^*, \|u\|_{[t_0,t]})\}. \end{aligned} \qquad (47)$$

We now show by contradiction that $|x_1(t)| < M$ for any $t \in [t_0, t_{\max})$. According to (46) and the definition of ν^x, we have that $|x_1(t_0)| < M$. Suppose that there exists $t_1 \in$

$(t_0, t_{\max})$ such that $|x_1(t_1)| = M$ and define $\bar{t} = \inf\big\{t \in (t_0, t_{\max}) : |x_1(t)| = M\big\}$, in view of (47) it holds that:

$$\begin{aligned}
\|x_1\|_{[t_0,\bar{t}]} &= |x_1(\bar{t})| \\
&\leq \max\big\{\beta_1(\theta^*, |x_1(t_0)|, 0), \gamma_1^2(\theta^*, \beta_2(|x_2(t_0)|, 0)), \\
&\qquad \gamma_1^2(\theta^*, \gamma_2^1(\|x_1\|_{[t_0,\bar{t}]})), \gamma_1^2(\theta^*, \gamma_2^u(\|u\|_{[t_0,\bar{t}]})), \\
&\qquad \gamma_1^u(\theta^*, \|u\|_{[t_0,\bar{t}]})\big\} \\
&= \max\big\{\beta_1(\theta^*, |x_1(t_0)|, 0), \gamma_1^2(\theta^*, \beta_2(|x_2(t_0)|, 0)), \\
&\quad \gamma_1^2(\theta^*, \gamma_2^1(M)), \gamma_1^2(\theta^*, \gamma_2^u(\|u_2\|_{[t_0,\bar{t}]})), \gamma_1^u(\theta^*, \|u\|_{[t_0,\bar{t}]})\big\}.
\end{aligned}$$

In view of (46) and since $\gamma_1^2(\theta^*, \gamma_2^1(M)) < M$ according to (9), we obtain the contradiction: $|x_1(\bar{t})| < M$. As a consequence, since x_1 is continuous, we have $|x_1(t)| < M$ for all $t \in [t_0, t_{\max})$. Similarly we can prove that $|x_2(t)| < M$ for any $t \in [t_0, t_{\max})$. Therefore, we can conclude that $t_{\max} = \infty$ by contradiction[1].

Step 3. Stability property (10).

We now show that conditions of Proposition 6 are satisfied. Note that, according to (9) and since $\gamma_1^2(\cdot, \gamma_2^1(\cdot))$ is increasing:

$$\begin{cases} \gamma_1^2(\theta^*, \gamma_2^1(s)) < s & \text{if } s \in [m, M] \\ \gamma_1^2(\theta^*, \gamma_2^1(s)) < \gamma_1^2(\theta^*, \gamma_2^1(m)) < m & \text{if } s \in [0, m). \end{cases} \tag{48}$$

Using (48) and condition 1) of Theorem 1, we have that for all $t \in [t_0, \infty)$:

$$\begin{aligned}
|x_1(t)| \leq \max\Big\{&\beta_1(\theta^*, |x_1(\tfrac{t+t_0}{2})|, \tfrac{t-t_0}{2}), \\
&\gamma_1^2(\theta^*, \|x_2\|_{[\frac{t+t_0}{2}, t]}), \gamma_1^u(\theta^*, \|u\|_{[\frac{t+t_0}{2}, t]})\Big\},
\end{aligned} \tag{49}$$

moreover, in view of conditions 1) and 2) of Theorem 1, (9) and the fact that $\|x_1\|_{[t_0,t]} < M$ according to Step 2, this holds:

$$\begin{aligned}
\|x_1\|_{[t_0,t]} &\leq \max\big\{\beta_1(\theta^*, |x_1(t_0)|, 0), \gamma_1^2(\theta^*, \beta_2(|x_2(t_0)|, 0)), \\
&\qquad \gamma_1^2(\theta^*, \gamma_2^u(\|u\|_{[t_0,t]})), \gamma_1^u(\theta^*, \|u\|_{[t_0,t]}), m\big\} \\
|x_2(t)| &\leq \max\Big\{\beta_2(|x_2(\tfrac{t+t_0}{2})|, \tfrac{t-t_0}{2}), \gamma_2^1(\|x_1\|_{[\frac{t+t_0}{2}, t]}), \\
&\qquad \gamma_2^u(\|u\|_{[\frac{t+t_0}{2}, t]})\Big\},
\end{aligned} \tag{50}$$

consequently, by including (50) into (49), and using the same arguments by symmetry for the variable x_2, we obtain:

$$\begin{aligned}
&\max\big\{|x_1(t)|, |x_2(t)|\big\} \leq \\
&\quad \max\Big\{\tilde{\beta}(\theta^*, \max\big\{|x_1(t_0)|, |x_2(t_0)|\big\}, t - t_0), \\
&\qquad \gamma^y(\theta^*, \|\max\{|x_1|, |x_2|\}\|_{[t_0,t]}), \\
&\qquad \gamma^u(\theta^*, \|u\|_{[t_0,t]}), \tilde{\sigma}^m(\theta^*, m)\Big\},
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
\tilde{\beta}(\theta, s, t) = \max\big\{&\beta_1(\theta, \beta_1(\theta, s, 0), \tfrac{t}{2}), \\
&\beta_1(\theta, \gamma_1^2(\theta, \beta_2(s, 0)), \tfrac{t}{2}), \\
&\gamma_1^2(\theta, \beta_2(\beta_2(s, 0), \tfrac{t}{2})), \\
&\gamma_1^2(\theta, \beta_2(\gamma_2^1(\beta_1(\theta, s, 0)), \tfrac{t}{2})) \\
&\beta_2(\beta_2(s, 0), \tfrac{t}{2}), \beta_2(\gamma_2^1(\beta_1(\theta, s, 0)), \tfrac{t}{2}), \\
&\gamma_2^1(\beta_1(\theta, \beta_1(\theta, s, 0), \tfrac{t}{2})), \\
&\gamma_2^1(\beta_1(\theta, \gamma_1^2(\theta, \beta_2(s, 0)), \tfrac{t}{2}))\big\},
\end{aligned} \tag{52}$$

$$\begin{aligned}
\gamma^y(\theta, s) &= \max\big\{\gamma_1^2(\theta, \gamma_2^1(s)), \gamma_2^1(\gamma_1^2(\theta, s))\big\}, && (53) \\
\gamma^u(\theta, s) &= \max\big\{\bar{\gamma}^u(\theta, s), \hat{\gamma}^u(s)\big\}, && (54) \\
\bar{\gamma}^u(\theta, s) &= \max\big\{\beta_1(\theta, \gamma_1^2(\theta, \gamma_2^u(s)), 0), \beta_1(\theta, \gamma_1^u(\theta, s), 0), \\
&\qquad \gamma_1^2(\theta, \beta_2(\gamma_2^1(\gamma_1^u(\theta, s)), 0)), \\
&\qquad \gamma_1^2(\theta, \beta_2(\gamma_2^u(s), 0)), \gamma_1^2(\theta, \gamma_2^u(s)), \\
&\qquad \gamma_1^u(\theta, s), \beta_2(\gamma_2^1(\gamma_1^u(\theta, s)), 0), \\
&\qquad \gamma_2^1(\beta_1(\theta, \gamma_1^2(\theta, \gamma_2^u(s)), 0)), \\
&\qquad \gamma_2^1(\beta_1(\theta, \gamma_1^u(\theta, s), 0)), \gamma_2^1(\gamma_1^u(\theta, s))\big\}, \\
\hat{\gamma}^u(s) &= \max\big\{\beta_2(\gamma_2^u(s), 0), \gamma_2^u(s)\big\}, && (55) \\
\tilde{\sigma}^m(\theta, s) &= \max\big\{\beta_1(\theta, s, 0), \gamma_1^2(\theta, \beta_2(s, 0)), \beta_2(s, 0), \\
&\qquad \gamma_2^1(\beta_1(\theta, s, 0))\big\}, && (56)
\end{aligned}$$

for $\theta \in (0, \vartheta)$ and $(s, t) \in \mathbb{R}^3_{\geq 0}$ and define

$$\begin{aligned}
\tilde{\gamma}^u(\theta, s) = \max\big\{&\tilde{\beta}(\theta, \gamma^y(\theta, \gamma^u(\theta, s)), 0), \tilde{\beta}(\theta, \gamma^u(\theta, s), 0), \\
&\gamma^u(\theta, s)\big\}.
\end{aligned} \tag{57}$$

We can see that all conditions of Proposition 6 are verified. Indeed, we identify $x = y = \max\big\{|x_1|, |x_2|\big\}$, for $(s, t) \in \mathbb{R}^2_{\geq 0}$, $\beta(s, t) = \tilde{\beta}(\theta^*, s, t)$, $\gamma^y(s) = \gamma^y(\theta^*, s)$, $\gamma^u(s) = \gamma^u(\theta^*, s)$, $c_y = \tilde{\sigma}^m(\theta^*, m)$, $\sigma^t(s) = 0$, $\sigma^x(s) = \tilde{\beta}(\theta^*, s, 0)$, $\sigma^y(s) = \gamma^y(\theta^*, s)$, $\sigma^u(s) = \gamma^u(\theta^*, s)$, $c_x = \tilde{\sigma}^m(\theta^*, m)$, $c = \max\big\{\tilde{\sigma}^m(\theta^*, m), m\big\}$, $\tilde{\gamma}^u(s) = \tilde{\gamma}^u(\theta^*, s)$. Inequalities (39) and (41) are satisfied according to (51), (40) in view of Step 2, (42) in view of (9), (43) in view of (8) since $\theta^* \leq \bar{\theta}$ and $\tilde{\sigma}^m, \nu^x, \nu^u$ are non-decreasing functions in θ. Consequently, in view of (45) and (46), by invoking Proposition 6, for $\eta \in (1, \infty)$ and with $\bar{\Delta} = \Delta$, there exists $\bar{\beta}_{\theta^*} \in \mathcal{KL}$ such that, for any $t \in [t_0, \infty)$:

$$\begin{aligned}
&\max\big\{|x_1(t)|, |x_2(t)|\big\} \leq \\
&\qquad \max\big\{\bar{\beta}_{\theta^*}(\max\big\{|x_1(t_0)|, |x_2(t_0)|\big\}, t - t_0), \\
&\qquad\qquad \tilde{\gamma}^u(\theta^*, \|u\|_{[t_0,t]}), \eta\tilde{\sigma}^m(\theta^*, m), \eta m\big\},
\end{aligned} \tag{58}$$

then, using the facts that $|(x_1, x_2)| \leq 2\max\{|x_1|, |y_2|\}$ and $\max\{|x_1|, |y_2|\} \leq |(x_1, x_2)|$, we have that:

$$\begin{aligned}
|(x_1(t), x_2(t))| \leq 2\max\big\{&\bar{\beta}_{\theta^*}(|(x_1(t_0), x_2(t_0))|, t - t_0), \\
&\tilde{\gamma}^u(\theta^*, \|u\|_{[t_0,t]}), \eta\tilde{\sigma}^m(\theta^*, m), \eta m\big\}.
\end{aligned} \tag{59}$$

The proof is completed by noting that $\delta^m(\theta^*, m) = 2\max\big\{\eta\tilde{\sigma}^m(\theta^*, m), \eta m\big\} \leq 2\max\big\{\eta\tilde{\sigma}^m(\bar{\theta}, m), \eta m\big\} \leq \varepsilon$ in view of (8) (since δ^m is non-decreasing in θ) and denoting $\beta_{\theta^*} = 2\bar{\beta}_{\theta^*}$ and $\sigma_{\theta^*} = 2\tilde{\gamma}^u(\theta^*, \cdot)$.

---

[1] If the states are affected by jumps, like in Sections III-B.2 and IV-B, our result applies as long as their norms do not increase at the jump instants.